

Multiple physical quantities sensors based on non-Hermitian topological systems with an impurity

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Spectra of non-Hermitian systems with skin effects are sensitive to the boundary conditions. In this work, we introduce a system composed of an impurity and a nonreciprocal Su-Schrieffer-Heeger (SSH) chain with odd number of lattice sites. The shift in the energy of the zero mode can be dramatically changed by adding two vanishingly small couplings between the impurity and two ends of the chain. Here, we propose a multiple physical quantities sensor with exponential-power-law amplification mechanism. Sensitivity of the proposed sensor exhibits an exponential scaling with SSH chain size and a power-law scaling with frequency of the impurity. This system is designed for detecting two small physical quantities, i.e., the impurity-chain coupling strengths. If and only if these two physical quantities exist simultaneously, the energy of zero mode will obviously change. Moreover, sensitivity is also robust against on-site disorder or dissipation of the impurity. Sufficiently strong disorder will change the localization of the eigenstate, and this leads to device failure. We also show the parameter range of the proposed sensor, and this range accurately corresponds to a nontrivial spectral winding number. The single physical quantity sensors suffered from repeated construction and low utilization rate. Thus, our findings pave a basic way toward sensors that couple multiple physical quantities to the boundary of the non-Hermitian topological chain.

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I. INTRODUCTION

In recent years, the development of sensors has made great progress, with research mainly focused on developing highly sensitive detection schemes. The basic physical mechanism of several types of sensors may be understood as a dramatical energy shift in response to a small external perturbation [1]. Non-Hermitian systems exhibit markedly different phenomena than their conventional Hermitian counterparts, such as exceptional points [2–4], where both eigenvalues and eigenstates coalesce. Exceptional points have also been suggested to be used for enhanced sensing [5–8], because a small perturbation can lift the degeneracy and lead to a visible splitting of the energy level. However, this type of sensor always requires precise tuning of multiple real parameters (e.g., gain-loss balance, geometric parameters, resonance frequencies, coupling phases, etc.), and the presence of disorders can deteriorate the sensor performance. So, it is necessary to design different types of sensors with high sensitivities and strong robustness.

In the last few years, the attention in non-Hermitian physics has been further enhanced due to its interesting topological properties [9–43]. The paradigmatic example is dubbed the

“non-Hermitian skin effect” [16]; i.e., all eigenstates of an open chain are localized near the boundary. A notable feature of the system with non-Hermitian skin effects is the sensitivity of their spectra to the boundary conditions. This feature opens up potential avenues for sensor applications [31–35]. Compared with using exceptional points to enhance sensing, this type of sensor has unique advantages. Boundary sensitivity of systems that grows exponentially with the size of the device can be regarded as a stable phenomenon independent of fine tuning [31]. Besides, this type of system also has a certain degree of robustness against disorder. Recently, it has been realized in photonic systems [34] and topological electrical circuits [35].

The impact of single local impurity on physical properties has also been the subject of much current interest in non-Hermitian topological areas [36–44]. In Ref. [36], it was first found that nonreciprocal impurities in a non-Hermitian chain can induce scale-free localized states. Besides, on-site impurities in non-Hermitian topological chains were discussed in Ref. [39]. It was shown that the impurity can act as an effective boundary condition for the system. On the other hand, superconducting qubits can be nonlocally coupled to a topological waveguide at multiple points and can act as multiple effective boundaries for the system [45–47]. The superconducting qubits described in these systems utilize only the two-level nature of the effective impurity. Hence, a new type of impurity was introduced into the non-Hermitian topological system and called the coupling impurity [48]. So, this begs the question that how spectra will change if an impurity couples to the two ends of the non-Hermitian topological chain. Furthermore, it

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is interesting to ask what role does couplings or impurities play in sensing.

Here, we develop a theory of non-Hermitian topological sensors with coupling impurities, and start from solving a one-dimensional system composed of a two-level system as an impurity and a nonreciprocal Su-Schrieffer-Heeger (SSH) chain. Interestingly, energy shift of the zero mode can be extraordinarily changed by adding two vanishingly small impurity-chain couplings, and sensitivity of the proposed sensor follows an exponential-power-law amplification mechanism. Besides, we show that this sensor can be used to detect two small coupling strengths g_1, g_2 together. Moreover, we find that sensitivity is also robust against on-site disorder or dissipation of the impurity. We also give the viable parameter range of the proposed non-Hermitian topological sensor with an impurity in the end.

The rest of the paper is organized as follows. In Sec. II, we present our model and derive the effective Hamiltonian. In Sec. III, we work out analytical result of the energy shift of zero modes by using non-Hermitian perturbation theory based on biorthogonal quantum mechanics. We also plot the numerical spectra to support our view. In Sec. IV, we study the robustness of the proposed sensor to the on-site disorder in the SSH chain and dissipation of impurity. In Sec. V, we explore the relation between the parameter range of sensors and the spectral winding number. Finally, we summarize our results in Sec. VI.

II. SYSTEM AND SPECTRA

As shown in Fig. 1(a), we start by considering a nonreciprocal SSH chain with open boundary conditions in real space, for which the Hamiltonian reads

$$H_{\text{SSH}} = \sum_{l=1}^{L-1} [(t_1 + \gamma_1) \hat{C}_{A,l}^\dagger \hat{C}_{B,l} + (t_1 - \gamma_1) \hat{C}_{B,l}^\dagger \hat{C}_{A,l} + (t_2 - \gamma_2) \hat{C}_{A,l+1}^\dagger \hat{C}_{B,l} + (t_2 + \gamma_2) \hat{C}_{B,l}^\dagger \hat{C}_{A,l+1}], \quad (1)$$

where the chain consists of odd number of lattice sites $2L - 1$ with an A site at both ends. The intracell hopping amplitudes are $t_1 \pm \gamma_1$ and intercell hopping amplitudes are $t_2 \pm \gamma_2$. The non-Hermiticity of the system is induced by the asymmetric hopping amplitudes ($\gamma_1 \neq 0$ or $\gamma_2 \neq 0$). $\hat{C}_{A(B),l}$ and $\hat{C}_{A(B),l}^\dagger$ denote the annihilation and creation operators for the sublattice site A(B) at unit cell l .

We next introduce a two-level impurity with frequency ω_0 coupled to the first and last sites of a nonreciprocal SSH chain with $2L - 1$ lattice sites. The total Hamiltonian for such a system is described by

$$H = H_{\text{SSH}} + \omega_0 \sigma_+ \sigma_- + H_I, \quad (2)$$

with

$$H_I = \sigma_+ (g_1 \hat{C}_{A,1} + g_2 \hat{C}_{A,L}) + \text{H.c.}, \quad (3)$$

where $\sigma_+ = \sigma_-^\dagger = |e\rangle\langle g|$ are the usual pseudospin ladder operators of the impurity, and $|g\rangle$ and $|e\rangle$ are the ground state and the excited state of the impurity, respectively. g_1 (g_2) denotes

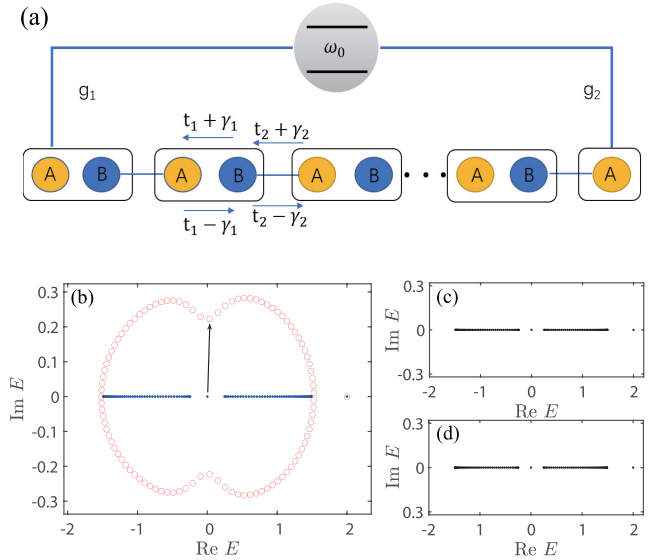


FIG. 1. (a) Sketch of the system. An impurity couples to the first and last sites of a nonreciprocal SSH chain with $2L - 1$ lattice sites. (b) Spectra of the system in complex plane. Blue filled points correspond to $g_1 = g_2 = 0$, and red open circles correspond to $g_1 = g_2 = 10^{-9}$. Black arrow marks the energy shift of zero mode. (c) Spectra with $g_1 = 10^{-9}$, $g_2 = 0$. (d) Spectra with $g_1 = 0$, $g_2 = 10^{-9}$. The results are obtained by numerically solving the time-independent Schrödinger equation in panels (b)–(d). The other parameters are $\omega_0 = 2$, $t_1 = 0.8$, $t_2 = 1$, $\gamma_1 = \gamma_2 = 0.5$, and $L = 50$.

the coupling strength of the impurity and the first (last) site of SSH chain.

