Work extraction with feedback control using limited resources

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Many physical, biological, and even social systems are faced with the problem of how to efficiently harvest free energy from an environment that can have many possible states, yet only have a limited number of harvesting protocols to choose among. We investigate this scenario by extending earlier work on using feedback control to extract work from nonequilibrium systems. Specifically, in contrast to that previous work on the thermodynamics of feedback control, we analyze the combined and separate effects of noisy measurements, memory limitations, and limitations on the number of possible work extraction protocols. Our analysis provides a general recipe to construct repertoires of allowed harvesting protocols that minimize the expected thermodynamic losses during free energy harvesting, i.e., that minimize expected entropy production. In particular, our results highlight that the benefits of feedback control over uninformed (random) actions extend beyond just the associated information gain, often by many orders of magnitude. Our results also uncover the effects of limitations on the number of possible harvesting protocols when there is uncertainty about the distribution over states of the environment.

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I. INTRODUCTION

A central challenge faced by essentially all living organisms is to interact intelligently with their nonequilibrium environment in order to extract enough free energy from it to survive. This requires them to acquire information concerning the state of their environment via measurements, to guide their behavior in those interactions [1]. Here, we consider a related requirement imposed by this need to extract free energy, concerning how organisms should best use their information to determine how to act on their environment.

Broadly speaking, this question of how best to extract free energy from an environment goes back very far in the scientific literature. A long line of scientists have been fascinated by the interplay between the probabilistic nature of the second law and the possibility to use information to improve thermodynamic operation [2]. Indeed, Maxwell's demon, first considered in the 19th century [3], and the first information engine using feedback control operations proposed by Szilard [4], are both examples of earlier work on this issue.

The development of stochastic thermodynamics in the last few decades provides the possibility of a far deeper investigation, since it is specifically designed to analyze the thermodynamics of systems far from thermal equilibrium (in contrast to conventional, equilibrium statistical physics) [5–8]. Indeed, as we show here, we can use stochastic thermodynamics to precisely formulate and then analyze feedback control scenarios [9] where the costs and benefits of using information can be assessed in a variety of situations, ranging from discrete measurements [10–15], to continuous monitoring [16–23], and from classical to quantum systems [24–29]. Experiments in a broad range of platforms from colloidal particles [30–34] and nanoelectronic devices [35–37], to DNA pulling experiments [21,38] and quantum systems [39,40], have tested and verified the role of information in work extraction [41].

In this paper, we focus on using feedback protocols to guide the extraction of work (i.e., the harvesting of free energy) from a nonequilibrium system of interest (e.g., the environment of an organism). We consider specifically the effects arising from noisy observation and from a limited number of possible actions on the nonequilibrium system, as well as limitations on the size of the memory used in the feedback control operation. Note in particular that limitations on the number of possible actions by the agent extracting work apply in many real-world scenarios. For example, they arise in switching on and off predefined potentials [30], suddenly raising the energy of a discrete level [35], moving the position of an optical trap [42,43], or swapping the polarity of photons [44]. More generally, such limitations apply to all biological organisms that are trying to harvest free energy from their environment.

Here, we show that such limitations can drastically reduce the efficiency of free energy extraction performed via feedback control. As a result, determining the optimal finite set of actions the controller can choose among, given such memory limitations and noise in measurements, is crucial in several applications. These include cases where the initial distribution of the system is not known and might be guessed

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from noisy observation. To analyze these issues, here we build on previous work demonstrating that optimal feedback control that maximizes work extraction can be achieved with reversible feedback protocols [45,46]. Such optimal control begins with a quench of the original Hamiltonian of the nonequilibrium system. The precise quench depends on the original distribution of system states and any (noisefree) measurement concerning that state. After this quench a quasistatic isothermal process is applied to the system, changing the Hamiltonian back to the original Hamiltonian, and changing the distribution to an equilibrium distribution with respect to that original Hamiltonian [9,47–50].

However, as mentioned, in real-world scenarios limitations often arise due to both imperfections in the measurements and because of severe constraints in the control operations that prevent the implementation of optimal protocols [51]. For example, if the relaxation timescales of the system are not fast enough with respect to the driving velocity, the quasistatic regime is lost [31,52]. In that case, optimal transport techniques are required to maximize the extractable work that usually involve numerical techniques [53–55]. In some cases, however, the problem can be directly solved in a linear response regime if the dynamics is sufficiently slow (i.e., close to but out of the quasistatic regime) [56–59]. Other limitations may arise from additional costs in the measurements [60–62] or in the feedback implementation [63,64]. Moreover, in many situations, the dynamical evolution of a system may show special symmetry or modularity properties that can affect its thermodynamics. In such cases, extra irreversibility may arise that can limit work extraction even when allowing generic thermodynamic processes under local detailed balance [65,66].

Our starting point is optimal feedback control processes, consisting of a sudden quench applied to the system Hamiltonian that depends on the measurement result, followed by quasistatic isothermal evolution that takes the Hamiltonian back to its original form. For simplicity, we focus on minimal modifications of such processes, to restrict the number of possible initial quenches. So, we impose no limitations on the relaxation process, despite this restriction on the quench process. In particular, no restrictions are imposed on the timescale of the relaxation (so quasistatic processes are allowed), and no extra symmetries are imposed on rate matrices and/or Hamiltonians. Moreover, although we impose such restrictions on the quenching Hamiltonians, we permit a continuum of Hamiltonians during that subsequent quasistatic relaxation process. Finally, note that in many real-world scenarios, the experimentalist has little (if any) direct control over rate matrices, beyond that arising via local detailed balance and their ability to change a system's energy spectrum. Accordingly (and in contrast to Refs. [65,66]), we assume no control of rate matrices whatsoever beyond that which arises automatically due to local detailed balance.

Our main results involve the derivation of universal expressions for the entropy production arising due to the control limitations, which limit the amount of extractable work (free energy) from generic nonequilibrium states. The general framework is outlined in Sec. II, starting from the ideal case and then introducing control limitations. That framework allows us to obtain in Sec. III precise conditions for the design of successful feedback control quenches that balance the eventual work costs with a net free energy extraction. There, we also determine the quantitative benefits of employing information with respect to scenarios where the quenching Hamiltonian is fixed or taken at random from a predefined set. In Sec. IV, we derive explicit expressions for the optimal form of the quenches that minimize entropy production and present our results for the construction of an optimal action repertoire. We discuss its implications in terms of the acquired information that is useful for feedback control. Our general results are particularized in various situations of interest in stochastic thermodynamics in Sec. V. Noisefree observation and observation with small errors are considered, as well as the case of a random initial distribution, all of which are illustrated using worked examples. Finally, in Sec. VI we provide a concluding summary and discussion. Details about the derivations of the main results are given in Appendixes B, C, D, and E, and Table I is a guide to the notation used in the manuscript.

II. FRAMEWORK

Consider a nonequilibrium system of interest from which work is to be extracted. It is described by a set of d_X variables $\mathbf{x} := \{x_i\}_{i=1}^{d_X}$ (we use bold letters to indicate vectors), with $x_i \in \mathbb{X}$, where the set $\mathbb{X}$ can be either discrete or continuous. We introduce an associated Hamiltonian function describing the energy of the system over this set of variables $H(\mathbf{x})$, which can be externally controlled by an external agent subjected to computational limitations that will be specified in detail later. The system is assumed to be initially in a nonequilibrium state described by distribution $\rho_0(\mathbf{x})$.

A feedback controller can extract work from the nonequilibrium state of the system by first measuring (all or some of) the system variables, and then performing a control operation over the system Hamiltonian conditioned on the obtained measurement result. The measurement provides the controller with a vector of measurement results $\mathbf{m} := \{m_i\}_{i=1}^{d_M}$ with $m_i \in \mathbb{M}$ (which can also be either discrete or continuous). The measurement results are affected by external noise as described by the (joint) conditional probability $p(\mathbf{m}|\mathbf{x})$ for obtaining outcome $\mathbf{m}$ given that the state of the system was $\mathbf{x}$. The (marginal) probability of obtaining the results $\mathbf{m}$ is therefore a function of the initial state $\rho_0(\mathbf{x})$ and the above conditional probabilities as given by

$$p(\mathbf{m}) = \sum_{\mathbf{x}} p(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x}). \tag{1}$$

After measurement, the best guess for the system distribution hence reads

$$\rho_{X|\mathbf{m}}(\mathbf{x}) := \rho(\mathbf{x}|\mathbf{m}) = \frac{p(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x})}{p(\mathbf{m})}, \tag{2}$$

as corresponds to Bayes' rule for updating probabilities when using $\rho_0(\mathbf{x})$ as the prior distribution. Note that we use $\rho_{X|\mathbf{m}}(\mathbf{x})$ as a shorthand notation to denote the conditional distribution $\rho(\mathbf{x}|\mathbf{m})$ for a given measurement result $\mathbf{m}$. Analogously, we will use in the following $p_{M|\mathbf{x}}(\mathbf{m})$ to denote the conditional probability $p(\mathbf{m}|\mathbf{x})$. In the case of a perfect measurement of the system variables (i.e., the measurement result m_i corresponds exactly to the system state variable x_i for all

TABLE I. Table of notation for the main symbols used in the paper.

Table of notation	
Symbol	Definition
$\mathcal{H}$	Set of quenching Hamiltonians $\{H_k\}_{k=1}^N$ for feedback control
L	Set of partitions $\{l(n)\}_{n=1}^N$ grouping the measurement outcomes
$\rho_0(\mathbf{x})$	Initial marginal distribution of the system states $\mathbf{x}$
$\rho_H^{\text{eq}}(\mathbf{x})$	Thermal equilibrium state for the system states $\mathbf{x}$ with Hamiltonian $H(\mathbf{x})$
$\rho_{X\|\mathbf{m}}(\mathbf{x})$	Conditional distribution of the system states $\mathbf{x}$ given measurement result $\mathbf{m}$
$\rho_n^{\text{opt}}(\mathbf{x})$	Distributions leading to optimal quenching Hamiltonians $H_n = -k_BT \log \rho_n^{\text{opt}}$
$p(\mathbf{m})$	Marginal distribution of the measurement outcomes $\mathbf{m}$
$p_{M\|\mathbf{x}}(\mathbf{m})$	Conditional distribution of measurement outcomes $\mathbf{m}$ for given system state $\mathbf{x}$
$n^*(\mathbf{m})$	Optimal choice for the quenching Hamiltonian given a measurement result ⇳
$p(n)$	Marginal distribution for obtaining partition $\ell(n)$ in the measurement
$p_{L\|\mathbf{x}}(n)$	Conditional probability distribution for obtaining partition $\ell(n)$ given that the system is in state $\mathbf{x}$
$\rho_0^{\text{cg}}(n)$	Marginal distribution for obtaining partition $\ell(n)$ in the noisefree scenario
p_α	Probability of component α in the uncertain initial distributions scenario
$\rho_0^{(\alpha)}(\mathbf{x})$	Component α of the initial marginal distribution in the uncertain initial distributions scenario
$\bar{\rho}_0(\mathbf{x})$	Effective mixture of marginal distributions in the uncertain initial distributions scenario
$S(\rho)$	Shannon entropy of distribution $\rho(\mathbf{x})$
$D(\rho\|\|\sigma)$	Kullback-Leibler divergence (relative entropy) between distributions $\rho(\mathbf{x})$ and $\sigma(\mathbf{x})$
$J_{\mathbf{p}}(\mathbf{P})$	Generalized Jensen-Shannon divergence among a weighted collection of distributions $\mathbf{P}$ with weights $\mathbf{p}$
$I_{X;M}$	Mutual information between the system states and measurement results
$I_{X;M\|L}$	Conditional mutual information between the system and measurement results given a set of partitions L
F_H^{eq}	Equilibrium free energy of the system for Hamiltonian $H(\mathbf{x})$
$\mathcal{F}_H(\rho)$	Nonequilibrium free energy of state ρ for Hamiltonian $H(\mathbf{x})$
$\Delta\mathcal{F}_H$	Available nonequilibrium free energy in the initial state, $\mathcal{F}_H(\rho_0) - F_H^{\text{eq}}$
$S_{\text{tot}}(\mathbf{m})$	Entropy production (free energy loss) in the feedback control process for measurement outcome $\mathbf{m}$
$\langle S_{\text{tot}}\rangle$	Average entropy production (average free energy loss) in the feedback control process after many runs
$W_{\text{ideal}}(\mathbf{m})$	Work extracted in the ideal feedback control protocol for measurement outcome $\mathbf{m}$
$\langle W_{\text{ideal}}\rangle$	Average work extracted in the ideal feedback control protocol after many runs
$W(\mathbf{m})$	Work extracted in the general feedback control protocol with limitations for measurement outcome $\mathbf{m}$
$\langle W\rangle$	Average work extracted in the general feedback control protocol after many runs
W_{cost}^q	Work cost for performing a predefined quench $H_q(\mathbf{x})$
$\langle W_{\text{cost}}\rangle$	Average work cost for performing the quenches after many runs
$\langle \Delta W\rangle$	Average work gain from feedback control with respect to a deterministic quench after many runs
$\langle \Delta W\rangle_r$	Average work gain from feedback control with respect to random quenching using distribution $r(q)$

i), the above expressions reduce to $p_{M|\mathbf{x}}(\mathbf{m}) = \prod_{i=1}^d \delta_{m_i,x_i}$, where $\delta_{m,x}$ denotes the Kronecker delta, and hence $p(\mathbf{m}) = \rho_0(\mathbf{m})$, implying that the state of the system becomes perfectly known, i.e., $\rho_{X|\mathbf{m}}(\mathbf{x}) = \prod_{i=1}^d \delta_{m_i,x_i}$. See Sec. V A for how our main results apply in this setting.

A. Ideal work extraction

We write the nonequilibrium free energy as the function

$$\mathcal{F}_H(\rho) := \langle H\rangle_\rho - k_BTS(\rho), \tag{3}$$

with $\langle H\rangle_\rho = \sum_{\mathbf{x}} H(\mathbf{x})\rho(\mathbf{x})$ the average energy of the system associated with distribution ρ and $S(\rho) = -\sum_{\mathbf{x}} \rho(\mathbf{x}) \ln \rho(\mathbf{x})$ its Shannon entropy. Notice that for equilibrium states $\mathcal{F}_H(\rho_H^{\text{eq}})$, the nonequilibrium free energy reduces to the traditional equilibrium (Helmholtz) free energy for Hamiltonian H, which we denote F_H^{eq}.

The maximum amount of work that can be extracted from the system, assuming knowledge of $\rho_{X|\mathbf{m}}(\mathbf{x})$, and access to an equilibrium reservoir at temperature T, is the difference in nonequilibrium free energies between such a state and the corresponding equilibrium state $\rho_H^{\text{eq}}(\mathbf{x}) = e^{-\beta H(\mathbf{x})}/Z$, with $Z = \sum_{\mathbf{x}} e^{-\beta H(\mathbf{x})}$ being the partition function, and $\beta = 1/(k_BT)$, where T is the temperature of the thermal reservoir connected to the system (or bath) [9]. That is,

$$W(\mathbf{m}) \leqslant \mathcal{F}_H(\rho_{X|\mathbf{m}}) - F_H^{\text{eq}}. \tag{4}$$

The feedback control protocol that achieves optimal work extraction [equality in Eq. (4)] can be constructed in a generic way by first instantaneously quenching the Hamiltonian to

$$H(\mathbf{x}) \to H_{\mathbf{m}}(\mathbf{x}) := -k_BT \ln \rho_{X|\mathbf{m}}(\mathbf{x}) - k_BT \ln Z_{\mathbf{m}}, \tag{5}$$

with $Z_{\mathbf{m}}$ an arbitrary constant (we can take $Z_{\mathbf{m}} = 1$ for all $\mathbf{m}$ without loss of generality), and then quasistatically modifying from $H_{\mathbf{m}}(\mathbf{x})$ back to the original Hamiltonian $H(\mathbf{x})$, while in contact with the external bath [9].

