

Stable supersolids and boselets in spin-orbit-coupled Bose-Einstein condensates with three-body interactions

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We explore the stability of supersolid striped waves, plane-wave boselets, and other extended states in one-dimensional spin-orbit-coupled Bose-Einstein condensates with repulsive three-body interactions (R3BIs), modeled by quintic terms in the framework of the corresponding Gross-Pitaevskii equations. In the absence of R3BIs, the extended states are susceptible to the modulational instability (MI) induced by the cubic attractive nonlinearity. Using the linearized Bogoliubov-de-Gennes equations, we identify multiple new types of MI, including baseband, passband, mixedband, and zero-wavenumber-gain ones, which give rise to deterministic rogue waves and complex nonlinear wave patterns. Our analysis reveals that R3BIs eliminate baseband and zero-wavenumber-gain MIs, forming, instead, phonon modes that enable stable boselets. Additionally, mixedband and passband MIs are suppressed, which results in a latticelike phonon-roton mode that supports a stable supersolid phase. These stable supersolids can be realized using the currently available ultracold experimental setup.

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I. INTRODUCTION

Spinor Bose-Einstein condensates (BECs) serve as a versatile platform for examining stable and unstable dynamics of matter waves under the action of intra- and intercomponent interactions [1–5]. In this context, the introduction of synthetic spin-orbit coupling (SOC) in binary BECs [6] has sparked interest in the creation of novel stable phases of quantum matter, particularly the stripe phase characterized by spontaneous breaking of the gauge and translational symmetries [7–13], leading to manifestations of superfluidity and

crystalline properties, ultimately resulting in the emergence of the supersolid phase [14–18]. These phases have been observed in solid helium [19–21] and dipolar BEC [22], as well as in SOC bosonic condensates [11–13]. However, in the presence of SOC, the supersolid phase is unstable even in the presence of a trapping potential [13], presenting a challenge to maintain stability under the action of attractive two-body interactions (A2BIs). The stability analysis is usually performed within the framework of linearized Bogoliubov-de-Gennes (BdG) equations for small perturbations added to the stationary state [23–29].

Modulational instability (MI) is a key factor in understanding the nonlinear dynamics of matter waves, such as soliton trains and domain patterns [1–5,30–32], particularly in binary BECs with SOC [33–36]. Besides matter waves [2,37–43], the MI analysis has significant implications across various fields [44], including nonlinear optics [45–52], photonics [53,54], liquid crystals [55], fluids [56–58], plasmas [59,60], and other discrete systems [61].

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In this context, it has been reported [62,63] that baseband modulational instability (BBMI) and zero-momentum-gain MI are primarily responsible for the formation of rogue waves (RWs). Additionally, passband modulational instability (PBMI) contributes to driving unstable nonlinear dynamics.

On the other hand, there has been significant interest that has emerged among the ultracold community to realize bright [37,64,65] and dark [38,39] solitons of BECs in the laboratory experiment. In particular, bright-soliton trains are known to be generated through MI from a plane-wave (PW) input, which has been experimentally demonstrated to exist in the quasi-one-dimensional BEC in the setup of ^7Li atomic gas [30,64], where the atomic interactions were tuned to be attractive using the Feshbach-resonance technique [66–68]. Characteristics of MI generated by the nonlinear evolution of MI of plane waves have been realized experimentally in fiber optics [69]. In BEC models that incorporate SOC and attractive cubic terms in the Gross-Pitaevskii equations (GPEs), stable one-dimensional (1D) solitons exhibit a smooth or striped inner structure within a specific range of chemical potential under trapping but become unstable without trapping [70], indicating a characteristic related to MI.

It has been shown that the inclusion of quartic self-repulsive terms in the GP equations, accounting for the Lee-Huang-Yang (LHY) correction to mean-field theory, stabilizes multidimensional self-trapped states in the form of quantum droplets [71–77]. With the LHY correction, such states are prone to thermal and dynamical instabilities [78–81].

The higher-order nonlinearity, represented by quintic terms in GPEs, significantly affects various characteristics of the BEC ground state. In the framework of the mean-field theory, the quintic terms naturally represent three-body interactions [82–84], although the applicability of this setting is limited by three-body losses [85]. Earlier studies of BEC models that incorporate quintic repulsive terms [86–91] have demonstrated that they may effectively control instabilities, leading to the formation of robust self-bounded bosonic droplets, known as *boselets*, without the need for a trapping potential [87]. Additionally, SOC also stabilizes 1D Townes solitons formed by quintic self-attraction [92] against the critical collapse [93,94]. Based on these insights, we propose a model aimed at stabilizing various quantum phases, in particular the stripe wave phase in SOC BECs [95], which otherwise is dynamically unstable, by incorporating three-body interactions, which are represented by quintic terms in the GPEs. Moreover, a stable supersolid phase, created by nonlinear excitations of SOC BECs, has yet to be identified, with limited studies conducted on achieving such a stable phase. In this paper, we demonstrate the emergence of both a stable superfluid *boselets* and a supersolid phase, incorporating repulsive three-body interactions (R3BI) alongside attractive two-body interactions (A2BI), into the BEC setting.