Then, we study the spectral structure of the system. In the single-excitation subspace, by numerically solving time-independent Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$, the spectra of the system in complex plane are shown in Figs. 1(b)–1(d). We first investigate the decoupling limit by setting $g_1 = g_2 = 0$ in Eq. (3), and the corresponding spectra (blue filled points) are shown in Fig. 1(b). In this case, impurity and SSH chain are separable. The main feature is that there is an isolated energy level $E = 2t_2$, i.e., frequency of the impurity ω_0 . Besides, SSH chain has a zero mode under open boundary conditions with odd number of lattice sites.

Having understood the system without couplings, we next add two vanishingly small impurity-chain couplings, $g_1 = g_2 = 10^{-9}$, to show spectral features (red open circles) of the system [Fig. 1(b)]. Note that vanishingly small couplings hardly change the eigenvalues with $E = \omega_0 = 2$, but dramatically change the other eigenvalues. This implies that this high sensitivity of energy to couplings could, for instance, find application for sensors. Besides, we show spectra with coupling strengths $g_1 = 10^{-9}$, $g_2 = 0$ in Fig. 1(c), and spectra with coupling strengths $g_1 = 0$, $g_2 = 10^{-9}$ in Fig. 1(d). It means that a single coupling does not obviously change the spectra of the system.

These features illustrate that the proposed sensor scheme depends on the unique properties of non-Hermitian systems. In the next section, we will mainly consider energy shift of the zero mode [black arrow in Fig. 1(b)] and derive interesting amplification mechanisms for energy shift in theory.

III. EXPONENTIAL-POWER-LAW AMPLIFICATION MECHANISM

To discuss this physical mechanism, we first recall basic elements of biorthogonal quantum mechanics [49]. In general, non-Hermitian Hamiltonian $\mathcal{H}$ has inequivalent right and left eigenstates $|u_{R,n}\rangle$ and $\langle u_{L,n}|$, respectively, such that its eigenvalue equations read

$$\begin{aligned}\mathcal{H}|u_{R,n}\rangle &= \mathcal{E}_n|u_{R,n}\rangle, \\ \langle u_{L,n}|\mathcal{H} &= \langle u_{L,n}|\mathcal{E}_n.\end{aligned}\quad (4)$$

The right and left eigenstates generally do not form an orthonormal relation with the standard inner product, but they satisfy the biorthogonal basis by demanding that

$$\langle u_{L,m}|u_{R,n}\rangle = \delta_{nm}, \quad (5)$$

$$H = \begin{pmatrix} 0 & t_1 + \gamma_1 & 0 & 0 & 0 & \cdots & g_1 \\ t_1 - \gamma_1 & 0 & t_2 + \gamma_2 & 0 & 0 & \cdots & 0 \\ 0 & t_2 - \gamma_2 & 0 & t_1 + \gamma_1 & 0 & \cdots & 0 \\ 0 & 0 & t_1 - \gamma_1 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & t_2 + \gamma_2 & 0 \\ \vdots & \vdots & \vdots & \ddots & t_2 - \gamma_2 & 0 & g_2 \\ g_1 & 0 & 0 & \cdots & 0 & g_2 & \omega_0 \end{pmatrix}. \quad (8)$$

When $g_1 = g_2 = 0$, Hamiltonian H of the system reduces to Hamiltonian H_0 with no couplings between SSH chain and impurity. We assume that the frequency of the impurity ω_0 is not zero. Apparently, the eigenstates of H_0 can be obtained by solving eigenequation of Hamiltonian H_{SSH} [Eq. (1)] for a nonreciprocal SSH chain with odd number of lattice sites $2L - 1$. It is to be expected that there is always a zero mode for H_{SSH} , whose right or left eigenstates have an exactly disappearing amplitude on all B sites, such that $|\psi_{R,0}\rangle \equiv \mathcal{N}_{\mathcal{R}}(1 \ 0 \ \mathcal{R} \ 0 \ \mathcal{R}^2 \ 0 \ \mathcal{R}^3 \ \cdots \ 0 \ \mathcal{R}^{L-1})^T$, $\langle\psi_{L,0}| \equiv \mathcal{N}_{\mathcal{L}}(\mathcal{L}^{1-L} \ 0 \ \mathcal{L}^{2-L} \ 0 \ \mathcal{L}^{3-L} \ 0 \ \mathcal{L}^{4-L} \ \cdots \ 0 \ 1)$ [14]. Since the system is decoupled with $g_1 = g_2 = 0$, the exact right and left eigenstates of the total Hamiltonian H_0 with eigenvalue $E = 0$ can be obtained by adding a zero at the end of $|\psi_{R,0}\rangle$ and $\langle\psi_{L,0}|$ as follows:

$$\begin{aligned}|\Psi_{R,0}^{(0)}\rangle &\equiv \mathcal{N}_{\mathcal{R}}(1 \ 0 \ \mathcal{R} \ 0 \ \mathcal{R}^2 \ 0 \ \mathcal{R}^3 \ \cdots \ 0 \ \mathcal{R}^{L-1} \ 0)^T, \\ \langle\Psi_{L,0}^{(0)}| &\equiv \mathcal{N}_{\mathcal{L}}(\mathcal{L}^{1-L} \ 0 \ \mathcal{L}^{2-L} \ 0 \ \mathcal{L}^{3-L} \ 0 \ \mathcal{L}^{4-L} \ \cdots \ 0 \ 1 \ 0),\end{aligned}\quad (9)$$

where $\mathcal{R} = -(t_1 - \gamma_1)/(t_2 + \gamma_2)$ and $\mathcal{L} = -(t_1 + \gamma_1)/(t_2 - \gamma_2)$. $u = \mathcal{R}, \mathcal{L}$ denote right and left eigenstates localized around lattice site $l = 1$ if $|u| < 1$, and around $l = 2L - 1$ if $|u| > 1$. It can be seen that the amplitude of the zero mode vanishes on B sites and impurity site. With Eq. (5), the normalization coefficient $\mathcal{N}_{\mathcal{L}}\mathcal{N}_{\mathcal{R}}$ is expressed as

$$\begin{aligned}\mathcal{N}_{\mathcal{L}}\mathcal{N}_{\mathcal{R}} &= \langle\Psi_{L,0}^{(0)}|\Psi_{R,0}^{(0)}\rangle \frac{(\mathcal{L}\mathcal{R} - 1)\mathcal{L}^{L-1}}{(\mathcal{L}\mathcal{R})^L - 1} \\ &= \frac{(\mathcal{L}\mathcal{R} - 1)\mathcal{L}^{L-1}}{(\mathcal{L}\mathcal{R})^L - 1}.\end{aligned}\quad (10)$$

Then, it is straightforward to calculate the energy shift of zero modes by using non-Hermitian perturbation theory [50]. The perturbed Hamiltonian is described by

$$\begin{aligned}\Delta H &= g_2(|2L\rangle\langle 2L - 1| + |2L - 1\rangle\langle 2L|) \\ &+ g_1(|2L\rangle\langle 1| + |1\rangle\langle 2L|).\end{aligned}\quad (11)$$

and the completeness relation

$$\sum_n |u_{R,n}\rangle\langle u_{L,n}| = \sum_n |u_{L,n}\rangle\langle u_{R,n}| = 1. \quad (6)$$

According to Eq. (4), an immediate and important consequence is that the eigenvalues of a non-Hermitian Hamiltonian are given by its expectation value with respect to the right and left eigenstates as follows:

$$\mathcal{E}_n = \langle u_{L,n}|\mathcal{H}|u_{R,n}\rangle. \quad (7)$$

Equation (7) is known as the biorthogonal expectation value and plays an important role in non-Hermitian physics.