Before justifying the choice of Eq. (5), we note that its physical feasibility can be questioned. Often in experiments, control apparatuses may have a limited number of externally adjustable parameters, allowing only partial manipulability of the Hamiltonian. In the most adversarial case, the distribution $\rho_{X|\mathbf{m}}(\mathbf{x})$ appearing in Eq. (5) can be an arbitrary probability distribution over $\mathbb{X}^{d_X}$, since we assume

arbitrary conditional probabilities $p(\mathbf{m}|\mathbf{x})$ for the measurement and the arbitrary initial nonequilibrium distribution $\rho_0(\mathbf{x})$. In that case, optimal work extraction requires complete manipulability of all microstate energy levels $H(\mathbf{x})$. For small (low-dimensional) discrete systems, this may be achievable to a high degree of approximation. However, it may become especially problematic in continuum cases, since the space of continuum probability densities is not traversable by simple finite-dimensional parametric families. Despite this idealization, we emphasize that the main focus of this study is the *removal* of a closely related idealization, namely, that of a perfectly tailored response quench $H_{\mathbf{m}}$ across *all possible measurements* $\mathbf{m} \in \mathbb{M}^{d_M}$ (we return to these considerations in Sec. III).

The rationale behind this procedure follows from the fact that the initial Hamiltonian quench instantaneously transforms the state $\rho_{X|\mathbf{m}}(\mathbf{x})$ into being an equilibrium state with respect to the quenched Hamiltonian $H_{\mathbf{m}}(\mathbf{x})$ (i.e., $\rho_{X|\mathbf{m}}(\mathbf{x}) = e^{-\frac{H_{\mathbf{m}}(x)}{k_B T}}/Z_{\mathbf{m}}$), after which the quasistatic relaxation step just becomes a thermodynamically reversible transformation where the state of the system is, at every time, in equilibrium with the bath. By following this procedure, all of the available free energy is extracted. We write the work extracted using this (ideal) feedback control protocol as

$$W_{\text{ideal}}(\mathbf{m}) = -\langle H_{\mathbf{m}}\rangle_{\rho_{X|\mathbf{m}}} + \langle H\rangle_{\rho_{X|\mathbf{m}}} + \mathcal{F}_{H_{\mathbf{m}}}(\rho_{X|\mathbf{m}}) - F_H^{\text{eq}} = \mathcal{F}_H(\rho_{X|\mathbf{m}}) - F_H^{\text{eq}}. \quad (6)$$

This formula follows from the fact that the free energy of the quenched state is $\mathcal{F}_{H_{\mathbf{m}}}(\rho_{\mathbf{m}}) = \langle H_{\mathbf{m}}\rangle_{\rho_{X|\mathbf{m}}} - k_B T S(\rho_{X|\mathbf{m}})$ and that during the reversible transformation the work extracted equals the (equilibrium) free energy change.

Notice that $\mathcal{F}_H(\rho_{X|\mathbf{m}})$ is a stochastic quantity depending on the specific outcomes obtained in the measurement, and hence so is $W_{\text{ideal}}(\mathbf{m})$. The associated average over measurement results is

$$\begin{aligned}\langle W_{\text{ideal}}\rangle &= \sum_{\mathbf{m}} p(\mathbf{m})\mathcal{F}_H(\rho_{X|\mathbf{m}}) - F_H^{\text{eq}} \\ &= \langle H\rangle_{\rho_0} - k_B T \sum_{\mathbf{m}} p(\mathbf{m}) S(\rho_{X|\mathbf{m}}) - F_H^{\text{eq}} \\ &= k_B T I_{X;M} + \mathcal{F}_H(\rho_0) - F_H^{\text{eq}}, \end{aligned} \quad (7)$$

where we identified $I_{X;M} := S(\rho_0) - \sum_{\mathbf{m}} p(\mathbf{m}) S(\rho_{X|\mathbf{m}})$ as the mutual information between the system variables and measurement outcomes. In the equation above, we used that the unconditional state of the system after measurement is just the initial state, $\sum_{\mathbf{m}} p(\mathbf{m})\rho_{X|\mathbf{m}}(\mathbf{x}) = \rho_0(\mathbf{x})$, which can be directly verified from the expression of $\rho_{X|\mathbf{m}}$ in Eq. (2) (a property of noninvasive measurements). Equation (7) tells us that the amount of work extractable from the environment is limited by two factors: (1) the amount of information that can be gathered from measuring the system of interest as given by the mutual information $I_{X;M}$ and (2) the amount of free energy contained in the initial nonequilibrium state of the environment $\rho_0(\mathbf{x})$ as compared to its equilibrium free energy, which we denote in the following $\Delta\mathcal{F}_H := \mathcal{F}_H(\rho_0) - F_H^{\text{eq}}$.

It is worth mentioning at this point that closing the cycle after the work extraction procedure requires a final step consisting in the erasure of the information gathered during the measurement of the system and employed for the implementation of the feedback control protocol [9]. Following generalized Landauer's principle, this erasure process has an unavoidable work cost that amounts, at least, to the mutual information $k_B T I_{X;M}$ in the reversible limit [11]. That means that the net effect of the work extraction procedure, once erasure costs of the memory have been included, is to transform the free energy difference from the initial nonequilibrium state $\rho_0(\mathbf{x})$ to the equilibrium one $\rho_H^{\text{eq}}(\mathbf{x})$ into useful work.

B. Introducing control limitations

In contrast to the ideal work extraction scenario described above, in many real-world scenarios the set of possible quenching Hamiltonians is finite and much smaller than the set of possible outcomes of the measurement. In addition, often in the real world that set is predefined and fixed ahead of time, and the only thing one can do is choose one of those fixed possibilities. We are interested in quantifying how much the ideal scenario presented above is affected when introducing such limitations affecting the control abilities of the agent performing work extraction.

More specifically, we consider a broad range of real-world scenarios characterized by the impossibility of the controller to perform the ideal quench introduced in Eq. (5), that leads to a postquench Hamiltonian $H_{\mathbf{m}}(\mathbf{x}) = -k_B T \ln \rho_{X|\mathbf{m}}(\mathbf{x})$. Instead, we consider the situation in which the feedback controller action is limited to choose one out of some finite set of quench Hamiltonians. We write that set as $\mathcal{H} := \{H_k(\mathbf{x})\}_{k=1}^{N}$ with $N < |\mathbb{M}^{d_M}|$ possible options. In this situation, after reading the measurement results $\mathbf{m}$, the feedback controller selects and applies one of the quenches in the repertoire $\mathcal{H}$ so as to maximize the amount of extractable work, according to the obtained measurement result. We denote the choice by the control protocol of which Hamiltonian to quench to in response to a given measurement result $\mathbf{m}$ by $n(\mathbf{m})$, or n for short. (We derive the optimal form of that function below.) Note that because $\mathbf{m}$ is random, in general so will n be. We will first assume the form of the elements $H_k(\mathbf{x})$ in $\mathcal{H}$ to be predefined and fixed, while in following sections we will also determine their optimal form.

In any case, as a consequence of the limitations in the available actions of the controller for the implementable (postquench) Hamiltonians, the state of the system after the quench will no longer be in equilibrium with the postquench Hamiltonian. In other words, $\rho_{X|\mathbf{m}}(\mathbf{x}) \neq -k_B T \ln H_{n(\mathbf{m})}(\mathbf{x})$. As a consequence, an irreversible relaxation from state $\rho_{X|\mathbf{m}}(\mathbf{x})$ to the equilibrium state $e^{-\beta H_n(\mathbf{x})}/Z_n$ (with $Z_n = \sum_x e^{-\beta H_n(\mathbf{x})}$) will unavoidably take place. Such relaxation will take place immediately after quenching and before the quasistatic process completing the work extraction protocol is finished.

In this scenario, the work extractable from the initial state using feedback control is reduced:

$$\begin{aligned} W(\mathbf{m}) &= -\langle H_n\rangle_{\rho_{X|\mathbf{m}}} + \langle H\rangle_{\rho_{X|\mathbf{m}}} + F_{H_n}^{\text{eq}} - F_H^{\text{eq}} \\ &= \mathcal{F}_H(\rho_{X|\mathbf{m}}) - F_H^{\text{eq}} - k_B T D\left(\rho_{X|\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_n}}{Z_n}\right), \end{aligned} \quad (8)$$

where, to reach the second line, we used the equality $F_{H_n}^{\rm eq} - \langle H_n \rangle_{\rho_{X|\mathbf{m}}} = k_B T \sum_{\mathbf{x}} \rho_{X|\mathbf{m}} \ln(\frac{e^{-\beta H_n}}{Z_n})$ and identified the Kullback-Leibler (KL) divergence between initial and final system probability densities,

$$D\left(\rho_{X|\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_n}}{Z_n}\right) = -S(\rho_{X|\mathbf{m}}) + \sum_{\mathbf{x}} \rho_{X|\mathbf{m}} \ln\left(\frac{e^{-\beta H_n}}{Z_n}\right). \tag{9}$$

[Recall that the Kullback-Leibler divergence $D(\rho||\sigma) \geqslant 0$ accounts for the distinguishability between probability distributions, being well defined if ρ has support on σ. It is non-negative $\forall \rho, \sigma$ and zero if and only if $\rho = \sigma$ [67].]

In the last equation [Eq. (8)] above, the first two terms are equal to the free energy extractable as work in the ideal case [cf. Eq. (6)]; i.e., $W_{\rm ideal}(\mathbf{m}) = \mathcal{F}_H(\rho_{X|\mathbf{m}}) - F_H^{\rm eq}$. The second term is proportional to the entropy production (or dissipation) generated in the irreversible postquench relaxation $\rho_{X|\mathbf{m}}(\mathbf{x}) \to e^{-\beta H_n(\mathbf{x})}/Z_n$:

$$S_{\rm tot}(\mathbf{m}) = D\left(\rho_{X|\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_n}}{Z_n}\right), \tag{10}$$

expressed as a Kullback-Leibler divergence [68]. We can hence rewrite Eq. (8) as $W(\mathbf{m}) = W_{\rm ideal}(\mathbf{m}) - k_B T S_{\rm tot}(\mathbf{m})$. It is worth noticing that the first term, $W_{\rm ideal}(\mathbf{m})$, does not depend on the actual quenching Hamiltonian chosen in the imperfect scenario $H_n(\mathbf{x})$. It is only the second, entropy production term $S_{\rm tot}(\mathbf{m})$ in Eq. (10) that depends on that Hamiltonian. Note in particular that if the two distributions in that second term are very different (as quantified by their KL divergence), the entropy production in the relaxation may be large enough so that the net work extracted in the process is negative. More explicitly, that will be the case whenever

$$D\left(\rho_{\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_n}}{Z_n}\right) \geqslant \mathcal{F}_H(\rho_{X|\mathbf{m}}) - \mathcal{F}_H\left(\rho_H^{\rm eq}\right), \tag{11}$$

leading to $W(\mathbf{m}) \leqslant 0$. This shows that a net amount of work might be *lost* in the work extraction process, if the quenching Hamiltonian $H_n(\mathbf{x})$ is not adequately chosen.

Repeating this entire process many times and taking the average of the work extracted in Eq. (8) over the different measurement results, we obtain the average work extracted under control limitations:

$$\langle W \rangle = k_B T I_{X;M} + \Delta\mathcal{F}_H - k_B T \langle S_{\rm tot} \rangle, \tag{12}$$

where we used the shorthand notation $\Delta\mathcal{F}_H := \mathcal{F}_H(\rho_0) - F_H^{\rm eq}$. In the above equation, the first two terms give the work extracted in the ideal case [cf. Eq. (7)] and the third one gives the average entropy production due to imperfect quenching, which reduces the amount of work extracted. That average entropy production over measurement results is given by a related calculation:

$$\langle S_{\rm tot} \rangle = \sum_{\mathbf{m}} p(\mathbf{m}) D\left(\rho_{X|\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_{n(\mathbf{m})}}}{Z_{n(\mathbf{m})}}\right). \tag{13}$$

These two calculations will be the basis of our main results. Again, erasure of the information employed for the feedback control in preparation for a next iteration of the entire process will have an unavoidable cost of $k_B T I_{X;M}$ imposed by the generalized Landauer's bound [9] as in the ideal case.

III. THE VALUE OF INFORMATION UNDER CONTROL LIMITATIONS

Given the general scenario described above for quasistatic work extraction using a limited set of feedback control quenches, we will first compare the situation of a prefixed set of quenching Hamiltonians $\mathcal{H} = \{H_k(\mathbf{x})\}_{k=1}^N$, where the controller is able to measure the system of interest and decide which action to take, to the case when no such informed choice can be made, and the controller may just either apply a deterministic Hamiltonian or perform a random choice in $\mathcal{H}$.

To make things more precise, we start by considering a partition of the entire space $\mathbb{M}^{d_M}$ of measurement results $\mathbf{m}$ into N disjoint sets $L := \{l(n)\}_n$ with $n = 1, \ldots, N$, such that $\cup_n \ell(n) = \mathbb{M}^{d_M}$ and $\ell(n) \cap \ell(n') = \varnothing$ for $n \neq n'$. In that way, every measurement result $\mathbf{m}$ can be unequivocally associated with a set $\ell(n)$ with label n. The probability that the measurement outcome $\mathbf{m}$ is inside the set $\ell(n)$ can then be written as

$$p(n) := \sum_{\mathbf{m} \in \ell(n)} p(\mathbf{m}). \tag{14}$$

Notice that by this point we have not imposed any property on the size of the different sets $\ell(n)$.

We then ascribe one of the possible quenching Hamiltonians $H_n(\mathbf{x}) \in \mathcal{H}$ to each set $\ell(n) \in L$. Since we are considering a finite partition, the computation of $\ell(n)$ that associates the Hamiltonians in the set $\mathcal{H}$ with the labels n (linked to the observations $\mathbf{m}$) is just a lookup table. This table returns the Hamiltonian $H_n(\mathbf{x})$ from the repertoire $\mathcal{H}$ verifying that $e^{-\beta H_n(\mathbf{x})}/Z_n$ is most similar to $\rho_{X|\mathbf{m}}(\mathbf{x})$ as quantified by KL divergence. Formally, the optimal lookup table is the function

$$n^*(\mathbf{m}) = \underset{k \in \{1,\ldots,N\}}{\rm argmin}\, D\left(\rho_{X|\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_k}}{Z_k}\right). \tag{15}$$

In this way, the labels n associated with the Hamiltonians $H_n(\mathbf{x})$ in $\mathcal{H}$ are determined by the minimization of the entropy production for given outcome $\mathbf{m}$ in Eq. (10) over all the possible entries in $\mathcal{H}$. In particular, the optimal value of the net extracted work is given by application of Eq. (15) in Eqs. (12) and (13).

To simplify notation, below we will always assume an optimal choice of the lookup table index $n^*(\mathbf{m})$ and will sometimes write just n or n^* to refer to the optimal choice $n^*(\mathbf{m})$. The context should always make the precise meaning clear.

A. Conditions for effective work extraction under control limitations

Having specified the way to choose the different quenching Hamiltonians $H_n(\mathbf{x})$ in the set $\mathcal{H}$ for a given measurement outcome $\mathbf{m}$, we may implement feedback control to extract the average amount of work given in Eq. (12). That implementation would make sense whenever a net work can be extracted from ρ_0 once the cost of information erasure is discounted. The requirement for successful feedback control is hence $k_B T \langle S_{\rm tot} \rangle < \Delta\mathcal{F}_H$. As we show in Appendix A, this inequality can be conveniently written as

$$\left\langle F_{H_n}^{\rm eq} \right\rangle - F_H^{\rm eq} > \langle W_{\rm cost} \rangle + k_B T I_{X;M}, \tag{16}$$

where $\langle F^{\rm eq}_{H_n}\rangle = \sum_n p(n) F^{\rm eq}_{H_n}$ is the average equilibrium free energy of the quenched Hamiltonian, and we identified the average work needed to implement the quenches $H \to H_n$ as

$$\langle W_{\rm cost}\rangle := \sum_{\mathbf{x}} \rho_0(\mathbf{x}) \left[\sum_n p_{L|\mathbf{x}}(n) H_n(\mathbf{x}) - H(\mathbf{x}) \right], \tag{17}$$

with $p_{L|\mathbf{x}}(n) := p(n|\mathbf{x}) = \sum_{\mathbf{m}\in\ell(n)} p_{M|\mathbf{x}}(\mathbf{m}|\mathbf{x})$ being the conditional probability of action n (quench to H_n) given the microstate $\mathbf{x}$. Equation (16) implies that a net work extraction is possible using feedback control if on average the (equilibrium) free energy difference between the quenching Hamiltonians and the original Hamiltonian H overcomes the combined work cost of implementing the quench and posterior erasure of the information employed in the feedback control.