In this work, we present a robust framework for the stabilization of quasi-1D quantum phases, specifically plane waves (PWs), and stripe waves (SWs), in the binary SOC BECs by means of R3BI. In the absence of R3BI, while A2BI is present, the baseband modulational instability (BBMI) and zero-momentum-gain MI drive the formation of deterministic

rogue waves (RWs) in the PW phase [62,63,96], while passband MI (PBMI) and mixedband MI (MBMI) in the SW phase give rise to nonlinear oscillatory waves [62,96]. We demonstrate that the quintic terms are responsible for the transition from the PW phase to a stable boselet configuration, where, in particular, the elimination of BBMI allows for the emergence of stable breathers. Furthermore, we find that, ultimately, the suppression of instabilities in the SW phase, characterized by stable latticelike phonon-roton minima, leads to the supersolid behavior [97,98]. These stable quantum phases may be promising candidates for the experimental investigation, with potential applications to spin-based quantum simulations, spin transport, topological insulators, the quantum spin-Hall effect, superconductivity, spintronics, and supersolids [99–104].

The paper is structured as follows. In Sec. II, we introduce the settings based on a system of mean-field coupled Gross-Pitaevskii equations (CGPEs) with the SOC terms and two- and three-body interactions. Stability analysis of the system, based on computation of the BdG excitation spectrum, is presented in Sec. III. Changes in the (in)stability for the particular set of interactions are investigated in Sec. IV. In Sec. V, we outline the proposal for the experimental demonstration of the predicted states of the quantum matter. The paper is concluded in Sec. VI.

II. MEAN-FIELD MODEL AND GOVERNING DYNAMICAL EQUATIONS

We consider binary BECs with equal Rashba and Dresselhaus SOC terms and the linear Rabi coupling between the components, assuming that a strong transverse confinement is imposed by a cigar-shaped trapping potential [105]. The mean-field energy functional for spin-orbit-coupled BECs incorporating the three-body interaction is $E = \int_{-\infty}^{+\infty} d^3x \mathcal{E}$, with the energy density [70,89,106,107]

$$\mathcal{E} = \Psi^\dagger H_{\text{sp}} \Psi + H_{2\text{B}} + H_{3\text{B}}. \quad (1)$$

Here, H_{sp} is the single-particle Hamiltonian that includes SOC and Rabi-coupling terms, while $H_{2\text{B}}$ and $H_{3\text{B}}$ are the two-body and three-body nonlinear interaction parts of the total Hamiltonian, respectively. The particular terms in the Hamiltonian are expressed as follows:

$$\begin{aligned} H_{\text{sp}} &= \frac{\mathbf{p}^2}{2m} + V(r) + \frac{\hbar k_L}{m} p_x \Sigma_z + \hbar \Omega \Sigma_x, \\ H_{2\text{B}} &= \frac{1}{2} [g_{\uparrow\uparrow} |\psi_\uparrow|^4 + g_{\downarrow\downarrow} |\psi_\downarrow|^4 + 2g_{\uparrow\downarrow} |\psi_\uparrow|^2 |\psi_\downarrow|^2], \\ H_{3\text{B}} &= \frac{1}{3} [\chi_{\uparrow\uparrow} |\psi_\uparrow|^6 + \chi_{\downarrow\downarrow} |\psi_\downarrow|^6 + 3\chi_{\uparrow\downarrow} |\psi_\uparrow|^2 |\psi_\downarrow|^2 (|\psi_\uparrow|^2 + |\psi_\downarrow|^2)]. \end{aligned} \quad (2)$$

Here, $\Psi = (\psi_\uparrow, \psi_\downarrow)^T$ is the pseudospin two-component wave function, $\mathbf{p} = -i\hbar(\partial_x, \partial_y, \partial_z)$ is the momentum operator, and $\Sigma_{x,y,z}$ are, respectively, the x , y , and z components of the 2×2 Pauli spin matrices. The atomic mass is m , while k_L and Ω denote the strengths of SOC and Rabi coupling, respectively. The trapping potential is $V(r)$, and intra- and interspecies two-body interaction strengths are $g_{\sigma\sigma} = 4\pi\hbar^2 N a_{\sigma\sigma}/m$ and $g_{\sigma\bar{\sigma}} = 4\pi\hbar^2 N a_{\sigma\bar{\sigma}}/m$, respectively, where $a_{\sigma\sigma}$ and $a_{\sigma\bar{\sigma}}$ are the corresponding s -wave scattering lengths, where $\sigma = \uparrow, \downarrow$ de-

