

Emulation of self-consistent non-Hermitian quantum formalisms

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Hermiticity has long been a cornerstone of quantum theory, ensuring the conservation of probability and real energy spectra. Recent advances in non-Hermitian physics have revealed that this requirement can be relaxed, giving rise to richer dynamics where the Hilbert space is endowed with a time-dependent inner product defined by a metric operator $\rho(t)$. This dynamical metric fundamentally reshapes quantum predictions, yet direct access to it on physical devices has remained elusive. Here, we bridge this gap by presenting general, platform-agnostic operator-dilation schemes that embed the metric formalism into closed Hermitian systems. Our proposed framework converts any time-dependent non-Hermitian Hamiltonian into a larger unitary evolution, bringing the theory from a purely mathematical curiosity into the experimentally testable realm. This permits the first controlled study of $\mathcal{PT}$ -symmetric and symmetry-broken regimes at arbitrarily long timescales. We demonstrate the power of this approach on a digital quantum simulator, showing that cutting-edge questions about non-Hermiticity are already within reach of today's noisy intermediate-scale quantum processors. Our results provide the first direct evidence of a dynamically evolving metric in a single qubit, establishing a new paradigm for simulating non-Hermitian quantum mechanics and opening the door to its realization in the laboratory.

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I. INTRODUCTION

Hermiticity of the Hamiltonian lies at the core of standard quantum mechanics. This allows for the interpretation of the Hamiltonian as an energy observable and ensures unitary time evolution. Under this assumption, both the inner product structure and the norms of states in the Hilbert space are preserved in time, allowing for a probabilistic interpretation of quantum mechanics [1]. In the past decade, there has been extensive focus on non-Hermitian Hamiltonians [2–9]. A non-Hermitian Hamiltonian that commutes with the parity-time-reversal operator ($\mathcal{PT}$) can possess a real spectrum [2,10], while exceptional points (EPs) and complex eigenvalues emerge in the spectra when this $\mathcal{PT}$ symmetry is spontaneously broken [3,11,12]. These features lead to a plethora of rich physics with no Hermitian counterparts [13–17].

The true richness of non-Hermitian quantum mechanics manifests in the time domain. As state norms are not preserved during nonunitary evolution, the oft-used approach to obtain well-defined expectation values is to simply normalize the time-evolved states with respect to their time-dependent norms. For the sake of clarity, we henceforth refer to this approach as the *norm method*. This approach correctly describes the non-Hermitian dynamics of quantum trajectories without

quantum jumps in open quantum systems. It leads to unconventional phenomena [18–24], such as faster-than-Hermitian evolution [7,14], violation of Lieb-Robinson bounds for information propagation [25], and quantum cloning [15], to name a few.

An alternative self-consistent framework, the *metric formalism* [5,8,9,26], allows for a consistent definition of probabilities and expectation values. This is a generalization of the better-known biorthogonal quantum mechanics [27]. For a system described by a general non-Hermitian Hamiltonian $H(t)$, the metric framework postulates the emergence of a modified Hilbert space ($\mathcal{H}_\rho, H(t)$) with a consistently determined dynamical inner product weighted by the metric $\langle \cdot, \cdot \rangle_{\text{metric}} := \langle \cdot | \rho(t) | \cdot \rangle$ [5,8,9,26]. This restores unitary evolution within the modified Hilbert space, resulting in a consistent probabilistic interpretation. The metric $\rho(t)$ is a Hermitian time-dependent operator that evolves according to [9]

$$i\dot{\rho}(t) = H^\dagger(t)\rho(t) - \rho(t)H(t), \quad (1)$$

which incorporates the dynamics of both $H(t)$ and $H^\dagger(t)$. It is straightforward to show that if $\rho(t)$ is the metric operator corresponding to $H(t)$, then $\rho^{-1}(t)$ is the metric operator of $H^\dagger(t)$. Further, this formalism works for both $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken Hamiltonians. We note that $\rho(t)$ is neither a metric in the strict sense of a map in a metric space nor does it correspond to the quantum geometric tensor discussed in Refs. [28,29]. The metric dynamics drastically alters the physics, as evinced by the restoration of information bounds in the quantum Brachistochrone problem [7] and the violation of quantum adiabaticity via defect freezing of $\mathcal{PT}$ -broken modes [30]. These studies show that this self-consistent formulation of quantum mechanics, encapsulated by the metric, harbors a phenomenological richness yet to be explored.

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An intriguing question is whether the metric formalism is purely of theoretical interest or can be realized in an experiment, especially when nature is described by effectively closed Hermitian systems or open systems. Earlier works, such as the Naimark dilation protocols [12,20,31,32], use what seem metriclike operators to encode a fully non-Hermitian Hamiltonian in a large Hermitian system, but it is important to note that these do not correspond to the metric formalism of quantum mechanics discussed here. In this work, we provide a pathway to realizing the metric framework of non-Hermitian quantum mechanics and obtain the first direct hardware-based measurement of the dynamical metric alongside other observables. Using the operator-dilation theorem [33], we present two new dilation schemes that embed the non-Hermitian system of interest in a larger Hermitian system. Both the norm method and the metric formalism emerge naturally in this embedding, and each can be probed via straightforward projective measurements. In a closed non-Hermitian system, the metric cannot be directly accessed [34] as the metric defines the inner product. Within the dilation framework, an observer whose access is confined to the reduced non-Hermitian system cannot measure the metric, whereas an observer with access to the enlarged system-ancilla space can reconstruct this metric operationally via joint measurements.

We implement our dilation protocols for a two-level non-Hermitian system on IBM Quantum's superconducting backend `ibm_Kyiv`. Using tomographic reconstructions, we track the time-resolved metric $\rho(t)$, which is a key ingredient in the self-consistent framework of non-Hermitian quantum mechanics. In doing so, we present the first physical implementation of generalized non-Hermitian quantum theory in both $\mathcal{PT}$ -symmetric and symmetry-broken regimes. Our work transforms this formerly theoretical construct into a practical quantum-simulation framework, enabling controlled studies of self-consistent non-Hermitian dynamics on today's noisy intermediate-scale quantum (NISQ) hardware and establishing simulators of generalized non-Hermitian quantum mechanics as a new frontier that bridges fundamental concepts with near-term devices.

The paper is organized as follows: In Sec. II, we present the theoretical framework underpinning the metric formalism, as well as the dilation theory. In Sec. III, we present the main results of this work, which encompass a general Dilation framework, the Biorthogonal Naimark dilation (BoNd), and the Generalized Biorthogonal Naimark dilation (GBoNd). In Secs. IV and V, we present our discussion and conclusion, respectively, touching on possible further studies and directions of interest to our framework and summarizing results of this work.

II. THEORY AND FRAMEWORK

A. $\mathcal{PT}$ symmetry

Relaxing the Hermiticity condition of the Hamiltonian H by allowing anti-Hermitian components introduces nonunitary time evolution, which leads to a nonconservation of the norm. However, a non-Hermitian Hamiltonian H can have real spectrum provided it has $\mathcal{PT}$ symmetry. In this case,

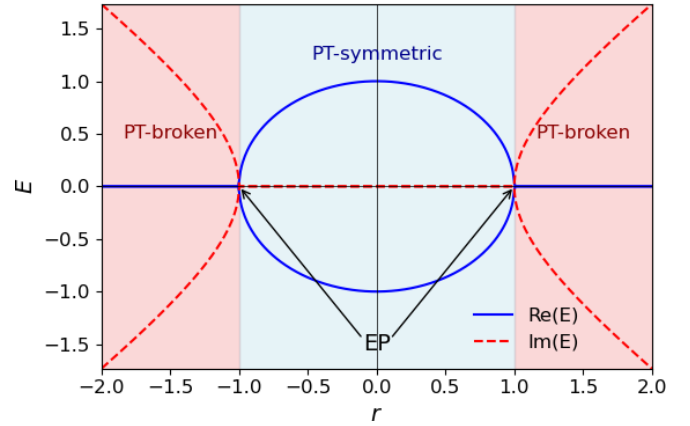


FIG. 1. The energy spectrum $E_{\pm} = \pm\sqrt{1-r^2}$ of a two-level non-Hermitian Hamiltonian $H = \sigma_x + ir\sigma_z$. The blue region highlights the $\mathcal{PT}$ -symmetry unbroken ($\mathcal{PT}$ -symmetric) regime, while the red region highlights the $\mathcal{PT}$ -symmetry broken ($\mathcal{PT}$ -broken) regime. At the values $r = \pm 1$, the quantum phase transition from the $\mathcal{PT}$ -symmetric to the $\mathcal{PT}$ -broken regime appears. This border is known as an exceptional point.

the eigenvectors of H are also eigenvectors of the discrete symmetry operator $\mathcal{PT}$, where $\mathcal{P}$ is a parity operator and $\mathcal{T}$ is the time-reversal operator [2]. However, at an EP, this $\mathcal{PT}$ symmetry can be spontaneously broken, wherein the energy spectrum becomes complex and the eigenvalues of H are no longer eigenvalues of $\mathcal{PT}$. An example of such a spectrum for a two-level system is shown in Fig. 1.

B. The metric formalism

As mentioned earlier, non-Hermiticity implies that the time propagator loses its unitarity and the state norm is no longer conserved. Expectation values of an observable are then redefined as [35]

$$\langle O(t) \rangle_{\text{norm}} = \frac{\langle \psi(t) | O(t) | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}. \quad (2)$$

We refer to this approach as the norm method. A more consistent approach is the metric framework, where the quantum evolution can be described in two physically different representations: (1) a conventional Hilbert space $(\mathcal{H}, h(t))$ with the Hermitian Hamiltonian $h(t)$ and standard inner product $\langle \cdot, \cdot \rangle := \langle \cdot | \cdot \rangle$, and (2) a modified Hilbert space $(\mathcal{H}_{\rho}, H(t))$ where evolution is governed by a non-Hermitian Hamiltonian $H(t)$, and the dynamical inner product is weighted by a Hermitian positive-definite time-dependent metric $\langle \cdot, \cdot \rangle_{\text{metric}} = \langle \cdot, \cdot \rangle_{\rho(t)} := \langle \cdot | \rho(t) | \cdot \rangle$, where $\rho(t)$ is the metric operator [5,8,26]. We can therefore define the expectation values with respect to $\langle \cdot, \cdot \rangle_{\text{metric}}$ as

$$\langle O(t) \rangle_{\text{metric}} = \langle \psi(t) | \rho(t) O(t) | \psi(t) \rangle, \quad (3)$$

where the expectation value is Hermitian when $\rho(t)O(t) = O^{\dagger}(t)\rho(t)$ is fulfilled. These two descriptions, which we refer to as the metric formalism, are equivalent to each other, but the physical interpretations differ: The norm method describes an open quantum system, while the metric formalism gives a description of a closed quantum system. The so-

called metric $\rho(t)$ evolves according to the time-dependent quasi-Hermiticity (tdqH) [Eq. (1)]. In our work, we set the initial condition of the metric operator to $\rho(t_0) = \mathbb{I}$, since we assume that these initial state can be reached through a unitary evolution. The metric can therefore be written as $\rho(t) = U_{H^\dagger}(t, t_0)U_H(t_0, t)$, where $U_H(t, t_0)$ and $U_{H^\dagger}(t, t_0)$ are the individual Schrödinger time evolution operators for $H(t)$ and $H^\dagger(t)$, respectively. Clearly, $\rho(t) = \mathbb{I}$ iff $H^\dagger(t) = H(t)$, implying that the definition of the dynamical norm reduces to the standard norm, $\langle \cdot, \cdot \rangle := \langle \cdot | \cdot \rangle = \langle \cdot | \rho(t) | \cdot \rangle |_{\rho(t)=\mathbb{I}}$, for Hermitian Hamiltonians. For a constant, $\mathcal{PT}$ -symmetric Hamiltonian with real eigenvalues, a constant solution to Eq. (1) exists [8,27,35,36], where $\rho_C H = H^\dagger \rho_C$ and $\dot{\rho}_C = 0$. However, other initial conditions, including $\rho(t_0) = \mathbb{I}$, impose a dynamical metric [6,9]. For further details on the connection between the dynamical metric $\rho(t)$ and the constant metric ρ_C , we refer to Appendix A.

