

Generalized geometric speed limits for quantum observables

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Leveraging quantum information geometry, we derive generalized quantum speed limits on the rate of change of the expectation values of observables. These bounds subsume and, for Hilbert space dimension ≥ 3 , tighten existing bounds—in some cases by an arbitrarily large multiplicative constant. Our theoretical results are supported by illustrative examples and an experimental demonstration using a superconducting qutrit. We also derive two upper bounds on the generalized quantum Fisher information in terms of the condition number of the density matrix. One of these bounds applies only to coherent dynamics and depends also on the variance of the Hamiltonian. The other bound depends also on the so-called Wigner-Yanase skew information. These bounds generalize well-known bounds on the symmetric logarithmic derivative quantum Fisher information and are tighter than the existing bounds for sufficiently mixed states (e.g., for sufficiently high temperature thermal states).

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I. INTRODUCTION

Energy-time uncertainty relations and the associated quantum speed limits were first formalized by Mandelstam and Tamm [1]. They provided a lower bound for the time $t^\perp$ for a quantum system in a pure state to reach an orthogonal state, given unitary evolution under some Hamiltonian. Since then, a variety of other bounds, all under the general heading of “quantum speed limits,” have been derived, providing bounds on the rate of change of quantum states [2–11] and observables [12–18], with applications to metrology [19,20], quantum thermodynamics [21–26], quantum control theory [27], the analysis of quantum algorithms [28], and many others [29].

Of particular note, García-Pintos *et al.* [13] derived a set of information-theoretic speed limits on the rate of change of expectation values of observables that hold for any probability-conserving dynamics. For a fixed observable A and a state ρ ,

$$|\dot{a}| \leq \Delta A \sqrt{\mathcal{I}^{\text{SLD}}}, \quad (1)$$

where $\dot{a} := \text{Tr}[A\dot{\rho}]$ (here, $\dot{\rho} := d\rho/dt$) is the velocity of the expectation value of A in the state

ρ , $(\Delta A)^2 := \text{Tr}(\rho A^2) - \text{Tr}(\rho A)^2$ is the variance of A , and $\mathcal{I}^{\text{SLD}}$ is the so-called symmetric logarithmic derivative (SLD) quantum Fisher information (QFI) with respect to time. The SLD QFI has a simple interpretation: It quantifies how much the state ρ changes as $t \rightarrow t + dt$.

Equation (1) consists of two components: (1) a dynamics-independent term, ΔA , that depends on the uncertainty of the observable in question with respect to the current state ρ ; and (2) an observable-independent term $\sqrt{\mathcal{I}^{\text{SLD}}}$ that depends only on the underlying state and the dynamics of the system. Both of these terms have geometric content, as they can each be associated with a natural notion of an inner product between Hermitian operators. In particular, considering ρ as a point in a one-dimensional manifold of quantum states parameterized by the time t , $\mathcal{I}^{\text{SLD}}$ is linked to a natural choice of distance between the points $\rho(t)$ and $\rho(t + dt)$. This distance measure is inherited from the Bures metric, defined on the manifold of positive-definite density matrices [30].

Stronger versions of Eq. (1) can be obtained by splitting A and $\mathcal{I}^{\text{SLD}}$ into their incoherent (diagonal) and coherent (off-diagonal) components in the eigenbasis of ρ [31]. In particular,

$$|\dot{a}| \leq (\Delta A_C) \sqrt{\mathcal{I}_C^{\text{SLD}}} + (\Delta A_I) \sqrt{\mathcal{I}_I^{\text{SLD}}}, \quad (2)$$

where the subscripts C and I denote the coherent and incoherent parts, respectively [13]. We have not yet defined precisely what the incoherent and coherent parts of $\mathcal{I}^{\text{SLD}}$ are, but, qualitatively, $\mathcal{I}_C^{\text{SLD}}$ ($\mathcal{I}_I^{\text{SLD}}$) quantifies how much the eigenvectors (eigenvalues) of ρ change as $t \rightarrow t + dt$.

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II. KEY RESULTS

Before delving into the mathematical details, we summarize the key results: First, we show that the bounds in Eqs. (1) and (2) are each but one example in a much richer family of geometric quantum speed limits. In particular, we leverage the fact that the choice of inner product on the (tangent space of the) manifold of quantum states that leads to $\mathcal{I}^{\text{SLD}}$ and ΔA is not unique [32,33]; thus, both the SLD QFI and the variance can be replaced with generalized quantities in Eqs. (1) and (2). Subject to the natural constraint that the associated Riemannian metric is contractive (i.e., distances shrink under noise), these generalized notions of QFI and of variance are characterized by a special set of functions f (defined rigorously below). We demonstrate that, for any valid choice of f ,

$$|\dot{a}| \leq (\Delta^f A) \sqrt{\mathcal{I}^f}, \quad (3)$$

and also

$$|\dot{a}| \leq (\Delta^f A_C) \sqrt{\mathcal{I}_C^f} + (\Delta A_I) \sqrt{\mathcal{I}_I^f}. \quad (4)$$

We dropped the f superscript on the incoherent terms because, for incoherent operators, the generalized QFI and variance are identical for all f . We prove that Eq. (4) is tighter than Eq. (3).

Equations (1) and (2) correspond to one particular choice of f . When minimized over valid choices of f , Eqs. (3) and (4) will, by construction, always be at least as tight as Eqs. (1) and (2). For Hilbert space dimension ≥ 3 and fully coherent dynamics, we will show that they can be tighter by an arbitrarily large multiplicative factor. These improvements come from taking advantage of the freedom in choosing a natural inner product on the manifold of quantum states to tighten the slack in the Cauchy-Schwarz inequality that underlies such bounds. Our results are supported by a toy example and an experimental demonstration using a superconducting qutrit. Finally, for arbitrary coherent dynamics driven by a Hamiltonian H and assuming incoherent dynamics driven by entanglement with an environment via a Hamiltonian H^{int} , we obtain a looser, but simpler, speed limit

$$|\dot{a}| \leq (\kappa_\rho + \kappa_\rho^{-1} - 2)^{-1/2} \Delta^f A_C \Delta H + 2 \Delta A_I \Delta H^{\text{int}}, \quad (5)$$

where κ_ρ is the condition number of ρ (i.e., the ratio of its largest eigenvalue to its smallest eigenvalue). Unlike Eq. (4), this bound does not depend directly on $\dot{\rho}$ and is expressed in terms of somewhat more accessible physical quantities.

Along the way to Eq. (5), we derive a bound on the coherent portion of any generalized QFI, in terms of the condition number κ_ρ of ρ ,

$$\mathcal{I}_C^f \leq (\kappa_\rho + \kappa_\rho^{-1} - 2)(\Delta H)^2, \quad (6)$$

which is of independent interest, as it is tighter than existing bounds for highly mixed states. Similar manipulations also yield the following generalization to arbitrary f of a well-known bound [34] on $\mathcal{I}^{\text{SLD}}$ in terms of the so-called Wigner-Yanase skew information $\mathcal{I}^{\text{WY}}$ [35]:

$$\mathcal{I}^f \leq 2(\kappa_\rho + \kappa_\rho^{-1}) \mathcal{I}^{\text{WY}}, \quad (7)$$

where $\mathcal{I}^{\text{WY}} := -(1/2)\text{Tr}([\sqrt{\rho}, H])^2$ is another special case of the generalized QFIs, notable because, unlike a usual notion

of energy variance, it picks out only the quantum (as opposed to classical) contributions to energy uncertainty [36].

III. MATHEMATICAL DETAILS

Let $\mathcal{M}_d$ denote the set of d -dimensional Hermitian matrices. Define the space of density operators $\mathcal{D} := \{\rho \in \mathcal{M}_d \mid \text{Tr } \rho = 1, \rho \geq 0\}$ over the d -dimensional Hilbert space $\mathcal{H}_d$. Restricting our attention to positive-definite $\rho \in \mathcal{D}$, we have a Riemannian manifold [37]; i.e., we can associate a Riemannian metric g_ρ with each point ρ on the manifold of positive-definite states, providing a notion of infinitesimal distance between two full rank states ρ and $\rho + d\rho$.

It is natural to require that g be contractive under noise; i.e., $g_{T\rho}(TA, TB) \leq g_\rho(A, B)$, where T is a completely positive trace-preserving map and A, B are elements of the tangent space at ρ (traceless Hermitian matrices of dimension d) [38]. As alluded to above, all such metrics can be identified with a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ known as a symmetric, normalized operator monotone function [32]. These functions f obey (1) continuity; (2) for matrices $A \geq B > 0$, $f(A) \geq f(B) > 0$ (operator monotonicity); (3) $xf(x^{-1}) = f(x)$, for $x > 0$ (symmetry); and (4) $f(1) = 1$ (normalization). Call the set of functions satisfying these properties $\mathbb{F}$.