notes the “up” and “down” components, and $\bar{\sigma}$ is the opposite state of σ . Further, the three-body interaction parameters are defined as $\chi_{\sigma\sigma} = \lambda_3^{\sigma\sigma} N^2$ and $\chi_{\sigma\bar{\sigma}} = \lambda_3^{\sigma\bar{\sigma}} N^2$, with $\lambda_3^{\sigma\sigma}$ and $\lambda_3^{\sigma\bar{\sigma}}$ representing the intra- and interspecies three-body coupling coefficients, respectively, which are functions of the complex scattering hypervolume D , whose real and imaginary parts determine the coupling constant and three-body loss, respectively [88,91].

To derive the quasi-1D dynamical equations, we assume that the condensate is confined by a strong axially symmetric transverse trap ($\omega_\perp \gg \omega_x$). Consequently, Eq. (1) reduces to the quasi-1D dimensionless form through the transformation [108,109] $x = a_\perp \bar{x}$, $y = a_\perp \bar{y}$, $z = a_\perp \bar{z}$, $t = \bar{t}/\omega_\perp$, and $\psi_{\uparrow,\downarrow}(x, y, z, t) = \bar{\psi}_{\uparrow,\downarrow}(x, t) \bar{\psi}_{\uparrow,\downarrow}(y, z) a_\perp^{-3/2}$, where

$$\bar{\psi}_\sigma(y, z) = A_y A_z \exp\left(-\frac{(\omega_y y^2 + \omega_z z^2)}{2} - \frac{i(\omega_y + \omega_z)t}{2}\right), \quad (3)$$

with $A_x = (\omega_y/2\pi)^{1/4}$, and $A_z = (\omega_z/2\pi)^{1/4}$. By inserting above equation in Eq. (1), integrating over y and z , and omitting the bar, we obtain the dimensionless form of the CGPEs:

$$i\partial_t \psi_\uparrow = \left[-\frac{1}{2} \partial_x^2 - ik_L \partial_x + V(x) + g|\psi_\uparrow|^2 + g_{\uparrow\downarrow}|\psi_\downarrow|^2 + \chi|\psi_\uparrow|^4 + \chi_{\uparrow\downarrow}|\psi_\downarrow|^2(|\psi_\downarrow|^2 + 2|\psi_\uparrow|^2) \right] \psi_\uparrow + \Omega \psi_\downarrow, \quad (4a)$$

$$i\partial_t \psi_\downarrow = \left[-\frac{1}{2} \partial_x^2 + ik_L \partial_x + V(x) + g_{\uparrow\downarrow}|\psi_\uparrow|^2 + g|\psi_\downarrow|^2 + \chi|\psi_\downarrow|^4 + \chi_{\uparrow\downarrow}|\psi_\uparrow|^2(|\psi_\uparrow|^2 + 2|\psi_\downarrow|^2) \right] \psi_\downarrow + \Omega \psi_\uparrow. \quad (4b)$$

The normalized dimensionless quantities are defined by rescaling, $k_L \rightarrow k_L a_\perp$, $\Omega \rightarrow \Omega/\omega_\perp$, $g = g_{\sigma\sigma} = 2a_{\sigma\sigma}N/a_\perp$, $g_{\sigma\bar{\sigma}} = 2a_{\sigma\bar{\sigma}}N/a_\perp$, $\chi = \chi_{\sigma\sigma} = \lambda_3^{\sigma\sigma} N^2/(3\pi^2 \omega_\perp a_\perp^6)$ and $\chi_{\sigma\bar{\sigma}} = \lambda_3^{\sigma\bar{\sigma}} N^2/(3\pi^2 \omega_\perp a_\perp^6)$. Here, $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the harmonic-oscillator strength, and $V(x) = v^2 x^2/2$, with the trap aspect ratio $v = \omega_x/\omega_\perp$, and $\omega_\perp = \sqrt{\omega_y \omega_z}$. In this paper, the dimensionless variables are used.

In the next section, we investigate the stability of the quantum phases, namely, PWs and SWs under the action of two- and three-body nonlinear interactions, using the BdG analysis. This approach suggests possibilities for the suppression of MI in these quantum phases.

III. COLLECTIVE-EXCITATION SPECTRUM IN THE PRESENCE OF TWO- AND THREE-BODY INTERACTIONS

In this section, we present the collective excitation spectrum of the interacting SOC binary BECs for the trapless case but with the Rabi coupling. Utilizing the BdG theory, we identify various MI types and elaborate the possibility of suppressing MI, aiming to achieve stable superfluid quantum phases.