We note that, via the factorization into its principal square root $\rho(t) = \eta^2(t)$ [9,30], $\rho(t)$ dictates a mapping to an equivalent Hermitian system in terms of a conventional Hilbert space $(\mathcal{H}, h(t))$. This Hilbert space features a standard inner product $\langle \cdot, \cdot \rangle := \langle \cdot | \cdot \rangle$, and the dynamics is governed by a Hermitian Hamiltonian $h(t)$ defined by

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\dot{\eta}(t)\eta^{-1}(t), \quad (4)$$

which can be shown to be Hermitian:

$$\begin{aligned} h^\dagger(t) &= \eta^{-1}(t)H^\dagger(t)\eta(t) - i\eta^{-1}(t)\dot{\eta}(t) \\ &= \eta^{-1}(t)[\rho(t)H(t)\rho^{-1}(t) + i\dot{\rho}(t)]\eta(t) - i\eta^{-1}(t)\dot{\eta}(t) \\ &= \eta(t)H(t)\eta^{-1}(t) + i\dot{\eta}(t)\eta^{-1}(t) = h(t), \end{aligned}$$

wherein the second equality comes from the tdqH [Eq. (1)]. Note that the transformation shown in Eq. (4) is given by a nonunitary matrix $\eta(t)$, which, unlike the typical rotating frame transformations found for closed quantum systems, does not preserve unitarity. The state evolution in both Hilbert spaces is governed by the Schrödinger equation: $i|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle$ and $i|\dot{\Psi}(t)\rangle = h(t)|\Psi(t)\rangle$, where the time-evolved states are related by $|\Psi(t)\rangle = \eta(t)|\psi(t)\rangle$. The state evolved with the Hermitian conjugate, $H^\dagger(t)$, is related to $|\psi(t)\rangle$ by the metric, i.e., $i\frac{d}{dt}(\rho(t)|\psi(t)\rangle) = H^\dagger(t)(\rho(t)|\psi(t)\rangle)$. Clearly, $\rho^{-1}(t)$ is the metric operator corresponding to $H^\dagger(t)$ as

$$\begin{aligned} i\rho^{-1}(t) &= -i\rho^{-1}(t)\dot{\rho}(t)\rho^{-1}(t) \\ &= -\rho^{-1}(t)H^\dagger(t) + H(t)\rho^{-1}(t) \\ &= H(t)\rho^{-1}(t) - \rho^{-1}(t)H^\dagger(t), \end{aligned}$$

which showcases the Hermitian conjugate symmetry that will play a role in our later consideration.

C. Non-Hermitian two-level systems

We now present the example of a two-level non-Hermitian system, where we can obtain analytical expressions for the metric operator $\rho(t)$, as well as the resulting Hermitian Hamiltonian $h(t)$. This example will then be later used to benchmark our dilation protocols discussed in Sec. III A. Consider a static non-Hermitian Hamiltonian of the form

$$H = \vec{V} \cdot \vec{\sigma} = a\hat{v} \cdot \vec{\sigma},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ is the vector containing the Pauli matrices σ_i , $i \in \{x, y, z\}$, $\vec{V} = a \cdot \hat{v}$, $\hat{v}$ is a complex vector with $\hat{v}^2 = 1$, $V \in \mathbb{C}^3$, and $\vec{\sigma}$ is the Pauli vector. If $\vec{V} \in \mathbb{R}^3$, the Hamiltonian remains Hermitian, whereas in the more general complex case it is non-Hermitian. For the case corresponding to the $\mathcal{PT}$ -symmetric region, where $a \in \mathbb{R}$, we can write the time propagator of this Hamiltonian as

$$U_H(t, t_0 = 0) = e^{-ita\hat{v} \cdot \vec{\sigma}} = \cos(at)\mathbb{I} - i\sin(at)\hat{v} \cdot \vec{\sigma}.$$

For $\mathcal{PT}$ -broken Hamiltonians, where $a \in i\mathbb{R}$, Sec. IIC still holds, but with the replacements: $\cos \rightarrow \cosh$, $\sin \rightarrow i\sinh$. The point $a = 0$ is the exceptional point that separates $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken regimes. The metric operator $\rho(t) = U_{H^\dagger}(t)U_H(-t)$ is now given by

$$\begin{aligned} \rho(t) &= (\cos^2(at) + \sin^2(at)(\hat{v}^* \cdot \hat{v}))\mathbb{I} \\ &\quad + (i\cos(at)\sin(at)(\hat{v} - \hat{v}^*) + i\sin^2(at)(\hat{v}^* \times \hat{v})) \cdot \vec{\sigma}, \end{aligned}$$

where $a \in \mathbb{R}$ or $a \in i\mathbb{R}$. We see that $\rho(t)$ is time periodic in the $\mathcal{PT}$ -symmetric region and its average over the period $T = \frac{2\pi}{a}$ yields the constant metric $\rho_C = \oint \rho(t)dt$ (see Appendix A):

$$\rho_C = \frac{a}{2}(1 + \hat{v}^* \cdot \hat{v})\mathbb{I} + \frac{a}{2}(i\hat{v}^* \times \hat{v}) \cdot \vec{\sigma}.$$

This satisfies the usual pseudo-Hermiticity condition, $\rho_C H = H^\dagger \rho_C$ [8,27,35,36]. For the choice $V = (1, 0, ir)^T$, the metric operator simplifies to

$$\begin{aligned} \rho(t) &= \left(\cos^2(wt) + \frac{(1+r^2)\sin^2(wt)}{w^2} \right) \mathbb{I} \\ &\quad + \frac{2\sin(wt)}{w} \left(\frac{\sin(wt)r}{w} \sigma_y - \cos(wt)\sigma_z \right), \end{aligned}$$

where $\beta = \tanh^{-1}(r)$ and $w = \sqrt{1-r^2}$. From the principal root of the metric operator $\eta(t)$, we can derive the Hermitian Hamiltonian counterpart $h(t)$:

$$h(t) = \frac{1-r^2}{1-r^2\cos(\omega t)} \sigma_x.$$

D. Dilation framework

The operator-dilation theorem permits the implementation of nonunitary evolutions with the help of an ancilla system and projective measurements. An example of a dilation scheme is the Naimark dilation [12,20,31,32] (see Appendix B for details), an established method in quantum information, where the total system-ancilla state takes the form

$$|\Psi_N(t)\rangle = |\psi(t)\rangle_s \otimes |0\rangle_a + \tilde{\eta}(t)|\psi(t)\rangle_s \otimes |1\rangle_a, \quad (5)$$

with $\tilde{\eta}(t) = \sqrt{\rho(t) - \mathbb{I}}$. This protocol requires an initial state preparation of the ancilla state, such that the total initial state becomes $|\Psi(t_0)\rangle = |\psi(t_0)\rangle_s \otimes (|0\rangle_a + \sqrt{\rho(t_0) - \mathbb{I}}|1\rangle_a)$, and the system state as well as the ancilla state is not entangled. Projecting onto the $|0\rangle_a$ state of the ancilla, one realizes the nonunitary evolution of $|\psi(t)\rangle_s$ corresponding to the usual norm method for non-Hermiticity. This dilation has been implemented in both solid-state and quantum circuit settings [12,20,31]. However, this implementation does not capture the metric dynamics, for $\tilde{\eta}(t)|\psi(t)\rangle$ cannot be

straightforwardly mapped to the dynamics of the Hamiltonian $h(t)$. Furthermore, $\tilde{\eta}(t)$, and consequently Eq. (B1), is generally ill-defined beyond a certain time, as $\rho(t) - \mathbb{I}$ is not guaranteed to be positive semidefinite in the $\mathcal{PT}$ -broken regime. The latter is attributable to the exponential decay of at least one of the eigenvalues of $\rho(t)$ or $\rho^{-1}(t)$ with time [37].