In many situations, the work needed for implementing the quenches $\langle W_{\rm cost}\rangle$ in Eq. (17) would be nonzero, at least in the absence of perfect observation. For example, in a popular feedback strategy for discrete systems that was implemented in the single-electron box Maxwell's demon of Refs. [35,36], the quenches in the feedback loop are performed by significantly raising the energy of a level when it is not occupied, hence incurring in zero cost. However, as soon as these measurements have some errors and the level might be occupied, there is a penalty cost in the quenching process that cannot be avoided [35,36]. In more idealized situations, such as the original Szilard engine, the work needed to make the quench is usually neglected $\langle W_{\rm cost}\rangle = 0$ in assuming that the control operations (there introducing partitions in the single-particle box) can be made, in principle, with no friction or other costs. Finally, there might be situations in which that work is negative, i.e., work is extracted already (or only) during the quench.

B. Work extraction from deterministic and random quenching operations

Now, consider a different scenario in which we try to extract work by using either a prespecified or a random quenching Hamiltonian $H_q \in \mathcal{H}$ that is actually independent of the measurement result $\mathbf{m}$. If the quenching Hamiltonian is fixed to H_q, the average entropy production in Eq. (13) simplifies to $\langle S_{\rm tot}\rangle = \sum_{\mathbf{m}} p(\mathbf{m}) D(\rho_{X|\mathbf{m}} || e^{-\beta H_q}/Z_q) = I_{X;M} + D(\rho_0 || e^{-\beta H_q})$, and hence the work that can be extracted in this case becomes

$$\begin{aligned} W_q &= \mathcal{F}_H(\rho_0) - F^{\rm eq}_H - k_B T D\left(\rho_0 \middle|\middle| \frac{e^{-\beta H_q}}{Z_q}\right) \\ &= F^{\rm eq}_{H_q} - F^{\rm eq}_H - W^q_{\rm cost}, \end{aligned} \tag{18}$$

and the whole expression becomes actually independent of $\mathbf{m}$. In the above equation, the last term stands for the work cost of the quench to H_q:

$$W^q_{\rm cost} = \sum_x \rho_0(x)[H_q(\mathbf{x}) - H(\mathbf{x})]. \tag{19}$$

Therefore, the deterministic quenching can lead to an effective work extraction when the free energy difference between the quenching and original Hamiltonians is greater than the cost to quench, $F^{\rm eq}_{H_q} - F^{\rm eq}_H > W^q_{\rm cost}$.

We note that above, if we assume $H_q = H_0 := -k_B T \ln \rho_0$ (and hence $Z_q = 1$), the third therm in the first line of Eq. (18) vanishes and we can extract the full free energy contained in ρ_0, namely, $W_q = \mathcal{F}_H(\rho_0) - F^{\rm eq}_H$ in accordance to optimal work extraction without feedback control [47–49]. However, the Hamiltonians in the set $\mathcal{H}$ would be typically optimized to extract work from a specific measurement outcome m, and hence we assume in the following that $H_0 \notin \mathcal{H}$.

Comparing the work extractable from imperfect feedback control (once the costs of information erasure has been subtracted) with W_q, we obtain the work gain due to using information in the quenching as

$$\begin{aligned} \langle \Delta W\rangle &:= \langle W\rangle - k_B T I_{X;M} - W_q \\ &= \sum_{n\neq q} p(n) F^{\rm eq}_{H_n} \\ &\quad + \sum_{\mathbf{x}} \rho_0(\mathbf{x}) \left[H_q(\mathbf{x}) - \sum_n p_{L|\mathbf{x}}(n) H_n(\mathbf{x}) \right], \end{aligned} \tag{20}$$

where we used that $H_q \in \mathcal{H}$. The first sum of equilibrium free energies in the above expression is always positive. However, $\langle \Delta W\rangle$ can in principle take any sign depending on how H_q and the set $\mathcal{H}$ is chosen, as well as on the level of noise in the measurement as captured by $p_{L|\mathbf{x}}(n)$. In particular, a net benefit in work extraction for the feedback control scenario requires that the energy after the quench q is not very low compared with the average after-quenching energy. This will always be the case in the situation in which the work cost of quenching is negligible [i.e., $\sum_{\mathbf{x}} \rho_0(\mathbf{x}) H \simeq \sum_{\mathbf{x}} \rho_0(\mathbf{x}) H_q$ for any q] and hence the second line in the above equation vanishes leading to $\langle \Delta W\rangle \geqslant 0$.

The expression in Eq. (20) hence implies that using feedback control may allow the feedback controller not only to gain an extra amount of work $k_B T I_{X;M}$ related to the information acquisition from the measurements (which needs to be paid back at the end of the protocol), but it may also increase our ability to extract work from the original distribution ρ_0 (whenever the optimal quench $H_0 := -k_B T \ln \rho_0$ is not available), enhancing the amount of work we can obtain with respect to the case of just using a predefined quench in $\mathcal{H}$.

If we now assume that the Hamiltonian H_q is sampled randomly from a distribution $r(q)$ independent from the measurement result, the average work extractable would be $\langle W\rangle_r = \sum_q r(q) W_q$ with W_q in Eq. (18). In that case, the difference between the work extracted when choosing the correct quench and a random one from $r(q)$ is

$$\begin{aligned} \langle \Delta W\rangle_r &= \langle W\rangle - k_B T I_{X;M} - \sum_q r(q) W_q \\ &= \sum_n [p(n) - r(n)] F^{\rm eq}_{H_n} \\ &\quad + \sum_{\mathbf{x}} \rho_0(\mathbf{x}) \sum_n [r(n) - p_{L|\mathbf{x}}(n)] H_n(\mathbf{x}). \end{aligned} \tag{21}$$

We note that if we choose the random distribution above to mimic the marginal distribution to obtain the action from the measurement outcomes, $r(q) = p(q)$, the first term in the above equation vanishes. Moreover, since

$p(n) = \sum_x p_{L|\mathbf{x}}(n)\rho_0(\mathbf{x})$ the second term is positive and we have a net gain $\langle \Delta W \rangle_p \geqslant 0$, remarking the convenience of using feedback control.

C. The "cheeseburger effect"

As a simple illustration of the value of information in feedback control scenarios, consider the following thought experiment. Suppose a cheeseburger is placed in either one of two boxes, with equal probability, so that a single bit of information is required to reveal the cheeseburger's location. If we only get one attempt to grab the cheeseburger, knowledge of which box it resides in doubles our energetic reward. So we increase our free energy extraction by an amount that is approximately 10^{26} times greater than $k_B T \log 2$, just from learning a single bit. This differs greatly from typical considerations about the "energetic value of a bit", e.g., arising from the analysis of Szilard's engine.

To understand this phenomenon, we first note that a cheeseburger's macrostate corresponds to a distribution of microstates representing the distinct arrangements of molecular ingredients composing that cheeseburger. We require two such distributions, corresponding to the two possible macrostates; we denote these distributions $\rho_A(\mathbf{x})$ for occupancy of the left box, and $\rho_B(\mathbf{x})$ for occupancy of the right box, with argument $\mathbf{x}$ representing a specific molecular arrangement. In the absence of information about which box contains the cheeseburger, its distribution would be $\rho_0(\mathbf{x}) = [\rho_A(\mathbf{x}) + \rho_B(\mathbf{x})]/2$.

We now suppose that the macroscopic position of the cheeseburger is accessible to measurement (e.g., by opening the boxes to see which one contains the cheeseburger), so that we identify the space $\mathbb{M}$ as consisting only of two possible discrete outcomes, left $m = A$ or right $m = B$, occurring with probabilities $p(A) = p(B) = 1/2$. Then by observing, say, $m = A$, the updated state of the system would be $\rho_{X|A}(\mathbf{x}) = \rho_A(\mathbf{x})$, and we may enact the ideal quenching Hamiltonian, $H_A(\mathbf{x}) = -k_B T \log \rho_A(\mathbf{x})$. Similarly, if we obtain $m = B$, the state of the system gets updated to $\rho_{X|B}(\mathbf{x}) = \rho_B(\mathbf{x})$ and the quench $H_B(\mathbf{x}) = -k_B T \log \rho_B(\mathbf{x})$ is performed. Notice that, unlike a measurement of the microstate (a distribution over $\mathbb{X}^{d_X}$), this is a measurement of the macrostate (a distribution over $\{A, B\}$). This is a similar measurement as in the case of Szilard's engine, where the measurement reveals which side of the box the particle is contained in, but not the specific location and momentum of the particle.

Let us then assume that the cheeseburger is in one of the two possible (Bayesian posterior) states $\rho_{X|m}$, for $m = \{A, B\}$. If the outcome m is accessible to the feedback controller, a work $W(m)$ as defined in Eq. (8) would be extracted, where in this case the optimal choice of quench in Eq. (15) is just $n^*(m) = m$. We assume that accessing the cheeseburger will allow for a positive work extraction, $W(m) > 0$ for both $m = A, B$. The average work that the controller would be able to extract is then $\langle W \rangle = \sum_m p(m) W(m) > 0$.

In contrast, if the controller has to randomly choose some action, i.e., without access to the outcome m, the amount of extractable work will be reduced. In particular, let us first consider the situation in which even if the quenching Hamiltonian is chosen at random, $q = \{A, B\}$, with arbitrary probabilities $r(q)$, the true measurement outcome m is revealed to the experimenter just before they take that action, so that the experimenter has the opportunity to halt the quench unless it is correct. [Physically, this models the fact that if you randomly reach behind a box and find that there is nothing there, you do not follow the rest of the work protocol of grabbing a (nonexistent) cheeseburger in your hand, moving that hand to your mouth, chewing, and then undertaking the full digestion process on that nonexistent cheeseburger.] For simplicity, suppose as well that we ignore the work expended by the experimenter before they aborted their quench, in those cases that they chose to do so. (In our model of reaching for a cheeseburger in a box, this means we ignore the work expended by reaching one's hand behind the wrong box.) In this case, when we average over measurement results, the work extracted is

$$\begin{aligned} \langle W' \rangle_r &= \sum_m p(m) \sum_q r(q) W(m)\, \delta_{m,q} \\ &= \sum_m p(m) r(m) W(m) > 0. \end{aligned} \tag{22}$$

The difference in extracted work between the two situations is therefore

$$\langle \Delta W' \rangle_r = \sum_m p(m) W(m) [1 - r(m)] > 0. \tag{23}$$

This difference is strictly positive since $r(m) \leqslant 1$ and $\sum_m r(m) = 1$. It can be even macroscopically large if the initial system is far enough from thermal equilibrium, so that $W(m)$ is large on macroscopic scales for all m. That is, if the cheeseburger contains enough processable energy, typically $W(m) \simeq 300$ cal for a standard cheeseburger.

An interesting variant of this analysis arises if we cannot halt an incorrect quench. In that case energetic penalties are incurred as a consequence of implementing a wrong quench, which may involve both the work cost of implementing the quench itself [see Eq. (19)] (the energy needed to pick up and eat the cheeseburger), and also the free energy lost in the subsequent relaxation to equilibrium (the putrefaction of the cheeseburger in the inaccessible box).

In this case, the average extracted work is

$$\langle W \rangle_r = \langle W' \rangle_r - \sum_m p(m) \sum_{q \neq m} r(q) W_q(m), \tag{24}$$

where the second term in the first line represents the total work lost due to the implementation of the quench H_k when the state of the system was $\rho_{X|m}$ (instead of $\rho_{X|k}$) and $W_q(m) = W(m) - k_B T D(\rho_{X|m} || \rho_{X|q})$, which in general can be either positive or negative.

The benefit of feedback control (i.e., observing and accurately quenching with the correct quench) in this situation is

$$\langle \Delta W \rangle_r = \langle \Delta W' \rangle_r - k_B T I_{X;M} + \sum_m p(m) \sum_{q \neq m} r(q) W_q(m), \tag{25}$$

which is consistent with Eq. (21) for noisefree observation of the macroscopic variable $m = n$ and no limitations in the available quenches, i.e., $H_n = -\ln \rho_{X|n}$. Note that in the above expression we have incorporated the *erasure cost*, $k_B T I_{X;M} = k_B T [S(\rho_0) - S(\rho_{X|A})/2 - S(\rho_{X|B})/2] =$

$k_B T S(p) = k_B T \log 2$. This is because, in contrast to the feedback control case, random quenching without observation requires no processing of information and hence a fair comparison requires taking that cost into account. The above equation tells us that despite having learned only one bit about m, the thermodynamic consequences for the distribution of the microstates $\mathbf{x} \in \mathbb{X}$ can be arbitrarily large. (See Ref. [69] for another, more complicated variant of this scenario.)

IV. OPTIMAL WORK EXTRACTION WITH CONTROL LIMITATIONS

Importantly, rather than taking an arbitrary collection of Hamiltonians, in many situations we would be interested in constructing an optimal set of quenching Hamiltonians $\mathcal{H}$ with a limited dimension N, given an initial distribution $\rho_0(\mathbf{x})$ and given the conditional probability $p_{M|\mathbf{x}}(\mathbf{m})$ characterizing the noisy measurement of the system.

As mentioned before, the measurement results $\mathbf{m}$ can be generically distributed into N disjoint but otherwise arbitrary sets $\ell(n)$ with $n = 1, \ldots, N$ to which one of the possible quenching Hamiltonians $H_n(\mathbf{x}) \in \mathcal{H}$ is attached by using the lookup table defined by the minimization in Eq. (15). Constructing the optimal set $\mathcal{H}$ requires that the H_n within each set $\ell(n)$ minimizes the expected entropy production across all such partition (see Fig. 1). The optimal form of $H_n(\mathbf{x})$ given a partition $L = \{\ell(n)\}_{n=1}^N$ is derived in Appendix B and reads

$$H_n(\mathbf{x}) = -k_B T \ln \rho_n^{\text{opt}}(\mathbf{x}), \tag{26}$$

where ρ_n^{opt} is a probabilistic mixture of postmeasurement posterior distributions $\rho_{X|\mathbf{m}}$, as

$$\rho_n^{\text{opt}}(\mathbf{x}) := \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\rho_{X|\mathbf{m}}(\mathbf{x}), \tag{27}$$

with $p(\mathbf{m}|n) = \mathbb{1}\{\mathbf{m} \in \ell(n)\}p(\mathbf{m})/p(n)$ the conditional probability of outcome $\mathbf{m}$ given knowledge of the set n to which it pertains. Here, $\mathbb{1}\{\mathbf{m} \in \ell(n)\}$ is the indicator function for whether $\mathbf{m}$ resides within the partition block $\ell(n)$, and recall that $p(n) = \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})$ is the probability that the measurement outcome $\mathbf{m}$ is inside the set $\ell(n)$. The distribution in Eq. (27) can be seen as mixed version of the postmeasurement distribution $\rho_{X|\mathbf{m}}(\mathbf{x})$ after averaging over the detailed information about the outcome $\mathbf{m}$ when that information is unavailable but we only have information about within which set n the outcome is.