These functions are important because they lead to a generalized mean $m^f(x, y) := xf(x^{-1}y)$ of a pair of either positive real numbers or positive-definite, commuting (super-)operators [39,40]. The properties of $\mathbb{F}$ imply various natural properties for the mean $m^f(x, y)$, e.g., $m^f(x, x) = x$ and $m^f(x, y) = m^f(y, x)$, as detailed in Appendix B.

In the context of quantum speed limits, consider $A = B = \dot{\rho}$; the distance $\mathcal{I}^f = 4g_\rho^f(\dot{\rho}, \dot{\rho})$ is called the generalized QFI with respect to time for the state ρ [41]. In particular [42,43],

$$\mathcal{I}^f := \text{Tr}[\dot{\rho} m^f(L_\rho, R_\rho)^{-1}(\dot{\rho})], \quad (8)$$

where $L_\rho(\cdot) := \rho(\cdot)$, $R_\rho(\cdot) := (\cdot)\rho$, and $m^f(L_\rho, R_\rho)$ are positive-definite super-operators and, thus, invertible. There exist particular choices of the function f — f_{SLD} and f_{RLD} —such that Eq. (8) corresponds to the SLD and RLD quantum Fisher information, respectively. The associated mean m^f is largest (smallest) over all $f \in \mathbb{F}$ for f_{SLD} (f_{RLD}).

We can also define a generalized variance as

$$(\Delta^f A)^2 := \text{Tr}[A_0 m^f(L_\rho, R_\rho)(A_0)], \quad (9)$$

where A is Hermitian and $A_0 := A - \text{Tr}(\rho A)$ [33]. For $f = f_{\text{SLD}}$, Eq. (9) is the usual variance.

Using these definitions, it is straightforward to derive Eqs. (3) and (4). To begin, define an f -dependent “logarithmic derivative” operator $\mathcal{L}^f := m^f(L_\rho, R_\rho)^{-1}(\dot{\rho})$. Note that $\mathcal{I}^f = (\Delta \mathcal{L}^f)^2$. Using that L_ρ and R_ρ are positive definite, $|\dot{a}| = |\text{Tr}[\dot{\rho} A]| = |\text{Tr}[\dot{\rho} A_0]| = |\text{Tr}[m^f(L_\rho, R_\rho)(\mathcal{L}^f) A_0]|$. Applying the Cauchy-Schwarz inequality gives Eq. (3).

Splitting A and $\mathcal{L}^f$ into coherent and incoherent parts in the eigenbasis of $\rho = \sum_j p_j |j\rangle\langle j|$, a similar derivation yields the bound in Eq. (4). Equation (4) is generically tighter than Eq. (3), coinciding if and only if $(\Delta^f A_C) \sqrt{\mathcal{I}_I^f} = (\Delta A_I) \sqrt{\mathcal{I}_C^f}$. See Appendix C.

IV. BOUND COMPARISON

While the speed limits specified by Eqs. (3) and (4) are of independent interest from the perspective of information geometry, to be of broader utility, there must exist triplets $(\rho, \dot{\rho}, A)$ where the tightest bound corresponds to $f \neq f_{\text{SLD}}$, so that the bounds are tighter than those of Ref. [13]. It is not immediately clear whether this is possible: on one hand, $\mathcal{I}^f \geq \mathcal{I}^{\text{SLD}}$ for all $f \in \mathbb{F}$; on the other, $\Delta^f A \leq \Delta^{\text{SLD}} A$ for all $f \in \mathbb{F}$. Tighter bounds depend on the tightening of the variance term winning out over the loosening of the QFI term.

As the incoherent term in Eq. (4) is identical for all f , to find improved bounds it is sufficient to consider only the coherent term. To this end, define the coherent ratio $\xi^f := (\Delta^f A_C)^2 \mathcal{I}_C^f / (\Delta^{\text{SLD}} A_C)^2 \mathcal{I}_C^{\text{SLD}}$. If $\xi^f < 1$, the generalized bound in Eq. (4) is tighter than the case with $f = f_{\text{SLD}}$ derived in Ref. [13].

While our results hold for all $f \in \mathbb{F}$, at times we will find it convenient to restrict to a natural one-parameter family of symmetric, normalized operator monotone functions that interpolates between the functions associated with the largest and smallest means m^f (f_{SLD} and f_{RLD} , respectively). In particular, in our examples below, we will consider a family $\mathbb{F}_\beta \subset \mathbb{F}$ defined in Ref. [44], specified by a parameter $\beta \in [-1, 1]$ such that $f_{\beta=1}(x) = (1+x)/2 = f_{\text{SLD}}$ and $f_{\beta=-1}(x) = 2x/(1+x) = f_{\text{RLD}}$ [45]. These limits of the one-parameter family are all that is needed to follow the main text, but Appendix A gives a full characterization of $\mathbb{F}_\beta$.

For a qubit, no improvement is possible over $f = f_{\text{SLD}}$ using the generalized bounds (see Appendix C 3). However, moving to larger systems, a qutrit ($d = 3$) is already sufficient to demonstrate two facts: (1) There are triplets $(\rho, \dot{\rho}, A)$ such that the generalized bounds with $f \neq f_{\text{SLD}}$ are tightest; and (2) for every $f_\beta \in \mathbb{F}_\beta$, there exists $(\rho, \dot{\rho}, A)$ such that the tightest bound, optimized over this family, is f_β .

Example

Consider a qutrit. In the eigenbasis of $\rho = \sum_j p_j |j\rangle\langle j|$, let $A_{02} = A_{12} = 0$, $v_{ij} := 2|(\dot{\rho})_{ij}|^2$, and $m_{ij}^f := m^f(p_i, p_j)$. Then, using expressions derived in Appendix C,

$$\xi^f = \frac{v_{01} + v_{02} \left(\frac{m_{01}^f}{m_{02}^f} \right) + v_{12} \left(\frac{m_{01}^f}{m_{12}^f} \right)}{v_{01} + v_{02} \left(\frac{m_{01}^{\text{SLD}}}{m_{02}^{\text{SLD}}} \right) + v_{12} \left(\frac{m_{01}^{\text{SLD}}}{m_{12}^{\text{SLD}}} \right)}. \quad (10)$$

Consider evolution under a Hamiltonian $H = i(\Omega/2)(|g\rangle\langle e| + |e\rangle\langle f|) + \text{H.c.}$, where $|g\rangle, |e\rangle, |f\rangle$ are the computational basis states. For simplicity, consider ρ diagonal in the computational basis, so that we can identify the computational basis with the eigenbasis of ρ . Thus, $v_{02} = 0$, so we can ignore the middle term in both the numerator and denominator of Eq. (10) and turn our attention to the third terms. Using that $m_{ij}^{\text{SLD}} \geq m_{ij}^f \forall f$, $\xi^f < 1$ if $p_1 \approx p_2$ (so that $m_{12}^{\text{SLD}} \approx m_{12}^f$, as $m^f(x, x + \epsilon) \approx x$ for all f if $\epsilon \ll 1$), and also $|p_0 - p_1|$ is sufficiently large (so that $m_{01}^{\text{SLD}} \gg m_{01}^f$).

This qualitative description is made precise in Fig. 1, where we minimize ξ^f over $f_\beta \in \mathbb{F}_\beta$ for equally spaced grid points (p_0, p_1, p_2) on the probability simplex with $p_j \in [0.0025, 0.9975] \forall j$. There are large regions of parameter

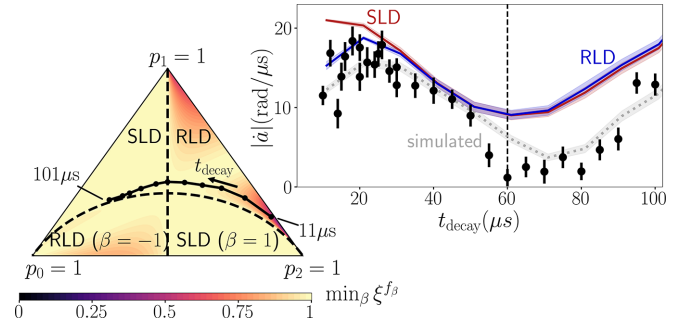


FIG. 1. For the qutrit example described in the text, ξ^f for the optimum $f_\beta \in \mathbb{F}_\beta$ as a function of the eigenvalues p_0, p_1, p_2 of the diagonal state ρ , plotted using barycentric coordinates. $\xi^f < 1$ corresponds to the bounds being tighter. In these regions, ξ^f is minimized for $f_{\beta=-1} = f_{\text{RLD}}$. The dashed lines in the main plot and the inset separate these regions from where there is no improvement over the SLD-based bounds. The states used in the experimental demonstration are prepared by letting the qutrit decay from $|f\rangle$ for a time $t_{\text{decay}} \in [11, 101] \mu\text{s}$ leading to the one-parameter family of states depicted with the black curve on the probability simplex. In the inset, the experimentally extracted value for $|\dot{a}|$ driving at a Rabi frequency of $\Omega = (2\pi)10 \text{ MHz}$ for a number of initial states along this trajectory is compared to the RLD and SLD bounds computed directly from experimentally determined states. The dotted gray line shows the simulated speed for these same states. Error bands are the standard deviation of the mean over 21 preparations of the states. For small t_{decay} , the f_{RLD} -based bound is significantly tighter than the old f_{SLD} -based bound.