III. MAIN RESULTS

In this work, we propose a new dilation scheme fundamentally different from the Naimark dilation schemes, which allows us to access all $\mathcal{PT}$ -symmetry regimes up to arbitrary timescales. To this end, we note that

$$U_H(t, t_0) = \eta^{-1}(t) U_h(t, t_0) \eta(t_0), \quad (6)$$

where $U_h(t, t_0)$ is the time propagator corresponding to $h(t)$ given in Eq. (4). Since $\eta(t)$ is the only nonunitary operation in Eq. (6), we can specifically define a dilation on $\eta(t)$. In other words, labeling the general dilation of $\eta(t)$ as $\zeta(t)$, we effectively implement

$$U_{\text{tot}}(t, t_0) = \zeta^{-1}(t) (U_h(t, t_0) \otimes \mathbb{I}_a) \zeta(t_0), \quad (7)$$

where the second system is a two-level ancilla belonging to the Hilbert space $\mathcal{A}$ and $\mathbb{I}_a : \mathcal{A} \rightarrow \mathcal{A}$ is the identity operator and $\zeta(t)$ is a unitary operator, making the time propagator $U_{\text{tot}}(t, t_0)$ unitary. The Hamiltonian associated to Eq. (7) is given by $H_{\text{tot}}(t) = \zeta^{-1}(t) (h(t) \otimes \mathbb{I}_a) \zeta(t) + i \dot{\zeta}^{-1}(t) \zeta(t)$, with the Hilbert space given by $(\mathcal{H} \otimes \mathcal{A}, H_{\text{tot}}(t))$. Although $\zeta(t)$ shows structural similarity to $\eta(t)$ in the definition of $h(t)$ in Eq. (4), $\zeta(t)$ is higher dimensional and unitary, while $\eta(t)$ is generally nonunitary. Therefore, the transformation given by $\zeta(t)$ preserves the Hermiticity. A general form of the dilation $\zeta^{-1}(t)$ in Eq. (7) is given by

$$\zeta_{C(t)}^{-1}(t) = \frac{1}{\sqrt{C(t)}} \begin{bmatrix} \eta^{-1}(t) & \sqrt{C(t)\mathbb{I} - \rho^{-1}(t)} \\ \sqrt{C(t)\mathbb{I} - \rho^{-1}(t)} & -\eta^{-1}(t) \end{bmatrix}, \quad (8)$$

where $C(t)$ is a system-dependent real scalar function that can be chosen to ensure that the dilation is always well defined. We note that this dilation is not unique, but it is valid for a general system, as long as the $C(t)$ is chosen to be greater than the largest eigenvalue of $\rho^{-1}(t)$. With the choice $C(t) = 1$ in Eq. (8), we recover the Naimark dilation. We emphasize that the general dilation $\zeta_{C(t)}(t)$, defined in Eq. (7), acts on the total system state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{C(t)}} (|\psi(t)\rangle |0\rangle + \sqrt{C(t)\rho(t) - \mathbb{I}} |\psi(t)\rangle |1\rangle) \quad (9)$$

to yield $\zeta_{C(t)}(t) |\Psi(t)\rangle = \eta(t) |\psi(t)\rangle |0\rangle$. This permits us to write the total time propagation $U_{\text{tot}}(t, t_0)$, as defined in Eq. (7), such that

$$\begin{aligned} U_{\text{tot}}(t, t_0) |\Psi(t_0)\rangle &= \zeta_{C(t)}^{-1}(t) (U_h(t, t_0) \otimes \mathbb{I}_a) \zeta_{C(t_0)}(t_0) |\Psi(t_0)\rangle \\ &= \zeta_{C(t)}^{-1}(t) (U_h(t, t_0) \otimes \mathbb{I}_a) \eta(t_0) |\psi(t_0)\rangle |0\rangle \\ &= \zeta_{C(t)}^{-1}(t) \eta(t) |\psi(t)\rangle |0\rangle = |\Psi(t)\rangle. \end{aligned}$$

Specifically for a two-level system, we can use the equality $\rho(t) + \rho^{-1}(t) = \text{tr}[\rho(t) + \rho^{-1}(t)] D^{-1} \mathbb{I}$, where $D = \text{tr}[\mathbb{I}]$ to

define a scalar $C(t) = \text{tr}[\rho(t) + \rho^{-1}(t)] D^{-1}$. This allows us to simplify the square-root term in Eq. (8) to $\sqrt{C(t)\mathbb{I} - \rho^{-1}(t)} = \eta^{-1}(t)$. It is important to mention that this simplification only works for the special cases, where $\rho(t) + \rho^{-1}(t) \propto \mathbb{I}$, which will not be fulfilled in the general case, but is fulfilled for the special case of two-level systems.

A. Generalized Biorthogonal Naimark dilation

Having defined a general dilation of $\eta(t)$ to implement nonunitary dynamics, we now turn to a scheme that allows us to access both interpretations $(\mathcal{H}_\rho, H(t))$ and $(\mathcal{H}, h(t))$ in the metric formalism. Ideally, we want to obtain a final system-ancilla state of the form

$$|\Psi(t)\rangle \propto |\psi(t)\rangle_s \otimes |0\rangle_a + \eta(t) |\psi(t)\rangle_s \otimes |1\rangle_a, \quad (10)$$

where $|\psi(t)\rangle$ and $\eta(t)$ are as defined previously. This implies that the time evolution of $|\Psi(t)\rangle$, which spans the Hilbert space $(\mathcal{H} \otimes \mathcal{A}, H_{s,a}(t))$, is governed by a non-Hermitian total Hamiltonian $H_{s,a}(t) = H(t) \otimes |0\rangle\langle 0|_a + h(t) \otimes |1\rangle\langle 1|_a$, where $H(t)$ is the non-Hermitian Hamiltonian of interest and $h(t)$ is the Hermitian counterpart as introduced in Eq. (4). Consequently, we need an additional dilation to be able to realize the desired time evolution with unitary gates.

To this end, we introduce the metric operator $\rho_G(t) = \rho(t) \otimes |0\rangle\langle 0|_a + \mathbb{I} \otimes |1\rangle\langle 1|_a$, where $\rho_G(t)$ and $\rho(t)$ are the solutions of Eq. (1) corresponding to $H_{s,a}(t)$ and $H(t)$, respectively. The principal root of $\rho_G(t)$ is given by $\eta_G(t) = \eta(t) \otimes |0\rangle\langle 0|_a + \mathbb{I} \otimes |1\rangle\langle 1|_a$. We now add a second ancilla $\mathcal{B}$ to obtain the dilation of $\eta_G(t)$, given by Eq. (8). In the rest of the paper, we focus on two-level systems and set $C(t) = \text{tr}[\rho_G(t) + \rho_G^{-1}(t)] D^{-1}$, where $D = \text{tr}[\mathbb{I}_s]$. With this choice, the expression for the dilation simplifies to

$$\zeta_G^{-1}(t) = \frac{1}{\sqrt{C(t)}} \begin{bmatrix} \eta_G^{-1}(t) & \eta_G(t) \\ \eta_G(t) & -\eta_G^{-1}(t) \end{bmatrix}, \quad (11)$$

where D is the dimension of the non-Hermitian Hilbert space $\mathcal{H}$ and $\zeta_G(t)$ is now defined in the Hilbert space $\mathcal{H} \otimes \mathcal{A} \otimes \mathcal{B}$. The total time evolution operator is now given by $U_{\text{tot}}(t, t_0) = \zeta_G^{-1}(t) (U_h(t, t_0) \otimes \mathbb{I}_{a,b}) \zeta_G(t_0)$. The implementation of this dilation is schematically shown in Fig. 2. Considering an initial state $|\Psi(t_0)\rangle = |\psi(t_0)\rangle_s \otimes |+\rangle_a \otimes |+\rangle_b$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the final state $|\Psi(t)\rangle = U_{\text{tot}}(t, t_0) |\Psi(t_0)\rangle$ implementing our GBoNd protocol is given by

$$\begin{aligned} |\Psi(t)\rangle &= \frac{1}{\sqrt{C(t)}} (|\psi(t)\rangle_s |00\rangle + \rho(t) |\psi(t)\rangle_s |10\rangle \\ &\quad + \eta(t) |\psi(t)\rangle_s (|11\rangle + |01\rangle)). \end{aligned} \quad (12)$$

In the above equation, for the sake of readability, we have dropped the subscripts $|ij\rangle_{a,b} := |ij\rangle$ and the tensor product.

Our dilation scheme encompasses the full array of non-Hermitian physics: the norm method as well as the self-consistent metric formulation. Specifically, the different projections of the ancillas provide a direct probabilistic access to time-evolved states corresponding to the Hamiltonians $H(t)$, $H^\dagger(t)$, and $h(t)$, namely, $|\psi(t)\rangle$, $\rho(t) |\psi(t)\rangle$, and $\eta(t) |\psi(t)\rangle$. Note that the first two correspond to the right and left states in biorthogonal quantum mechanics [27] (see Appendix A). A particular advantage of our dilation protocol

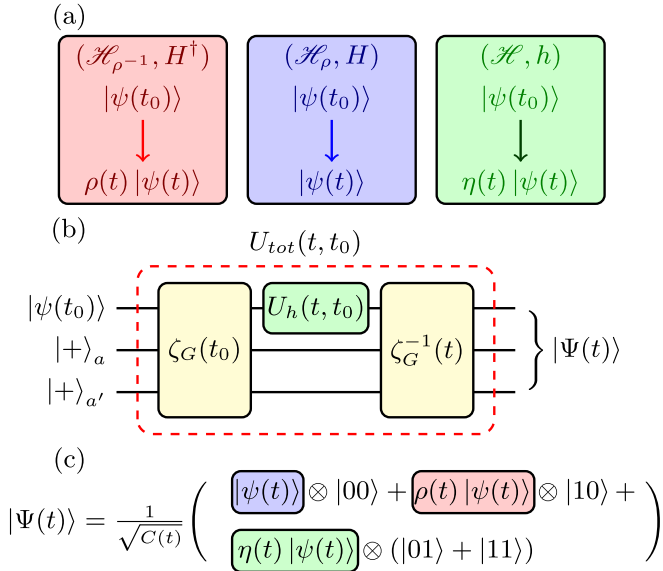


FIG. 2. (a) Schematic of the Hilbert spaces with their corresponding states evolving according to $H(t)$, $H^\dagger(t)$, and $h(t)$, respectively. The metric $\rho(t)$ and its principal root $\eta(t)$ serve as the mappings between the states. (b) The circuit layout of the Generalized Biorthogonal Naimark dilation, which consists of a system qubit and two ancilla qubits. This protocol can be understood as a unitary evolution $U_h(t, t_0)$ sandwiched by nontrivial dilations $\zeta_G(t)$ of $\rho_G^{1/2}(t) = \eta_G(t)$ as defined in Eq. (11). (c) The final state $|\Psi(t)\rangle$ obtained in the GBoNd protocol, where $C(t)$ is a normalization factor. The colors of the boxes indicate the components of the total states evolving under different Hamiltonians, as shown in panel (a). In the postselection after measuring the ancilla states, the effectively normalized states are measured: $|\psi(t)\rangle / |\psi(t)\rangle$, $\rho(t)|\psi(t)\rangle / |\rho(t)\psi(t)\rangle$, and $\eta(t)|\psi(t)\rangle$, with the third being normalized by construction.

is that via $|\psi(t)\rangle$ and $\rho(t)|\psi(t)\rangle$, we can directly measure the dynamical metric operator $\rho(t)$ using state tomography on the single states in postselection. This in turn allows us to define a self-contained protocol under a feedback loop, where the total state can evolve under $H_{\text{tot}}(t)$ for a small time step Δt . We then derive an approximation of the metric $\rho(t + \Delta t)$ and approximate the next $H_{\text{tot}}(t + \Delta t)$ with the help of the approximate metric to perform another time step on the total state (see Appendix E).