Given the optimal form of the elements of the set $\mathcal{H}$ in Eq. (26), we can now compute the average entropy production (or work dissipated) in Eq. (13) for a given choice of partition L:

$$\begin{aligned}\langle S_{\text{tot}}\rangle &= \sum_{\mathbf{m}} p(\mathbf{m}) D\big(\rho_{X|\mathbf{m}} \big|\big| \rho_n^{\text{opt}}\big) \\ &= \sum_{\mathbf{m}} p(\mathbf{m}) \sum_{\mathbf{x}} \rho_{X|\mathbf{m}}(\mathbf{x}) \ln \frac{\rho_{X|\mathbf{m}}(\mathbf{x})}{\rho_n^{\text{opt}}(\mathbf{x})} \\ &= \sum_{n} p(n) \left(S(\rho_n^{\text{opt}}) - \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n) S(\rho_{X|\mathbf{m}}) \right), \end{aligned} \tag{28}$$

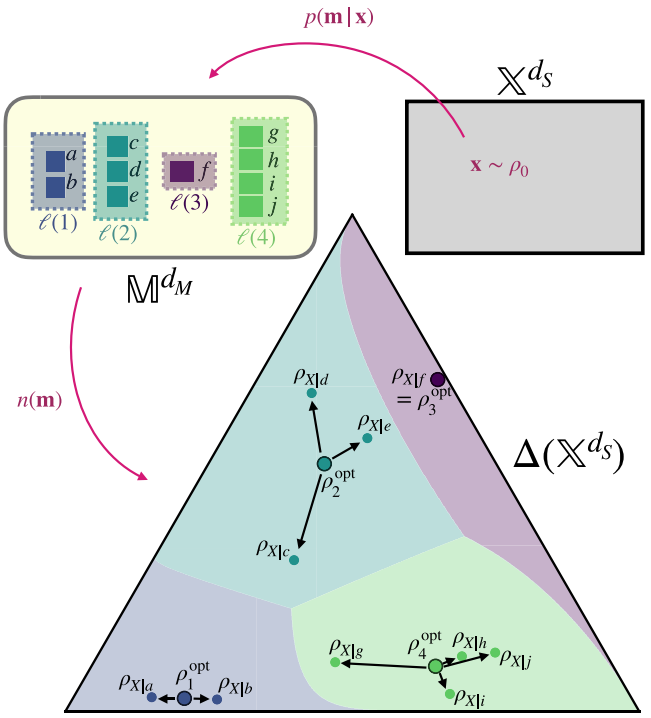


FIG. 1. Illustration of the construction of an optimal quenching repertoire $\mathcal{H}$. We group all possible postmeasurement distributions $\rho_{X|\mathbf{m}}$ in the simplex $\Delta(\mathbb{X}^{d_S})$ into N sets according to a partition $\ell(n)$. The partition encodes a lookup table $n(\mathbf{m})$ from possible measurement results $\mathbf{m}$ (colored squares) in the space $\mathbb{M}^{d_M}$ to the set of possible quenching actions. Measurement results $\mathbf{m}$ arise from the nonequilibrium system of interest (big gray square) by sampling from a noisy channel $p(\mathbf{m}|\mathbf{x})$, where system microstate $\mathbf{x} \in \mathbb{X}^{d_S}$ is distributed as ρ_0. The optimal quench ρ_n^{opt} (large dots) within a given set $\ell(n)$ (green shaded area) is chosen by finding the distribution that minimizes the average entropy production (Kullback-Leibler divergence) when relaxing from all other postmeasurement states within that set (small dots). Equation (33) is used to obtain H_n^{opt} from ρ_n^{opt}. Shaded regions depict a Voronoi-like partitioning of $\Delta(\mathbb{X}^{d_S})$ according to KL divergence from $\{\rho_n\}_{n=1}^N$. In the illustration, $N = 4$, and the partition is $\{a, b\}, \{c, d, e\}, \{f\}, \{g, h, i, j\}$.

where we took the optimal index choice n as the one verifying $\mathbf{m} \in \ell(n)$ for every outcome of the measurement. The above equation [Eq. (28)] is one of our main results that relates the entropy production in the imperfect work extraction process to the increase in average Shannon entropy due to coarse graining the information about the system state over the interval $\ell(n)$ for $n = 1, \ldots, N$.

We note that in Eq. (28), the optimal quenching distribution $\rho_n^{\text{opt}}(\mathbf{x})$ is a mixture of the postmeasurement distributions $\rho_{X|\mathbf{m}}(\mathbf{x})$ across $\mathbf{m}$ weighted by $p(\mathbf{m}|n)$ [cf. Eq. (27)], and that the second term inside the parenthesis in the last line of Eq. (28) is the mean entropy of $\rho_{X|\mathbf{m}}$ over the same mixing distribution $p(\mathbf{m}|n)$. As a consequence, the entropy production can be rewritten as

$$\langle S_{\text{tot}}\rangle = \sum_n p(n) J_{p(\mathbf{m}|n)}(\{\rho_{X|\mathbf{m}}\}_{\mathbf{m}\in\ell(n)}), \tag{29}$$

with $J_{\mathbf{p}}(\mathbf{P}) := S(Q) - \sum_i p_i S(P_i) \geqslant 0$ denoting the generalized Jensen-Shannon divergence (JSD) among a weighted collection of distributions $\mathbf{P} = \{P_i\}_i$ with (normalized) weights $\mathbf{p} = \{p_i\}_i$ [70]. Here, $Q(\mathbf{x}) = \sum_i p_i P_i(\mathbf{x})$ denotes the mixture distribution of $\{P_i(\mathbf{x})\}_i$ with weights $\{p_i\}_i$. [Note, $\sum_i p_i = 1$ and $\sum_{\mathbf{x}} P_i(\mathbf{x}) = 1$.] In Eq. (29), we identified $P_i(\mathbf{x}) = \rho_{X|\mathbf{m}}(\mathbf{x})$ and $p_i = p(\mathbf{m}|n)$ with $i = \mathbf{m} \in \ell(n)$, leading to $Q(\mathbf{x}) = \rho_n^{\text{opt}}(\mathbf{x})$, cf. Eq. (28). The Jensen-Shannon divergence and its generalizations have been employed in a variety of fields ranging from bioinformatics [71] to the social sciences [72] and more recently in machine learning [73]. The above result in Eq. (29) implies that the optimal partition is one that minimizes the generalized JSD among the distributions grouped into a common partition block $\ell(n)$, averaged over the action n taken.

A second, illuminating way of rewriting $\langle S_{\text{tot}} \rangle$ in Eq. (28) is in terms of informational quantities. A key fact is that the distribution $\rho_n^{\text{opt}}(\mathbf{x})$ in Eq. (27) is the marginal probability of finding the system in state $\mathbf{x}$ given that the measurement result is inside the partition $\ell(n)$. More precisely, we can rewrite the entropy production as the conditional mutual information [67] between the system state and the measurement result, given the choice of quenching Hamiltonian according to the partitions $\ell(n)$:

$$\begin{aligned}\langle S_{\text{tot}} \rangle &= \sum_n p(n) \sum_{\mathbf{x}} \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\rho_{X|\mathbf{m}}(\mathbf{x}) \ln\left(\frac{\rho_{X|\mathbf{m}}(\mathbf{x})}{\rho_n^{\text{opt}}(\mathbf{x})}\right) \\ &= \sum_n p(n) \sum_{\mathbf{x}} \sum_{\mathbf{m}\in\ell(n)} P(\mathbf{x},\mathbf{m}|n) \ln\left(\frac{P(\mathbf{x},\mathbf{m}|n)}{\rho_n^{\text{opt}}(\mathbf{x})p(\mathbf{m}|n)}\right) \\ &= I_{X;M|L} \geqslant 0, \end{aligned} \tag{30}$$

where in the second line we expanded the joint conditional probability $P(\mathbf{x},\mathbf{m}|n) = p(\mathbf{m}|n)\rho_{X|\mathbf{m}}(\mathbf{x})$, and inside the logarithm we multiplied and divided by $p(\mathbf{m}|n)$. The last line is reached by identifying the conditional mutual information, which for three generic random variables X, Y, and Z in standard notation reads $I_{X;Y|Z} = \sum_z p_Z(z) \sum_{y,x} p_{X,Y|Z}(x,y|z) \ln(\frac{p_{X,Y|Z}(x,y|z)}{p_{X|Z}(x|z)p_{Y|Z}(y|z)})$. Here, X corresponds to the system state taking values $\mathbf{x}$, Y to the measurement result M taking values $\mathbf{m}$, and variable Z corresponds to the number of partition (which we denoted as L) taking values $n = 1, \ldots, N$, and whose marginal probability is $p(n)$ [equivalent to p_Z]. The probability $P(\mathbf{x},\mathbf{m}|n)$ in Eq. (30) is the joint probability of having state $\mathbf{x}$ in the system and measuring $\mathbf{m}$, given the result is inside the partition $\ell(n)$ [equivalent to $p_{X,Y|Z}$], $\rho_n^{\text{opt}}(x)$ is the (optimal) postquench system distribution conditioned on partition n [equivalent to $p_{X|Z}$], and $p(\mathbf{m}|n)$ is the probability of obtaining the result $\mathbf{m}$ conditioned on partition n [equivalent to $p_{Y|Z}$].

The conditional mutual information above can be interpreted as the amount of statistical correlations between the state of the system and the measurement result that remain once the partition n to which the measurement result pertains [$\mathbf{m} \in \ell(n)$] is known. This quantity would be exactly zero if the measurement result is completely determined by the partition (e.g., if one has only one possible measurement result in each partition) but nonzero otherwise. Note that this effect arises *only* because of the restriction on the number of possible actions. Without such a restriction, this information-theoretic loss of the maximal amount of free energy the agent can extract would not exist.

Equation (30) says that reducing the dissipation in the feedback process amounts to reducing as much as possible the conditional mutual information $I_{X;M|L}$ between $\mathbf{x}$ and $\mathbf{m}$ given that the measurement result $\mathbf{m}$ is in the set $\ell(n)$. That would require the postmeasurement distribution $\rho_{X|\mathbf{m}}(\mathbf{x})$ to be as independent as possible of the measurement result $\mathbf{m}$ within every partition $\ell(n) \in L$. Notice though that such a condition cannot be satisfied without spoiling the possibility of performing feedback control, which in general requires correlations between $\mathbf{m}$ and $\mathbf{x}$, cf. Eq. (12). In other words, zero entropy production requires that the measurements provide no other information than the one necessary to deduce the actual set n to which the measurement belongs. In that case, the distributions $\rho_{X|\mathbf{m}}(\mathbf{x})$ only depend on n, which would imply actually $\rho_n^{\text{opt}}(\mathbf{x}) = \rho_{X|\mathbf{m}}(\mathbf{x})$. That condition would be (trivially) satisfied when the number of possible quenching Hamiltonians coincides with the number of measurement results, $N = |\mathbb{M}^{d_M}|$, as expected.

By plugging Eq. (30) into Eq. (12), we obtain the average extractable work:

$$\langle W \rangle = k_B T [I_{X;M} - I_{X;M|L}] + \Delta\mathcal{F}_H. \tag{31}$$

The term $I_{X;M} - I_{X;M|L}$ is sometimes called the "interaction information." In general, it can be either positive or negative [67]. Therefore, an efficient measurement generating high correlations between $\mathbf{m}$ and $\mathbf{x}$ in general improves work extraction, in that it provides more information $I_{X;M}$ and in the limit of perfect correlation allows the extraction of all the free energy in the initial state. At the same time, spurious correlations as characterized by the conditional mutual information $I_{X;M|L}$ may spoil the ability to extract work by increasing the entropy production.

Taking into account the erasure of information after the feedback control protocol implies, nonetheless, that the optimal choice of partition corresponds to the one minimizing $I_{X;M|L}$. In other words, it corresponds to one in which the measurements generate the minimum mutual information between the measurement outcomes and the system state within the partitions $\ell(n)$, hence allowing the correct choice of the quenching Hamiltonians $H_n(\mathbf{x}) = -k_B T \ln \rho_n^{\text{opt}}(\mathbf{x})$ but avoiding, as much as possible, the correlations that cannot be used in the feedback control operation.

Finally, we can straightforwardly obtain an upper bound on the entropy production as

$$\sum_n p(n) S\big(\rho_n^{\text{opt}}\big) \geqslant \langle S_{\text{tot}} \rangle, \tag{32}$$

which follows from the concavity of entropy, $S(\rho_n^{\text{opt}}) \geqslant \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n) S(\rho_{X|\mathbf{m}}) \geqslant 0$. The above upper bound establishes, respectively, a lower bound on the average work that can be extracted, cf. Eq. (12).

V. SOME APPLICATIONS

In the following, we describe various situations that may arise in different contexts fitting within the general framework introduced above. In these applications, the work extraction

procedure always uses feedback control, but the kind of measurements employed and the limitations vary.

A. Feedback control with memory limitations and noisefree measurements

As a standard scenario of general interest in stochastic thermodynamics, we consider in the following the case in which the measurement results $m_i \in \mathbf{m}$ are associated one to one with the system variables $x_i \in \mathbf{x}$ and hence $\mathbb{M} = \mathbb{X}$ and $d_X = d_M$. Moreover, even if we are allowed to optimally choose the set of quenching Hamiltonians in the set $\mathcal{H}$, we assume operation with a finite memory where the number of quenches N is generically much smaller than the state space of the system $N < |\mathbb{X}^{d_X}|$, and hence likewise smaller than the possible measurement results $|\mathbb{M}^{d_M}| = |\mathbb{X}^{d_X}|$.

In this context, we here assume the case of ideal (perfect) measurements on the system where $p(\mathbf{m}) = \rho_0(\mathbf{x})$ and $\rho_{X|\mathbf{m}}(\mathbf{x}) = \delta_{\mathbf{m},\mathbf{x}} = \prod_{i=1}^{d} \delta_{m_i,x_i}$. The optimal quench set $\mathcal{H}$, according to Eqs. (26) and (27) then reduces to

$$H_n(\mathbf{x}) = -k_B T \ln\left(\frac{\rho_0(\mathbf{x})}{p_0^{\mathrm{cg}}(n)} \mathbb{1}\{\mathbf{x} \in \ell(n)\}\right), \tag{33}$$

where $\mathbb{1}\{\mathbf{x} \in \ell(n)\}$ is the indicator function within the partition $\ell(n)$, and $p_0^{\mathrm{cg}}(n) := \sum_{x \in \ell(n)} \rho_0(\mathbf{x})$ is the coarse-grained version of $\rho_0(\mathbf{x})$ according to partition $\ell(n)$. That is, the above quenching Hamiltonian takes finite values in the interval $\mathbf{x} \in \ell(n)$ and it is infinite otherwise. Here, above the optimal postquench distribution $\rho_n^{\mathrm{opt}}(\mathbf{x})$ is just the expression inside the parenthesis.

In this case of ideal measurements, a more intuitive result can be obtained from Eq. (28) (see Appendix C), namely,

$$\begin{aligned}\langle S_{\mathrm{tot}} \rangle &= \sum_{\mathbf{m}} \rho_0(\mathbf{m}) D(\delta_{\mathbf{m},\mathbf{x}} || e^{-\beta H_n}) \\ &= \sum_{n} p_0^{\mathrm{cg}}(n) S(\rho_n^{\mathrm{opt}}) = S(\rho_0) - S(p_0^{\mathrm{cg}}),\end{aligned} \tag{34}$$

which is just the drop in Shannon entropy when coarse graining the original start distribution $\rho_0(\mathbf{x})$ over the partitions $\ell(n)$. The above equation indicates that the extra dissipation (work lost) in this case is entirely due to the loss of resolution in the initial distribution ρ_0 due to the coarse graining imposed by the partition $\ell(n)$ over the finite memory of the controller.

Noting that $p_0^{\mathrm{cg}}(n)$ is a probability distribution with N distinct outcomes, we have $S(p_0^{\mathrm{cg}}) \leqslant \log N$, which is attained if and only if the partition $\ell(n)$ is such that p_0^{cg} is uniformly distributed. Therefore, we obtain the bound

$$\langle S_{\mathrm{tot}} \rangle \geqslant S(\rho_0) - \log N, \tag{35}$$

across all possible partitions $\ell(n)$ with fixed dimension N. As a result, we see that the optimal design for the distribution of measurement outcomes consists of a partition $\ell(n)$ that is probabilistically equitable with respect to the initial distribution ρ_0. That is,

$$p_0{}^{\mathrm{cg}}(n) = \sum_{x \in \ell(n)} \rho_0(\mathbf{x}) = \sum_{x \in \ell(n')} \rho_0(\mathbf{x}) = p_0{}^{\mathrm{cg}}(n'), \tag{36}$$

for all n, n' in the set, in which case the bound in Eq. (35) is attained. Additionally, we notice it may be possible to further tighten the bound above by examination of the optimal choice of partition given ρ_0.

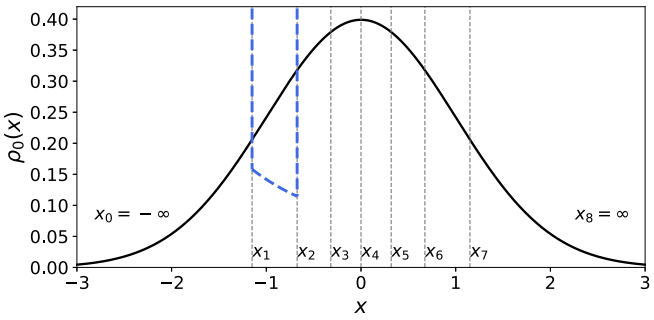


FIG. 2. Optimal partition $\ell(n)$ for a particle in one dimension using $N = 8$ contiguous regions, $(x_0, x_1]$, $(x_1, x_2]$, …, (x_7, x_8) according to Eq. (37). The solid line corresponds to the initial distribution, here taken to be $\rho_0(\mathbf{x}) = e^{-x^2/2}/\sqrt{2\pi}$ while the blue dashed curve corresponds to the quenching Hamiltonian $H_n(\mathbf{x})$ in Eq. (33) for $n = 2$ (not normalized, for visibility), equaling ∞ outside of $(x_1, x_2]$.