In the following, we discuss how the biorthogonal quantum mechanics can be used to calculate the energy shift of the zero mode for the non-Hermitian system. Consider system (2) with $2L - 1$ lattice sites and one site for impurity, for which the Hamiltonian in the single-excitation subspace expanded in the local site basis satisfies

Using second-order perturbation theory in g , we obtain (see Appendix A for details)

$$\begin{aligned}\Delta E_0 &\approx \langle\Psi_{L,n}^{(0)}|\Delta H|\Psi_{R,0}^{(0)}\rangle \\ &+ \sum_{j \neq 0} \frac{\langle\Psi_{L,j}^{(0)}|\Delta H|\Psi_{R,0}^{(0)}\rangle\langle\Psi_{L,0}^{(0)}|\Delta H|\Psi_{R,j}^{(0)}\rangle}{E_0^{(0)} - E_j^{(0)}}.\end{aligned}\quad (12)$$

$|\Psi_{R,j}^{(0)}\rangle$ and $\langle\Psi_{L,j}^{(0)}|$ denote the other right and left eigenstates of H_0 , except for the zero mode. Simple algebra with Eqs. (9) and (11) shows that the first-order energy shift is zero and independent of the coupling strengths g_1 and g_2 . Besides, we find that only one eigenstate in $|\Psi_{R,j}^{(0)}\rangle$ makes a contribution to the solution of second-order energy shift described by

$$|\Psi_{R,j=2L}^{(0)}\rangle \equiv (0 \ 0 \ 0 \ 0 \ \cdots \ 0 \ 0 \ 1)^T, \quad (13)$$

with eigenvalues $E_{2L}^{(0)} = \omega_0$. The other eigenstates of Hamiltonian H_0 have no contribution to the energy shift of zero modes (see Appendix B for details), and can be obtained by

calculating bulk states $|\psi\rangle$ of Hamiltonian H_{SSH} with odd number of lattice sites $2L - 1$ using nonBloch band theory [15,16].

The eigenstates of Hamiltonian H_0 can be written as $|\Psi_{j \neq 2L}^{(0)}\rangle = |\psi\rangle|g\rangle$, where $|\psi\rangle = \sum_i |\phi^{(i)}\rangle$ with each $|\phi^{(i)}\rangle$ taking the exponential form: $(\phi_{A,l}^{(i)}, \phi_{B,l}^{(i)}) = \beta_l^i (\phi_A^{(i)}, \phi_B^{(i)})$. For a two-band system, there are only two $\beta_{i=1,2}$ corresponding to the same eigenvalues, and they satisfy

$$\beta_1^L = \beta_2^L \quad (14)$$

for the SSH model with odd number of lattice sites. We find that Eq. (14) is independent of the length of the chain (see Appendix B for details), and this is a crucial result. It is clearly shown that these eigenstates are exponentially localized at the boundary whenever $|\beta_{1,2}| \neq 0$ even for a short chain. However, the system with even number of lattice sites in previous studies [16] requires more strict conditions (a long chain limit) to obtain this result as Eq. (14). According to our subsequent research, exponential localization of the eigenstate will determine the sensitivity of the sensor. Hence, odd number of lattice sites are more appropriate for the proposed sensor (see Appendix C for details). Besides, we generally consider the system with odd number of lattice sites $2L - 1$ that stabilize a single zero mode to avoid coalescence [31]. However, for the system with even number of lattice sites, a pair of zero modes leads to deviations from the proposed exponential-power-law amplification mechanism of sensors.

In brief, substituting Eq. (13) into Eq. (12), we obtain the energy shift of the zero mode:

$$\Delta E_0 \approx \frac{(1 - \mathcal{LR})(g_1 + g_2 \mathcal{R}^{L-1})(g_1 + g_2 \mathcal{L}^{L-1})}{[(\mathcal{LR})^L - 1]\omega_0}, \quad (15)$$

which is the key result of this work. Specifically, for the large L limit, Eq. (15) reduces to

$$\begin{aligned} \Delta E_0 &\approx \frac{\mathcal{R}^L + \mathcal{L}^L}{\omega_0} g_1 g_2 \\ &\approx \frac{\max[\mathcal{R}, \mathcal{L}]^L}{\omega_0} g_1 g_2 \end{aligned} \quad (16)$$

for $|\mathcal{LR}| < 1$ and

$$\begin{aligned} \Delta E_0 &\approx \frac{\mathcal{R}^{-L} + \mathcal{L}^{-L}}{\omega_0} g_1 g_2 \\ &\approx \frac{\min[\mathcal{R}, \mathcal{L}]^{-L}}{\omega_0} g_1 g_2 \end{aligned} \quad (17)$$

for $|\mathcal{LR}| > 1$, where $\max[\mathcal{R}, \mathcal{L}]$ and $\min[\mathcal{R}, \mathcal{L}]$ select the maximum and minimum values as sorted by the absolute value.

In fact, this system is designed for detecting two small physical quantities g_1, g_2 together. If and only if these two physical quantities exist simultaneously, the energy of zero mode is going to change dramatically. As long as one of these is zero, such as $g_1 = 0$ or $g_2 = 0$, the energy of zero mode will not change obviously, because the impurity coupling to only one end of SSH chain does not significantly change the boundary conditions of the system from open to closed ones. Besides, these two couplings (g_1, g_2) between the impurity

and the SSH chain have played an important role in the proposed sensor. According to Eqs. (16) and (17), and as shown in Figs. 1(c) and 1(d), it is clearly seen that the single coupling scenario cannot be realized by sensing.

Of course, it is possible to detect one of the coupling strengths separately. Setting one of two coupling strengths to a finite constant quantity, another small coupling strength can be detected. However, the single physical quantity sensors suffered from a redundant sensing system, repeated construction, and low utilization rate. Thus, our findings pave a basic way toward sensors that couple multiple physical quantities to the boundary of systems. We also show a design scheme for general systems that couples in series two or more impurities to the first and last sites of a nonreciprocal SSH chain in Appendix D.

For a finite small product of two coupling strengths g_1, g_2 , the energy E_0 of zero mode is then found to exhibit an exponential-power-law sensitivity S :

$$S = \frac{\partial E_0}{\partial g_1 g_2} \approx \frac{e^{\alpha L}}{\omega_0}, \quad (18)$$

where $\alpha = \log(\max[\mathcal{R}, \mathcal{L}])$ when $|\mathcal{LR}| < 1$ and $\alpha = -\log(\min[\mathcal{R}, \mathcal{L}])$ when $|\mathcal{LR}| > 1$. Note that only when $|\mathcal{L}| > 1$ and $|\mathcal{R}| < 1$ (or $|\mathcal{L}| < 1$ and $|\mathcal{R}| > 1$), the crucial parameter $\alpha > 0$. When $|\mathcal{L}| > 1$ and $|\mathcal{R}| > 1$ (or $|\mathcal{L}| < 1$ and $|\mathcal{R}| < 1$), the parameter $\alpha < 0$. This result implies that ΔE_0 follows an exponential scaling with SSH chain size $2L - 1$ and a power-law scaling with impurity frequency ω_0 amplification mechanism when $|\mathcal{L}| > 1$ and $|\mathcal{R}| < 1$ (or $|\mathcal{L}| < 1$ and $|\mathcal{R}| > 1$). According to Eq. (9), one finds that this parameter range is tied to a localization effect; i.e., the right and left eigenstates of zero mode localize at different ends of the chain. In a word, the shift in the energy of the zero mode ΔE_0 can be dramatically changed by adding two vanishingly small couplings under a long chain and a small frequency of the impurity conditions.