We now implement the two-ancilla GBoNd protocol in a quantum circuit, providing the first realization that enables direct physical access to the metric formalism and its dynamical inner product. To this end, we consider a single two-level system described by the non-Hermitian Hamiltonian

$$H = \sigma_x + ir\sigma_z. \quad (13)$$

This Hamiltonian has a spectrum $\pm\sqrt{1-r^2}$, which hosts exceptional points at $r = \pm 1$. For $|r| > 1$, the $\mathcal{PT}$ symmetry is spontaneously broken and the spectrum consists of complex conjugate pairs.

The total time propagator $U_{\text{tot}}(t, t_0)$, as shown in Fig. 2, can be realized with the help of an approximate quantum compiler [38] in a circuit layout as a sequence of universal gates and rotations. Our simulation is performed on the quantum processing unit ibm_Kyiv provided by the IBM. We choose

a circuit depth of 3 to approximate the evolution operator $U_{\text{tot}}(t, t_0)$ and use gate-twirling options to minimize errors. A minimum of $N = 10^3$ shots at each time step was considered to extract the states corresponding to the different dynamics H , $H^\dagger$, and $h(t)$. However, due to the probabilistic nature of our protocol, the true number of shots used is $N_{\text{eff}} = N/p_0$, where p_0 is the lowest projection probability corresponding to the desired ancilla configuration. Beyond such projective measurements, tomographic approaches can be used to directly obtain the metric operator. For further details regarding this tomography, we refer to Appendix D.

We focus on the three spin expectation values defined by

$$\langle \sigma_i \rangle = \frac{\langle \chi(t) | \sigma_i | \chi(t) \rangle}{\langle \chi(t) | \chi(t) \rangle}, \quad (14)$$

where $|\chi(t)\rangle \in \{|\psi(t)\rangle, \rho(t)|\psi(t)\rangle, \eta(t)|\psi(t)\rangle\}$, and $i \in \{x, y, z\}$. Our hardware simulation results for these spin expectation values are shown in Fig. 3. Note that the three states manifest different behaviors as expected. The expectation values evaluated in $|\psi(t)\rangle$ represent the norm method valid for dissipative systems, while that with $\eta(t)|\psi(t)\rangle$ represents the self-consistent metric formalism. We see that in all three states, the $\mathcal{PT}$ -symmetric regime is characterized by oscillatory behavior while asymptotic behaviors are observed in the $\mathcal{PT}$ -broken regimes. Clearly, the GBoNd protocol easily accesses the $\mathcal{PT}$ -broken regime $|r| > 1$, making it more versatile than the Naimark dilation [12,31,32]. In both regimes, we see that the observables calculated in the metric formalism exhibit qualitatively different behaviors when compared to those using the norm method. In particular, at the exceptional points $r = \pm 1$, $\langle \sigma_y \rangle \rightarrow 0$ and $\langle \sigma_z \rangle \rightarrow -1$ in the metric formalism, while the inverse happens in the norm method. The fundamentally different physical behaviors predicted by the two approaches highlight the possibility of novel phenomena in the self-consistent non-Hermitian realm. The dependence of the size of the error bars on r reflects ancilla projection probabilities, which can become small, and hence reduce the effective number of successful measurement events. Consequently, finite-shot statistics and hardware noise lead to larger statistical uncertainties for the corresponding expectation values.

B. Biorthogonal Naimark dilation

Having demonstrated a general dilation protocol using two ancillas, we turn to a simplified, one-ancilla scheme based on a dilation of $\eta(t)$. To this end, we replace $\eta_G(t)$ by $\eta(t)$ in Eq. (11) and set $C(t) = \text{tr}[\rho(t) + \rho^{-1}(t)]D^{-1}$, where $D = \text{tr}[\mathbb{I}_s]$. This allows us to define the total time propagator analogously to GBoNd, $U_{\text{tot}}(t, t_0) = \zeta^{-1}(t)(U_h(t, t_0) \otimes \mathbb{I})\zeta(t_0)$. Taking the initial state $|\Psi(t_0)\rangle = |\psi(t_0)\rangle_s \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, we have the time-evolved state

$$|\Psi(t)\rangle = \frac{1}{\sqrt{C(t)}}(|\psi(t)\rangle_s |0\rangle + \rho(t)|\psi(t)\rangle_s |1\rangle). \quad (15)$$

This dilation encompasses the states $|\psi(t)\rangle$ and $\rho(t)|\psi(t)\rangle$, which evolve with the Hamiltonians H and $H^\dagger$, respectively. They correspond to the right and left states in biorthogonal quantum mechanics [27]. Similarly to GBoNd, the BoNd protocol enables the access to the $\mathcal{PT}$ -broken regime for

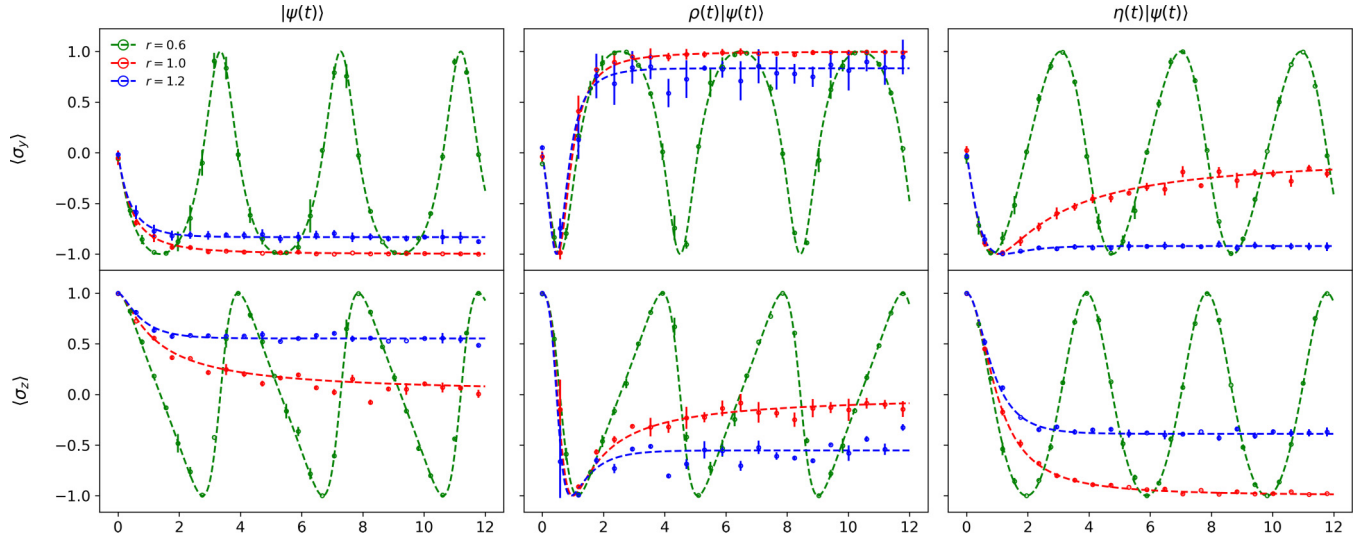


FIG. 3. The spin expectation values $\langle \sigma_y \rangle$ and $\langle \sigma_z \rangle$ calculated using the time-evolved states $|\psi(t)\rangle$ / $|\psi(t)\rangle$, $\rho(t)|\psi(t)\rangle$ / $|\rho(t)\psi(t)\rangle$ (corresponding to the norm method with H and $H^\dagger$, respectively), and $\eta(t)|\psi(t)\rangle$ (representing the metric formalism). The states are evolved using the GBoNd protocol with the non-Hermitian Hamiltonian defined in Eq. (13) and the initial condition $|\psi(t_0)\rangle = |0\rangle$. The time is in units of [s] = 1, and does not equate to physical time in the quantum circuit (see Appendix F). The dots and the corresponding error bars are measurement results from the IBM digital quantum device, while the dashed lines are the analytical solutions. For our Hamiltonian, $\langle \sigma_x \rangle = 0$ in all three states at all times. The figure demonstrates that a fixed, shallow-depth circuit faithfully reproduces the full non-Hermitian dynamics over arbitrary evolution times, including in the $\mathcal{PT}$ -symmetry broken regime.

arbitrary times. This is contrasted with the Naimark dilation [12,31,39], which is ill-defined for arbitrary times, since the total dilated Hamiltonian contains divergent terms like $\tilde{\eta}^{-1}(t)$. Reference [31] worked around the problem by redefining the prefactor C at every time step in the $\mathcal{PT}$ -broken regime. Therefore, the protocol samples $|\psi(t)\rangle$ from multiple total dilations, which means the total protocol is not a single Hermitian evolution.

To directly obtain $\rho(t)$, we implement the BoNd protocol for the Hamiltonian H in Eq. (13) with the same circuit setup used for GBoNd. Using state tomography on the wavefunction in Eq. (15), we obtain the states $|\psi(t)\rangle$, $\rho(t)|\psi(t)\rangle$ and the operator $\rho(t)/\text{tr}[\rho(t)]$. The latter two are shown in Fig. 4. As with GBoNd, there is good agreement with the analytical results in both $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken regimes. However, note that the BoNd does not permit the direct measurement of the observables within the metric formalism, contrary to GBoNd.

Figure 4 highlights an important aspect of the metric $\rho(t)$: It is time periodic in the $\mathcal{PT}$ -symmetric regime. Defining $\rho_C = \oint \rho(t)dt = e^{\beta\sigma_y}$, $\tanh(\beta) = r$, where $\oint$ denotes the time average over a period, we see that $\oint (H^\dagger \rho(t) - \rho(t)H)dt = \oint i \frac{d}{dt} \rho(t)dt = 0$. This implies that ρ_C , a stationary solution of Eq. (1), defines the pseudo-Hermiticity similarity transformation $H^\dagger \rho_C = \rho_C H$ expected in the $\mathcal{PT}$ -symmetric region of a constant Hamiltonian [8] (see Appendix A for further details). This reiterates that the dynamical metric encodes the stationary similarity transformation, making it the self-consistent approach for generic non-Hermitian Hamiltonians. Note that for general Hamiltonians H in the $\mathcal{PT}$ -symmetric regime, the time propagator is typically quasiperiodic. This also requires the usage of the more general dilation scheme given in Eq. (8), where the choice of $C(t)$

will be system specific. Nonetheless, the quasiperiodicity can be used to define $\rho'_C = \int_{t_0}^{t_1} dt \rho(t)$ where $t_1 \gg t_0$, such that $|H^\dagger \rho'_C - \rho'_C H| = |\rho(t_0) - \rho(t_1)| < \epsilon$ for some appropriately small bound $\epsilon > 0$.