As a simple example of interest, let us consider a Brownian particle in one dimension described by its position x in $\mathbb{R}$ with initial density $\rho_0(x)$. This situation has been often implemented in the laboratory using colloidal particles trapped by laser beams in the context of stochastic thermodynamics [30–32,41]. We assume perfect observation, so that $p(m|x) = \delta(m - x)$, leading to the nth optimal quenching Hamiltonian in the set $\mathcal{H}$ as given in Eq. (33), with $\rho_0^{\mathrm{cg}} = \int_{\ell(n)} \rho_0(x')dx'$ and $\{\ell(n)\}$ a partition of $\mathbb{R}$. For simplicity, we now make the restriction that $\ell(n)$ is a *contiguous* partition, i.e., that each partition block has the form $\ell(n) = (x_{n-1}, x_n]$ for some $x_0 < x_1 < \cdots < x_N$, with N size of the Hamiltonian repertoire, and with the last partition block having the form $\ell(N) = (x_{N-1}, x_N)$ rather than $(x_{N-1}, x_N]$. We take $x_0 = -\infty$ and $x_N = \infty$ so that $\cup_n \ell(n) = \mathbb{R}$; if we took any $x_0 > -\infty$ or $x_N < \infty$, our quenching repertoire would have smaller support than ρ_0, yielding unnecessary dissipation.

In this case, the unique way to achieve optimal quenching is to set partition $\ell(n)$ as N intervals in $\mathbb{R}$ with boundaries x_k for $k = 0, 1, 2, \ldots, N$ at

$$x_k = F_0^{-1}\left(\frac{n}{N}\right), \tag{37}$$

with $F_0(x) = \int_{-\infty}^{x} \rho_0(x')dx'$ denoting the cumulative probability of $X < x$ under ρ_0 (see Appendix C for a proof).

The optimal partition $\ell(n)$ is shown in Fig. 2 for $N = 8$ possible quenches. The optimal quenches inherit the form of the initial distribution ρ_0 within each interval as illustrated with the blue dashed line in the second partition ($n = 2$).

B. Work extraction from random initial distributions

A second related but different scenario of interest is the case in which the initial distribution $\rho_0(\mathbf{x})$ of the system is not completely known to the feedback controller.

Uncertainty in the initial distribution amounts to a *random macrostate*, from which the randomness of the microstates given the macrostate is subsidiary. In this setting, the measurement is *about* the distribution (i.e., information regarding

which of several possible distributions may describe the system), rather than *of* the distribution (i.e., a noisy measurement of the system microstate).

In particular, we assume that the initial distribution $\rho_0^{(\alpha)}(\mathbf{x})$ is sampled from a finite set labeled by α according to some prefixed probability p_α, with $\sum_\alpha p_\alpha = 1$. In this case, we consider that the feedback controller performs some (imperfect) measurement of the system concerning the value of α of the original distribution, rather than the state $\mathbf{x}$. Then, the conditional probability capturing the effect of external noise is of the form $p(m|\alpha)$, where now the result of the measurement m is a single number related to the values of the parameter α, occurring with (marginal) probability $p(m) = \sum_\alpha p(m|\alpha)p_\alpha$. Such a situation is to be compared with the cases in which no information about the parameter α is available, implying an effective mixture

$$\bar{\rho}_0 = \sum_\alpha p_\alpha \rho_0^{(\alpha)}, \tag{38}$$

and should also be compared to the case in which perfect knowledge of the initial distribution $\rho_0^{(\alpha)}$ is assumed.

After performing the noisy measurement of α, the controller can gain knowledge of the initial distribution of the system. The conditional probability of obtaining a possible α value given the measurement result m is $p(\alpha|m) = p_\alpha p(m|\alpha)/p(n)$, from which the updated system distribution $\rho_{X|m}(x)$ can be computed as

$$\rho_{X|m}(\mathbf{x}) = \sum_\alpha p(\alpha|m)\rho_0^{(\alpha)}(\mathbf{x}). \tag{39}$$

Notice that the controller obtains perfect knowledge of the initial distribution if the measurement is noisefree, i.e., when $p(m|\alpha) = \mathbb{1}\{m = \alpha\}$ leading to $\rho_{X|m}(\mathbf{x}) = \rho_0^{(\alpha(m))}$ with $\alpha(m)$ denoting the parameter unequivocally related to the measurement outcome. On the other hand, zero knowledge from the measurement is obtained if $p(m|\alpha)$ is independent of α, so that we obtain $\rho_{X|m}(\mathbf{x}) = \bar{\rho}_0(\mathbf{x})$.

Assuming no further limitations in the memory or action of the controller, the repertoire of Hamiltonians defining the quench set $\mathcal{H} = \{H_m(\mathbf{x})\}$ to optimally extract work from the initial state is given by the updated states after measurement, i.e., $H_m(\mathbf{x}) = -\ln \rho_{X|m}(\mathbf{x})$. As a consequence of the mismatch between that distribution and the "true" initial distribution, entropy production will be generated during the feedback control protocol also in this case. Such entropy production is as before related to the relaxation from the (generally unknown) distribution $\rho_0^{(\alpha)}$ to the equilibrium state $e^{-\beta H_m(\mathbf{x})}$ after the quench. Following our previous arguments, the average entropy production due to the uncertainty in the parameter α is therefore given by

$$\langle S_{\text{tot}} \rangle = \sum_\alpha p_\alpha \sum_m p(m|\alpha) \mathrm{D}\big(\rho_0^{(\alpha)} \big|\big| \rho_{X|m}\big) \geqslant 0, \tag{40}$$

with equality if and only if the channel $p(m|\alpha)$ is noisefree, i.e., if $\rho_{X|m}(\mathbf{x}) = \rho_0^{(\alpha(m))}(\mathbf{x})$. On the other hand, such entropy production is maximized in the zero-knowledge case, leading to $\langle S_{\text{tot}} \rangle = \sum_\alpha p_\alpha D(\rho_0^{(\alpha)} || \bar{\rho}_0)$.

We now consider the combination of the situations of random initial distributions as described above and control limitations. Since each measurement outcome m is unambiguously related to an optimal postquench equilibrium distribution $\rho_{X|m}(\mathbf{x})$, our previous logic will apply. If there are $|\mathbb{M}|$ possible measurements $m \in \{1, \ldots, |\mathbb{M}|\}$, and N possible quenching Hamiltonians, $n \in \{1, \ldots, N\}$, where $N < |\mathbb{M}|$, then as before we need to design a repertoire and lookup table $n(m)$ mapping from the measurement results to the associated quenches in $\mathcal{H}$. Given a partition $L = \{\ell(n)\}_{n=1}^N$, similar calculations to those of Appendix B apply to obtain an optimal quenching Hamiltonian $H_n(\mathbf{x}) = -k_B T \ln \rho_n^{\text{opt}}(\mathbf{x})$ with

$$\rho_n^{\text{opt}}(\mathbf{x}) = \sum_\alpha p(\alpha|n)\rho_0^{(\alpha)}(\mathbf{x}), \tag{41}$$

with $p(\alpha|n) = p_\alpha p(n|\alpha)/p(n)$, where $p(n) = \sum_\alpha p(n|\alpha)$ and $p(n|\alpha) = \sum_{m\in\ell(n)} P(m|\alpha)$. Details are reported in Appendix E, where we also show the equivalence of Eq. (41) above and Eq. (27), with use of the measurement-based expression [Eq. (39)]. That is, we show that we may equivalently express the form of the optimal postquench equilibrium ρ_n^{opt} as a probabilistic mixture of $\rho_{X|\mathbf{m}}$ across $\mathbf{m}$ (given n) or as a probabilistic mixture of $\rho_0^{(\alpha)}$ across α (given n).

Then, likewise the average entropy production under control limitations reads

$$\langle S_{\text{tot}} \rangle = \sum_\alpha p_\alpha \sum_m p(m|\alpha) D\big(\rho_0^{(\alpha)} \big|\big| \rho_{n^*(m)}^{\text{opt}}\big), \tag{42}$$

which also exhibits a representation in terms of JSD and conditional mutual information akin to Eqs. (29) and (30), respectively.

In particular, under the optimal action, $\langle S_{\text{tot}} \rangle$ can be expressed as the average of a weighted multidistributional Jensen-Shannon divergence, namely,

$$\langle S_{\text{tot}} \rangle = \sum_n p(n) J_{p_{\alpha|n}}\big(\{\rho_0^{(\alpha)}\}_\alpha\big). \tag{43}$$

The JSD in Eq. (43) at any given n is the dissipation associated with action n in response to the noisy measurement of α. Accordingly, optimal partitions will cluster nearby distributions in the probability simplex (see Fig. 3 for an illustrative example).

Additionally, akin to Eq. (30), we can re-express $\langle S_{\text{tot}} \rangle$ in terms of a conditional mutual information between the system micro and macro states for a given a partition, $\langle S_{\text{tot}} \rangle = I_{X,A;L}$, with X, A, L denoting the random variables whose values are denoted $\mathbf{x}$, α, n (see Appendix E for details). We note that here the macrostates α play the role of the measurement results $\mathbf{m}$ in expression (30). Therefore, we can conclude that minimum dissipation in this scenario requires a minimum amount of extra correlations between microstates and macrostates, other than those strictly needed to perform the choice of the optimal action.

We demonstrate this scenario with a numerical example of a generic system with three discrete states (or energy levels), obtaining the optimal partitions for a set of possible initial nonequilibrium distributions indexed by $\alpha = 1, \ldots, 10$. We denote these distributions as $\{\rho_0^{(\alpha)}\}_{\alpha=1}^{10}$, with each $\rho_0^{(\alpha)}$ being a three-outcome distribution (for visualizability in the unit simplex). Moreover, we consider equal probabilities for each of them, $p_\alpha = 1/10$ for all α. We then numerically scan across all distinct partitionings of the ten distributions (macrostates)

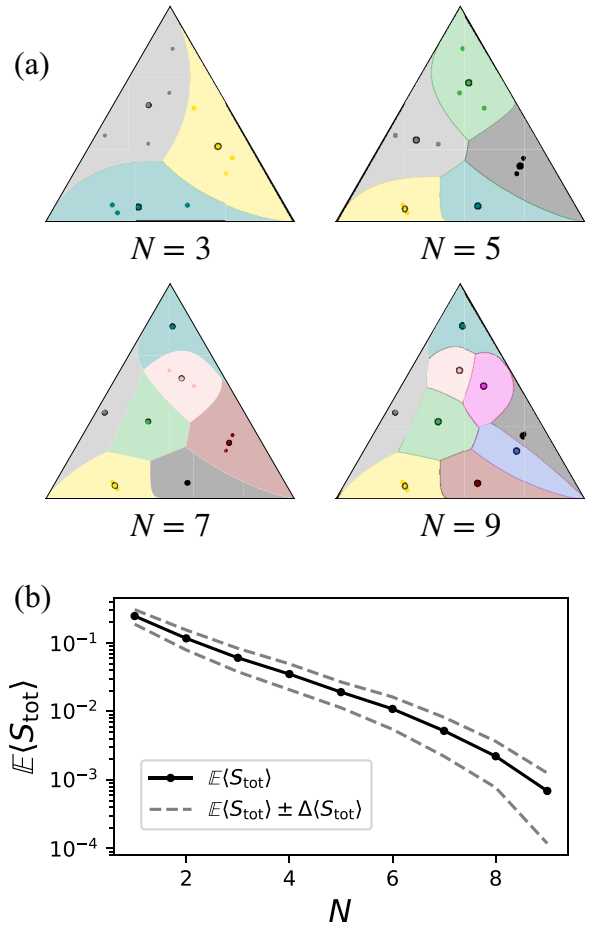


FIG. 3. (a) Optimal partitions and postquench equilibria for $N = 3, 5, 7, 9$ actions and ten possible initial distributions. The distributions $\{\rho_0^{(\alpha)}\}_{\alpha=1}^{10}$ are depicted as small dots, colored according to their assigned partition; large dots of corresponding colors represent the optimal quenching choices $\rho_n^{\rm opt}$. Additionally, regions are shaded based on the minimization of the KL divergence between $\{\rho_n^{\rm opt}\}_{n=1}^{N}$ and the points therein, akin to a Voronoi diagram [74,75]. (b) Expected dissipation under optimal partition as a function of the number of actions N when the ground truth distributions $\{\rho_0^{(\alpha)}\}_{\alpha=1}^{10}$ are sprinkled randomly into the unit 3-simplex. $\mathbb{E}\langle S_{\rm tot}\rangle$ and $\Delta\langle S_{\rm tot}\rangle$ denote the mean and standard deviation of $\langle S_{\rm tot}\rangle$ in Eq. (42).

into N response actions, computing the dissipation under the optimal choice in Eq. (41).

In Fig. 3(a), we depict the optimal partition for different number of allowed actions (memory size limitations) $N = 3, 5, 7, 9$. The different colored areas visually represent which quenching Hamiltonian will be chosen for the different distributions $\rho_0^{(\alpha)}$ (represented as small dots). The optimal quenching distributions for each partition [Eq. (41)] are represented by the larger dots. In Fig. 3(b), we visualize the dissipation resulting from the optimal action repertoire with a number of available actions $N < 10$. We show the mean $\mathbb{E}[\langle S_{\rm tot}\rangle]$ and standard deviation $\Delta[\langle S_{\rm tot}\rangle]$ of the entropy production in the feedback control process across numerous samples of the choice of initial distributions. As can be observed, when N grows from 1 to 9, the dissipation decreases confirming the optimal choice for the partitions. Each of the ten distributions is placed uniformly at random in the probability simplex, and each measurement outcome is assumed to have equal probability $1/10$. The averages and standard deviations are with respect to 20 iterations of the whole process.

VI. CONCLUSIONS AND DISCUSSION

In this work, we have developed a thermodynamic framework to describe the interplay of measurement and feedback control operation in systems with restricted capabilities preventing completely optimal free energy extraction.

We formulated this problem by considering a nonequilibrium distribution that can be noisily measured, in response to which one of a limited number of actions can be taken, in the form of *quenches*, determining when extraction of a net amount of work remains possible. In particular, the problem boils down to the minimization of the expected Kullback-Leibler divergence between the would-be optimal response and the actual response resorted to; this requires a partition of scenarios into response actions, and in addition, a specific form of those response actions given the partition. We derived an explicit expression for the latter, finding that the optimal postquench equilibrium is a probabilistic mixture of would-be optimal quenches across the scenarios grouped together into a common response. Furthermore, we obtained an expression for the dissipated work in the case of optimal quenching, finding that it equals the average across possible actions of the multidistributional Jensen-Shannon divergence among the distributions whose mixture composes the optimal action. This reduced the general problem of optimal response construction to a simpler and more specific search problem across partitions of scenarios into actions.

Additionally, we have considered a variety of different situations of restricted and permitted measurements and actions, including the employment of random quenching strategies or deterministic quenching strategies. We obtained the expected work extraction in these situations and derived expressions comparing free energy extraction capacity across several such scenarios, discussing the favorability of different strategies in different situations. That allowed us to explore and explain the fact that a single bit's thermodynamic worth can far exceed $k_BT \log 2$, exemplified by a cheeseburger residing uncertainly in one of two locations—when learning that one bit, we may gain half a cheeseburger's worth of calories in comparison to the expectation in the case of choosing the location at random. Numerous further extensions are possible involving adjustments to the precise setup we analyzed; many of them fit directly within the framework—for instance, analysis of multiple nonequilibrium samples or multiple noisy measurements thereof, including non-i.i.d. scenarios, would amount to altered choices of measurement channel $p(\mathbf{m}|\mathbf{x})$ and ground truth distribution.

The type of control limitations considered here are arguably ubiquitous in experimental settings of feedback control in nanosystems, where a complete control of the system of interest is generally not possible and the intrinsic stochastic behavior due to the impact of thermal fluctuations makes it difficult to have perfect measurements [51]. In such situations the type of configurations from which work could be optimally or even efficiently extracted is reduced [42,43,76,77]

and many experiments exploring the thermodynamics of small systems typically restrict themselves to the case of initial equilibrium states, i.e., as in Maxwell's demon setup. The theory developed here should help to provide a precise description of work extraction from arbitrary nonequilibrium initial states under noisy observation and lack of perfect control.