In Fig. 2, we show the shift in the energy of the zero mode $|\Delta E_0|$ due to addition of two same impurity-chain couplings $g_1 = g_2 = g$ with various values of $2L - 1$. The other parameters are set as $\omega_0 = 2, t_1 = 0.8, t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$. It is clear that the analytical results (empty circles) given by Eq. (15) are in good agreement with the numerical results (solid lines) in a wide range of g . For a fixed parameter $\omega_0 = 2$, numerical energy shift $|\Delta E_0|$ exhibits an exponential scaling with system size $2L - 1$ until it approaches the continual band. For example, setting $g = 10^{-9}$, we show that the zero mode merges into the continual band with $2L - 1 = 99$ in Fig. 1(b), so in this case the analytical results are not in agreement with the numerical results in Fig. 2. In the end, when reaching the scale set by the bulk gap, all solid lines stop growing at basically similar $|\Delta E_0|$ independent of the coupling strength g , as shown in Fig. 2 [31].

In Fig. 3, we show the energy shift $|\Delta E_0|$ as a function of $2L - 1$ with different parameters ω_0 . The other parameters are chosen as $g_1 = g_2 = 10^{-9}, t_1 = 0.8, t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$. Numerical energy shift $|\Delta E_0|$ exhibits an exponential scaling with $2L - 1$, and a power-law scaling with ω_0 until the energy shift of zero mode particularly approaches the impurity frequency ω_0 . In this case, all numerical results (solid lines)

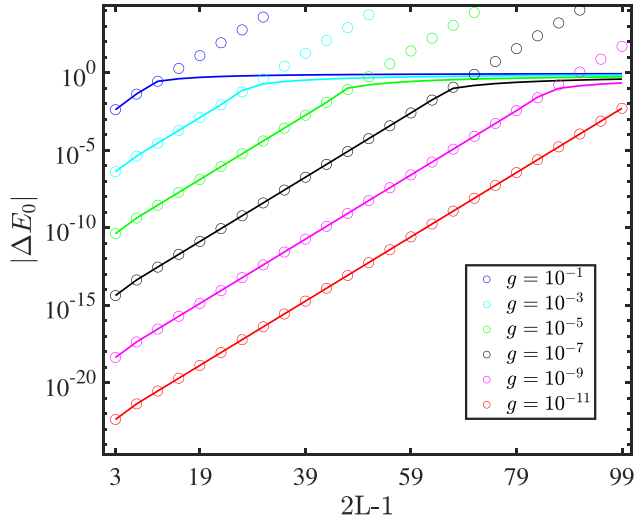


FIG. 2. Energy shift $|\Delta E_0|$ follows exponential scaling with chain size $2L - 1$ for system (2) with different values of $g_1 = g_2 = g$. Solid lines are exact numerical eigenvalues, which are in striking agreement with the analytical results given by Eq. (15) (empty circles). The parameters are set as $\omega_0 = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$.

deviate from the perturbative analytical results (empty circles) given by Eq. (15), because energy degeneracy undermines the applicability of the nondegenerate perturbation theory. Then, numerical energy shift exhibits a smaller exponential scaling with $2L - 1$ than the nondegenerate case.

In short, lower impurity frequency and longer chain length greatly improve sensor sensitivity. The degeneracy of energy gives rise to the existence of the inflection point.

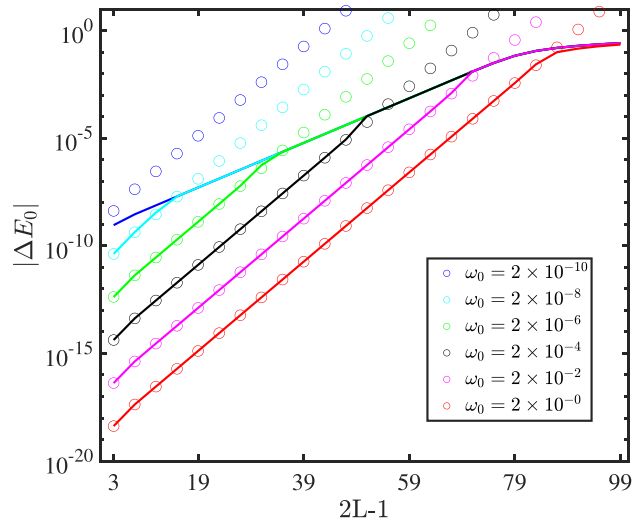


FIG. 3. Energy shift $|\Delta E_0|$ follows an exponential scaling with system size $2L - 1$ and a power-law scaling with impurity frequency ω_0 for system (2). Solid lines are exact numerical eigenvalues, and empty circles represent the analytical results given by Eq. (15). The parameters are chosen as $g_1 = g_2 = 10^{-9}$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$.

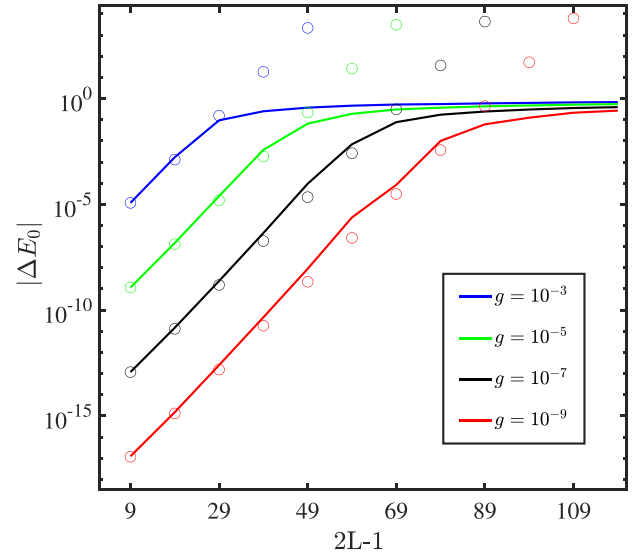


FIG. 4. Mean energy shift $|\Delta E_0|$ for the system in the presence of normal on-site disorder with different values of $g_1 = g_2 = g$. Solid lines are the results averaged for 50 disorder realizations. In contrast, empty circles represent the analytical results without disorder given by Eq. (15). The parameters are chosen as $S = 0.2$, $\omega_0 = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$.

IV. ROBUSTNESS OF THE SYSTEM

We further study the robustness of the proposed sensor to the on-site disorder in the SSH chain and dissipation of impurity. To this end, the Hamiltonian in Eq. (2) of the proposed sensor can be rewritten as

$$H' = H'_{\text{SSH}} + (\omega_0 - i\Gamma)\sigma_+\sigma_- + H_I, \quad (19)$$

with

$$H_I = \sigma_+(g_1\hat{C}_{A,1} + g_2\hat{C}_{A,L}) + \text{H.c.} \quad (20)$$

and

$$\begin{aligned} H'_{\text{SSH}} = & \sum_{l=1}^{L-1} [(t_1 + \gamma_1)\hat{C}_{A,l}^\dagger\hat{C}_{B,l} + (t_1 - \gamma_1)\hat{C}_{B,l}^\dagger\hat{C}_{A,l} \\ & + (t_2 - \gamma_2)\hat{C}_{A,l+1}^\dagger\hat{C}_{B,l} + (t_2 + \gamma_2)\hat{C}_{B,l}^\dagger\hat{C}_{A,l+1}] \\ & + \sum_{l=1}^L [V_l\hat{C}_{A,l}^\dagger\hat{C}_{A,l} + V_l\hat{C}_{B,l}^\dagger\hat{C}_{B,l}], \end{aligned} \quad (21)$$

where Γ denotes the decay rate of the impurity and V_l denotes an on-site disorder potential. Here, we set $V_l = SR_l$, where S is the disorder strength and R_l is a normal random number.