IV. DISCUSSION

We have demonstrated that the dilation protocols introduced in this work completely capture the norm method as well as the metric formalisms used to describe the time evolution of a non-Hermitian system. Our general dilation framework shows that both formalisms are related to each other via a unitary operator $\zeta(t)$, which is completely determined by the metric operator $\rho(t)$ and a scalar function $C(t)$. Our scheme is valid for any finite-dimensional non-Hermitian system, regardless of its $\mathcal{PT}$ symmetry, where a problem-specific scalar function $C(t)$ needs to be specified. In Appendix E, we further show that the dilation framework allows for a quantum protocol based on a closed feedback loop, where the state is evolved under a given total Hamiltonian instance, the metric operator is reconstructed via state tomography, and the resulting metric is subsequently used to update the total Hamiltonian. This protocol can therefore handle non-Hermitian dynamics without recourse to analytical solutions, since it relies fully on the quantum simulator and state tomography.

Future studies could focus on the implications that our dilations could have in quantum computing with non-Hermitian dynamics, where restrictions such as the no-cloning theorem [15] and evolution speed bounds endemic to Hermitian dynamics [7,14] can be circumvented. Another interesting direction would include the scaling of our dilations to more complex systems, such as higher-dimensional generalizations

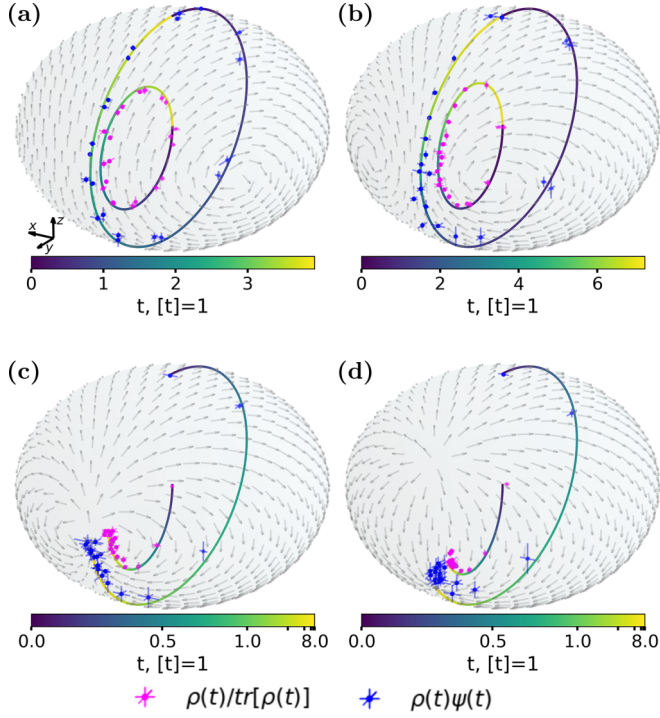


FIG. 4. The evolutions of the normalized state $\rho(t)|\psi(t)\rangle/|\rho(t)\psi(t)\rangle$ and metric $\rho(t)/\text{tr}[\rho(t)]$ under the non-Hermitian Hamiltonian $H^\dagger = \sigma_x - ir\sigma_z$, as defined in Eq. (13), with (a) $r = 0.6$, (b) $r = 0.9$, (c) $r = 1$, and (d) $r = 1.2$. The vector field points toward $(\partial_t |\chi(t)\rangle \langle \chi(t)|)dt$, where $|\chi(t)\rangle$ is a pure state on the Bloch sphere. The points are measurement results obtained from a physical circuit using the BoNd protocol, while the lines are the analytical solutions. The metric components were multiplied by a factor $f = 0.8$ for $r \geq 0.9$ to distinguish the two lines. The figure illustrates that the BoNd protocol enables simultaneous access to both state and metric dynamics with a fixed, shallow circuit, including across the $\mathcal{PT}$ -symmetry breaking point.

of our framework, where $\rho(t) + \rho^{-1}(t) \propto \mathbb{I}$ is either constructed or not a constraint for (G)BoNd. On the experimental side, estimating the overhead of state tomography for the implementation of the closed feedback loop protocol would be of interest for bigger systems.

V. CONCLUSION

Using novel dilation protocols, we show how different formulations of non-Hermitian quantum mechanics can be embedded into larger unitary dynamics, enabling controlled emulation of both $\mathcal{PT}$ -symmetric and symmetry-broken regimes. Implemented on a digital NISQ quantum simulator, our schemes achieve a high degree of accuracy as compared to analytic solutions with only three layers of gates. Their constant ancilla cost, shallow depth, and robustness to noise make them broadly applicable across platforms, establishing simulations of generalized non-Hermitian quantum mechanics as a powerful new frontier. This first physical realization transforms a theoretical construct into a practical framework for exploring the rich physics of speed limits [25], skin effects [40], beyond-Hermitian field theories [6], and the geometry of dynamical Hilbert spaces [7]

on today's hardware, to name a few. Our work thus greatly expands the experimentally accessible scope of quantum mechanics and provides strong impetus for future exploration of non-Hermitian physics, from many-body phenomena to curved-space analogies, seeding new paradigms of fundamental and applied research.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: BIORTHOGONAL QUANTUM MECHANICS

Given a $\mathcal{PT}$ -symmetric Hamiltonian H , it is possible to define the right and left states $|\psi(t)\rangle_R$ and $|\psi(t)\rangle_L$ that evolve with H and $H^\dagger$, respectively [2,27]. These correspond exactly to $|\psi(t)\rangle_R = |\psi(t)\rangle$ and $|\psi(t)\rangle_L = \rho(t)|\psi(t)\rangle$. If we define the left and right eigenstates, we can also see that in the $\mathcal{PT}$ -symmetric regime the Hamiltonian operator, and any operator, can be decomposed into a biorthogonal basis:

$$H|E_i^R\rangle = E_i^R|E_i^R\rangle, \quad (A1)$$

$$H^\dagger|E_j^L\rangle = E_j^L|E_j^L\rangle, \quad (A2)$$

$$\langle E_j^L|E_i^R\rangle = \delta_{ij} \Rightarrow H = \sum_i E_i^R|E_i^R\rangle\langle E_i^L|. \quad (A3)$$

This is how one typically tackles non-Hermiticity for constant non-Hermitian Hamiltonians. One defines a biorthogonal basis pair $\{|E_i^R\rangle, |E_i^L\rangle\}_{i \in 0, \dots, N}$, where N is the number of energy states. For a finite N , one can also derive the constant metric solution ρ_C for a constant $\mathcal{PT}$ -symmetric Hamiltonian H , which can be calculated as

$$U_H(t, t_0) = \sum_i e^{-i(t-t_0)E_i^L}|E_i^L\rangle\langle E_i^R|, \quad (A4)$$

$$\Rightarrow \rho(t) = \sum_{i,j} e^{-i(t-t_0)(E_i^L - E_j^R)}|E_i^L\rangle\langle E_j^L||\langle E_i^R|E_j^R\rangle, \quad (A5)$$

$$\rho_C := \lim_{T \rightarrow \infty} \frac{1}{T - t_0} \int_{t_0}^T \rho(t) dt, \quad (A6)$$

$$\Rightarrow \rho_C = \sum_i |E_i^L\rangle\langle E_i^L||\langle E_i^R|E_i^R\rangle \quad (A7)$$

$$= \sum_i |E_i^L\rangle\langle E_i^L|. \quad (A8)$$

ρ_C is the unique constant metric that fulfills

$$\rho_C H^\dagger = H \rho_C, \quad (A9)$$

$$\rho_C|E_i^R\rangle = |E_i^L\rangle. \quad (A10)$$

Note that this metric can only be found in the $\mathcal{PT}$ -symmetric regime of a constant Hamiltonian (since the eigenvectors also need to be constant), and it is the same object we described in the main text for the one-qubit case.

APPENDIX B: NAIMARK DILATION

In the existing Naimark-based scheme implemented in Ref. [12], the total system-ancilla state takes the form [12]

$$|\Psi_N(t)\rangle = |\psi(t)\rangle_s \otimes |0\rangle_a + \tilde{\eta}(t) |\psi(t)\rangle_s \otimes |1\rangle_a, \quad (\text{B1})$$

where $|\psi(t)\rangle_s$ evolves nonunitarily according to a finite-dimensional non-Hermitian Hamiltonian $H_s(t)$. The metric $\rho(t)$ corresponding to this evolution can be obtained by solving Eq. (1) using $H = H_s$. We can then identify $\tilde{\eta}(t) = \sqrt{\rho(t) - \mathbb{I}}$ in Eq. (B1). Projecting onto the $|0\rangle_a$ state of the ancilla, one obtains the normalized state $\frac{|\psi(t)\rangle_s}{\| |\psi(t)\rangle_s \|}$ corresponding to the usual division-by-norm method. The evolution of the total state $|\Psi_N(t)\rangle$ is governed by a dilated Hamiltonian $H_{s,a}(t)$, which is not uniquely determined. For example, one can define

$$H_{s,a}(t) = \Lambda(t) \otimes I + \Gamma(t) \otimes \sigma_z, \quad (\text{B2})$$

where

$$\Lambda(t) = \{H_s(t) + [i\dot{\tilde{\eta}}(t) + \tilde{\eta}(t)H_s(t)]\tilde{\eta}(t)\}\rho^{-1}(t), \quad (\text{B3})$$

$$\Gamma(t) = i[H_s(t)\tilde{\eta}(t) - \tilde{\eta}(t)H_s(t) - i\dot{\tilde{\eta}}(t)]\rho^{-1}(t). \quad (\text{B4})$$

However, this implementation is not viable for the study of metric dynamics for two reasons. First, the modified square root $\tilde{\eta}(t) = \sqrt{\rho(t) - \mathbb{I}}$ cannot be straightforwardly mapped to the metric square root, $\eta(t) = \sqrt{\rho(t)}$. Therefore, the dynamics of the state $\tilde{\eta}(t) |\psi(t)\rangle$, obtained by projecting onto the $|1\rangle_a$ state of the ancilla, cannot be directly used to study the dynamics under the Hermitian mapping $h(t)$ given by Eq. (2). Furthermore, $\tilde{\eta}(t)$, and consequently Eq. (B1), are generally ill-defined beyond a certain time in the $\mathcal{PT}$ -broken regime. This is due to the fact that $\rho(t) - \mathbb{I}$ is not guaranteed to be positive semidefinite in this regime. This stems from the exponential decay of at least one of the eigenvalues of $\rho(t)$ or $\rho^{-1}(t)$ with time [37]. We illustrate the second point above using the model considered in Ref. [12]. The non-Hermitian Hamiltonian of interest is given by

$$H_s = \sigma_x + ir\sigma_y, \quad (\text{B5})$$

where $r \in \mathbb{R}$. For the values $|r| \leq 1$, the system is in the $\mathcal{PT}$ -symmetric regime with real eigenvalues $\pm\omega = \pm\sqrt{1-r^2}$. The spectrum features exceptional points at $r = \pm 1$, beyond which the $\mathcal{PT}$ symmetry is spontaneously broken and the energy eigenvalues become complex conjugate pairs, $\pm i\tilde{\omega} = \pm i\sqrt{r^2 - 1}$.