Our findings have relevance to many real-world systems, including physical, biological, and social in which action limitations strongly impact the free energy extraction process. Indeed these systems do not need to be necessarily small [78]. For instance, numerous biological systems efficiently harvest free energy under stochastic environmental conditions with constraints on possible actions. For example, individual organisms with a finite number of innate behaviors, e.g., a bacterium or an insect, must selectively apply these behaviors under a multitude of conditions [79,80]. Evolutionary pressures hence arguably favor the development of effective action repertoires, both in terms of the actions themselves and in terms of the mapping of scenarios to actions. Housing a perfect innate response for all situations would be exorbitantly costly; instead, it is advantageous to develop multipurpose behaviors [81].

In addition, such limitations are abundant in social systems. For example, many modern hunter-gatherer groups maintain social norms that help facilitate collective behavior, including the deterrence of free riders and the encouragement of demand sharing (i.e., sharing food according to need [82]). Thus, when an individual hunter-gatherer finds food or is successful in a hunt, their energy extraction actions are limited: They might eat some of the food immediately, but must save the rest to share with their camp. Further, while most hunter-gatherer populations store or preserve food, the extent of this storage is typically limited [83,84]. In contrast, individuals living in farming societies rely on being able to freely choose among multiple actions. Namely, to consume food immediately, or store it for later use. The increasingly complex technology available to farming and urbanized societies [85] suggests that having access to a larger action repertoire may be crucial to explaining the massive increase in per-person energy use between the ancient and modern world [86].

We emphasize that this work addresses scenarios subject to several key idealizations. We examined the effect of having access to only a limited number of Hamiltonians, that might be fixed (Sec. III) or arbitrarily tunable, allowing us their optimization (Sec. IV). There are many rich ways to extend the analysis in this paper, to capture even more features of real-world systems that try to extract free energy from their environment. We have implicitly restricted the set of Hamiltonians accessible during the relaxation of protocol by assuming that the quasistatic trajectories do not pass through Hamiltonians that would be energetically preferable over any Hamiltonian from the quenching repertoire; future works could consider additional restrictions on the quasistatic relaxation protocol, e.g., allowing only access to a finite set of Hamiltonians akin to the quenching repertoire. Another obvious extension is to consider the case where there is more than just one agent who is extracting the free energy. In this extension, the different agents would run quench-then-relax protocols on distributions involving different physical variables in the environment. It might also be that each agent observes some features of importance to some other ones, and so would need to communicate their information to the other agents, incurring a thermodynamic cost. Another avenue of future research arises from the fact that in this paper considering only a single agent we do *not* model the thermodynamic cost of the physical system that computes what action to take based on a given measurement.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATION OF SUCCESSFUL FEEDBACK CONTROL CONDITION

Here, we show how the condition $k_B T \langle S_{\text{tot}} \rangle < \Delta \mathcal{F}_H$ can be rewritten in the form of Eq. (16). Our strategy will be to successively rewrite the left-hand side of the inequality involving $\langle S_{\text{tot}} \rangle$, ultimately reaching Eq. (16).

First, we note that

$$\langle S_{\text{tot}} \rangle = \sum_{\mathbf{m}} p(\mathbf{m}) D\left(\rho_{X|\mathbf{m}} \middle|\middle| \frac{e^{-\beta H_{n(\mathbf{m})}}}{Z_{n(\mathbf{m})}} \right), \tag{A1}$$

with $\beta = 1/k_B T$. Consider also the relation

$$\begin{aligned} -\log \frac{e^{-\beta H_n(\mathbf{x})}}{Z_n} &= \beta H_n(\mathbf{x}) + \log Z_n \\ &= \beta \big(H_n(\mathbf{x}) - F_{H_n}^{\text{eq}} \big), \end{aligned} \tag{A2}$$

where the equilibrium free energy of the postquench equilibrium after action n is

$$F_{H_n}^{\text{eq}} = -k_B T \log Z_n. \tag{A3}$$

We also note that

$$\begin{aligned} p(\mathbf{m}) \rho_{X|\mathbf{m}}(x) \log \rho_{X|\mathbf{m}}(x) &= p(\mathbf{m}, x) \log \frac{p(\mathbf{m}, \mathbf{x})}{p(\mathbf{m})} \\ &= p(\mathbf{m}, \mathbf{x}) \log \frac{p(\mathbf{m}, \mathbf{x})}{p(\mathbf{m}) \rho_0(\mathbf{x})} \\ &\quad + p(\mathbf{m}, \mathbf{x}) \log \rho_0(\mathbf{x}), \end{aligned} \tag{A4}$$

having denoted $p(\mathbf{x}, \mathbf{m})$ as the joint distribution of microstate and measurement outcome. By using $\sum_{\mathbf{m}} p(\mathbf{m}, \mathbf{x}) = \rho_0(\mathbf{x})$, alongside the definitions of Shannon entropy, Kullback-Leibler divergence, and mutual information, application of Eqs. (A2) and (A4) to Eq. (A1) yields

$$\langle S_{\text{tot}} \rangle = I_{X;M} - S(\rho_0) + \beta \sum_{\mathbf{x},\mathbf{m}} p(\mathbf{x}, \mathbf{m}) \big[H_{n(\mathbf{m})}(\mathbf{x}) - F_{H_{n(\mathbf{m})}}^{\text{eq}} \big]. \tag{A5}$$

We next add and subtract $\langle H\rangle_{\rho_0} = \sum_{\mathbf{x}} \rho_0(\mathbf{x})H(\mathbf{x})$ to identify the nonequilibrium free energy $\mathcal{F}_H(\rho_0)$, and introduce the action distribution $p(n) = \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})$ and action-conditional microstate distribution $p(\mathbf{x}|n) = \sum_{\mathbf{m}\in\ell(n)} p_{X|\mathbf{m}}(\mathbf{x})p(\mathbf{m})$, and multiply by k_BT. Then, Eq. (A5) becomes

$$k_BT\,\langle S_{\rm tot}\rangle = k_BT I_{X;M} + \mathcal{F}_H(\rho_0) - \sum_n p(n)F^{\rm eq}_{H_n} + \sum_n p(n)\sum_{\mathbf{x}} p(\mathbf{x}|n)(H_n(\mathbf{x}) - H(x)). \quad \text{(A6)}$$

Identifying the second line of Eq. (A6) as the work cost $\langle W_{\rm cost}\rangle$ associated with the quench, we have

$$k_BT\,\langle S_{\rm tot}\rangle = k_BT I_{X;M} + \mathcal{F}_H(\rho_0) - \langle F^{\rm eq}_{H_n}\rangle + \langle W_{\rm cost}\rangle, \quad \text{(A7)}$$

with shorthand $\langle F^{\rm eq}_{H_n}\rangle = \sum_n p(n)F^{\rm eq}_{H_n}$. As such, the original inequality $k_BT\,\langle S_{\rm tot}\rangle < \Delta\mathcal{F}_H$ is now expressed as

$$k_BT I_{X;M} + \mathcal{F}_H(\rho_0) - \langle F^{\rm eq}_{H_n}\rangle + \langle W_{\rm cost}\rangle < \Delta\mathcal{F}_H. \quad \text{(A8)}$$

Finally, noting that the excess nonequilibrium free energy is $\Delta\mathcal{F}_H = \mathcal{F}_H(\rho_0) - F^{\rm eq}_H$, we arrive at the desired equation [Eq. (16)]. We note that the work cost $\langle W_{\rm cost}\rangle$ can equivalently be expressed with the reverse order of averaging,

$$\langle W_{\rm cost}\rangle = \sum_{\mathbf{x}} \rho_0(\mathbf{x})\left[\sum_n p_{L|\mathbf{x}}(n)H_n(\mathbf{x}) - H(\mathbf{x})\right], \quad \text{(A9)}$$

as per Eq. (17).

APPENDIX B: OPTIMAL SET OF QUENCHING HAMILTONIANS FOR A GIVEN PARTITION

Here, we derive what the optimal quenching Hamiltonians are for a given lookup table encoded by partition of observations into actions. In doing so, we reduce the general problem of repertoire construction to the simpler (yet still nontrivial) problem of finding an optimal partition.

Recall, the postmeasurement would-be optimal quench (under no limitations) is the Bayesian posterior $\rho_{X|\mathbf{m}}(\mathbf{x}) = \rho_0(\mathbf{x})P(\mathbf{m}|\mathbf{x})/p(\mathbf{m})$, with $p(\mathbf{m}) = \sum_{\mathbf{x}} \rho_0(\mathbf{x})P(\mathbf{m}|\mathbf{x})$. The observation $\mathbf{m}$ is grouped into the action indexed by $n^*(\mathbf{m})$. The inverse of this mapping defines a partition $\ell(n)$ into blocks indexed by n. In response to any observation $\mathbf{m}$ in a given partition block $\ell(n)$, a common response is employed, namely, the quenching Hamiltonian H_n, or equivalently, the associated equilibrium distribution $\rho^{\rm opt}_n = e^{-\beta H_n}/Z_n$.

The mean dissipated work $\langle S_{\rm tot}\rangle$ is the mean Kullback-Leibler divergence between the would-be optimal quench $\rho_{X|\mathbf{m}}$ and the actual quench resorted to, $\rho^{\rm opt}_{n^*(\mathbf{m})}$. That is,

$$\begin{aligned}\langle S_{\rm tot}\rangle &= \sum_{\mathbf{m}} p(\mathbf{m})D\big(\rho_{X|\mathbf{m}}\big|\big|\rho^{\rm opt}_{n^*(\mathbf{m})}\big)\\ &= \sum_n p(n)\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\mathrm{D}\big(\rho_{X|\mathbf{m}}\big|\big|\rho^{\rm opt}_n\big),\end{aligned} \quad \text{(B1)}$$

with $p(n) = \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})$ being the coarse-grained probability of performing action n, i.e., the probability that $\mathbf{m}\in\ell(n)$. We seek the repertoire $\mathcal{H} = \{H_n\}_n$ corresponding to the collection of distributions $\{\rho^{\rm opt}_n\}_n$ minimizing $\langle S_{\rm tot}\rangle$, where we take the partition as fixed.

Note,

$$\frac{\mathrm{d}\langle S_{\rm tot}\rangle}{\mathrm{d}\rho^{\rm opt}_n(\mathbf{x})} = p(n)\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\frac{\mathrm{d}D\big(\rho_{X|\mathbf{m}}\big|\big|\rho^{\rm opt}_n\big)}{\mathrm{d}\rho^{\rm opt}_n(\mathbf{x})}, \quad \text{(B2)}$$

with

$$\frac{\mathrm{d}D\big(\rho_{X|\mathbf{m}}\big|\big|\rho^{\rm opt}_n\big)}{\mathrm{d}\rho^{\rm opt}_n(\mathbf{x})} = -\frac{\rho_{X|\mathbf{m}}(\mathbf{x})}{\rho^{\rm opt}_n(\mathbf{x})}. \quad \text{(B3)}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}\langle S_{\rm tot}\rangle}{\mathrm{d}\rho^{\rm opt}_n(\mathbf{x})} &= -p(n)\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\frac{\rho_{X|\mathbf{m}}(\mathbf{x})}{\rho^{\rm opt}_n(\mathbf{x})}\\ &= -\frac{\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})\rho_{X|\mathbf{m}}(\mathbf{x})}{\rho^{\rm opt}_n(\mathbf{x})}.\end{aligned} \quad \text{(B4)}$$

Introducing a set of Lagrange multipliers $\{\lambda_n\}_n$, we consider Lagrangian

$$L = \langle S_{\rm tot}\rangle + \sum_n \lambda_n \sum_{\mathbf{x}} \rho^{\rm opt}_n(\mathbf{x}), \quad \text{(B5)}$$

with the latter terms included to enforce that all $\{\rho^{\rm opt}_n\}$ are normalized. Minimizing with respect to any given one of the variables $\{\rho^*_n(\mathbf{x})\}_{n,\mathbf{x}}$, we have

$$0 = \frac{\partial L}{\partial\rho^{\rm opt}_n(\mathbf{x})} = -\frac{\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})\rho_{X|\mathbf{m}}(\mathbf{x})}{\rho^{\rm opt}_n} + \lambda_n, \quad \text{(B6)}$$

from which we have

$$\rho^{\rm opt}_n(\mathbf{x}) = \frac{1}{\lambda_n}\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})\rho_{X|\mathbf{m}}(\mathbf{x}). \quad \text{(B7)}$$

The values of $\{\lambda_n\}_n$ are determined by normalization:

$$\begin{aligned}\lambda_n &= \sum_{\mathbf{x}}\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})\rho_{X|\mathbf{m}}(\mathbf{x})\\ &= \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})\\ &= p(n),\end{aligned} \quad \text{(B8)}$$

and thus

$$\begin{aligned}\rho^{\rm opt}_n(\mathbf{x}) &= \frac{1}{p(n)}\sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m})\rho_{X|\mathbf{m}}(x)\\ &= \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\rho_{X|\mathbf{m}}(\mathbf{x}).\end{aligned} \quad \text{(B9)}$$

Thus, $\rho^{\rm opt}_n(\mathbf{x})$ is a probabilistic mixture of the would-be optimal responses $\rho_{X|\mathbf{m}}(\mathbf{x})$, with mixture distribution being that of $\mathbf{m}$ conditional on action n being taken. For the associated quenching Hamiltonian, $H_n(\mathbf{x}) = -k_BT\log\rho^{\rm opt}_n(\mathbf{x})$. Therefore, the problem of finding an optimal repertoire is reduced to the problem of finding an optimal partition.

APPENDIX C: THE CASE OF PERFECT OBSERVATION

Herein, we consider the case of perfect observation, in which case $\rho_{X|\mathbf{m}}(\mathbf{x}) = \delta_{\mathbf{m},\mathbf{x}}$. The distribution of measurements

is therefore

$$p(\mathbf{m}) = \sum_{\mathbf{x}} \delta_{\mathbf{x},\mathbf{m}} \rho_0(\mathbf{x}) = \rho_0(\mathbf{m}), \tag{C1}$$

with $\mathbf{m}$ a valid argument of ρ_0 because $\mathbb{M}^{d_M} = \mathbb{X}^{d_S}$ to accommodate perfect observation. In this case, the distribution representing the nth partition block is given by

$$\rho_n^{\text{opt}}(\mathbf{x}) = \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n)\delta_{\mathbf{m},\mathbf{x}} = p(\mathbf{x}|n). \tag{C2}$$

The conditional distribution of measurements given that they lead to action n is proportional to ρ_0, windowed to support $\ell(n)$. Namely,

$$p(\mathbf{x}|n) = \frac{\mathbb{1}\{\mathbf{x} \in \ell(n)\}\rho_0(\mathbf{x})}{p^{\text{cg}}(n)}, \tag{C3}$$

with $p_0^{\text{cg}}(n) := \sum_{x\in\ell(n)} \rho_0(\mathbf{x})$ being the coarse-grained version of $\rho_0(\mathbf{x})$. Note that the partition block $\ell(n)$ of possible measurements $\mathbf{m} \in \ell(n)$ is now also a partition of possible environmental states $\mathbf{x} \in \ell(n)$, since $\mathbb{M}^{d_M} = \mathbb{X}^{d_S}$ under perfect observation.

The mean dissipated work is given by

$$\langle S_{\text{tot}} \rangle = \sum_n p^{\text{cg}}(n) J_{p(\mathbf{m}|n)}(\{\rho_{X|\mathbf{m}}\}), \tag{C4}$$

with, in this case,

$$\begin{aligned} J_{p(\mathbf{m}|n)}(\{\rho_{X|\mathbf{m}}\}) &= S\big(\rho_n^{\text{opt}}\big) - \sum_{\mathbf{m}\in\ell(n)} p(\mathbf{m}|n) S(\delta_{\mathbf{x},\mathbf{m}}) \\ &= -\sum_{\mathbf{x}\in\ell(n)} \frac{\rho_0(\mathbf{x})}{p^{\text{cg}}(n)} \log \frac{\rho_0(\mathbf{x})}{p^{\text{cg}}(n)}. \end{aligned} \tag{C5}$$

Therefore,

$$\begin{aligned} \langle S_{\text{tot}} \rangle &= \sum_n p(n) \left(-\sum_{\mathbf{x}\in\ell(n)} \frac{\rho_0(\mathbf{x})}{p^{\text{cg}}(n)} \log \frac{\rho_0(\mathbf{x})}{p^{\text{cg}}(n)} \right) \\ &= -\sum_n \sum_{\mathbf{x}\in\ell(n)} \rho_0(\mathbf{x}) \log \rho_0(\mathbf{x}) + \sum_n p^{\text{cg}}(n) \log p^{\text{cg}}(n) \\ &= S(\rho_0) - S(p^{\text{cg}}). \end{aligned} \tag{C6}$$

Noting that p^{cg} is a probability distribution with N distinct possible outcomes, we have

$$S(p^{\text{cg}}) \leqslant \log N, \tag{C7}$$

with $\log N$ achieved if and only if the partition $\ell(y)$ is such that p^{cg} is uniformly distributed across the set of N possible actions. We call such a partition (probabilistically) *equitable*. Therefore, we arrive at the bound

$$\langle S_{\text{tot}} \rangle \geqslant S(\rho_0) - \log N. \tag{C8}$$

At any given finite N and for a given distribution ρ_0, partitions achieving this bound may or may not exist.