A. Stability analysis using the BdG theory

The stability of states produced by Eq. (4) is investigated using the BdG equations derived from the linearization of Eq. (4). We assume the conservation of the total density, $n = n_\uparrow + n_\downarrow$, and equal chemical potentials μ of both components, due to the presence of the linear coupling. In the absence of the trapping potential, $V(x) = 0$, the perturbed uniform wave function is expressed as [26,110]

$$\psi_\sigma = e^{-i\mu t} (\sqrt{n_\sigma} + \delta\psi_\sigma),$$

where n_σ are the uniform densities of the components, and small perturbations are introduced as

$$\delta\psi_\sigma = u_\sigma e^{i(kx - \omega t)} + v_\sigma^* e^{-i(kx - \omega^* t)},$$

with u_σ and v_σ being the perturbation amplitudes, k the real wavenumber, and ω the eigenfrequency, which may be complex. The presence of $\text{Im}(\omega) \neq 0$ indicates the presence of MI. We obtained the eigenvalue spectrum through the energy minimization, defining the Rabi-coupling strength as $\Omega = \Omega e^{i\theta}$, which yields $\Omega = -\Omega$ for $\theta = \pi$ [36,110]. Substituting the perturbed wave functions in Eq. (4) yields

$$(\mathcal{L} - \omega I)(u_\uparrow, v_\uparrow, u_\downarrow, v_\downarrow)^T = 0, \quad (5)$$

$$\mathcal{L} \equiv \begin{pmatrix} H_\uparrow^+ & M_\uparrow & R_\uparrow - \Omega & R_\uparrow \\ -M_\uparrow & -H_\uparrow^- & -R_\uparrow & -R_\uparrow + \Omega \\ R_\downarrow - \Omega & R_\downarrow & H_\downarrow^- & M_\downarrow \\ -R_\downarrow & -R_\downarrow + \Omega & -M_\downarrow & -H_\downarrow^+ \end{pmatrix}, \quad (6)$$

where I is the 4×4 unit matrix, and $\mathcal{L}$ is the BdG matrix, with

$$\begin{aligned} H_\sigma^\pm &= \frac{k^2}{2} \pm k_L k + 2gn_\sigma + g_{\uparrow\downarrow}n_{\bar{\sigma}} + 3\chi n_\sigma^2 + \chi_{\uparrow\downarrow}n_{\bar{\sigma}}^2 \\ &\quad + 4\chi_{\uparrow\downarrow}n_\sigma n_{\bar{\sigma}} - \mu_\sigma, \\ M_\sigma &= (g + 2\chi n_\sigma + 2\chi_{\uparrow\downarrow}n_{\bar{\sigma}})n_\sigma, \\ R_\sigma &= (g_{\uparrow\downarrow} + 2\chi_{\uparrow\downarrow}n_{\bar{\sigma}} + 2\chi_{\uparrow\downarrow}n_\sigma)\sqrt{n_\sigma n_{\bar{\sigma}}}, \\ \mu_\sigma &= gn_\sigma + g_{\uparrow\downarrow}n_{\bar{\sigma}} + \chi n_\sigma^2 + \chi_{\uparrow\downarrow}n_{\bar{\sigma}}^2 \\ &\quad + 2\chi_{\uparrow\downarrow}n_\sigma n_{\bar{\sigma}} - \Omega\sqrt{n_\sigma n_{\bar{\sigma}}}/n_\sigma. \end{aligned} \quad (7)$$

Equation (5) leads to the eigenvalue equation $\omega^4 - \Lambda\omega^2 + \Delta = 0$ with solutions

$$\omega_\pm^2 = (\Lambda \pm i\sqrt{4\Delta - \Lambda^2})/2, \quad (8)$$

for the underlying symmetric uniform states, with $n_\uparrow = n_\downarrow = 1/2$, in the cases of $g = g_{\uparrow\downarrow}$ and $\chi = \chi_{\uparrow\downarrow}$. Here, we define

$$\begin{aligned} \Lambda &= (k^4/2 + \Omega)(k^4/2 + X + \Omega) + k^2 k_L^2 + \Omega(\Omega - Y), \\ \Delta &= [(k^2/2 + 2\Omega)(k^2/2 + X - Y) - k^2 k_L^2 + 2\Omega(Y - X)] \\ &\quad \times [(k^2/2 + 2\Omega)(k^2/2 + X + Y) - k^2 k_L^2], \end{aligned} \quad (9)$$

with

$$X \equiv g + \chi + \chi_{\uparrow\downarrow}, \quad Y \equiv g_{\uparrow\downarrow} + 2\chi_{\uparrow\downarrow}. \quad (10)$$

In the next section, using Eq. (8), we explore various MI types in the presence of A2BIs and further analyze methods to eliminate them for achieving stable quantum phases.