space where the generalized bounds are tighter than existing bounds. Here, the optimal f_β is $f_{-1} = f_{\text{RLD}}$. For $p_0 = \epsilon^3$, $p_1 = \epsilon$, $p_2 = 1 - p_0 - p_1$, we have $\xi^f = \mathcal{O}(\epsilon)$, indicating that arbitrarily large multiplicative improvements over existing bounds are possible by taking $\epsilon \rightarrow 0$ (see Appendix D for details). As the incoherent term in Eq. (4) is the same for all f , this arbitrarily large multiplicative improvement depends crucially on the unitarity of the dynamics. This example is also well suited for a proof-of-principle experimental demonstration. Leaving the details of the experimental setup to Appendix E, we prepare a superconducting qutrit in a diagonal mixed state ρ by letting the qutrit naturally decay for a time t_{decay} from the $|f\rangle$ state. We then consider driving both the $|g\rangle \leftrightarrow |e\rangle$ and $|e\rangle \leftrightarrow |f\rangle$ transitions much faster than the decoherence timescale, so that we can assume that the decay is negligible, as in the theoretical toy example described above. We then extract the speed $|\dot{a}|$ of the Pauli-X observable in the $\{|g\rangle, |e\rangle\}$ subspace from estimates of the rate of change of the populations of these states. We repeat this for many initial states prepared via the natural decay of the qutrit and compare the measured speeds to the f_{SLD} - and f_{RLD} -based bounds, computed directly from the experimentally determined states at $10 \mu\text{s}$ intervals along the decay trajectory, as shown in the inset of Fig. 1. While both bounds constrain the measured speeds within experimental error for small t_{decay} , the f_{RLD} -based bound is significantly tighter than the f_{SLD} -based bound.

The problem becomes richer if $v_{01}, v_{02}, v_{12} \neq 0$. Here, the optimal choice of $f_\beta \in \mathbb{F}_\beta$ can occur for any $\beta \in [-1, 1]$ as one moves through the parameter space, demonstrating fact (2) above. An example is shown in Appendix F.

V. SATURATION OF BOUNDS

As the upper bounds come from the Cauchy-Schwarz inequality, Eqs. (3) and (4) are tight when $A_0 \propto \mathcal{L}^f$. In the eigenbasis of ρ , this corresponds to the condition $(A_0)_{ij} = \gamma[\dot{\rho}_{ij}/m^f(p_i, p_j)]$ for some constant γ . In the example above, A has only a single pair of nonzero components A_{ij}, A_{ji} ; thus, if the bounds are tight for any f , they are tight for all f . In general, however, the saturation of the speed limits is f -dependent. In fact, one can construct examples where the SLD-based bound is loose, while, with an appropriate choice of f , the generalized bounds are tight (see Appendix G for an example). The saturation conditions can also be leveraged to determine what choice of speed limit saturating dynamics (as given by a choice of $f \in \mathbb{F}$) maximizes $|\dot{a}|$ (see Appendix H).

Bounds using energy uncertainties. It is desirable to also derive quantum speed limits that do not require complete information of the state and dynamics, as with the generalized QFIs. To this end, we derive Eqs. (6) and (7) (see Appendix I) bounding the generalized QFI in terms of the condition number κ_ρ of ρ . Unlike calculating $\mathcal{I}^f$ directly, these bounds do not require knowledge of $\dot{\rho}$. Equation (6) is similar to the well-known bound for the SLD QFI $\mathcal{I}^{\text{SLD}} \leq 4(\Delta H)^2$ [30], but holds for any f . Furthermore, while the SLD bound is tight for pure states, our bound is tighter for highly mixed states. That is, for a maximally mixed state, Eq. (6) is, correctly, zero for all f while $4(\Delta H)^2$ is finite. Similar manipulations yield the bound in Eq. (7) in terms of the Wigner-Yanase skew information [35,36,46]. This bound generalizes another well-known bound $\mathcal{I}^{\text{SLD}} \leq 2\mathcal{I}^{\text{WY}}$ [34]. As we demonstrate explicitly in Appendix I3, either bound can be tighter than the other in certain regimes.

Returning to speed limits, consider unitary dynamics driven via a Hamiltonian H and assume any nonunitary dynamics are due to entanglement with an environment via a Hamiltonian H^{int} consisting of all terms with support on both the system and the environment. Then, plugging Eq. (6) and $\mathcal{I}_I \leq 4(\Delta H^{\text{int}})^2$ [13] into Eq. (4) yields the speed limit in Eq. (5). Equation (5) is tightest $f = f_{\text{RLD}}$. However, for all f , if $\kappa_\rho < 3 + 2\sqrt{2} =: \kappa^*$, then Eq. (5) is tighter than existing similar bounds [13]:

$$|\dot{a}| \leq 2\Delta A_C \Delta H + 2\Delta A_I \Delta H^{\text{int}}. \quad (11)$$

The condition $\kappa_\rho < \kappa^*$ holds for nearly fully mixed states; for instance, a thermal state $\rho \propto \exp(-\beta H)$ with $\beta < \log(\kappa^*)/\|H\|_s$, where $\|H\|_s$ is the seminorm defined as the difference between the largest and smallest eigenvalues of H .

VI. DISCUSSION

Leveraging the tools of quantum information geometry, we prove quantum speed limits for observables that are generically tighter than previously derived bounds [13] and generalize some well-known bounds on the SLD QFI. The family of inner products we use have also been used to derive tighter bounds for multiparameter quantum metrology [47], generalized uncertainty relations [48,49], and quantum speed limits on the evolution of states [6,50], and to characterize entropic flow in quantum thermodynamics [51]. We expect these inner products and the bounds in this work can also be

applied in other settings where the SLD QFI is used [52], e.g., for analyzing quantum machine learning models [53,54] for parameter estimation problems [55].

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DATA AVAILABILITY

Tabular data and code for examples are available upon request.

APPENDIX A: SUMMARY OF KEY DEFINITIONS

In this section, we provide, in Table I, a summary of the key quantities and their mathematical definitions. In these

TABLE I. Definitions of key mathematical quantities.

Quantity	Mathematical expression
$\dot{a}$	$\text{Tr}(\dot{\rho}A)$
$(\Delta A)^2$	$\text{Tr}(\rho A^2) - \text{Tr}(\rho A)^2$
A_0	$A - \text{Tr}(\rho A)$
$m^f(x, y)$	$xf(x^{-1}y)$
$a(\Delta^f A)^2$	$\text{Tr}[A_0 m^f(L_\rho, R_\rho)(A_0)]$
$\mathcal{L}^f$	$m^f(L_\rho, R_\rho)^{-1}(\dot{\rho})$
$\mathcal{I}^f$	$\text{Tr}[\dot{\rho} m^f(L_\rho, R_\rho)^{-1}(\dot{\rho})]$ $= \text{Tr}[\mathcal{L}^f m^f(L_\rho, R_\rho)(\mathcal{L}^f)]$

^aEquals $(\Delta A)^2$ for $f = f_{\text{SLD}}$.

TABLE II. Important examples of $f \in \mathbb{F}_\beta$ and the associated means.

Name	β	$f(x)$	$m^f(x, y)$
SLD	1	$\frac{1+x}{2}$	$\frac{x+y}{2}$
Wigner-Yanase	$\frac{1}{2}$	$\frac{(1+\sqrt{x})^2}{4}$	$\frac{(\sqrt{x}+\sqrt{y})^2}{4}$
RLD	-1	$\frac{2x}{1+x}$	$\frac{2xy}{x+y}$

definitions, as in the main text, A denotes a general Hermitian observable.

We also provide the complete mathematical description of the one-parameter family of operator monotone functions $\mathbb{F}_\beta$ used in the main text. In particular,

$$f_\beta(x) := \begin{cases} \frac{\beta(1-\beta)(x-1)^2}{(x^\beta-1)(x^{1-\beta}-1)}, & \beta \in [-1, \frac{1}{2}) \setminus \{0\}, \\ \frac{x-1}{\log x}, & \beta = 0, \\ \left(\frac{1+x^\beta}{2}\right)^{1/\beta}, & \beta \in [\frac{1}{2}, 1]. \end{cases} \quad (\text{A1})$$

Some important examples from this family and their associated means are summarized in Table II.