APPENDIX D: DETAILS ON THE PARTICLE IN $\mathbb{R}$ EXAMPLE

Herein, we show that in the setting explored in the example of Sec. V A (one dimensional, perfect observation, contiguous partition, ρ_0 has full support on $\mathbb{R}$), the unique way to achieve optimal quenching is to set partition boundaries at the positions given in Eq. (37).

Recall that an optimal partition must be probabilistically equitable, so that the distribution over measurements is maximally entropic: $S(p(m)) = \log|\mathbb{M}| = \log N$. That is, we want the total probability of landing in $\ell(n)$ to be $1/N$, for each $n \in \{1, \ldots, N\}$.

By the fact that $\rho_0(x) > 0$ for all $x \in \mathbb{R}$, we have $\frac{d}{dx}F_0(x) > 0$ for all $x \in \mathbb{R}$. Therefore, $F_0(x)$ is monotonically increasing and uniquely invertible; we denote $F_0^{-1} : [0, 1] \to \mathbb{R}$ as the inverse of F_0. Recall, $x_0 = -\infty$, as we argued that our quenching Hamiltonian repertoire $\mathcal{H}$ should (collectively) reproduce the support of ρ_0. The location of x_1 is thus uniquely determined by the condition that $\mathbb{P}(X < x_1) = 1/N$. That is,

$$\begin{aligned} \int_{-\infty}^{x_1} \rho_0(x)dx = F_0(x_1) &= \frac{1}{N} \\ \Rightarrow x_1 &= F_0^{-1}\left(\frac{1}{N}\right). \end{aligned} \tag{D1}$$

Now, by induction we can obtain $x_2, \ldots, x_N$. Suppose the inductive hypothesis that for some $n \in \{2, \ldots, N\}$ we have $x_{n-1} = F_0^{-1}(\frac{n-1}{N})$. Then, by monotonicity of F_0, and the optimality requirement of $\mathbb{P}(X \in \ell(n)) = 1/N$, the uniquely determined position of x_n will satisfy

$$\int_{x_{n-1}}^{x_n} \rho_0(x)dx = F_0(x_n) - F_0(x_{n-1}) = \frac{1}{N}. \tag{D2}$$

By hypothesis, $x_{n-1} = F_0^{-1}(\frac{n-1}{N})$, and thus $F_0(x_{n-1}) = \frac{n-1}{N}$. Therefore, our requirement on x_n [Eq. (D2)] becomes

$$\begin{aligned} F_0(x_n) - \frac{n-1}{N} &= \frac{1}{N} \\ \Rightarrow F_0(x_n) &= \frac{n}{N}, \end{aligned} \tag{D3}$$

from which by solving for x_n we obtain Eq. (37). Since we showed earlier that $x_1 = F_0^{-1}(1/N)$ [in Eq. (D1)], we have by induction that Eq. (37) holds for all $n \in \{2, \ldots, N\}$. [Including the case of $n = N$ in which we recover the already assumed $x_N = \infty$ at $F_0^{-1}(1)$, since $F(\infty) = 1$.] See Fig. 2 for an example of the optimal partition boundary locations in the case where $\rho_0(x)$ is a standard normal density.

We stress that the uniqueness of this solution is lost if (1) the full support requirement fails, in which case regions of $\rho_0(x) = 0$ may be present, corresponding to regions of $F_0(x) = \text{const}$ (in which boundaries may be freely moved, retaining a partition identical in function). Uniqueness is also lost if (2) the contiguity requirement is dropped. We conjecture that this degeneracy vanishes in generic cases of noisy measurement, in which case localized noise will favor partitions with smaller boundaries.

APPENDIX E: ADDITIONAL CALCULATIONS FOR AN ARBITRARY RANDOM INITIAL DISTRIBUTION

Herein, we provide details on the derivation of results for the scenario of feedback action based on a noisy measurement of a distribution-specifying parameter α, considered in Sec. V B. The parameter α, sampled with probability p_α,

specifies the initial nonequilibrium distribution $\rho_0^{(\alpha)}$. Then, a noisy measurement m of α is produced with probability $p(m|\alpha)$, after which the feedback action $n = n^\star(m)$ is employed in response. Here, again a lookup table $n^\star$ determines a partition $\{\ell(n)\}_n$ by $m \in \ell(n) \Rightarrow n^\star(m) = n$.

First, we note that the total entropy production in Eq. (42) can be re-expressed as follows:

$$\begin{aligned}\langle S_{\text{tot}}\rangle &= \sum_\alpha p_\alpha \sum_m P(m|\alpha) D\big(\rho_0^{(\alpha)}\big|\big|\rho_{n(m)}^{\text{opt}}\big) \\ &= \sum_\alpha p_\alpha \sum_n p(n|\alpha) D\big(\rho_0^{(\alpha)}\big|\big|\rho_n\big),\end{aligned} \tag{E1}$$

with $p(n|\alpha) = \sum_{m\in\ell(n)} P(m|\alpha)$. Let us then define the conditional probability that parameter α was sampled given that action n is taken:

$$p_{\alpha|n} = \frac{p(n|\alpha)p_\alpha}{p(n)}, \tag{E2}$$

with $p(n) = \sum_\alpha p_\alpha p(n|\alpha)$ the marginal probability of taking action n. Then, continuing from Eq. (E1), we obtain

$$\begin{aligned}\langle S_{\text{tot}}\rangle &= \sum_{\alpha,n} p_\alpha p(n|\alpha) D\big(\rho_0^{(\alpha)}\big|\big|\rho_n^{\text{opt}}\big) \\ &= \sum_n p(n) \sum_\alpha p_{\alpha|n} D\big(\rho_0^{(\alpha)}\big|\big|\rho_n^{\text{opt}}\big).\end{aligned} \tag{E3}$$

The optimal postquench equilibrium ρ_n^{opt} minimizes the summand $\sum_\alpha p_{\alpha|n} D(\rho_0^{(\alpha)}||\rho_n^{\text{opt}})$ in the above equation [Eq. (E3)], and thus by the derivation of Appendix B, we have that the solution has a probabilistic mixture form:

$$\rho_n^{\text{opt}} = \sum_\alpha p_{\alpha|n}\rho_0^{(\alpha)}, \tag{E4}$$

which was reported as Eq. (41). Note the difference between Eqs. (27) and (E4) above; we will reconcile these below [see Eq. (E17)].

In the optimal scenario of Eq. (41), the dissipated work becomes

$$\begin{aligned}\langle S_{\text{tot}}\rangle &= \sum_n p(n) \sum_\alpha p_{\alpha|n} D\left(\rho_0^{(\alpha)}\middle|\middle|\sum_\alpha p_{\alpha|n}\rho_0^{(\alpha)}\right) \\ &= \sum_n p(n) \sum_\alpha p_{\alpha|n} \sum_x \rho_0^{(\alpha)}(\mathbf{x}) \log \frac{\rho_0^{(\alpha)}(\mathbf{x})}{\rho_n^{\text{opt}}(\mathbf{x})} \\ &= \sum_n p(n)\left(S\big(\rho_n^{\text{opt}}\big) - \sum_\alpha p_{\alpha|n} S\big(\rho_0^{(\alpha)}\big)\right) \\ &= \sum_n p(n) J_{p_{\alpha|n}}\big(\{\rho_0^{(\alpha)}\}_\alpha\big).\end{aligned} \tag{E5}$$

The latter expresses that the dissipation is the average weighted multidistributional JSD among the possible ground truth distributions $\{\rho_0^{(\alpha)}\}_\alpha$ as weighted by the posterior probability of α given n.

Additionally, we can re-express $\langle S_{\text{tot}}\rangle$ in terms of a conditional mutual information. The conditional mutual information among three random variables X, Y, Z is given by the general expression

$$I_{X,Y;Z} = \sum_z P(z) \sum_{x,y} P(x,y|z) \log \frac{P(x,y|z)}{P(x|z)P(y|z)}. \tag{E6}$$

We will show below that $\langle S_{\text{tot}}\rangle = I_{X,A;\mathcal{L}}$, with shorthand $X, A, \mathcal{L}$ for the random variables whose values are being denoted $\mathbf{x}, \alpha, n$. First, we define the joint distribution of $(\alpha, \mathbf{x})$ given n, denoted

$$p(\alpha, \mathbf{x}|n) = p_{\alpha|n}\rho_0^{(\alpha)}(\mathbf{x}). \tag{E7}$$

The conditional distribution of $\mathbf{x}$ given n is

$$\sum_\alpha p(\alpha, \mathbf{x}|n) = \sum_\alpha p_{\alpha|n}\rho_0^{(\alpha)}(\mathbf{x}) = \rho_n^{\text{opt}}(\mathbf{x}). \tag{E8}$$

The overall probability of action n is

$$\begin{aligned}p(n) &= \sum_{\alpha,m} p_\alpha P(m|\alpha)\delta_{n,n^\star(m)} \\ &= \sum_\alpha p_\alpha \sum_{m\in\ell(n)} P(m|\alpha) \\ &= \sum_\alpha p_\alpha p(n|m);\end{aligned} \tag{E9}$$

i.e., it equals the optimal distribution for the quenching Hamiltonian in Eq. (E4). Therefore, we have

$$\begin{aligned}I_{X,A;L} &= \sum_n p(n) \sum_{\alpha,\mathbf{x}} p(\alpha,\mathbf{x}|n) \log \frac{p(\alpha,\mathbf{x}|n)}{p_{\alpha|n}\rho_n^{\text{opt}}(\mathbf{x})} \\ &= \sum_n p(n)\left(\sum_{\alpha,\mathbf{x}} p_{\alpha|n}\rho_0^{(\alpha)}(\mathbf{x}) \log \frac{\rho_0^{(\alpha)}(\mathbf{x})}{\sum_\alpha p_{\alpha|n}\rho_0^{(\alpha)}(\mathbf{x})}\right) \\ &= \sum_n p(n)\left(S\big(\rho_n^{\text{opt}}\big) - \sum_\alpha p_\alpha S\big(\rho_0^{(\alpha)}\big)\right),\end{aligned} \tag{E10}$$

reproducing Eq. (E5). Thus, the mean dissipated work is the conditional mutual information between the postquench equilibrium state $\mathbf{x}$ and the ground truth index α, with conditionality on the action taken n.

As a consistency check, let us reconcile Eq. (41) with Eq. (27) of the manuscript, to demonstrate the compatibility of the measurement-based expressions with the case of arbitrary initial distributions. Given the measurement of m, we have a posterior over α of the form

$$p(\alpha|m) = \frac{P(m|\alpha)p_\alpha}{p(m)} = \frac{P(m|\alpha)p_\alpha}{\sum_\alpha P(m|\alpha)p_\alpha}, \tag{E11}$$

from which we can calculate the postmeasurement distribution as

$$\rho_{X|m}(\mathbf{x}) = \sum_\alpha p(\alpha|m)\rho_0^{(\alpha)}(\mathbf{x}). \tag{E12}$$

Thus, by following the reasoning of Appendix B, the nth optimal action should result in a postquench equilibrium of the form

$$\rho_n^{\text{opt}}(\mathbf{x}) = \sum_m p(m|n)\rho_{X|m}(\mathbf{x}). \tag{E13}$$

Equation (E13) is of the form of a mixture distribution across measurement outcomes, as opposed to the expression derived above [Eq. (E4)], which is a mixture distribution across the hidden parameter α. We next show the equivalence of the two.

Rewriting Eq. (E13),

$$\begin{aligned}\rho_n^{\text{opt}}(\mathbf{x}) &= \sum_m p(m|n)\rho_{X|m}(\mathbf{x}) \\ &= \sum_m \frac{\mathbb{1}\{m \in \ell(n)\}p(m)}{p(n)}\rho_{X|m}(\mathbf{x}) \\ &= \frac{\sum_{m\in\ell(n)} p(m)\rho_{X|m}(\mathbf{x})}{\sum_{m\in\ell(n)} p(m)}. \end{aligned} \tag{E14}$$

Now, noting that $p(m) = \sum_\alpha P(m|\alpha)p_\alpha$, we have

$$\begin{aligned}\sum_{m\in\ell(n)} p(m) &= \sum_{m\in\ell(n)}\sum_\alpha P(m|\alpha)p_\alpha \\ &= \sum_\alpha \left(\sum_{m\in\ell(n)} P(m|\alpha)\right)p_\alpha \\ &= \sum_\alpha p(n|\alpha)p_\alpha = p(n). \end{aligned} \tag{E15}$$

Similarly, we also have

$$\begin{aligned}\sum_{m\in\ell(n)} p(m)\rho_{X|m}(\mathbf{x}) &= \sum_{m\in\ell(n)} p(m)\left(\sum_\alpha p(\alpha|m)\rho_0^{(\alpha)}(\mathbf{x})\right) \\ &= \sum_{m\in\ell(n)}\sum_\alpha P(m|\alpha)p_\alpha\rho_0^{(\alpha)}(\mathbf{x}) \\ &= \sum_\alpha p_\alpha\rho_0^{(\alpha)}(\mathbf{x})\sum_{m\in\ell(n)} P(m|\alpha) \\ &= \sum_\alpha \rho_0^{(\alpha)}(\mathbf{x})p(n|\alpha)p_\alpha, \end{aligned} \tag{E16}$$

Therefore, replacing the above two expressions on Eq. (E14), we obtain

$$\begin{aligned}\rho_n^{\text{opt}}(\mathbf{x}) &= \sum_m p(m|n)\rho_{X|m}(\mathbf{x}) \\ &= \frac{\sum_\alpha \rho_0^{(\alpha)}(\mathbf{x})p(n|\alpha)p_\alpha}{p(n)} \\ &= \sum_\alpha \rho_0^{(\alpha)}(\mathbf{x})p(\alpha|n), \end{aligned} \tag{E17}$$

with $p(\alpha|n) = p(n|\alpha)p_\alpha/p(n)$.

Relation to original measurement scheme

Finally, we show how the original measurement scenario considered in Sec. II is equivalent to the random initial distributions scenario when $\alpha \sim p_\alpha$ is assumed to be a random microstate $\mathbf{x} \sim \rho_0$, again passed through a noisy channel [$P(\mathbf{m}|\mathbf{x})$ rather than $P(m|\alpha)$], and with the associated set of ground truth distributions $\{\rho_0^{(\alpha)}\}_\alpha$ being set equal to $\{\delta_{\mathbf{x},\mathbf{x}'}\}_\mathbf{x}$.

Under these circumstances ($\alpha \to \mathbf{x}$, $p_\alpha \to \rho_0(\mathbf{x})$, $\rho_0^{(\alpha)}(\mathbf{x}) \to \rho_0^{(\mathbf{x})}(\mathbf{x}') = \delta_{\mathbf{x},\mathbf{x}'}$), we have

$$p(\alpha|m) \to p(\mathbf{x}|\mathbf{m}) = \frac{P(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x})}{\sum_\mathbf{x} P(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x})} = \frac{P(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x})}{p(\mathbf{m})}, \tag{E18}$$

and thus the posterior is

$$\begin{aligned}\rho_{X|\mathbf{m}}(\mathbf{x}') &= \sum_\alpha p(\alpha|\mathbf{m})\rho_0^{(\alpha)}(\mathbf{x}') \\ &= \sum_\mathbf{x} \frac{P(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x})}{p(\mathbf{m})}\delta_{\mathbf{x},\mathbf{x}'} \\ &= \frac{P(\mathbf{m}|\mathbf{x})\rho_0(\mathbf{x}')}{p(\mathbf{m})}, \end{aligned} \tag{E19}$$

i.e., the posterior with respect to the initial microstate distribution ρ_0. Note that in this scenario, the initial average microstate distribution, given in general by Eq. (38), becomes identical to the microstate distribution itself:

$$\bar{\rho}_0 = \sum_{\mathbf{x}_0} \rho_0(\mathbf{x}_0)\delta_{\mathbf{x}_0,\mathbf{x}} = \rho_0. \tag{E20}$$

[1] G. Tkačik and W. Bialek, Information processing in living systems, Annu. Rev. Condens. Matter Phys. **7**, 89 (2016).