We first consider the robustness of the proposed sensor to the on-site disorder in the SSH chain. In Fig. 4, we show mean energy shift $|\Delta E_0|$ as a function of lattice size $2L - 1$ in the presence of on-site disorder with different values of $g_1 = g_2 = g$. Solid lines are the results averaged for 50 disorder realizations. In contrast, the empty circle represents the analytical result without disorder. Disorder satisfies a normal distribution with the disorder strength $S = 0.2$. The other parameters are the same as in Fig. 2. Note that for a wide range of coupling strengths g , the energy shift of zero modes always exhibits a nearly exponential growth with fluctuations in the

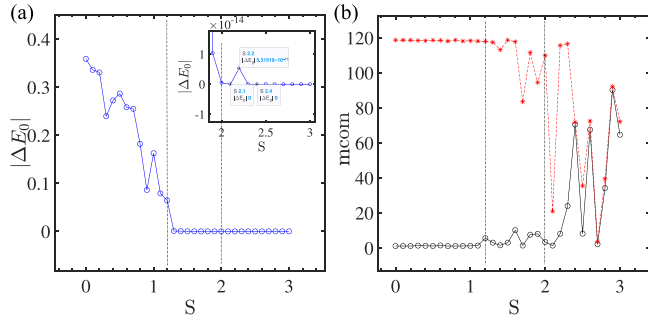


FIG. 5. (a) Energy shift $|\Delta E_0|$ for the system as a function of disorder strengths S with lattice size $2L - 1 = 119$, and an enlarged zoom in shown in inset (a). (b) mcom as a function of disorder strengths S with lattice size $2L - 1 = 119$. The black circle represents the mcom for the right eigenstate, and the red star represents the mcom for the left eigenstate. The parameters shared by the two panels are $g_1 = g_2 = 10^{-9}$, $\omega_0 = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$.

presence of disorder. It is clearly seen that obvious fluctuations do not significantly reduce the sensitivity of the sensor, because the topological protection makes the zero mode more robust to the disorder. So, the energy shift of the zero mode can also be dramatically changed by adding two vanishingly small couplings in a wide range of parameters g . It means that our proposed sensor has not only high sensitivities but also strong robustness.

Of course, a sufficiently strong disorder would be devastating to the system. Next, we show the energy shift $|\Delta E_0|$ for the system as a function of disorder strengths S with lattice size $2L - 1 = 119$ in Fig. 5(a). The other parameters are $g_1 = g_2 = 10^{-9}$, $\omega = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$. It is clearly shown that energy shift $|\Delta E_0|$ gradually decreases or even becomes zero as the disorder strength S increases. This is a foregone conclusion, because on-site disorder will play a major role in the system and break original topological structure (localization of the left and right eigenstates) as S increases. We have also shown that the localization of right and left eigenstates of zero mode localiz at different ends of the chain is the reason for the exponential growth of the energy shift in the previous section. So what happens to the localization of the eigenstates in the disorder?

The localization behavior of the eigenstate can be easily quantified by the mean center of mass (mcom) of the amplitude squared of eigenstates $|\Psi_{R(L),0}\rangle$ as follows:

$$\text{mcom} = \frac{\sum_{\ell=1}^{N-1} \ell \mathcal{A}(\ell)}{\sum_{\ell=1}^{N-1} \mathcal{A}(\ell)}, \quad (22)$$

with

$$\mathcal{A}(\ell) = |\langle \ell | \Psi_{R(L),0} \rangle|^2. \quad (23)$$

Here, we are only interested in the wave functions living at the SSH chain, so population on the impurity is not considered.

As shown in Fig. 5(b), we plot mcom as a function of disorder strength S with lattice size $2L - 1 = 119$. The other parameters are the same as those in Fig. 5(a). The black circle represents the mcom for the right eigenstate, and the red star represents the mcom for the left eigenstate. From this plot,

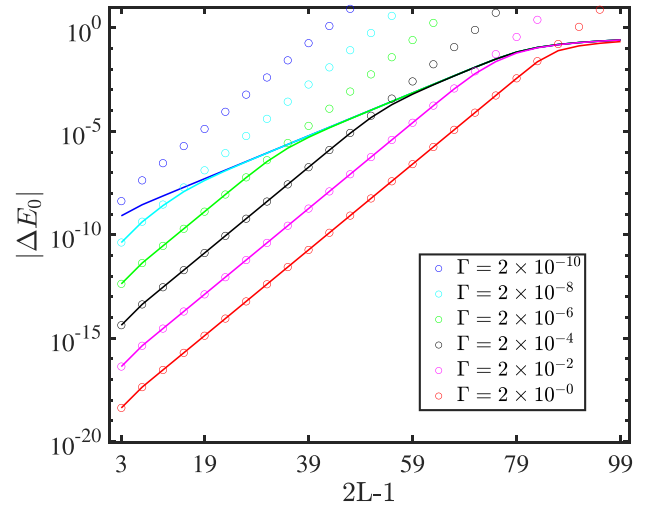


FIG. 6. Energy shift $|\Delta E_0|$ as a function of the system size $2L - 1$ with different decay rates of impurity Γ . Solid lines are exact numerical eigenvalues, and empty circles represent the analytical results given by Eq. (15). The parameters are chosen as $g_1 = g_2 = 10^{-9}$, $t_1 = 0.8$, $t_2 = 1$, $\gamma_1 = \gamma_2 = 0.5$, and $\omega_0 = 0$.

one can clearly see that in the limit of small disorder strength S ($0 < S < 1.2$), right eigenstate $|\Psi_{R,0}\rangle$ localizes near the first site of SSH chain, and left eigenstate $|\Psi_{L,0}\rangle$ localizes near the last site of SSH chain. Hence, energy shift $|\Delta E_0|$ is still obvious as shown in Fig. 5(a), and there is little negative effect on the sensitivity of the proposed sensor within this range of disorder strength. Then, making the disorder strength S larger ($1.2 < S < 2$) will overtly change the localization of the right and left eigenstates away from the two boundary sites of the chain. However, it is also roughly distributed to the left and right of the center in the chain. Hence, energy shift $|\Delta E_0|$ might be a very small value, as shown in Fig. 5(a). In the end, the system exhibits a different localization behavior in the limit of large disorder strength S ($S > 2$). For example, when $S = 2.4$, mcom of the left and right eigenstates almost coincide, which means that localization of the two eigenstates may be similar. On-site potential disorder will always dominate, leading eigenstates to localize on the basis of Anderson localization [51]. Hence, energy shift $|\Delta E_0|$ is zero when $S = 2.4$, as shown in Fig. 5(a).

In this sense, these results indicate that the sensitivity of sensors averaging many disorder realizations for different g follows similar rules to those without disorder. It is emphasized that the strong disorder changes the localization of the eigenstate, and this leads to device failure.

We further consider the robustness of the proposed sensor to the dissipation of impurity. As shown in Fig. 6, we plot energy shift $|\Delta E_0|$ as a function of the system size $2L - 1$ with different decay rates of impurity Γ . Solid lines are exact numerical eigenvalues, and empty circles represent the analytical results given by Eq. (15). It is clear that the analytical results are in good agreement with the numerical results. In the limit of short size of SSH chain, energy shift $|\Delta E_0|$ follows a power-law scaling with the decay rate of impurity Γ . This indicates that energy shift $|\Delta E_0|$ will decrease significantly as the decay rate of impurity increases. In this case, the negative

impact on the sensitivity of sensor is great. However, the good news is that this adverse factor can be almost completely ignored as the size of the chain increases. For example, when $\Gamma = 2$, energy shift $|\Delta E_0|$ can be significantly changed when the size of SSH chain $2L + 1 = 99$, and this is no less than other cases with lower decay rates. Hence, the system is robust to dissipation of impurity in the limit of long size of the system.

V. DESIRABLE PARAMETER RANGE AND SPECTRAL WINDING NUMBER

In the previous section, we have shown that energy shift ΔE_0 can be dramatically changed by adding vanishingly small impurity-chain couplings when $|\mathcal{L}| > 1$ and $|\mathcal{R}| < 1$ (or $|\mathcal{L}| < 1$ and $|\mathcal{R}| > 1$). Taking t_1, t_2 to be positive, this leads to the parameter γ_1 satisfying

$$\begin{aligned} \gamma_1 < \gamma_2 + t_1 + t_2 & \quad \gamma_1 < -\gamma_2 - t_1 + t_2 \\ \gamma_1 > -\gamma_2 - t_1 + t_2 & \quad \text{or} \quad \gamma_1 > -\gamma_2 + t_1 - t_2 \\ \gamma_1 > \gamma_2 - t_1 - t_2 & \quad \gamma_1 < \gamma_2 - t_1 - t_2 \end{aligned} \quad (24)$$

whenever $|\mathcal{L}| > 1$ and $|\mathcal{R}| < 1$, and

$$\begin{aligned} \gamma_1 < \gamma_2 + t_1 + t_2 & \quad \gamma_1 < -\gamma_2 - t_1 + t_2 \\ \gamma_1 < -\gamma_2 + t_1 - t_2 & \quad \text{or} \quad \gamma_1 > -\gamma_2 + t_1 - t_2 \\ \gamma_1 > \gamma_2 - t_1 - t_2 & \quad \gamma_1 > \gamma_2 + t_1 + t_2 \end{aligned} \quad (25)$$

whenever $|\mathcal{L}| < 1$ and $|\mathcal{R}| > 1$. This is the desirable parameter range of the proposed high-sensitivity sensor.