APPENDIX B: FROM OPERATOR MONOTONE FUNCTIONS TO MEANS

In this section, we review how the properties of the operator monotone functions $\mathbb{F}$ imply certain natural properties of the associated means. Recall that $m^f(x, y) := xf(x^{-1}y)$. Here, we consider $x, y \in \mathbb{R}^+$ or as commuting, positive-definite (super-)operators. As described in the main text, the operator monotone functions $f \in \mathbb{F}$ have the following properties:

- (1) Continuity.
- (2) For matrices $A \geq B > 0$, $f(A) \geq f(B) > 0$ (operator monotonicity).
- (3) $xf(x^{-1}) = f(x)$, for $x > 0$ (symmetry).
- (4) $f(1) = 1$ (normalization).

These properties of $f \in \mathbb{F}$ imply the following (nonexhaustive) list of properties for the associated means:

- (1') $m^f(x, y)$ is continuous.
- Follows directly from property (1) and the definition of $m^f(x, y)$.
- (2') $m^f(x, x) = x$.
- Follows from property (4):

$$m^f(x, x) = xf(x^{-1}x) = xf(1) = x.$$

- (3') $m^f(x, y) = m^f(y, x)$.

Follows from property (3):

$$m^f(x, y) = \frac{x}{xy^{-1}} f(y^{-1}x) = yf(y^{-1}x) = m^f(y, x).$$

- (4') $x \leq y \Rightarrow x \leq m^f(x, y) \leq y$.

Follows from properties (2), (4) and (3'):

$$m^f(x, y) = xf(x^{-1}y) \geq xf(1) = x,$$

$$m^f(x, y) = yf(y^{-1}x) \leq yf(1) = y.$$

- (5') $x \leq x', y \leq y' \Rightarrow m^f(x, y) \leq m^f(x', y')$.

Follows from properties (2), (4), and (3') almost identically to the demonstration of property (4').

APPENDIX C: COHERENT-INCOHERENT BOUND

In this section, we derive the split coherent-incoherent bound of Eq. (4), repeated here for convenience:

$$|\dot{a}| \leq (\Delta^f A_C) \sqrt{\mathcal{I}_C^f} + (\Delta A_I) \sqrt{\mathcal{I}_I}. \quad (\text{C1})$$

We also demonstrate how to express the quantities in this bound in the eigenbasis of $\rho = \sum_j p_j |j\rangle\langle j|$ and use these expressions to show that the split coherent-incoherent bound in Eq. (C1) is generically tighter than the nonsplit bound in Eq. (3).

1. Proving the coherent-incoherent bound

To derive Eq. (C1), note that

$$\begin{aligned} \dot{a} &= \text{Tr}[m^f(L_\rho, R_\rho)(\mathcal{L}_I^f + \mathcal{L}_C^f)(A_{0,I} + A_{0,C})] \\ &= \text{Tr}[m^f(L_\rho, R_\rho)(\mathcal{L}_I^f)A_{0,I}] + \text{Tr}[m^f(L_\rho, R_\rho)(\mathcal{L}_C^f)A_{0,C}], \end{aligned} \quad (\text{C2})$$

where we use that the cross terms vanish under the trace. The subscripts C and I denote the coherent and incoherent parts of each operator in the eigenbasis of ρ , respectively. The bound in Eq. (C1) then follows immediately by applying the Cauchy-Schwarz inequality.

Therefore, it remains to explicitly show that the cross terms indeed vanish. That is, we must show that

$$\text{Tr}[m^f(L_\rho, R_\rho)(\mathcal{L}_I^f)A_{0,C}] = 0, \quad (\text{C3})$$

$$\text{Tr}[m^f(L_\rho, R_\rho)(\mathcal{L}_C^f)A_{0,I}] = 0. \quad (\text{C4})$$

To show this, we will use the matrix representation of the super-operators $L_\rho = \rho \otimes I$ and $R_\rho = I \otimes \rho^T$, which act on vectorized versions of the regular operators. That is, for an operator O , we denote its vectorized form in an arbitrary basis as

$$|O\rangle := \sum_{i,j} O_{ij} |i\rangle \otimes |j\rangle = \sum_{i,j} O_{ij} |ij\rangle, \quad (\text{C5})$$

where we use the rounded “ket” $|\cdot\rangle$ to denote vectors in the super-operator vector space.

Specifically, in the eigenbasis of $\rho = \sum_j p_j |j\rangle\langle j|$, we can write $L_\rho = \sum_{ij} p_i |ij\rangle\langle ij|$ and $R_\rho = \sum_{ij} p_j |ij\rangle\langle ij|$. It is then easy to show, in this basis, that

$$\begin{aligned} m^f(L_\rho, R_\rho) &:= L_\rho f(L_\rho^{-1} R_\rho) \\ &= \sum_{ij} p_i f(p_i^{-1} p_j) |ij\rangle\langle ij| \\ &= \sum_{ij} m^f(p_i, p_j) |ij\rangle\langle ij|, \end{aligned} \quad (\text{C6})$$

where we used that $m^f(p_i, p_j) := p_i f(p_i^{-1} p_j)$. Consequently,

$$\begin{aligned} m^f(L_\rho, R_\rho)(\mathcal{L}_I^f) &= \sum_{ij} m^f(p_i, p_j) |ij\rangle \langle ij| \sum_k (\mathcal{L}_I^f)_{kk} |kk\rangle \\ &= \sum_i (\mathcal{L}_I^f)_{ii} m^f(p_i, p_i) |ii\rangle \\ &= \sum_i p_i (\mathcal{L}_I^f)_{ii} |ii\rangle, \end{aligned} \quad (\text{C7})$$

from which it is easy to show Eq. (C3), as $A_{0,C}$ is off-diagonal in this basis. Similar computations allow one to show Eq. (C4).

2. Generalized quantities in state eigenbasis

For analysis of the bounds, it is helpful to derive expressions for the elements of the split coherent-incoherent bound in Eq. (4) [alternatively, Eq. (C1)] in the eigenbasis of $\rho = \sum_j p_j |j\rangle \langle j|$. In particular, it holds that

$$(\Delta^f A_C)^2 = \sum_{i \neq j} |(A_0)_{ij}|^2 m^f(p_i, p_j), \quad (\text{C8})$$

$$\mathcal{I}_C^f := (\Delta \mathcal{L}_C^f)^2 = \sum_{i \neq j} |(\dot{\rho})_{ij}|^2 \frac{1}{m^f(p_i, p_j)}, \quad (\text{C9})$$

$$(\Delta A_I)^2 = \sum_i p_i |(A_0)_{ii}|^2, \quad (\text{C10})$$

$$\mathcal{I}_I := (\Delta \mathcal{L}_I^f)^2 = \sum_j p_j \left(\frac{d \log p_j}{dt} \right)^2. \quad (\text{C11})$$

Recall, we drop the f superscript on the incoherent terms because, for incoherent operators, the generalized quantum Fisher information and variance are identical for all f [56]. Furthermore, $\mathcal{I}_I$ is simply the classical Fisher information of the probability distribution specified by the eigenvalues $\{p_j\}_{j=0}^{d-1}$ of ρ .

We now sketch the derivation for these expressions. Using Eq. (C6),

$$\begin{aligned} m^f(L_\rho, R_\rho)(A_0) &= \sum_{ij} m^f(p_i, p_j) |ij\rangle \langle ij| \sum_{kl} (A_0)_{kl} |kl\rangle \\ &= \sum_{ij} (A_0)_{ij} m^f(p_i, p_j) |ij\rangle. \end{aligned} \quad (\text{C12})$$

Undoing the vectorization and plugging into Eq. (7) yields

$$\begin{aligned} (\Delta^f A)^2 &:= \text{Tr}[A_0 m^f(L_\rho, R_\rho)(A_0)] \\ &= \text{Tr} \left[\sum_{ijkl} (A_0)_{kl} |k\rangle \langle l| (A_0)_{ij} m^f(p_i, p_j) |i\rangle \langle j| \right] \end{aligned}$$

$$= \sum_{ij} |(A_0)_{ij}|^2 m^f(p_i, p_j). \quad (\text{C13})$$

Restricting A to its coherent component $A \rightarrow A_C$ yields Eq. (C8). Similarly, considering $A \rightarrow A_I$ and using that $m^f(p_i, p_i) = p_i$ yields Eq. (C10). Essentially identical manipulations prove Eqs. (C9) and (C11) starting from the definition of $\mathcal{I}^f$ in Eq. (6) and using that $\dot{\rho} = m^f(L_\rho, R_\rho)(\mathcal{L}^f)$.