[2] H. Leff and A. F. Rex, *Maxwell's Demon 2 Entropy, Classical and Quantum Information, Computing* (CRC Press, Boca Raton, 2002).

[3] J. Maxwell, *Theory of Heat*, *Text-Books of Science* (Longmans, Green, and Company, London, 1872).

[4] L. Szilard, Über die entropieverminderung in einem thermodynamischen system bei eingriffen intelligenter wesen, Z. Phys. **53**, 840 (1929).

[5] K. Sekimoto, *Stochastic Energetics* (Springer, Berlin, 2010), Vol. 799.

[6] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, Rep. Prog. Phys. **75**, 126001 (2012).

[7] C. Jarzynski, Equalities and inequalities: Irreversibility and the second law of thermodynamics at the nanoscale, Annu. Rev. Condens. Matter Phys. **2**, 329 (2011).

[8] L. Peliti and S. Pigolotti, *Stochastic Thermodynamics: An Introduction* (Princeton University Press, Princeton and Oxford, 2021).

[9] J. M. Parrondo, J. M. Horowitz, and T. Sagawa, Thermodynamics of information, Nat. Phys. **11**, 131 (2015).

[10] H. Touchette and S. Lloyd, Information-theoretic limits of control, Phys. Rev. Lett. **84**, 1156 (2000).

[11] T. Sagawa and M. Ueda, Minimal energy cost for thermodynamic information processing: Measurement and information erasure, Phys. Rev. Lett. **102**, 250602 (2009).

[12] F. J. Cao and M. Feito, Thermodynamics of feedback controlled systems, Phys. Rev. E **79**, 041118 (2009).

[13] T. Sagawa and M. Ueda, Generalized Jarzynski equality under nonequilibrium feedback control, Phys. Rev. Lett. **104**, 090602 (2010).

[14] J. M. Horowitz and S. Vaikuntanathan, Nonequilibrium detailed fluctuation theorem for repeated discrete feedback, Phys. Rev. E **82**, 061120 (2010).

[15] D. Abreu and U. Seifert, Thermodynamics of genuine nonequilibrium states under feedback control, Phys. Rev. Lett. **108**, 030601 (2012).

[16] T. Sagawa and M. Ueda, Fluctuation theorem with information exchange: Role of correlations in stochastic thermodynamics, Phys. Rev. Lett. **109**, 180602 (2012).

[17] S. Deffner and C. Jarzynski, Information processing and the second law of thermodynamics: An inclusive, Hamiltonian approach, Phys. Rev. X **3**, 041003 (2013).

[18] J. M. Horowitz, T. Sagawa, and Juan, M. R. Parrondo, Imitating chemical motors with optimal information motors, Phys. Rev. Lett. **111**, 010602 (2013).

[19] H. Sandberg, J.-C. Delvenne, N. J. Newton, and S. K. Mitter, Maximum work extraction and implementation costs for nonequilibrium Maxwell's demons, Phys. Rev. E **90**, 042119 (2014).

[20] P. P. Potts and P. Samuelsson, Detailed fluctuation relation for arbitrary measurement and feedback schemes, Phys. Rev. Lett. **121**, 210603 (2018).

[21] M. Ribezzi-Crivellari and F. Ritort, Large work extraction and the Landauer limit in a continuous Maxwell demon, Nat. Phys. **15**, 660 (2019).

[22] M. Debiossac, D. Grass, J. J. Alonso, E. Lutz, and N. Kiesel, Thermodynamics of continuous non-Markovian feedback control, Nat. Commun. **11**, 1360 (2020).

[23] T. Tohme, V. Bedoya, C. di Bello, L. Bresque, G. Manzano, and E. Roldán, Gambling carnot engine, Phys. Rev. Lett. **135**, 067101 (2025).

[24] T. Sagawa and M. Ueda, Second law of thermodynamics with discrete quantum feedback control, Phys. Rev. Lett. **100**, 080403 (2008).

[25] K. Jacobs, Second law of thermodynamics and quantum feedback control: Maxwell's demon with weak measurements, Phys. Rev. A **80**, 012322 (2009).

[26] K. Funo, Y. Watanabe, and M. Ueda, Integral quantum fluctuation theorems under measurement and feedback control, Phys. Rev. E **88**, 052121 (2013).

[27] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: Enhanced work extraction from quantum correlations, npj Quantum Inf. **3**, 12 (2017).

[28] G. Manzano, F. Plastina, and R. Zambrini, Optimal work extraction and thermodynamics of quantum measurements and correlations, Phys. Rev. Lett. **121**, 120602 (2018).

[29] B. Annby-Andersson, F. Bakhshinezhad, D. Bhattacharyya, G. De Sousa, C. Jarzynski, P. Samuelsson, and P. P. Potts, Quantum Fokker-Planck master equation for continuous feedback control, Phys. Rev. Lett. **129**, 050401 (2022).

[30] S. Toyabe, T. Sagawa, M. Ueda, E. Muneyuki, and M. Sano, Experimental demonstration of information-to-energy conversion and validation of the generalized Jarzynski equality, Nat. Phys. **6**, 988 (2010).

[31] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz, Experimental verification of Landauer's principle linking information and thermodynamics, Nature (London) **483**, 187 (2012).

[32] É. Roldán, I. A. Martínez, J. M. R. Parrondo, and D. Petrov, Universal features in the energetics of symmetry breaking, Nat. Phys. **10**, 457 (2014).

[33] T. Admon, S. Rahav, and Y. Roichman, Experimental realization of an information machine with tunable temporal correlations, Phys. Rev. Lett. **121**, 180601 (2018).

[34] G. Paneru and H. Kyu Pak, Colloidal engines for innovative tests of information thermodynamics, Adv. Phys.: X **5**, 1823880 (2020).

[35] J. V. Koski, V. F. Maisi, J. P. Pekola, and D. V. Averin, Experimental realization of a Szilard engine with a single electron, Proc. Natl. Acad. Sci. USA **111**, 13786 (2014).

[36] J. V. Koski, V. F. Maisi, T. Sagawa, and J. P. Pekola, Experimental observation of the role of mutual information in the nonequilibrium dynamics of a Maxwell demon, Phys. Rev. Lett. **113**, 030601 (2014).

[37] K. Chida, S. Desai, K. Nishiguchi, and A. Fujiwara, Power generator driven by Maxwell's demon, Nat. Commun. **8**, 15301 (2017).

[38] M. Rico-Pasto, R. K. Schmitt, M. Ribezzi-Crivellari, J. M. R. Parrondo, H. Linke, J. Johansson, and F. Ritort, Dissipation reduction and information-to-measurement conversion in DNA pulling experiments with feedback protocols, Phys. Rev. X **11**, 031052 (2021).

[39] P. A. Camati, John, P, S. Peterson, T. B. Batalhão, K. Micadei, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental rectification of entropy production by Maxwell's demon in a quantum system, Phys. Rev. Lett. **117**, 240502 (2016).

[40] Y. Masuyama, K. Funo, Y. Murashita, A. Noguchi, S. Kono, Y. Tabuchi, R. Yamazaki, M. Ueda, and Y. Nakamura, Information-to-work conversion by Maxwell's demon in a superconducting circuit quantum electrodynamical system, Nat. Commun. **9**, 1291 (2018).

[41] S. Ciliberto, Experiments in stochastic thermodynamics: Short history and perspectives, Phys. Rev. X **7**, 021051 (2017).

[42] T. K. Saha, J. N. E. Lucero, J. Ehrich, D. A. Sivak, and J. Bechhoefer, Maximizing power and velocity of an information engine, Proc. Natl. Acad. Sci. USA **118**, e2023356118 (2021).

[43] T. K. Saha, Joseph N. E. Lucero, J. Ehrich, D. A. Sivak, and J. Bechhoefer, Bayesian information engine that optimally exploits noisy measurements, Phys. Rev. Lett. **129**, 130601 (2022).

[44] M. D. Vidrighin, O. Dahlsten, M. Barbieri, M. S. Kim, V. Vedral, and I. A. Walmsley, Photonic Maxwell's demon, Phys. Rev. Lett. **116**, 050401 (2016).

[45] J. M. Horowitz and J. M. R. Parrondo, Thermodynamic reversibility in feedback processes, Europhys. Lett. **95**, 10005 (2011).

[46] J. M. Horowitz and J. M. R. Parrondo, Designing optimal discrete-feedback thermodynamic engines, New J. Phys. **13**, 123019 (2011).

[47] H.-H. Hasegawa, J. Ishikawa, K. Takara, and D. Driebe, Generalization of the second law for a nonequilibrium initial state, Phys. Lett. A **374**, 1001 (2010).

[48] K. Takara, H. H. Hasegawa, H. H. Hasegawa, and D. J. Driebe, Generalization of the second law for a transition

between nonequilibrium states, Phys. Lett. A **375**, 88 (2010).
[49] M. Esposito and C. V. den Broeck, Second law and Landauer principle far from equilibrium, Europhys. Lett. **95**, 40004 (2011).
[50] J. Anders and V. Giovannetti, Thermodynamics of discrete quantum processes, New J. Phys. **15**, 033022 (2013).
[51] D. A. S. Johan du Buisson and J. Bechhoefer, Performance limits of information engines, Adv. Phys.: X **9**, 2352112 (2024)
[52] S. Dago, S. Ciliberto, and L. Bellon, Adiabatic computing for optimal thermodynamic efficiency of information processing, Proc. Natl. Acad. Sci. USA **120**, e2301742120 (2023).
[53] T. Schmiedl and U. Seifert, Optimal finite-time processes in stochastic thermodynamics, Phys. Rev. Lett. **98**, 108301 (2007).
[54] E. Aurell, C. Mejía-Monasterio, and P. Muratore-Ginanneschi, Optimal protocols and optimal transport in stochastic thermodynamics, Phys. Rev. Lett. **106**, 250601 (2011).
[55] T. Kamijima, A. Takatsu, K. Funo, and T. Sagawa, Optimal finite-time Maxwell's demons in Langevin systems, Phys. Rev. Res. **7**, 023159 (2025).
[56] D. A. Sivak and G. E. Crooks, Thermodynamic metrics and optimal paths, Phys. Rev. Lett. **108**, 190602 (2012).
[57] P. R. Zulkowski, D. A. Sivak, G. E. Crooks, and M. R. DeWeese, Geometry of thermodynamic control, Phys. Rev. E **86**, 041148 (2012).
[58] K. Proesmans, J. Ehrich, and J. Bechhoefer, Finite-time Landauer principle, Phys. Rev. Lett. **125**, 100602 (2020).
[59] M. Scandi, D. Barker, S. Lehmann, K. A. Dick, V. F. Maisi, and M. Perarnau-Llobet, Minimally dissipative information erasure in a quantum dot via thermodynamic length, Phys. Rev. Lett. **129**, 270601 (2022).
[60] Y. Guryanova, N. Friis, and M. Huber, Ideal projective measurements have infinite resource costs, Quantum **4**, 222 (2020).
[61] T. Debarba, G. Manzano, Y. Guryanova, M. Huber, and N. Friis, Work estimation and work fluctuations in the presence of non-ideal measurements, New J. Phys. **21**, 113002 (2019).
[62] X. Linpeng, L. Bresque, M. Maffei, A. N. Jordan, A. Auffèves, and K. W. Murch, Energetic cost of measurements using quantum, coherent, and thermal light, Phys. Rev. Lett. **128**, 220506 (2022).
[63] J. M. Horowitz and M. Esposito, Thermodynamics with continuous information flow, Phys. Rev. X **4**, 031015 (2014).
[64] J. Ehrich, S. Still, and D. A. Sivak, Energetic cost of feedback control, Phys. Rev. Res. **5**, 023080 (2023).
[65] A. Kolchinsky and D. H. Wolpert, Work, entropy production, and thermodynamics of information under protocol constraints, Phys. Rev. X **11**, 041024 (2021).
[66] A. Kolchinsky and D. H. Wolpert, Entropy production given constraints on the energy functions, Phys. Rev. E **104**, 034129 (2021).
[67] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. (John Wiley and Sons, Hoboken, New Jersey, 2012).
[68] R. Kawai, J. M. R. Parrondo, and C. Van den Broeck, Dissipation: The phase-space perspective, Phys. Rev. Lett. **98**, 080602 (2007).
[69] A. Kolchinsky, I. Marvian, C. Gokler, Z.-W. Liu, P. Shor, O. Shtanko, K. Thompson, D. Wolpert, and S. Lloyd, Maximizing free energy gain, Entropy **27**, 91 (2025).
[70] J. Lin, Divergence measures based on the Shannon entropy, IEEE Trans. Inf. Theory **37**, 145 (1991).
[71] G. E. Sims, S.-R. Jun, G. A. Wu, and S.-H. Kim, Alignment-free genome comparison with feature frequency profiles (FFP) and optimal resolutions, Proc. Natl. Acad. Sci. USA **106**, 2677 (2009).
[72] S. Klingenstein, T. Hitchcock, and S. DeDeo, The civilizing process in London's old Bailey, Proc. Natl. Acad. Sci. USA **111**, 9419 (2014).
[73] E. Englesson and H. Azizpour, Generalized jensen-shannon divergence loss for learning with noisy labels, in *Proceedings of the 35th International Conference on Neural Information Processing Systems, NIPS '21* (Curran Associates Inc., Red Hook, NY, USA, 2024).
[74] K. Onishi and H. Imai, Voronoi diagram in statistical parametric space by Kullback-Leibler divergence, in *Proceedings of the Thirteenth Annual Symposium on Computational Geometry* (ACM, New York, 1997), pp. 463–465.
[75] F. Nielsen, On Voronoi diagrams on the information-geometric cauchy manifolds, Entropy **22**, 713 (2020).
[76] O. Maillet, P. A. Erdman, V. Cavina, B. Bhandari, E. T. Mannila, J. T. Peltonen, A. Mari, F. Taddei, C. Jarzynski, V. Giovannetti, and J. P. Pekola, Optimal probabilistic work extraction beyond the free energy difference with a single-electron device, Phys. Rev. Lett. **122**, 150604 (2019).
[77] Joseph N. E. Lucero, J. Ehrich, J. Bechhoefer, and D. A. Sivak, Maximal fluctuation exploitation in Gaussian information engines, Phys. Rev. E **104**, 044122 (2021).
[78] J. M. R. Parrondo, The Szilard engine revisited: Entropy, macroscopic randomness, and symmetry breaking phase transitions, Chaos **11**, 725 (2001).
[79] X. Jiang and Y. Pan, Neural control of action selection among innate behaviors, Neurosci. Bull. **38**, 1541 (2022).
[80] A. B. Barron, K. N. Gurney, L. F. Meah, E. Vasilaki, and J. A. Marshall, Decision-making and action selection in insects: Inspiration from vertebrate-based theories, Front. Behav. Neurosci. **9**, 216 (2015).
[81] W. J. Bell, Searching behavior patterns in insects, Annu. Rev. Entomol. **35**, 447 (1990).
[82] H. M. Lewis, L. Vinicius, J. Strods, R. Mace, and A. B. Migliano, High mobility explains demand sharing and enforced cooperation in egalitarian hunter-gatherers, Nat. Commun. **5**, 5789 (2014).
[83] I. Armit and B. Finlayson, Hunter-gatherers transformed: The transition to agriculture in northern and western europe, Antiquity **66**, 664 (1992).
[84] M. J. Hamilton, B. Buchanan, J. Lobo, and R. Walker, Food storage, mobility, and the density-dependence of hunter-gatherer movement ecology, J. Archaeol. Sci.: Rep. **54**, 104441 (2024).
[85] M. J. Hamilton, R. S. Walker, and B. Buchanan, Institutional complexity emerges from socioecological complexity in small-scale human societies, Proc. Biol. Sci. **291**, 20240246 (2024).
[86] F. Krausmann, H. Weisz, and N. Eisenmenger, Transitions in sociometabolic regimes throughout human history, in *Social Ecology: Society-Nature Relations Across Time and Space*, edited by H. Haberl, M. Fischer-Kowalski, F. Krausmann, and V. Winiwarter (Springer, Cham, 2016), pp. 63–92.