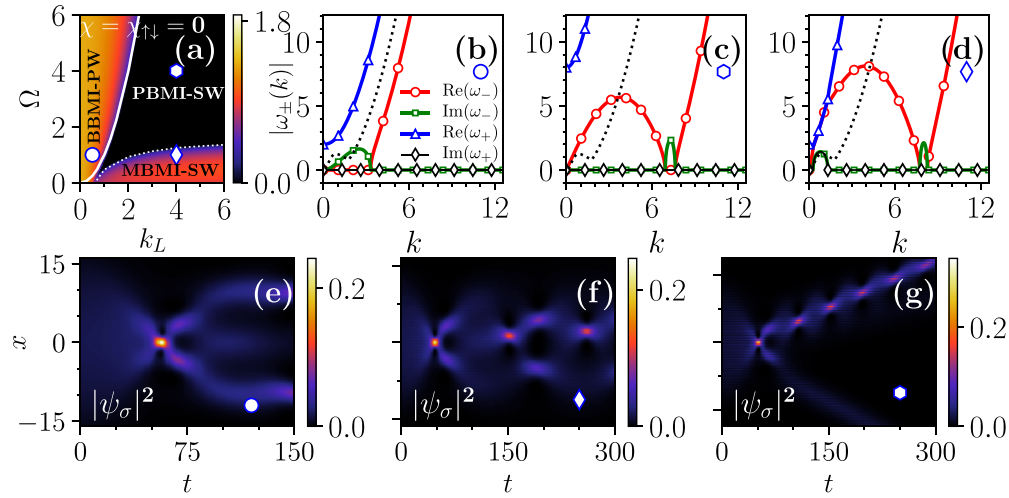


FIG. 1. (a) shows characteristics of MI gain (G_-) in the (k_L, Ω) plane for fixed A2BI strengths, $g = g_{\uparrow\downarrow} = -2$, with the perturbation wavenumber $k = 1$ and R3BI strengths $\chi = \chi_{\uparrow\downarrow} = 0$. The solid line denotes the phase-transition boundary ($\Omega = k_L^2$) between the PW and SW phases, while the dotted line indicates the transition from MBMI to PBMI as Ω varies in the SW phase. In (b)–(d) eigenspectra show different types of MI, viz. BBMI, PBMI, and MBMI, respectively, each exhibiting a distinct nonlinear dynamical behavior. The spectra correspond to the phase plot in (a), as indicated by identical markers. Solid lines and symbols denote the analytical and numerical results, respectively. Black dotted lines indicate the Feynman dispersion derived from the structure factor. For a dense superfluid, the Feynman energy (black-dotted lines) matches the excitation energy only in the phonon limit ($k \rightarrow 0$); beyond this limit, the two energies diverge. (e)–(g) illustrate the evolution of ground-state densities ($|\psi_{\sigma}|^2$) corresponding to the points marked in Fig. 1(a). Under effective attractive interactions with $X + Y < 0$, the unstable phases are monitored by simulating the evolution of the ground-state solution after quenching the interactions and gradually ramping the trap down from $t = 0$ till $t = 20$. In the present work, we consider symmetric inputs, resulting in outputs that are also symmetric with respect to the spin-up and spin-down components. Therefore, we present the evolution plots for the single component.

B. Different MI phases in the presence of attractive two-body interactions

From Eq. (8), we calculate the MI gain, $G_{\pm} = |\text{Im}(\omega_{\pm})|$. Scalar and spinor BECs generally show MI driven by A2BI [3,5], while SOC and Rabi coupling can significantly alter this behavior. We also aim to explore how R3BI terms can affect MI of SOC binary BECs.

To investigate the stability of the perturbed SOC BECs in the (k_L, Ω) plane, we consider the MI gain G_- with $g = g_{\uparrow\downarrow} = -2$. In this case, the self-attractive SOC BECs exhibit three distinct types of MI in the absence of R3BI, as shown in Fig. 1(a). Specifically, we identify the following MI species: (i) The baseband MI (BBMI), characterized by $|\text{Im}(\omega)| \neq 0$ at $|k| > 0$ and $|\text{Im}(\omega)| = 0$ at $k = 0$, which appears in the PW state at $\Omega \geq k_L^2$, as illustrated in Fig. 1(b). (ii) The passband MI (PBMI), characterized by $|\text{Im}(\omega)| \neq 0$ at

$$|k| > |k_{\min}| > 0,$$

separated from $k = 0$, with a gain growth starting from $k_{\min} \neq 0$, as depicted in Fig. 1(c). (iii) The mixedband-MI (MBMI), a combination of BBMI and PBMI with conditions $|k_{\text{BBMI}}| > 0$ and $|k_{\text{PBMI}}| > |k_{\min}|$, as shown in Fig. 1(d).