3. All bounds are equivalent for a qubit

The results of the previous section can be used to demonstrate a simple proposition, the results of which were described in the main text.

Proposition C1. For a qubit, $(\Delta^f A_C) \sqrt{\mathcal{I}_C^f} = (\Delta^{\text{SLD}} A_C) \sqrt{\mathcal{I}_C^{\text{SLD}}}$ for all $f \in \mathbb{F}$.

Proof. In both Eqs. (C8) and (C9), there is a single pair of equal terms for a qubit. Thus, working in the eigenbasis of ρ , $(\Delta^f A_C) \sqrt{\mathcal{I}_C^f} = 2 |(A_0)_{01}| |\dot{\rho}_{01}|$ for all f , proving the result. ■

4. Coherent-incoherent bound is a generically tighter upper bound

As stated in the main text, the coherent-incoherent bound of Eq. (4) [alternatively, Eq. (C1)] is always at least as tight as the nonsplit bound of Eq. (3),

$$|\dot{a}| \leq (\Delta^f A) \sqrt{\mathcal{I}^f}. \quad (\text{C14})$$

To prove this, we start from Eq. (C13) for the generalized quantum f -variance of A in the eigenbasis of the state ρ . We then split $A_0 := A - \text{Tr}(\rho A)$ into its coherent and incoherent parts and simplify the resulting expression as follows:

$$\begin{aligned} (\Delta^f A)^2 &= \sum_{ij} |(A_0)_{ij}|^2 m^f(p_i, p_j) \\ &= \sum_{ij} [|(A_0)_{ij}|^2 + |(A_0)_{ij}|^2] m^f(p_i, p_j) \\ &= (\Delta^f A_I)^2 + (\Delta^f A_C)^2 \\ &= (\Delta A_I)^2 + (\Delta^f A_C)^2, \end{aligned} \quad (\text{C15})$$

where in the last line we drop the f subscript for the variance of the incoherent operator A_I , as for diagonal operators the variance evaluates to the usual variance for all f .

Therefore, the ratio of the right-hand side of Eq. (C14) and the right-hand side of Eq. (C1) can be computed to be

$$\begin{aligned} \frac{(\Delta^f A) \sqrt{\mathcal{I}^f}}{(\Delta^f A_C) \sqrt{\mathcal{I}_C^f} + (\Delta A_I) \sqrt{\mathcal{I}_I}} &= \frac{[(\Delta A_I)^2 + (\Delta^f A_C)^2] [\mathcal{I}_C^f + \mathcal{I}_I]}{[(\Delta^f A_C) \sqrt{\mathcal{I}_C^f} + (\Delta A_I) \sqrt{\mathcal{I}_I}]^2} = \sqrt{1 + \frac{(\Delta^f A_C)^2 \mathcal{I}_I + (\Delta A_I)^2 \mathcal{I}_C^f - 2(\Delta^f A_C)(\Delta A_I) \sqrt{\mathcal{I}_C^f \mathcal{I}_I}}{[(\Delta^f A_C) \sqrt{\mathcal{I}_C^f} + (\Delta A_I) \sqrt{\mathcal{I}_I}]^2}} \\ &= \sqrt{1 + \frac{[(\Delta^f A_C) \sqrt{\mathcal{I}_I} - (\Delta A_I) \sqrt{\mathcal{I}_C^f}]^2}{[(\Delta^f A_C) \sqrt{\mathcal{I}_C^f} + (\Delta A_I) \sqrt{\mathcal{I}_I}]^2}} \geq 1. \end{aligned} \quad (\text{C16})$$

Thus, the coherent-incoherent bound in Eq. (C1) is tighter than the nonsplit bound in Eq. (C14). The two are equal if and only if $(\Delta^f A_C)\sqrt{\mathcal{I}_I} = (\Delta A_I)\sqrt{\mathcal{I}_C^f}$. This precisely mirrors the results derived in Appendix F of Ref. [13] for the special case $f = f_{\text{SLD}}$.

APPENDIX D: ARBITRARILY LARGE MULTIPLICATIVE IMPROVEMENTS FROM GENERALIZED BOUNDS

In this section, we demonstrate the claim that arbitrarily large multiplicative improvements in the bounds are possible compared to the existing f_{SLD} -based bounds of Ref. [13]. In particular, we show this by example by demonstrating that the quantity ξ^f can be made arbitrarily small in the qutrit example of the main text.

It is simple to compute that for this example $v_{01} = (1/2)\Omega^2(p_1 - p_0)^2$ and $v_{02} = (1/2)\Omega^2(p_2 - p_1)^2$. Then, using Eq. (8) for $f = f_{\text{RLD}}$, the fact that $m_{ij}^{\text{RLD}} = 2p_i p_j / (p_i + p_j)$ and $m_{ij}^{\text{SLD}} = (p_i + p_j)/2$, and a bit of elementary algebra, we have that

$$\xi^{\text{RLD}} = \frac{p_1 + p_2}{p_2(p_0 + p_1)} \times \left[\frac{(p_1 - p_0)^2(p_0 + p_1)p_2 + (p_2 - p_1)^2 p_0(p_1 + p_2)}{(p_1 - p_0)^2(p_1 + p_2) + (p_2 - p_1)^2(p_0 + p_1)} \right]. \quad (\text{D1})$$

Consider $p_0 = \epsilon^3$, $p_1 = \epsilon$, $p_2 = 1 - \epsilon - \epsilon^3$ for some small ϵ . Then,

$$\xi^{\text{RLD}} = \epsilon + \mathcal{O}(\epsilon^2), \quad (\text{D2})$$

proving the desired result. Such states correspond to the lower right corner near $p_2 = 1$ in Fig. 1.

APPENDIX E: EXPERIMENTAL DETAILS

In this section, we provide details behind the experimental demonstration discussed in the qutrit example described in the main text. We use a superconducting transmon qutrit [57] to collect the data presented in the proof-of-concept measurement in Fig. 1. The qutrit is a standard X-mon style device with a Ta ground plane on a sapphire substrate and a Al/AIO_x/Al junction. The qutrit is coupled to a coplanar waveguide resonator for dispersive readout of the qutrit state [58,59]. A dedicated drive line is used to apply coherent control on the qutrit (see Ref. [60] for further details on the measurement setup). The relevant qutrit parameters are listed in Table III.

We initialize the qutrit in the $|f\rangle$ state using two consecutive π pulses on the $|g\rangle$ - $|e\rangle$ and $|e\rangle$ - $|f\rangle$ transitions. By varying the delay time between the state initialization and the beginning of the experiment, we utilize the natural depolarization of the excited states to generate purely diagonal mixed states in the $|g\rangle$, $|e\rangle$, $|f\rangle$ basis. We fit the population dynamics to a simple decay chain model given by the Bateman equation [61,62] to verify our initialization scheme and qutrit state readout as shown in Fig. 2. These data are used to determine population values and uncertainties used in Fig. 1.

To carry out the experiment shown in Fig. 1, we initialize the qutrit in a mixed state via the depolarization technique

TABLE III. Qutrit parameters for the device used in the experimental measurements. $\nu_{ge(e f)}$ is the transition frequency between the $|g\rangle$ and $|e\rangle$ ($|e\rangle$ and $|f\rangle$) states of the qutrit. $\Gamma_{eg(fe)}$ is the measured decay rate from the $|e\rangle$ ($|f\rangle$ to $|e\rangle$) state (defined as $1/T_1$). $\Omega_{ge(e f)}$ is the Rabi rate applied to the $|g\rangle$ to $|e\rangle$ ($|e\rangle$ to $|f\rangle$) transition.

Parameter	Value
ω_{ge}	$2\pi \times 2.78 \text{ GHz}$
ω_{ef}	$2\pi \times 2.63 \text{ GHz}$
Ω_{ge}	$2\pi \times 10 \text{ MHz}$
Ω_{ef}	$2\pi \times 10 \text{ MHz}$
Γ_{eg}	14.7 ms^{-1}
Γ_{fe}	18.4 ms^{-1}

described above. After the state is initialized, we estimate $\dot{X}_{ge}$ for the specified Hamiltonian, $H = i(\Omega/2)(|g\rangle\langle e| + |e\rangle\langle f|) + \text{H.c.}$, by applying simultaneous drives for a variable amount of time on the $|g\rangle$ - $|e\rangle$ transition with drive strength Ω_{ge} and $|e\rangle$ - $|f\rangle$ transition with drive strength Ω_{ef} about the Pauli Y axis of the Bloch spheres associated with these subspaces. We measure $\langle X_{ge} \rangle$ by applying a $\pi/2$ pulse before qutrit state readout to map the observable onto the population of the qutrit state: $\langle X_{ge} \rangle = P_e - P_g$, where $P_{g,(e)}$ is the extracted population of the $|g\rangle$ ($|e\rangle$) qutrit level. We measure the rate of change of the populations by varying the drive time and fitting a line to the short time dynamics (0–16 ns evolution time) of $P_{g,e}$. We use the population rates of change to calculate the speed $\dot{X}_{ge} = \dot{P}_e - \dot{P}_g$ under the measurement mapping.