We solve Eq. (1) using H_s to obtain the exact dynamics of the metric $\rho(t)$. We use the initial condition $\rho(t=0) = \eta_0^2 \mathbb{I}$ for reasons we will see below. Since $\rho(t)$ is Hermitian by construction, we can express it in terms of the Pauli matrices

$$\rho_k(t) = \rho_{0,k}(t)\mathbb{I} + \sum_{j=x,y,z} \rho_{j,k}(t)\sigma_j, \quad (\text{B6})$$

where its components are given by

$$\begin{aligned} \rho_0(t) &= \frac{\eta_0^2}{\omega^2} (1 - r^2 \cos(2\omega t)) = \frac{\eta_0^2}{\tilde{\omega}^2} (r^2 \cosh(2\tilde{\omega} t) - 1), \\ \rho_y(t) &= \frac{\eta_0^2}{\omega^2} 2r \sin^2(\omega t) = \frac{\eta_0^2}{\tilde{\omega}^2} 2r \sinh^2(\tilde{\omega} t), \\ \rho_z(t) &= -\frac{\eta_0^2}{\omega} r \sin(2\omega t) = -\frac{\eta_0^2}{\tilde{\omega}} r \sinh(2\tilde{\omega} t), \end{aligned} \quad (\text{B7})$$

and $\rho_x(t) = 0$. The determinant of the metric is constant, i.e., $\det(\rho(t)) = \rho_0(t)^2 - (\rho_y(t)^2 + \rho_z(t)^2) = \eta_0^4$.

For the square-root decomposition $\tilde{\eta}(t) = \sqrt{\rho(t) - \mathbb{I}}$ to exist, the eigenvalues of $\rho(t) - \mathbb{I}$ have to be non-negative. The eigenvalues are given by

$$\lambda_{\pm} = (\rho_0 - 1) \pm \sqrt{\rho_y^2 + \rho_z^2} = (\rho_0 - 1) \pm \sqrt{\rho_0^2 - \eta_0^4}, \quad (\text{B8})$$

where the first term $\rho_0 - 1 = \frac{(\eta_0^2 - 1) + r^2(1 - \eta_0^2 \cos(2\omega t))}{\omega^2} = \frac{(\eta_0^2 - 1) + r^2(\eta_0^2 \cosh(2\tilde{\omega} t) - 1)}{\tilde{\omega}^2}$. Using this expression, we can see that $\min(\lambda_+) = \eta_0^2 - 1$ for all values of r , and this occurs at time $t^* = \frac{n\pi}{\omega}$, $n \in \mathbb{Z}$, for $|r| \leq 1$ and $t^* = 0$ for $|r| > 1$. From this, we conclude that $\lambda_+ \geq 0$ as long as $\eta_0 \geq 1$. We perform the same analysis for the eigenvalue λ_- . For $|r| \leq 1$, we find that the minimum of λ_- occurs at $t^* = \frac{(2n+1)\pi}{2\omega}$, $n \in \mathbb{Z}$, with the value $\min(\lambda_-) = \frac{\eta_0^2}{\omega^2} (1 - |r|^2) - 1$, which means that $\lambda_- \geq 0$ as long as $\eta_0^2 \geq \frac{1-r^2}{(1-|r|)^2}$. Therefore, in the $\mathcal{PT}$ -symmetric regime, the dilation scheme can be valid for arbitrarily long times by tuning the initial value η_0 . However, the issue stems from the eigenvalue λ_- in the $\mathcal{PT}$ -broken regime. From the expression

$$\begin{aligned} \lambda_- &= -1 + \frac{\eta_0^2}{\tilde{\omega}^2} \left[r^2 \cosh(2\tilde{\omega} t) - 1 - |r| \sqrt{\cosh(2\tilde{\omega} t) - 1} \right. \\ &\quad \left. \times \sqrt{(r^2 - 1) + (r^2 \cosh(2\tilde{\omega} t) - 1)} \right], \end{aligned} \quad (\text{B9})$$

we see that $\lim_{t \rightarrow \infty} \lambda_- = -1$. Therefore, the dilation scheme given by Eq. (B1) is not well defined for arbitrarily long times, as $\lim_{t \rightarrow \infty} \rho(t) - \mathbb{I}$ is not positive semidefinite. One possible fix of this issue is to only consider time evolution up to a finite time $t_{\max}$. Defining

$$\begin{aligned} f_r(t) &:= r^2 \cosh(2\tilde{\omega} t) - 1 - |r| \sqrt{\cosh(2\tilde{\omega} t) - 1} \\ &\quad \times \sqrt{(r^2 - 1) + (r^2 \cosh(2\tilde{\omega} t) - 1)}, \end{aligned} \quad (\text{B10})$$

we can choose an initial condition

$$\eta_0^2 \geq \frac{r^2 - 1}{f_r(t_{\max})} \quad (\text{B11})$$

to ensure that Eq. (B1) is valid until $t = t_{\max}$. In the $\mathcal{PT}$ -symmetry broken regime, the function $f_r(t_{\max})$ becomes exponentially small, forcing us to choose large η_0 values. The initial condition of the total state will therefore be $|\Psi_N(t_0)\rangle = |\psi(t_0)\rangle_s \otimes (|0\rangle_a + \sqrt{\eta_0^2 - 1} |1\rangle_a) \approx |\psi(t_0)\rangle_s \otimes |1\rangle$, and hence, the weight of the ancilla $|1\rangle$ will dominate in the evolution (see Fig. 5 for an example of the Naimark dilation breaking down).

Using a concrete model studied in Ref. [12], we have demonstrated that the Naimark dilation breaks down at an arbitrarily long time in the $\mathcal{PT}$ -broken regime. This stems from the breakdown of the positive semidefiniteness of the term $\rho(t) - \mathbb{I}$, which appears in the dilated state given by Eq. (B1). We note that the invalidity of the dilation scheme in the $\mathcal{PT}$ -broken regime is a general result shown in Ref. [41]. The scheme could still be valid for a finite time in the

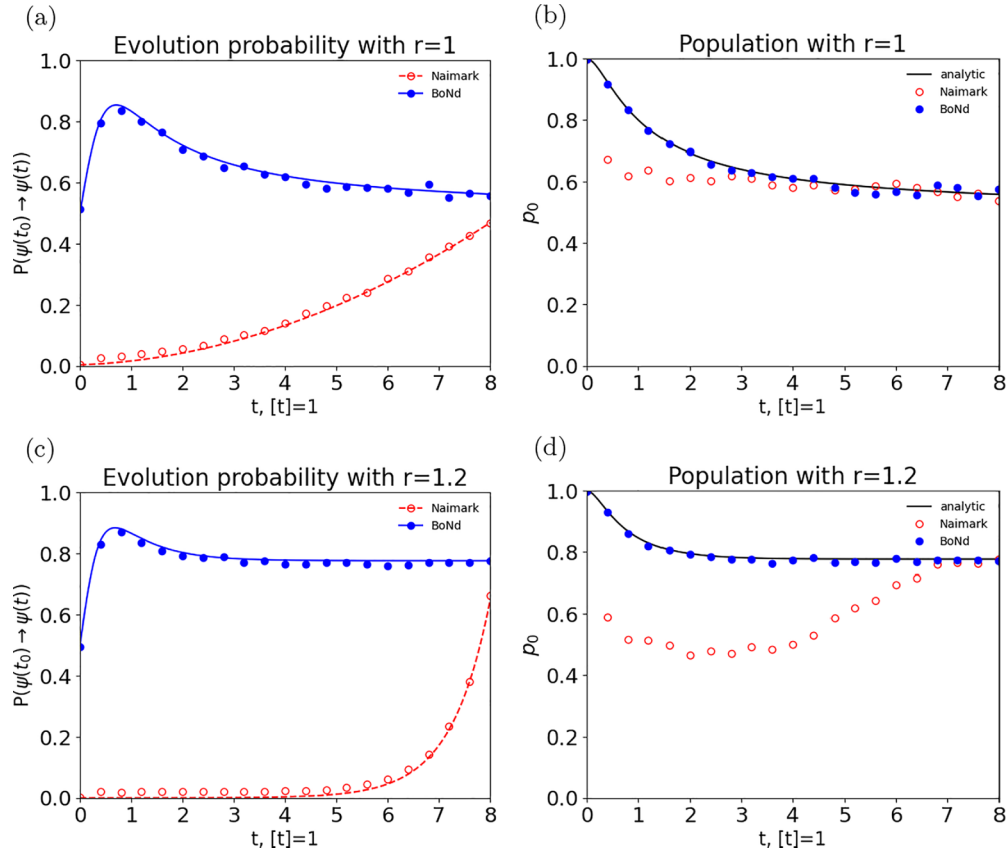


FIG. 5. Local simulation of the population dynamics p_0 [panels (b) and (d)] and the evolution probability $P(\psi(t_0) \rightarrow \psi(t))$ [panels (a) and (c)] under the Hamiltonian $H = \sigma_x + i r \sigma_z$. The simulation incorporates a depolarization error $p_1 = 0.001$ and two-qubit gate error $p_2 = 0.01$. The error bars are not visible since the shots in the simulation amount to at least 1000 per data point. The Naimark dilation protocol breaks down at a $\mathcal{PT}$ -symmetry broken regime (complex spectrum).

$\mathcal{PT}$ -broken regime, where the timescale of validity is varied through the initial condition. In contrast, the dilation scheme is valid for arbitrarily long times in the $\mathcal{PT}$ -symmetric regime, though an appropriate initial condition has to be chosen to ensure the square-root decomposition of $\rho(t) - \mathbb{I}$ exists. Nevertheless, the dynamics of the dilated state cannot be mapped directly to the metric framework due to the term $\tilde{\eta}(t) = \sqrt{\rho(t)} - \mathbb{I}$. Therefore, while the Naimark dilation scheme [12] is suitable for studying the division-by-norm evolution in the $\mathcal{PT}$ -symmetric regime, it cannot be used as a probe of the self-consistent formalisms of non-Hermitian quantum mechanics.

APPENDIX C: NAIMARK DILATION VERSUS BoNd AND GBoNd

In Fig. 5, the population dynamics and the evolution probability $P(\psi(t_0) \rightarrow \psi(t))$ are compared for the two different dilation protocols. At the exceptional point and in the $\mathcal{PT}$ -symmetry broken regime ($r \geq 1$), the Naimark dilation breaks down due to the low evolution probability. This indicates that the amplitude corresponding to the $|0\rangle$ ancilla is negligible. Since the state of interest is also attached to the $|0\rangle$ ancilla, the Naimark dilation protocol will not be capable of reproducing the correct evolution dynamics under minimal noise. This effect will worsen for higher r values. On the other hand, the

BoNd protocol not only captures the dynamics, but is well defined for any arbitrary time, while the Naimark dilation will only achieve a maximal time in the $\mathcal{PT}$ -broken regime. See Table I for a comparison of the different dilations.