Interestingly, this desirable parameter range also corresponds to the parameter range with a nontrivial spectral winding number w [11]:

$$w \equiv \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \partial_k \ln \det H_{\text{SSH}}(k), \quad (26)$$

with

$$\begin{aligned} H_{\text{SSH}}(k) = & (t_1 + \gamma_1) \hat{C}_{A,k}^\dagger \hat{C}_{B,k} + (t_1 - \gamma_1) \hat{C}_{B,k}^\dagger \hat{C}_{A,k} \\ & + (t_2 - \gamma_2) e^{-ik} \hat{C}_{A,k}^\dagger \hat{C}_{B,k} + (t_2 + \gamma_2) e^{ik} \hat{C}_{B,k}^\dagger \hat{C}_{A,k}, \end{aligned} \quad (27)$$

where $H_{\text{SSH}}(k)$ represents the Hamiltonian of the SSH model in momentum spaces, and $\hat{C}_{A(B),k}^\dagger$ and $\hat{C}_{A(B),k}$ are the creation and annihilation operators, respectively. The nontrivial winding number gives the number of times the complex eigenvalues encircle the origin of the complex plane.

In Fig. 7, we show the phase diagram of the non-Hermitian SSH model as a function of γ_1 and γ_2 given by Eq. (26). According to Fig. 7, we can obtain the parameter range of a nontrivial spectral winding number:

$$w = \begin{cases} 1, & (|\mathcal{L}| > 1, |\mathcal{R}| < 1), \\ -1, & (|\mathcal{L}| < 1, |\mathcal{R}| > 1). \end{cases} \quad (28)$$

It is clear that the parameter range of a nontrivial spectral winding number given by Eq. (28) is in full agreement with the parameter range of the proposed sensors given by Eqs. (24) and (25).

Next, we will analyze this result in detail. The non-Hermitian topological invariant w characterizes the bulk band structure of the system. Hence, phase transition occurs at the gap-closing points. Bloch bands

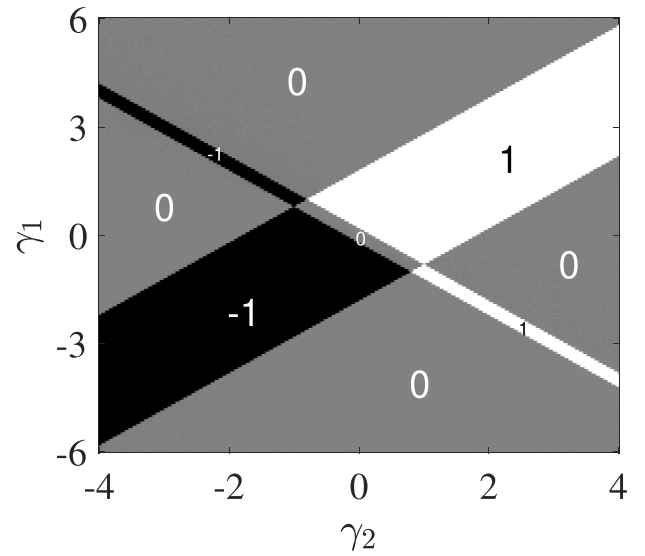


FIG. 7. Phase diagram of the non-Hermitian SSH model as a function of γ_1 and γ_2 given by Eq. (26). The colors from shallow to deep represent $w = 1, 0$, and -1 as marked. The parameters are chosen as $t_1 = 0.8$ and $t_2 = 1$.

of Hamiltonian $H(k)$ of the SSH model in momentum spaces given by Eq. (27) are expressed as $E_{\pm}(k) = \sqrt{[t_1 + \gamma_1 + (t_2 - \gamma_2)e^{-ik}][t_1 - \gamma_1 + (t_2 + \gamma_2)e^{-ik}]}$. The energy gap closes when $t_1 \pm \gamma_1 = -(t_2 \mp \gamma_2)[k = 0]$ or $t_1 \pm \gamma_1 = t_2 \mp \gamma_2[k = \pi]$. Due to this $H(k)$, gap-closing points are located where zero modes migrate from one end to the other with $|\mathcal{R}| = |(t_1 - \gamma_1)/(t_2 + \gamma_2)| = 1$ or $|\mathcal{L}| = |(t_1 + \gamma_1)/(t_2 - \gamma_2)| = 1$ [14,16]. Crucially, $|\mathcal{L}| < 1, |\mathcal{R}| > 1$ (or $|\mathcal{L}| > 1, |\mathcal{R}| < 1$) is the desirable parameter range of the proposed sensor [31]. Hence, the desirable parameter range of the sensor is precisely in agreement with the parameter range with a nontrivial spectral winding number w .

VI. DISCUSSION AND CONCLUSIONS

We have studied that an impurity couples to the first and last sites of a nonreciprocal SSH chain with odd number of lattice sites. We mainly find that two vanishingly small impurity-chain couplings can lead to a visible shift in the energy of the zero mode, but a single coupling does not obviously change the spectra. Such a setup is interesting and can be applied to detect the multiple impurity-chain coupling strengths together. The result indicates that energy shift follows an exponential scaling with SSH chain size and a power-law scaling with impurity frequency, and the corresponding analytical results are given out. This notable feature is tied to a localization effect; i.e., the right and left eigenstates of zero mode localize at different ends of the chain. We demonstrate the robustness of our sensors against perturbations. However, sufficiently strong disorder will change the localization of the eigenstate, and this leads to device failure. We also show the desirable parameter range of sensors in the end.

This system can be observed in various experimental settings, including photonic systems [34] and electrical circuits [35]. Our primary focus lies in electrical circuits. The

nonreciprocal coupling can be achieved by connecting a capacitor in series with a voltage follower [28]. Due to the virtual open and virtual short-circuit conditions between the inverting input and noninverting input pins, the current at one side of the capacitor is blocked, while it remains uninfluenced at the other side. In addition, the impurity can be thought of as a node in the circuit [35]. Couplings between the impurity and chain are achieved by connecting several different capacitors, which are much smaller than other capacitors used in the system. To be precise, multiple physical quantities refer to the couplings (g_1, g_2 , etc.). In the circuit system, the corresponding detected quantities are the capacitances. It is well established that variations in the distance between the two plates, the dielectric constant of the insulating medium, and the environmental temperature can all induce changes in capacitances. This implies that capacitances (or couplings g_1, g_2 , etc.) are, in fact, direct detected quantities. Quantities such as the dielectric and the distance between the plates are indirect detected quantities. Additionally, other indirect detected quantities with measurement values can also be developed. For this specific experimental scheme, energy gains need to be prevented. To achieve this, a constant imaginary shift is added to all sites, which corresponds to a passive configuration involving loss only [52]. Importantly, this correction does not affect the localization or existence of boundary modes.

In the optical experimental platform, this model can also be thought of as an array of coupled optical ring resonators [31,34]. The coupling between impurity and chain may be represented by any optically impenetrable region imposing a tunneling barrier. The variation of coupling strength then only needs to change the refractive index of the impenetrable medium [31].

In summary, our findings pave a basic way toward sensors that couple multiple physical quantities to the boundary of the non-Hermitian topological systems. Obviously, the sensor can also be implemented in a wider scenario.