We collect data points with interleaved $|g\rangle$, $|e\rangle$, $|f\rangle$ reference measurements for population extraction. For each point, we collected 20 000 averages. We compute uncertainties of fitted $\dot{P}_{g,e}$ from the linear least-squares fit of the populations. These uncertainties are used to obtain the error bars on $\dot{X}_{ge}$ shown in Fig. 1. Data available upon request.

APPENDIX F: ANY BOUND CAN BE TIGHTEST

In this section, we provide an example demonstrating the claim in the main text that for every $f_\beta \in \mathbb{F}_\beta$ there exists

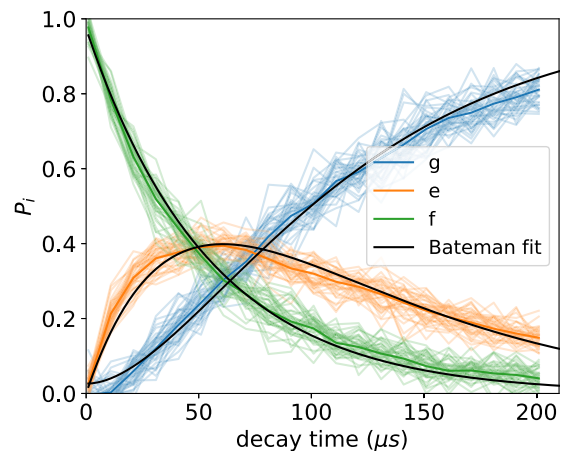


FIG. 2. Repeated measurements of qutrit populations after variable decay time from the $|f\rangle$ state. Solid lines are the fit to the Bateman equations with decay rates listed in Table III. Data are used to calibrate the state preparation technique used in Fig. 1.

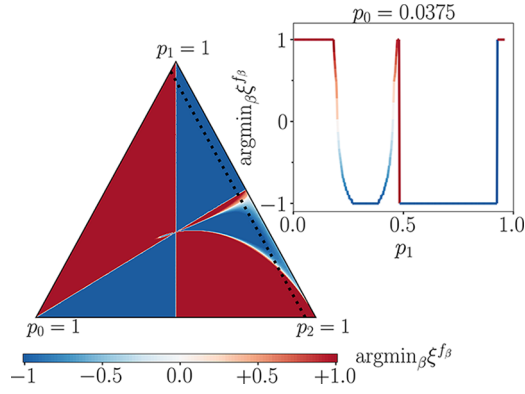


FIG. 3. For a qutrit, optimal $f_\beta \in \mathbb{F}_\beta$ to minimize the coherent ratio ξ^{f_β} [Eq. (8)] with $v_{01}, v_{02}, v_{12} \neq 0$ as a function of the eigenvalues $\{p_j\}_{j=0}^2$ of ρ . In the inset, the optimal β is shown as a function of p_1 for fixed $p_0 = 0.0375$ (along dotted line in main figure).

$(\rho, \dot{\rho}, A)$ such that the tightest bound, optimized over this family, is f_β . This example is similar to the qutrit example of the main text. Let the observable of interest is $A = |g\rangle\langle e| + |e\rangle\langle g|$, and, as in that example, we focus on diagonal ρ , so that this corresponds to $A = |0\rangle\langle 1| + |1\rangle\langle 0|$.

Consider driving the system with the Hamiltonian $H = -(i2\pi)(2.5 \text{ MHz})(|0\rangle\langle 1| + 4|1\rangle\langle 2| + |0\rangle\langle 2|) + \text{H.c.}$ In Fig. 3, we show the β that minimizes the coherent ratio ξ^{f_β} as a function of the eigenvalues $\{p_j\}_{j=0}^2$ of ρ , revealing a rich structure in the f_β yielding the tightest bound.

APPENDIX G: SATURATION OF BOUNDS

In this section, we provide additional details on when the bounds considered in this work are saturated.

Starting with Eq. (3), this speed limit is tight when $A_0 \propto \mathcal{L}^f$. To better understand this condition, it is, again, helpful to work in the eigenbasis of ρ . In the super-operator vector space,

$$\begin{aligned} \mathcal{L}^f &:= m^f(L_\rho, R_\rho)^{-1}(\dot{\rho}) \\ &= \sum_{ij} \frac{1}{m^f(p_i, p_j)} |ij\rangle\langle ij| \left(\sum_{kl} \dot{\rho}_{kl} |kl\rangle \right) \\ &= \sum_{ij} \frac{\dot{\rho}_{ij}}{m^f(p_i, p_j)} |ij\rangle\langle ij|. \end{aligned} \quad (\text{G1})$$

So, Eq. (3) is tight if (and only if)

$$(A_0)_{ij} = \gamma \frac{\dot{\rho}_{ij}}{m^f(p_i, p_j)}, \quad (\text{G2})$$

for all i, j and some constant γ . Observe the f dependence of this saturation condition. That is, given some fixed observable A , dynamics, and a state ρ , the bounds are generically saturable only for a particular choice of f . There are special cases where the f dependence is eliminated: For instance, if everything is fully incoherent, the fact that $m^f(p_i, p_i) = p_i$ for all f removes the f dependence.

Clearly, if and only if $A_0 \propto \mathcal{L}^f$, then also $A_C \propto \mathcal{L}_C^f$ and $A_I \propto \mathcal{L}_I$ (where we drop the f subscript because it is irrelevant

for incoherent terms). So, this condition also implies the saturation of the split coherent-incoherent upper bound in Eq. (4).

Note that, for the example in the main text (a qutrit with a fully coherent observable and only $a_{01} \neq 0$), if the bounds are saturated for any f , they are saturated for all f . In particular, $|\dot{a}| = (\Delta^f A) \sqrt{\mathcal{I}^f} = (\Delta^f A_C) \sqrt{\mathcal{I}_C^f}$ for all f in this example if and only if $v_{01} \neq 0$ and $v_{12} = v_{02} = 0$. This is because only A_{01}, A_{10} are nonzero for this choice of observable; thus, the saturability condition in Eq. (G2) is especially simple, and the f dependence can be absorbed into the constant γ . If there is more than a single relevant pair i, j in the saturability condition of Eq. (G2), then we can recover the f dependence.

For the case of purely coherent dynamics driven by a Hamiltonian H , it is informative to rewrite Eq. (G2) as

$$A_{ij} = -i\gamma \frac{H_{ij}(p_j - p_i)}{m^f(p_i, p_j)}, \quad (\text{G3})$$

where we use that $\dot{\rho} = -i[H, \rho]$ and H_{ij} are the matrix elements of H in the eigenbasis of ρ .

Example of saturating the bounds

In the main text, we claim one can construct examples where the SLD-based bounds are loose, but one of our generalized bounds can be tight for the appropriate choice of f . Here, we show such an example.

To begin, as γ is independent of i and j , Eq. (G3) implies the necessary condition for the saturation of the bound in Eq. (4) for purely coherent dynamics:

$$\frac{H_{ij}(p_j - p_i)}{A_{ij}m^f(p_i, p_j)} = \frac{H_{kl}(p_l - p_k)}{A_{kl}m^f(p_k, p_l)} \quad (\text{G4})$$

for all i, j, k, l such that $A_{ij}, H_{ij}, A_{kl}, H_{kl} \neq 0$. This condition is not sufficient as Eq. (G3) must also hold for matrix elements where $A_{ij} = 0$ and/or $H_{ij} = 0$. Let us assume for simplicity that these conditions are satisfied trivially so that, when $H_{ij} = 0$, also $A_{ij} = 0$.

Define $x := p_i/p_j$, $y := p_k/p_l$, and $c = H_{kl}A_{ij}/(H_{ij}A_{kl})$. Then, we have the necessary condition(s):

$$cf(x) - f(y) = cyf(x) - xf(y), \quad (\text{G5})$$

where $x, y > 0$. Generically, $c \in \mathbb{C}$, but it is clear (as $x, y, f(x), f(y) > 0$) that, unless $c \in \mathbb{R}$, Eq. (G5) has no solutions. So, we restrict our attention to the case that $c \in \mathbb{R}$.

Consider two qubits with $H = X_1 X_2$ and let $A = Y_1 X_2 - X_1 Y_2/2$. We have one nontrivial saturation condition in this example corresponding to an identification of the indices $i \rightarrow 00, j \rightarrow 11, k \rightarrow 01$, and $l \rightarrow 10$. The associated $c = 1/3$. Also, $x := p_{00}/p_{11}$ and $y := p_{01}/p_{10}$.