In addition, we reinforce the analytical findings related to the excitation spectrum by numerically solving the BdG equations, which also allow us to derive the eigenvectors as a function of the wave vector k . To begin this process, we consider a grid in real space that spans the range of $[-1000, +1000]$, utilizing a step size of $\delta x = 0.05$. This

choice of the grid size ensures a detailed mapping of the physical system under consideration. Following this, we apply the Fourier collocation method [111,112], where we numerically execute the Fourier transformation of the BdG equations. This procedure results in a truncated reduced BdG matrix, which encompasses the essential features of the system. We then proceed to diagonalize this matrix using the LAPACK package [113], which is renowned for its efficiency in handling such computational tasks. In terms of the momentum space, we focus on the modes in the range of $[-50, +50]$ in the k direction, employing a grid step size of $\delta k = 0.0628$. This carefully chosen momentum space grid allows us to achieve an accurate representation of the system's behavior in the momentum space.

Following the BdG analysis of MI, we proceed to demonstrate the dynamical (in)stability by numerically solving the full GPEs system, aiming to obtain the ground states, using Gaussian profiles as a seed wave function. Here, we present the dynamical results produced by the numerical solution of CGPEs (4) for SOC BECs. First, we determine the ground states using the imaginary-time propagation (ITP) method [108,109]. Subsequently, we evolve the ground state wave function through the real-time propagation (RTP). For both ITP and RTP, we have employed the split-step Crank-Nicholson scheme [108,109]. In this work, we used the grid size of $[-250, +250]$ with a spatial step of $\Delta x = 0.025$, and time steps of $\Delta t = 10^{-4}$ for ITP and $\Delta t = 5 \times \Delta x^2$ for RTP.

Further, we follow the procedure detailed in Refs. [64,65], where the ground state was first produced by means of ITP and subsequently quenched in RTP by switching the cubic