ACKNOWLEDGMENT

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PERTURBATION CORRECTION TO THE EIGENVALUE

We now calculate perturbation correction to the eigenvalue. The non-Hermitian Hamiltonian H can be rewritten as

$$H = H_0 + \Delta H = H_0 + gH', \quad (\text{A1})$$

where H_0 is the non-Hermitian unperturbed Hamiltonian, whose left and right eigenstates are $\langle \Psi_{L,n}^{(0)} |$ and $|\Psi_{R,n}^{(0)}\rangle$, respectively. The total Hamiltonian H satisfies

$$H|\Psi_{R,n}\rangle = E_{R,n}|\Psi_n\rangle. \quad (\text{A2})$$

Expanding in powers of g and collecting the same power terms gives

$$(E_n^{(0)} - H_0)|\Psi_{R,n}^{(0)}\rangle = 0, \quad (\text{A3a})$$

$$(E_n^{(0)} - H_0)|\Psi_{R,n}^{(1)}\rangle + (E_n^{(1)} - H')|\Psi_{R,n}^{(0)}\rangle = 0, \quad (\text{A3b})$$

$$(E_n^{(0)} - H_0)|\Psi_{R,n}^{(2)}\rangle + (E_n^{(1)} - H')|\Psi_{R,n}^{(1)}\rangle + E_n^{(2)}|\Psi_{R,n}^{(0)}\rangle = 0, \quad (\text{A3c})$$

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Equation (A3a) is the unperturbed case. Equation (A3b) gives the first-order energy shift if we take the inner product with $\langle \Psi_{L,0}^{(0)} |$, so that

$$E_n^{(1)} = \langle \Psi_{L,n}^{(0)} | H' | \Psi_{R,n}^{(0)} \rangle. \quad (\text{A4})$$

Similarly, Eq. (A3c) gives the second-order energy shift if we take the inner product with $\langle \Psi_{L,j}^{(0)} |$, and $j \neq n$. We have

$$E_n^{(2)} = \sum_{j \neq n} \frac{\langle \Psi_{L,j}^{(0)} | H' | \Psi_{R,n}^{(0)} \rangle \langle \Psi_{L,n}^{(0)} | H' | \Psi_{R,j}^{(0)} \rangle}{E_n^{(0)} - E_j^{(0)}}. \quad (\text{A5})$$

APPENDIX B: EIGENSTATES $|\Psi^{(0)}\rangle$ OF UNPERTURBED HAMILTONIAN H_0

When $g = 0$, there is no coupling between the impurity and SSH chain. So, the right eigenstate of unperturbed Hamiltonian H_0 with $E = \omega_0$ in the local site basis obviously takes the form

$$|\Psi_{R,j=2L}^{(0)}\rangle \equiv (0 \quad 0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0 \quad 1)^T. \quad (\text{B1})$$

It is straightforward to calculate the energy shift of zero mode as Eq. (15) by using only this eigenstate with $E = \omega_0$.

Besides, the form of other bulk states of Hamiltonian H_0 can be obtained by calculating bulk states $|\psi\rangle$ of Hamiltonian H_{SSH} with odd number of lattice sites $2L - 1$. In the bulk of chain, it satisfies

$$\begin{aligned} (t_2 - \gamma_2)\psi_{B,l-1} + (t_1 + \gamma_1)\psi_{B,l} &= E\psi_{A,l}, \\ (t_1 - \gamma_1)\psi_{A,l} + (t_2 + \gamma_2)\psi_{A,l+1} &= E\psi_{B,l}. \end{aligned} \quad (\text{B2})$$

Setting $|\psi\rangle = \sum_i |\phi^{(i)}\rangle$ with each $|\phi^{(i)}\rangle$ taking the exponential form (omitting the i index temporarily): $(\phi_{A,l}, \phi_{B,l}) = \beta^l(\phi_A, \phi_B)$. Then, we have

$$\begin{aligned} (t_2 - \gamma_2)\beta^{-1}\phi_B + (t_1 + \gamma_1)\phi_B &= E\phi_A, \\ (t_1 - \gamma_1)\phi_A + (t_2 + \gamma_2)\beta\phi_A &= E\phi_B. \end{aligned} \quad (\text{B3})$$

Therefore, it yields

$$a\beta^2 + b\beta + c = 0, \quad (\text{B4})$$

where $a = (t_1 + \gamma_1)(t_2 + \gamma_2)$, $b = t_1^2 - \gamma_1^2 + t_2^2 - \gamma_2^2 - E^2$, and $c = (t_1 - \gamma_1)(t_2 - \gamma_2)$, and we can obtain

$$\beta_1\beta_2 = c/a = \frac{(t_1 - \gamma_1)(t_2 - \gamma_2)}{(t_1 + \gamma_1)(t_2 + \gamma_2)}. \quad (\text{B5})$$

Restoring the i index in $\phi^{(i)}$, the two equivalent equations are written as

$$\phi_A^{(i)} = \frac{E}{(t_1 - \gamma_1) + (t_2 + \gamma_2)\beta_i} \phi_B^{(i)}, \quad (\text{B6a})$$

$$\phi_B^{(i)} = \frac{E}{(t_1 + \gamma_1) + (t_2 - \gamma_2)\beta_i^{-1}} \phi_A^{(i)}. \quad (\text{B6b})$$

The general solution is written as a linear combination: $(\psi_{A,l}, \psi_{B,l}) = \beta_1^l(\phi_A^{(1)}, \phi_B^{(1)}) + \beta_2^l(\phi_A^{(2)}, \phi_B^{(2)})$. With the boundary conditions

$$\begin{aligned} (t_1 + \gamma_1)\psi_{B,1} - E\psi_{A,1} &= 0, \\ (t_2 - \gamma_2)\psi_{B,L-1} - E\psi_{A,L} &= 0, \end{aligned} \quad (\text{B7})$$

$$|\Psi_R^{(0)}\rangle = \sum_i (\beta_i \phi_A^{(i)} \quad \beta_i \phi_B^{(i)} \quad \beta_i^2 \phi_A^{(i)} \quad \beta_i^2 \phi_B^{(i)} \quad \dots \quad \beta_i^{L-1} \phi_A^{(i)} \quad \beta_i^{L-1} \phi_B^{(i)} \quad \beta_i^L \phi_A^{(i)} \quad 0)^T,$$

and the form of the corresponding left bulk state $|\Psi_L^{(0)}\rangle$ is similar. Simple algebra shows that

$$\frac{\langle \Psi_L^{(0)} | \Delta H | \Psi_R^{(0)} \rangle \langle \Psi_L^{(0)} | \Delta H | \Psi_R^{(0)} \rangle}{E_0^{(0)} - E} = 0. \quad (\text{B11})$$

Note that this result is independent of the parameters $\beta_i, \phi_{A(B)}^{(i)}$ of bulk states $|\Psi_R^{(0)}\rangle$. The above equation remains true as long as the amplitude at the last site of the bulk state $|\Psi_R^{(0)}\rangle$ is zero. Hence, all bulk states of Hamiltonian H_0 have no contribution to the energy shift of zero modes.