It is fairly straightforward to work out that, in this setting, with $f(t) = f_{\text{SLD}}(t) = (1+t)/2$, Eq. (G5) becomes

$$x = \frac{1+2y}{2+y}. \quad (\text{G6})$$

As a concrete example, $p_{00} = 0.3, p_{01} = 0.4, p_{10} = 0.1, p_{11} = 0.2$, corresponding to $y = 4$ and $x = 3/2$, obey this saturation condition.

In contrast to the first example, valid solutions to the condition for $f(t) = f_{\text{RLD}}(t) = 2t/(1+t)$ do not correspond

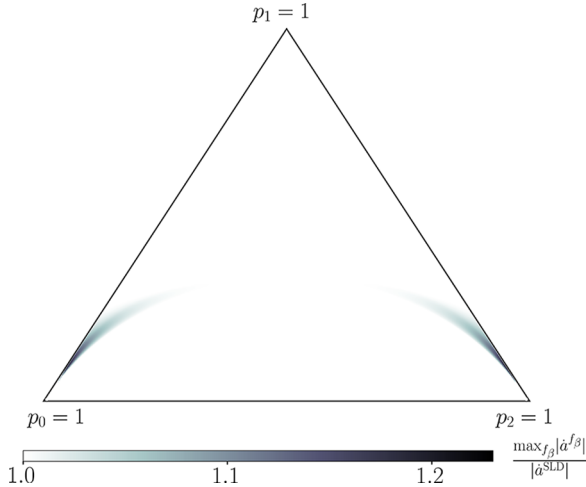


FIG. 4. Plot of the ratio of the speed with the optimal $H_{\rho,A}^{f,\text{fast}}$ to the speed with $H_{\rho,A}^{f_{\text{SLD}},\text{fast}}$ as a function of the eigenvalues p_0, p_1 of ρ . Note the bands of states where $f \neq f_{\text{SLD}}$ yield larger $|\dot{a}|$.

to those for the condition in Eq. (G6). In particular, here, Eq. (G5) becomes

$$x = \frac{y^2 - 1 \pm \sqrt{34y^2 + y^4 + 1}}{6y}. \quad (\text{G7})$$

Any solutions to this equation with $x, y > 0$ correspond to states that satisfy the generalized quantum speed limit associated with f_{RLD} . For instance, $p_{00} = (\sqrt{89} - 3)/20$, $p_{01} = 0.4$, $p_{10} = 0.1$, $p_{11} = (13 - \sqrt{89})/20$, corresponding to $y = 4$, $x = (15 + 3\sqrt{89})/24$, satisfy Eq. (G7), but clearly not Eq. (G6). Therefore, we have constructed an example where the generalized bounds can be tight when the SLD-based bounds are not.

APPENDIX H: “FAST” HAMILTONIANS

The saturation conditions considered in the previous section can also be leveraged to determine what choice of speed limit saturating dynamics (as given by a choice of $f \in \mathbb{F}$) maximizes the speed $|\dot{a}|$.

As the f dependence of this problem depends only on the coherent part of the bounds [see Eq. (4)], we restrict to Hamiltonian dynamics to emphasize the role of f . Then, using that $\dot{\rho} = -i[H, \rho]$ and assuming that ρ has a nondegenerate spectrum, the “fast” Hamiltonian

$$H_{\rho,A}^{f,\text{fast}} = -\frac{i}{\gamma} \sum_{j \neq k} \frac{m^f(p_j, p_k)}{p_j - p_k} A_{jk} |j\rangle\langle k|, \quad (\text{H1})$$

for some constant γ , yields speed-limit saturating dynamics—i.e., $|\dot{a}| = (\Delta^f A_C) \sqrt{\mathcal{I}_C^f}$. The optimal f that maximizes $|\dot{a}|$ will be state and observable dependent. Using $H_{\rho,A}^{f,\text{fast}}$ for $f \neq f_{\text{SLD}}$ can lead to larger $|\dot{a}|$ than f_{SLD} alone.

In particular, consider a qutrit with the observable of interest as $A = |0\rangle\langle 1| + |1\rangle\langle 2| + \text{H.c.}$ For this observable and a range of different states ρ , we optimize over the “fast” Hamiltonians $H_{\rho,A}^{f,\text{fast}}$ with respect to functions $f \in \mathbb{F}_B$ [see Eq. (A1)] for fixed spectral norm. In Fig. 4, we plot the ratio

of the speed with the optimal $H_{\rho,A}^{f,\text{fast}}$ from the result of this maximization to the speed with $H_{\rho,A}^{f_{\text{SLD}},\text{fast}}$ as a function of the eigenvalues p_0, p_1 of ρ . Observe that there is a whole band of states where the optimal driving Hamiltonian is derived from $f \neq f_{\text{SLD}}$. Therefore, we have a demonstration that the generalized bounds can be used to derive faster driving Hamiltonians than the usual SLD-based bounds. Here, the driving Hamiltonians based on the generalized bounds can increase $|\dot{a}|$ by up to approximately 20% relative to the SLD-based drives.

These “fast” Hamiltonians may be hard to implement for large systems, but it has been suggested that they could guide the construction of alternative Hamiltonians built from a limited set of controls [13]. However, it is not clear that aiming to saturate these bounds is easier than just directly optimizing for the observable of interest as the same information is needed regarding the state and dynamics. This is one motivation for considering looser bounds that require somewhat less complete information about the system, as considered in Appendix I.

APPENDIX I: BOUNDS ON GENERALIZED QUANTUM FISHER INFORMATION

1. A bound in terms of energy variance and condition number

In this section, we show that at the cost of loosening the bounds we can obtain bounds similar to Eq. (4) that depend on physical quantities more easily accessible and easily interpretable than the quantum Fisher informations $\mathcal{I}^f$. Namely, we derive Eq. (5). This bound can be tighter by a constant factor than existing similar bounds from Ref. [13]. Along the way, we also derive Eq. (6).

To begin, we use $\dot{\rho} = -i[\rho, H]$ and Eq. (C9) to write

$$\mathcal{I}_C^f = \sum_{i \neq j} \frac{(p_i - p_j)^2}{m^f(p_i, p_j)} |H_{ij}|^2 \leq \sum_{i \neq j} \frac{(p_i - p_j)^2}{m^{\text{RLD}}(p_i, p_j)} |H_{ij}|^2, \quad (\text{I1})$$

where, in the second line, we use $m^f(p_i, p_j) \geq m^{\text{RLD}}(p_i, p_j)$ for all $f \in \mathbb{F}$. Then, using the definitions of the means $m^{\text{RLD}}(p_i, p_j)$ and $m^{\text{SLD}}(p_i, p_j)$, we find

$$\begin{aligned} \mathcal{I}_C^f &= \sum_{i \neq j} \frac{(p_i - p_j)^2 (p_i + p_j)}{2p_i p_j} |H_{ij}|^2 \\ &\leq \max_{p_i, p_j} \frac{(p_i - p_j)^2}{p_i p_j} \sum_{i \neq j} \frac{p_i + p_j}{2} |H_{ij}|^2 \\ &= \max_{p_i, p_j} \frac{(p_i - p_j)^2}{p_i p_j} (\Delta H_C)^2 \\ &\leq \max_{p_i, p_j} \frac{(p_i - p_j)^2}{p_i p_j} (\Delta H)^2. \end{aligned} \quad (\text{I2})$$

In the next-to-last line we use Eq. (C8), and in the last line we use that $(\Delta H_C)^2 \leq (\Delta H)^2$.

Now, letting $\kappa_{ij} := p_j/p_i$,

$$\max_{p_i, p_j} \left(\frac{(p_i - p_j)^2}{p_i p_j} \right) = \max_{\kappa_{ij}} \left(\frac{1}{\kappa_{ij}} + \kappa_{ij} - 2 \right). \quad (\text{I3})$$

This expression is maximized by either making κ_{ij} as large or small as possible (note that the expression above is identical with $\kappa_{ij} \rightarrow \kappa_{ij}^{-1}$). Without loss of generality, let us take $p_j \geq p_i$. Then the maximum $\kappa_{ij} = \max_{p_i, p_j} (p_j/p_i)$ is the condition number κ_ρ of ρ . Therefore,

$$\mathcal{I}_C^f \leq \left(\kappa_\rho + \frac{1}{\kappa_\rho} - 2 \right) (\Delta H)^2. \quad (I4)$$

One natural limit of this bound is when $\kappa_\rho = 1$ (for a maximally mixed state). Here, the bound becomes zero, matching expectations. In the other limit, as we approach pure states, $\kappa_\rho \rightarrow \infty$ and the bound becomes loose.