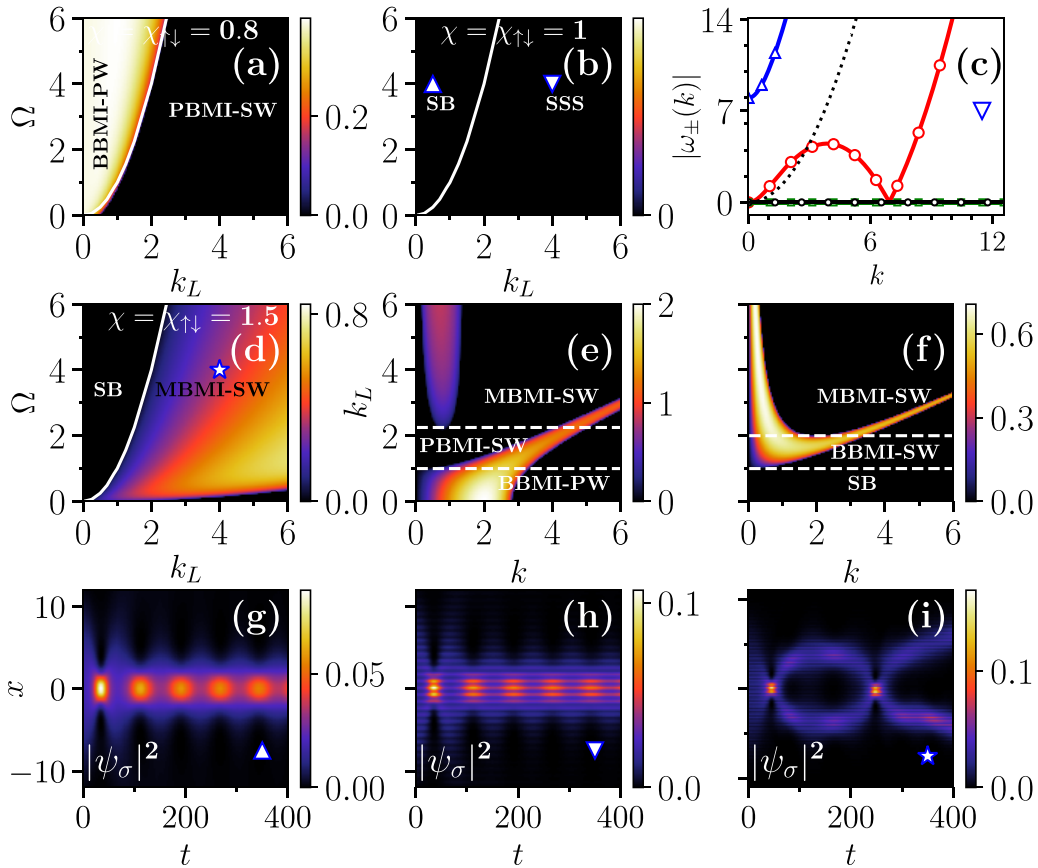


FIG. 2. (a)–(c) illustrate the nature of MI in the (k_L, Ω) plane for fixed A2BI strengths $g = g_{\uparrow\downarrow} = -2$, with perturbation wavenumber $k = 1$ and varying R3BI strengths: (a) $\chi = \chi_{\uparrow\downarrow} = 0.8$, and (b) $\chi = \chi_{\uparrow\downarrow} = 1$. (c) depicts the excitation spectrum of the stable supersolid phase in (b), showing a roton-phonon latticelike state for the balanced effective interactions $X + Y = 0$, which stabilizes the unstable supersolid phase. (d) presents the instabilities for the effective repulsive interactions, with $X + Y > 0$, including the three-body interactions, with $\chi = \chi_{\uparrow\downarrow} = 1.5$. The solid line denotes the phase-transition boundary ($\Omega = k_L^2$) between the PW and SW phases. For fixed $\Omega = 1$, (e) and (f) show the emergence of distinct MI gain bands in the cases displayed in Figs. 1(a) and 2(d), with the dashed line distinguishing different MI regimes. Panels (g)–(i) illustrate the evolution of ground-state densities ($|\psi_\sigma|^2$) corresponding to the points marked in (b), (c). When the system's effective interaction is modified to $X + Y \geq 0$, the dynamics of the ground state is excited by imprinting a periodic density modulation with wavenumber k , cf. Ref. [72].

terms in the GPEs from repulsion to attraction. Simultaneously, the trapping potential is gradually removed, so that $V(x) = 0$ at $t \geq 20$. Under such conditions, in the absence of R3BI and for $\Omega \geq k_L^2$, we observe the emergence of deterministic RWs due to the effect of BBMI in the initial PW phase. Notably, the cubic attraction induces RW-like dynamics [114–116], leading to abrupt localization of BEC with a large amplitude, which then fragments into two soliton trains that eventually decays, as shown in Fig. 1(e). The emergence of an RW-like feature at $t = 57$ is evident in Fig. 1(e), where the density attains a maximum value ≈ 0.266 , which is 15 times higher than the initial density, $|\psi_\sigma|_{t=0}^2 = 0.018$, being consistent with previous findings [114]. The initial exponential growth indicates the onset of the instability, cf. Refs. [117,118]. Overall, we find that BBMI in the attractive SOC BECs does not inherently lead to the emergence of RWs and soliton trains, as observed in other trapped systems [114]. Additionally, considering the BBMI phase and evolving the ground state with attractive interactions under the trapping potential, we observe the emergence of chaotic spatiotemporal patterns. This complex evolution suggests a transition

towards a regime resembling the soliton turbulence, cf. Ref. [119,120].

However, in the regime with $k_L^2 > \Omega$, both PBMI and MBMI occur in the SW phase. Specifically, for relatively small $\Omega < \mathcal{R}$, where $\mathcal{R} \equiv a - (b + a)c/(c + k_L)$, with $a = 1.6$, $b = 2.7$, and $c = 0.35$, the system exhibits MBMI. For $k_L^2 > \Omega \gtrsim \mathcal{R}$, PBMI is observed, as shown in Fig. 1(a) (the dotted line). Notably, PBMI is characterized by a gap, while MBMI exhibits a gapless instability-avoided-crossing (IAC) region between ω_- and ω_+ . In the MBMI phase, ω_+ mode is particularly sensitive to perturbations, resulting in $G_+ \neq 0$ and leading to RWs accompanied by nonlinear oscillations (cf. Ref. [62]), as shown in Fig. 1(f). Furthermore, PBMI is associated with the formation of breathers that propagate in $\pm x$ directions, as displayed in Fig. 1(g).

Here, we identify distinct types of MI and their corresponding dynamical behaviors, which are characterized by their instability bands, as exhibited by the eigenvalue spectra. The respective spectra are displayed in Figs. 1(b) and 1(c), demonstrate excellent agreement between the analytical and numerical results.

C. Effect of the three-body interaction on MI

In this section, we examine the impact of three-body interactions on the system's stability, gradually increasing their strength while maintaining the two-body attractive interaction at a fixed level.