APPENDIX D: METRIC FROM STATE TOMOGRAPHY

The metric operator $\rho(t)$, which defines the physical inner product of the non-Hermitian system, can be extracted directly from the total dilated state $|\Psi(t)\rangle$. In this Appendix, we detail the tomographic protocol employed in the Main Text to reconstruct $\rho(t)$ from measurable quantities. By performing state tomography, we can recover the metric of any two-level subsystem. To understand the degrees of freedom required for this reconstruction, we begin by examining the determinant of $\rho(t)$:

$$\det(\rho(t)) = \det(e^{2\tilde{r}(t)\tilde{\sigma}}) = e^{2\text{tr}[\tilde{r}(t)\tilde{\sigma}]} = 1, \quad (\text{D1})$$

or alternatively for a finite-dimensional non-Hermitian Hamiltonian $H(t)$:

$$\det(\rho(t)) = \det(U_H^\dagger(t, t_0)U_H(t_0, t)) \quad (\text{D2})$$

$$= e^{-2i \int dt (\text{tr}[H_a(t)])}, \quad (\text{D3})$$

$$\Rightarrow \text{tr}[H_a(t)] = 0 \Leftrightarrow \det(\rho(t)) = 1. \quad (\text{D4})$$

TABLE I. Comparison of the physical regimes accessible with different dilation schemes. The table indicates whether each protocol can describe no-jump evolution, the $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken regimes, and the metric-framework formulation. Only the Generalized Biorthogonal Naimark dilation provides simultaneous access to all these regimes.

	No-jump evolution	$\mathcal{PT}$ -symmetric	$\mathcal{PT}$ -broken	Metric framework
Naimark	✓	✓	✗	✗
BoNd	✓	✓	✓	✗
GBoNd	✓	✓	✓	✓

Hence, we only require that $H_a(t) = (H(t) - H^\dagger(t))/2$ is traceless for the conservation of the determinant. Because $\rho(t) = \rho^\dagger(t)$ and $\det(\rho(t)) = 1$, there are $(2^{2M} - 1)|_{M=1} = 3$ degrees of freedom, where M is the number of qubits. Performing state tomography on $|\psi(t)\rangle/|\psi(t)|$ and $\rho(t)|\psi(t)\rangle/|\rho(t)\psi(t)|$ is enough to derive $\rho(t)$ for the simple case of a qubit.

For the derivation of the metric from the state tomography, we need to consider the probability to project onto the 1 ancilla or the 0 ancilla. This can be interpreted as an evolution probability $P(\psi(t_0) \rightarrow \psi(t))$ or $P(\psi(t_0) \rightarrow \rho(t)\psi(t))$. We can write these probabilities for this specific dilation as

$$P(\psi(t_0) \rightarrow \psi(t)) = \frac{1}{2^{1-M}\text{tr}[\rho(t)]} \langle \psi(t) | \psi(t) \rangle, \quad (\text{D5})$$

$$P(\psi(t_0) \rightarrow \rho(t)\psi(t)) = \frac{1}{2^{1-M}\text{tr}[\rho(t)]} \langle \psi(t) | \rho(t)^2 | \psi(t) \rangle. \quad (\text{D6})$$

The mapping of the metric $\rho(t)$ follows from the tomography of the states:

$$\frac{|\psi(t)\rangle}{\sqrt{\langle \psi(t) | \psi(t) \rangle}} = \frac{|\psi(t)\rangle}{|\psi(t)|}, \quad \frac{\rho(t)|\psi(t)\rangle}{\sqrt{\langle \psi(t) | \rho^2(t) | \psi(t) \rangle}} = \frac{\rho(t)|\psi(t)\rangle}{|\rho(t)\psi(t)|}. \quad (\text{D7})$$

We can derive the factor $\frac{|\psi(t)|}{|\rho(t)\psi(t)|}$ as

$$\frac{|\psi(t)|}{|\rho(t)\psi(t)|} = \left(\frac{P(\psi(t_0) \rightarrow \psi(t))}{P(\psi(t_0) \rightarrow \rho(t)\psi(t))} \right)^{1/2}. \quad (\text{D8})$$

We can treat the problem in terms of Pauli operators, that is, by setting

$$\rho(t) = \rho_0(t)\mathbb{I} + \vec{\rho}(t) \cdot \vec{\sigma}, \quad (\text{D9})$$

$$v(t) = \frac{|\psi(t)\rangle \langle \psi(t)|}{\langle \psi(t) | \psi(t) \rangle} = \frac{1}{2}(\mathbb{I} + \vec{v}(t) \cdot \vec{\sigma}), \quad (\text{D10})$$

$$\beta(t) = \frac{\rho(t)|\psi(t)\rangle \langle \psi(t)| \rho(t)}{\langle \psi(t) | \rho^2(t) | \psi(t) \rangle} = \frac{1}{2}(\mathbb{I} + \vec{b}(t) \cdot \vec{\sigma}), \quad (\text{D11})$$

$$\beta(t)\rho^{-1}(t) = \rho(t)v(t) - \frac{|\psi(t)|^2}{|\rho(t)\psi(t)|^2}. \quad (\text{D12})$$

The problem can then be studied in terms of Pauli vectors, which gives us an intuitive insight into the metric. Using the definition of $P(\psi(t_0) \rightarrow (\rho(t)\psi(t)))$ and the identities of the metric $\det(\rho(t)) = 1$ and $\text{tr}[\rho(t)] = \text{tr}[\rho^{-1}(t)]$, we can derive an analytical expression for $\rho(t)$ by comparing real factors on the resulting Pauli vectors from Eq. (D12) and using Eq. (D8):

$$\frac{\vec{\rho}(t)}{\rho_0(t)} = P(\psi(t_0) \rightarrow \rho(t)\psi(t))\vec{b}(t), \quad (\text{D13})$$

$$-P(\psi(t_0) \rightarrow \psi(t))\vec{v}(t), \quad (\text{D14})$$

$$\rho_0^2(t) = |\vec{\rho}(t)|^2 + 1 = \left(1 - \left(\frac{\vec{\rho}(t)}{\rho_0(t)}\right)^2\right)^{-1}, \quad (\text{D15})$$

$$\Rightarrow \frac{\rho(t)}{\text{tr}[\rho(t)]} = P(\psi(t_0) \rightarrow \rho(t)\psi(t))\beta(t), \quad (\text{D16})$$

$$-P(\psi(t_0) \rightarrow \psi(t))v(t), \quad (\text{D17})$$

$$+P(\psi(t_0) \rightarrow \psi(t))\mathbb{I}, \quad (\text{D18})$$

see Fig. 6 for an example of the evolution probabilities. Since this method works for any possible initial state regardless of its purity, we have a closed-form implementation of the non-Hermitian dynamics. From Eq. (D15), we can derive $\rho_0(t)$, so $\rho(t)$ can be completely mapped out. Therefore, we can simulate the state and the metric in parallel. In the bigger system case, we might choose appropriate initial conditions like the maximally mixed state. In that case, we first map out the metric and then implement the dynamics with $H_{\text{tot}}(t)$.

APPENDIX E: DISCRETE IMPLEMENTATION OF BoNd

The nontrivial dynamics of the metric $\rho(t)$ described by Eq. (1) does not always permit an analytical expression, away from the simple case of two-level systems discussed in the Main Text. The total Hamiltonian $H_{\text{tot}}(t)$ is also time dependent due to the generally time-dependent metric $\rho(t)$. Even when an analytical solution is not available, the isomorphism between $|\psi(t)\rangle$ and $\rho(t)|\psi(t)\rangle$ allows us to derive the metric from a tomographic state measurement at every time point. We can use this fact to create a feedback loop between the time-evolved state and the time-dependent total Hamiltonian in order to simulate the non-Hermitian dynamics in a completely closed protocol. We have to take into account that the measurements we are able to take for $\rho(t)$ are discrete in time. For our initial purposes, we simply want to show the straightforward implementation of the dynamics with a simple protocol. We can implement a first time step by propagating the state with

$$|\Psi(t_0 + \Delta t)\rangle \approx e^{-iH_{\text{tot}}(t_0)\Delta t} |\Psi(t_0)\rangle, \quad (\text{E1})$$

where $\rho(t_0) = \mathbb{I}$. For later times, we can implement the total propagation:

$$H'_{\text{tot}}(t_0 + m\Delta t) = H_{\text{tot}}(t_0 + m\Delta t)|_{\rho(t)=\rho'(t)}, \quad (\text{E2})$$

$$U_{\text{tot}}(t_0 + n\Delta t, t_0) \approx \prod_{m=n}^0 e^{-iH'_{\text{tot}}(t_0+m\Delta t)\Delta t}, \quad (\text{E3})$$

where $\rho'(t)$ is the metric measured from the tomography and n is the number of time steps Δt . The exponentials can be calculated numerically, since we are working with