APPENDIX C: THE CASE WITH EVEN NUMBER OF LATTICE SITES

As depicted in Fig. 8, we consider an impurity coupled to the first and last sites of a nonreciprocal SSH chain with even number of lattice sites, whose Hamiltonian can be written as

$$H^E = H_{\text{SSH}}^E + \omega_0 \sigma_+ \sigma_- + H_I^E, \quad (\text{C1})$$

with

$$\begin{aligned} H_{\text{SSH}}^E &= \sum_{l=1}^L [(t_1 + \gamma_1) \hat{C}_{A,l}^\dagger \hat{C}_{B,l} + (t_1 - \gamma_1) \hat{C}_{B,l}^\dagger \hat{C}_{A,l}] \\ &+ \sum_{l=1}^{L-1} [(t_2 - \gamma_2) \hat{C}_{A,l+1}^\dagger \hat{C}_{B,l} + (t_2 + \gamma_2) \hat{C}_{B,l+1}^\dagger \hat{C}_{A,l+1}] \end{aligned} \quad (\text{C2})$$

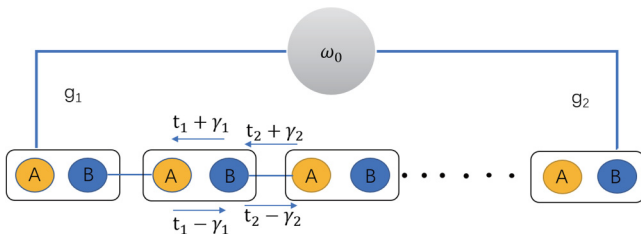


FIG. 8. Sketch of the system. An impurity coupled to the first and last sites of a nonreciprocal SSH chain with $2L$ lattice sites.

and making use of Eq. (B6b), the following results are obtained:

$$\begin{aligned} (t_2 - \gamma_2)\phi_A^{(1)} + (t_2 - \gamma_2)\phi_A^{(2)} &= 0, \\ (t_1 + \gamma_1)\beta_1^L \phi_A^{(1)} + (t_1 + \gamma_1)\beta_2^L \phi_A^{(2)} &= 0. \end{aligned} \quad (\text{B8})$$

Obviously, $\phi_A^{(1)} = -\phi_A^{(2)}$, and we have $\beta_1^L = \beta_2^L$. Combined with Eq. (B5), we have

$$|\beta_1| = |\beta_2| = \sqrt{\left| \frac{(t_1 - \gamma_1)(t_2 - \gamma_2)}{(t_1 + \gamma_1)(t_2 + \gamma_2)} \right|}, \quad (\text{B9})$$

which is independent of the length of the chain. In fact, one can numerically solve the eigenenergies of an open chain and obtain $\beta_i(E)$ from Eq. (B4).

So, any one of the right bulk states $|\Psi_R^{(0)}\rangle$ of H_0 with the eigenvalue E can be expressed in the following form:

$$\dots \quad \beta_i^{L-1} \phi_A^{(i)} \quad \beta_i^{L-1} \phi_B^{(i)} \quad \beta_i^L \phi_A^{(i)} \quad 0)^T, \quad (\text{B10})$$

and

$$H_I^E = g_1 \sigma_+ \hat{C}_{A,1} + g_2 \sigma_+ \hat{C}_{B,L} + \text{H.c.}, \quad (\text{C3})$$

where a nonreciprocal SSH chain with $2L$ lattice sites is described by H_{SSH}^E . H_I^E is the interaction Hamiltonian between the impurity and SSH chain.

In Fig. 9, we show numerical energy shift of the zero mode $|\Delta E_0|$ as a function of lattice sizes $2L$ with various values of $g_1 = g_2 = g$. The parameters are set as $\omega_0 = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$. It is clear that energy shift $|\Delta E_0|$ exhibits a roughly exponential scaling with system size as the length of the chain increases. However, in the limit of short size of SSH chain, $|\Delta E_0|$ may be zero. The reason is that eigenstates may not be exponentially localized at the boundary when $|\beta_{1,2}| \neq 0$ for a short chain. It is crucial that exponential localization of the eigenstate will determine the

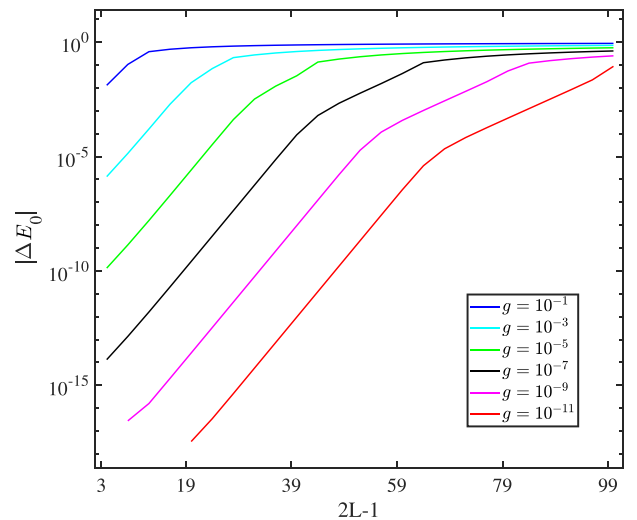


FIG. 9. Energy shift of the zero mode $|\Delta E_0|$ as a function of lattice sizes $2L$ with various values of $g_1 = g_2 = g$. The parameters are set as $\omega_0 = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$.

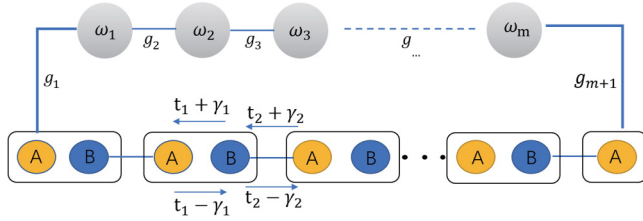


FIG. 10. Sketch of the system. Multiple impurities coupled in series to the first and last sites of a nonreciprocal SSH chain with $2L - 1$ lattice sites.

sensitivity of the sensor. Hence, even number of lattice sites are not fully appropriate for the proposed sensor.

APPENDIX D: THE CASE WITH MULTIPLE IMPURITIES

As depicted in Fig. 10, we consider multiple impurities coupled in series to the first and last sites of a nonreciprocal SSH chain with $2L - 1$ lattice sites, whose Hamiltonian can be described by

$$H^M = H_{\text{SSH}} + H_{\text{Imp}} + H_I^M, \quad (\text{D1})$$

with

$$H_{\text{Imp}} = \sum_{n=1}^m \omega_n \sigma_{n,+} \sigma_{n,-} + \sum_{n=2}^m [g_n \sigma_{n-1,+} \sigma_{n,-} + \text{H.c.}] \quad (\text{D2})$$

and

$$H_I^M = g_1 \sigma_{1,+} \hat{C}_{A,1} + g_{m+1} \sigma_{m,+} \hat{C}_{A,L} + \text{H.c.}, \quad (\text{D3})$$

where a nonreciprocal SSH chain with $2L - 1$ lattice sites is described by H_{SSH} . H_{Imp} denotes the Hamiltonian of multiple impurities coupled in series, and H_I^M is the interaction Hamiltonian between impurities and chains.

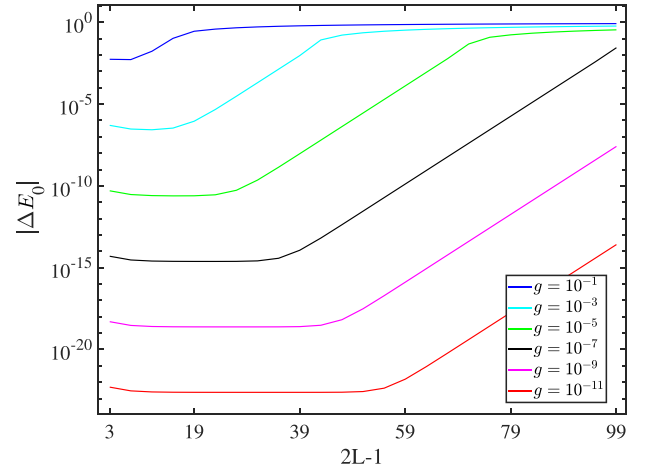


FIG. 11. Energy shift of the zero mode $|\Delta E_0|$ as a function of lattice sizes $2L - 1$ with various values of $g_1 = g_2 = g_3 = g$. The parameters are set as $\omega_{1,2} = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$.

Now, we consider a simple case with two impurities and three physical quantities g_1, g_2, g_3 to be simultaneously detected. In Fig. 11, we show numerical energy shift of the zero mode $|\Delta E_0|$ due to addition of three small physical quantities $g_1 = g_2 = g_3 = g$ with various values of $2L - 1$. The other parameters are set as $\omega_{1,2} = 2$, $t_1 = 0.8$, $t_2 = 1$, and $\gamma_1 = \gamma_2 = 0.5$. Note that in the limit of short size of SSH chain, energy shift $|\Delta E_0|$ does not change with the length of the chain. However, energy shift $|\Delta E_0|$ exhibits an exponential scaling with system size $2L - 1$ as the length of the chain increases. This indicates that our approach is feasible for the detection of more physical quantities simultaneously, but it may consume more resources, such as the length of the chain. This provides a different idea for multiple physical quantity detection.

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