As a final simplification (and further loosening of the bound), we can use that $4(\Delta H)^2 \leq \|H\|_s^2$, where the seminorm is defined as the difference in the maximum and minimum eigenvalues of H [63]. This gives us a bound that depends only on the maximum and minimum eigenvalues of ρ and H :

$$\mathcal{I}_C^f \leq \frac{1}{4} \left(\kappa_\rho + \frac{1}{\kappa_\rho} - 2 \right) \|H\|_s^2. \quad (I5)$$

Finally, if there are incoherent dynamics and we assume the source of this nonunitary dynamics is entanglement with an environment via a Hamiltonian H^{int} that includes all terms with support on both the system and environment then, as proven in Ref. [13],

$$\mathcal{I}_I \leq 4(\Delta H^{\text{int}})^2 \leq \|H^{\text{int}}\|_s^2, \quad (I6)$$

where this variance is calculated on the joint state of the system and the environment.

Putting together Eqs. (I4) and (I6) gives the quantum speed limit:

$$|\dot{a}| \leq \left(\kappa_\rho + \frac{1}{\kappa_\rho} - 2 \right)^{1/2} \Delta^f A_C \Delta H + 2\Delta A_I \Delta H^{\text{int}}. \quad (I7)$$

As $\Delta^f A \leq \Delta A$ for all f , the tightest bound of this form will always correspond to $f = f_{\text{RLD}}$. Equation (I7) can be tighter than the existing bounds [13],

$$|\dot{a}| \leq 2\Delta A_C \Delta H + 2\Delta A_I \Delta H^{\text{int}}. \quad (I8)$$

For Eq. (I7) to be tighter than Eq. (I8), we require that

$$\frac{\kappa_\rho + \frac{1}{\kappa_\rho} - 2}{4} < \frac{(\Delta A_C)^2}{(\Delta^{\text{RLD}} A_C)^2}. \quad (I9)$$

Note that, for any state or observable, Eq. (I9) is satisfied if

$$\kappa_\rho + \kappa_\rho^{-1} - 2 < 4 \Rightarrow \kappa_\rho < 3 + 2\sqrt{2}, \quad (I10)$$

as the smallest $(\Delta A_C)^2/(\Delta^{\text{RLD}} A_C)^2$ can be 1.

As a particular scenario, consider the example in the main text. Recall in this example we consider a qutrit with coherent dynamics driven by a Hamiltonian $\propto (i|0\rangle\langle 1| + i|1\rangle\langle 2|) + \text{H.c.}$ and the observable of interest is $A = A_C = |0\rangle\langle 1| + |1\rangle\langle 0|$. In Fig. 5, we plot the ratio of the bound in Eq. (I7) with $f = f_{\text{RLD}}$ to the bound in Eq. (I8) as a function of the eigenvalues p_0, p_1, p_2 of ρ .

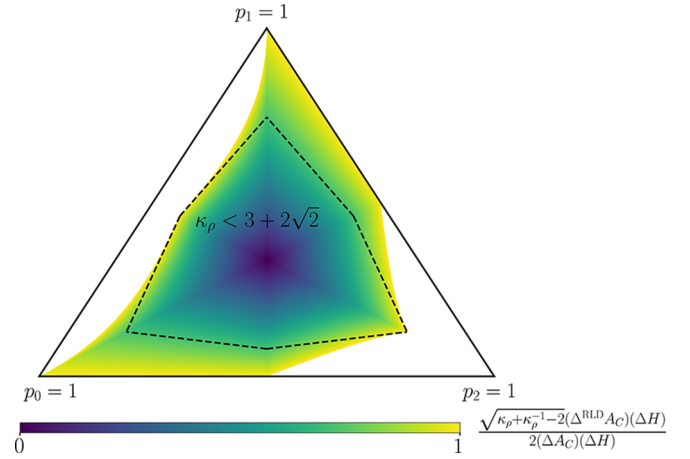


FIG. 5. For the qutrit example of the main text, plot of the ratio of the bound in Eq. (I7) with $f = f_{\text{RLD}}$ to the bound in Eq. (I8) as a function of the eigenvalues p_0, p_1, p_2 of ρ , plotted in barycentric coordinates. The region where $\kappa < 3 + 2\sqrt{2}$ is marked by the dashed lines. In this region, the bound is guaranteed to be tighter for any observable A . However, there is also a large (state, and observable-dependent) region of parameter space beyond this region where the bound is still tighter.

2. A bound in Terms of Wigner-Yanase skew information

Similar manipulations to the previous section allow us to derive another interesting bound on the generalized quantum Fisher information for arbitrary f in terms of the so-called Wigner-Yanase skew information [35]. The result will be Eq. (7).

In particular, starting with Eq. (II), we obtain the bound

$$\begin{aligned} I_C^f &\leq 4 \max_{p_i, p_j} \frac{1}{m^{\text{RLD}}(p_i, p_j)} \sum_{i \neq j} m^{\text{WY}}(p_i, p_j) |H_{ij}|^2 \\ &= 4 \max_{ij} \frac{1}{m^{\text{RLD}}(p_i, p_j)} \mathcal{I}_C^{\text{WY}}, \end{aligned} \quad (II1)$$

where, from Table II,

$$m^{\text{WY}}(p_i, p_j) = \frac{(\sqrt{p_i} + \sqrt{p_j})^2}{4}, \quad (II2)$$

and we used that $(p_i - p_j)^2 \leq (\sqrt{p_i} + \sqrt{p_j})^2$ for $p_i, p_j \in [0, 1]$. Here, $\mathcal{I}_C^{\text{WY}}$ is the coherent part of the Wigner-Yanase skew information.

As the incoherent part of the generalized quantum Fisher information is the same for all f , Eq. (II1) implies that

$$I^f \leq 4 \max_{p_i, p_j} \frac{1}{m^{\text{RLD}}(p_i, p_j)} \mathcal{I}^{\text{WY}}. \quad (II3)$$

As in the previous section, let $\kappa_{ij} := p_j/p_i$. Then, the prefactor is

$$\begin{aligned} 4 \max_{p_i, p_j} \frac{1}{m^{\text{RLD}}(p_i, p_j)} &= 2 \max_{p_i, p_j} \left(\frac{p_i + p_j}{p_i p_j} \right) \\ &= 2 \max_{p_i, p_j} (\kappa_{ij} + \kappa_{ij}^{-1}) = 2 \left(\kappa_\rho + \frac{1}{\kappa_\rho} \right), \end{aligned} \quad (II4)$$

where κ_ρ is the condition number of ρ .

Thus, we obtain the following bound on the generalized quantum Fisher information in terms of the Wigner-Yanase skew information and the condition number [Eq. (7)]:

$$I^f \leq 2 \left(\kappa_\rho + \frac{1}{\kappa_\rho} \right) \mathcal{I}^{\text{WY}}. \quad (\text{I15})$$

Similar to the previous section, this can be viewed as a generalization of a known bound [34] relating the SLD quantum Fisher information and the Wigner-Yanase skew information [35]:

$$\mathcal{I}^{\text{SLD}} \leq 2 \mathcal{I}^{\text{WY}}. \quad (\text{I16})$$

Note that Eq. (I15) is looser than the known bound in Eq. (I16) for $f = f_{\text{SLD}}$. This is to be expected, however, as our bound holds for all f , and $\mathcal{I}^{\text{SLD}}$ is the smallest of the generalized quantum Fisher informations.

3. Bound comparison

Either one of the bounds derived in the previous two subsections can be tighter than the other. To allow for a direct comparison to Eq. (I4), let us ignore the incoherent part of the

latter bound in Eq. (7):

$$I_C^f \leq 2 \left(\kappa_\rho + \frac{1}{\kappa_\rho} \right) \mathcal{I}_C^{\text{WY}}. \quad (\text{I17})$$

For convenience, we also rewrite Eq. (I4) here:

$$\mathcal{I}_C^f \leq \left(\kappa_\rho + \frac{1}{\kappa_\rho} - 2 \right) (\Delta H)^2. \quad (\text{I18})$$

To see that either bound can be tight while the other is not, it is sufficient to consider two scenarios: First, consider a full-rank state that is an incoherent mixture in the energy eigenbasis. Essentially by definition, $\mathcal{I}_C^f = 0$, and, indeed, the bound in Eq. (I17) is zero as $\mathcal{I}_C^{\text{WY}} = 0$. On the other hand, $\Delta H \neq 0$ and κ_ρ can be quite large, so Eq. (I17) can be much tighter than Eq. (I18).

In the other direction, consider a state $\rho \propto \exp(-\beta Q)$ for some sufficiently small β and Q not diagonal in the eigenbasis of H . For small β , the bound in Eq. (I18) can be made arbitrarily small. On the other hand, Eq. (I17) does not approach zero as the condition number term ≥ 4 , and $\mathcal{I}_C^{\text{WY}}$ will not vanish as Q is not diagonal in the eigenbasis of H .

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