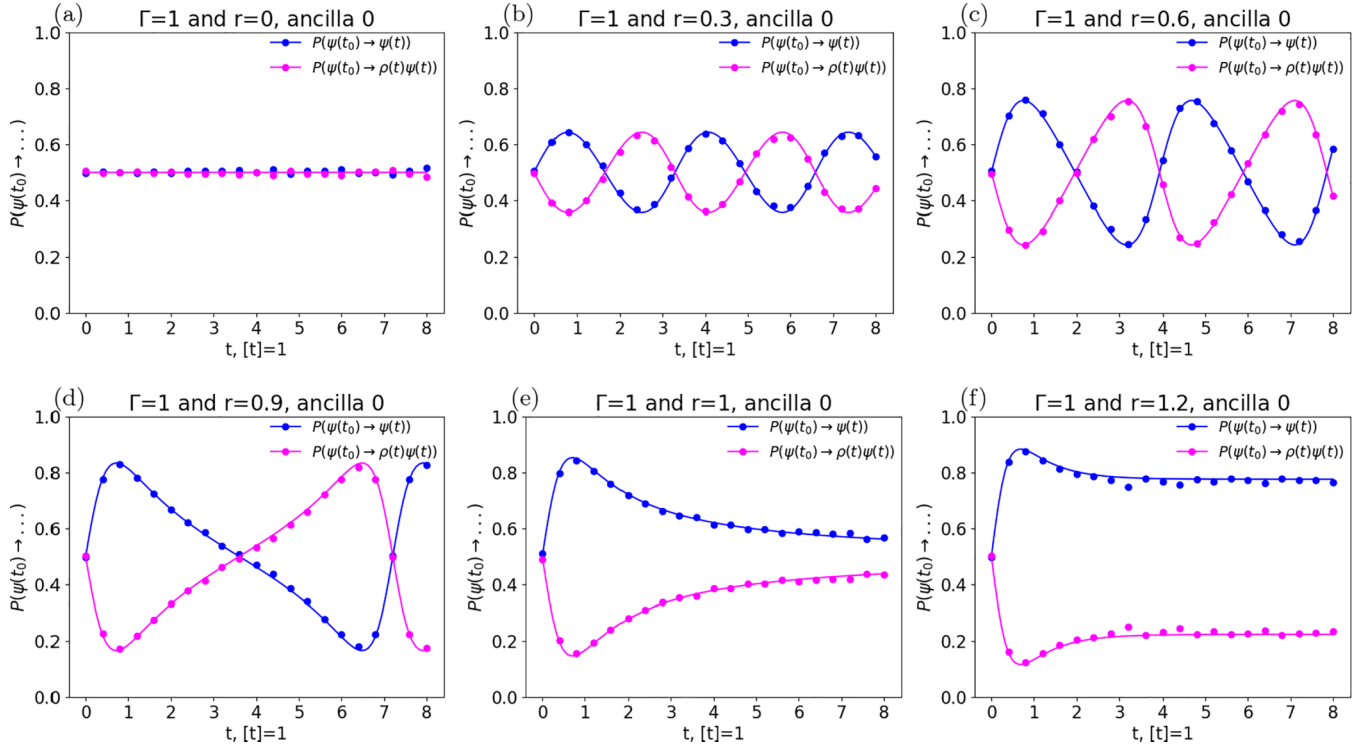


FIG. 6. The evolution probabilities $P(\psi(t_0) \rightarrow \psi(t))$ and $P(\psi(t_0) \rightarrow \rho(t)\psi(t))$ of the BoNd protocol evolving under the Hamiltonian $H = \Gamma\sigma_x + ir\sigma_z$ [Eq. (11)], which equate to the projection probabilities of the two distinct ancilla states. The measured evolution probabilities for different r values are depicted as points. The analytical evolution probabilities are the solid lines. The initial condition is $|\psi(t_0)\rangle = |0\rangle$.

four-dimensional matrices. To analyze the error buildup from this Trotterization, we must consider the structure of $\rho(t)$:

$$i\frac{d}{dt}\rho(t) = H^\dagger(t)\rho(t) - \rho(t)H(t), \quad (\text{E4})$$

$$\begin{aligned} \Rightarrow \rho(t) &= U_{H^\dagger}(t)U_H(-t) \\ &= \left(\cos(\sqrt{1-r^2}t) + \frac{(1+r^2)\sin^2(\sqrt{1-r^2}t)}{1-r^2} \right) \mathbb{I} \\ &\quad + \frac{2\sin(\sqrt{1-r^2}t)}{\sqrt{1-r^2}} \left(\frac{\sin(\sqrt{1-r^2}t)r}{\sqrt{1-r^2}} \sigma_y \right. \\ &\quad \left. - \cos(\sqrt{1-r^2}t) \sigma_z \right). \end{aligned} \quad (\text{E5})$$

The intrinsic error to this approximation can be assumed to come from higher than linear Δt terms, since the Trotterization accounts for up to linear terms. The Taylor expansion for

small Δt values is

$$\begin{aligned} \rho(\Delta t) &= \left(1 + 2r^2\Delta t^2 - \frac{2}{3}r^2(1-r^2)\Delta t^4 + \mathcal{O}(\Delta t^6) \right) \mathbb{I} \\ &\quad + \left(2r\Delta t^2 - \frac{2}{3}(1-r^2)r\Delta t^4 + \mathcal{O}(\Delta t^6) \right) \sigma_y \\ &\quad + \left(-2\Delta t + \frac{4(1-r^2)\Delta t^3}{3} + \mathcal{O}(\Delta t^5) \right) \sigma_z. \end{aligned} \quad (\text{E6})$$

Therefore, the leading error will be of order $\mathcal{O}(r^2\Delta t^2)$ coming from the $\mathbb{I}$ term and $\mathcal{O}(r\Delta t^2)$ from the σ_y term. This means that the total accumulated error will be of order $\mathcal{O}(Tr^2\Delta t + Tr\Delta t)$, where $T = n\Delta t$ is the total simulated time. We are, however, picking up an error not only in the metric $\rho(t)$, but also in the total Hamiltonian $H_{\text{tot}}(t)$, which will be of the same magnitude per step as the error from the metric. This will also reflect in the time propagator. Overall, the error we gain is going to be of magnitude $\mathcal{O}(T^2r^2 + T^2r)$, which will bring our optimal operating range to $T^2|r| \ll 1$ and $T^2r^2 \ll 1$. The

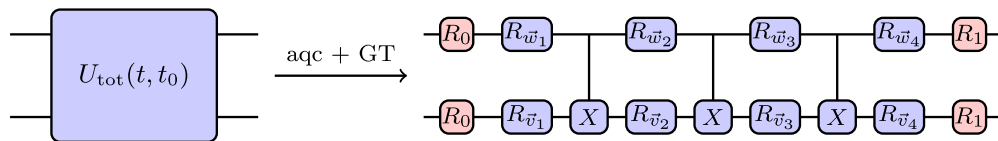


FIG. 7. The approximate quantum compiler (aqc) takes a unitary operation and turns it into an approximate operation as described in Ref. [38]. In a depth of 3, the circuit will implement three control operations between single-qubit rotations R_{v_i} and R_{w_j} shown in blue. The circuit shown approximates a two-qubit total unitary $U_{\text{tot}}(t, t_0)$ on the left. On the right, we see the approximate circuit with gate twirling (GT) shown in red as random gates $R_{0,1}$ at the beginning and the end of the circuit, effectively evening the noise in the circuit.

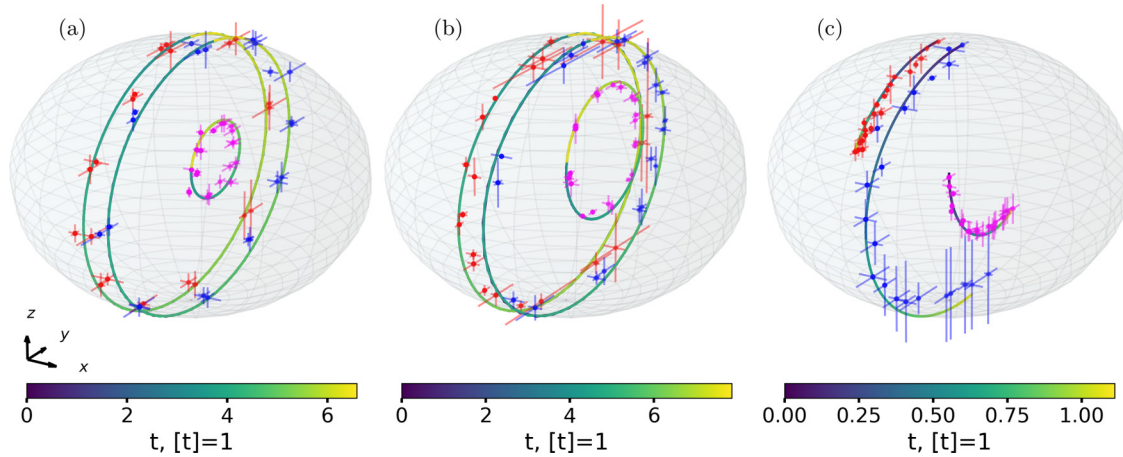


FIG. 8. Results of the state tomography for different r values of the discrete BoNd protocol implementing the Hamiltonian $H = \sigma_x + ir\sigma_z$: (a) $r = 0.3$, (b) $r = 0.6$, and (c) $r = 0.9$. The normalized states $|\psi(t)\rangle / |\psi(t)|$ and $\rho(t) |\psi(t)\rangle / |\rho(t)\psi(t)|$, as well as the normalized metric $\rho(t)/\text{tr}[\rho(t)]$, are presented in red, blue, and magenta, respectively. The color gradient on the analytical curves represents the time evolution.

error accumulation can be interpreted as a binomial expansion, leading to $O(n^2(\Delta t^2 r^2 + \Delta t^2 r)) = O(T^2 r^2 + T^2 r)$.

It is worth noting that this discrete method is not the most efficient nor the most exact method possible for implementing a time-dependent Hamiltonian evolution. One could consider implementing a Magnus expansion on an interpolation of $H'_{\text{tot}}(t)$ to better approximate the dynamics, so that we can get rid of the $O(T^2 r^2) + O(T^2 r)$ term, or let every time step converge onto a more accurate $\rho'(t)$ to neglect the error from the Hamiltonian. The primary advantage of this implementation is its simplicity and the preservation of unitarity. Rather than showcasing a numerical method, our goal is to illustrate the intrinsic metric nature of the BoNd evolution as shown in Fig. 8.

In this specific implementation, the unitaries $U_{\text{tot}}(t, t_0)$ are already known analytically. These can be implemented in the circuit with the help of the package `qiskit.synthesis.unitary.aqc`, which is able to approximate the given unitary as a sequence of universal gates and easy-to-implement rotations. We measured the metric from state tomography at each time step and we redefined a total time propagator based on the measured metric, which is of greater value and interest for the studies on self-consistent non-Hermitian dynamics. The quantum processing unit (QPU) `ibm_kyiv` was used for all the simulations, and the derived error bars were simulated based on the noise model of the quantum circuit.

APPENDIX F: QUANTUM CIRCUIT

In all implementations presented in the main text, the dilation unitaries $U_{\text{tot}}(t, t_0)$ [as defined in Eq. (6)] were computed analytically in advance. These unitaries encode the effective non-Hermitian dynamics within a fully unitary framework.

To realize them on quantum hardware, we employed the `qiskit.synthesis.unitary.aqc` module, which approximates a given unitary as a sequence of universal gates and single-qubit rotations. This decomposition enables the construction of digital quantum circuits that faithfully reproduce the desired evolution, as illustrated in Fig. 7.

For all ancilla measurements, we set the minimal number of measurements to be of at least $N = 10^3$. The QPU `ibm_kyiv` was used for all the simulations with a circuit depth of 3 and with gate twirling to make the average error a depolarizing error.

Most of the noise appearing in the simulation results is expected from the nature of the quantum circuit. One can see that the noise becomes more prominent when the evolution probability is low. The variation is derived via a local simulation using the noise model provided by the IBM and the optimal implementation of the time evolution. It is worth mentioning that the noise model and the actual noise in the QPU differ depending on many factors.

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