Adding the quintic R3BI suppresses MI, exhibiting the transformation of MBMI into weaker PBMI, as illustrated in Fig. 2(a). When R3BI is introduced under condition $X + Y < 0$, with $g = g_{\uparrow\downarrow} = -2$ and $\chi = \chi_{\uparrow\downarrow} = 0.8$ [see Eq. (10)], both BBMI and PBMI emerge. The PBMI is further characterized by a roton instability, defined as a well pronounced dip in the excitation spectrum with $\text{Im}(\omega_-) \neq 0$. The roton minimum can lead to a stable supersolid if $\text{Im}(\omega_-) = 0$. As R3BI strength increases, the instability associated with BBMI and PBMI becomes attenuated, approaching the PW-to-SW transition line ($\Omega = k_L^2$), where R3BI inhibits MI. Further enhancement of R3BI fully eliminates MI for the balanced interaction, under condition $X + Y = 0$, with fixed A2BI strengths $g = g_{\uparrow\downarrow} = -2$ and R3BI coefficient $\chi = \chi_{\uparrow\downarrow} = 1.0$ [see Eq. (10)]. Thus both the PW and SW phases exhibit stable phonon and latticelike phonon-roton modes with $\text{Im}(\omega) = 0$, as shown in the phase plot depicted in Fig. 2(b) and the respective stable supersolid excitation spectrum in Fig. 2(c). These modes are responsible for the formation of stable boselet and supersolid phases, respectively. However, for the repulsive effective interaction with $X + Y > 0$, the stability is preserved only in the PW phase, while the SW one is subject to MBMI, see Fig. 2(d).

We further explore MI in the (k, k_L) plane at a fixed Rabi strength $\Omega = 1$, highlighting distinct behaviors for $\chi = \chi_{\uparrow\downarrow} = 0$ and 2, as illustrated in Figs. 2(e) and 2(f), respectively. Initially, for $X + Y < 0$, both the PW and SW phases exhibit MI, leading to a series of novel findings. In particular, for $\Omega \geq k_L^2$, the BBMI is present in the PW phase for $0 < k < 3$, extending to $k_L^2 = \Omega$. The PBMI emerges only in the interval of $\Omega < k_L^2 \lesssim 2\Omega$, while, beyond this range, only MBMI is observed [see Fig. 2(e)]. For $X + Y > 0$, comparing Figs. 2(f) and 2(e) yield several key insights: (i) the disappearance of BBMI and formation of stable boselets; (ii) the transition from PBMI to BBMI; (iii) the persistence of MBMI's characteristics.

In the presence of R3BI with coefficients $\chi = \chi_{\uparrow\downarrow} = 1$, the stable PW phase resembles a superfluid boselet, exhibiting stable breather dynamics [72], as illustrated in Fig. 2(g). Conversely, the SW phase displays a latticelike phonon-roton minimum softening, which is attributable to the effect of R3BI [see Fig. 2(c)]. The rotons are considered as a “soft mode,” signaling the system's approach to crystallization into a supersolid phase [7–9,11,97,98]. This mode facilitates the establishment of the stable supersolid phase, see Fig. 2(h). The excitation spectrum reveals the emergence of a roton minimum without instability [$\text{Im}(\omega_-) = 0$], indicating the appearance of a stable supersolid in the SOC BECs [10,13,121–123]. Our analysis confirms that the system stays in a dense superfluid phase, rather than a gaseous one, as validated by the consideration of the Feynman energy [124]. For $X + Y > 0$, a stable boselet forms, while the MBMI in the supersolid phase induces nonlinear oscillations and divergence in the $\pm x$ directions, as shown in Fig. 2(i). In addition to these MI scenarios, we observe one with the MI gain at the

zero wavenumber, characterized by $|\text{Im}(\omega)|_{k \rightarrow 0} \neq 0$, a phenomenon that was absent in previously considered scenarios where $|\text{Im}(\omega)|_{k \rightarrow 0} = 0$ took place. In this case, the maximum gain occurs at $k = 0$, while the minimum gain is observed at $|k| < k_{\text{max}}$, cf. Refs. [62,63]. This specific MI gain arises exclusively from the strong interspecies interaction under the condition of $X - Y + 2\Omega < 0$, precipitating the emergence of deterministic RWs.

D. Modified MI phases without Rabi coupling $\Omega = 0$

The previously analyzed nature of MI changes significantly when Rabi coupling is set to $\Omega = 0$. For $\chi = \chi_{\uparrow\downarrow} = 0$, BBMI-PW emerges in both eigenspectra $\omega_{\pm}$, in contrast, for $\Omega \neq 0$ ω_+ remains stable. At $\Omega = 0$, MBMI-SW exhibits IAC behavior starting from $k = 0$, while for $\Omega \neq 0$, mixedband and passband MIs emerge. These MIs are suppressed by quintic terms for $\chi = \chi_{\uparrow\downarrow} = 1$. It was found that two gapless Goldstone modes, along with roton-phonon latticelike states, contribute to the emergence of the supersolid phase. For $\chi = \chi_{\uparrow\downarrow} > 1$, the stable boselet transforms into BBMI, while MBMI exhibits PBMI-roton instability, all maintaining a gapless nature with $G_+ = 0$. This fact emphasizes the importance of Ω in determining the nature of MI in the ω_+ state. Therefore, the physical mechanism for the stable supersolid created in the SOC BECs [125] significantly differs from dipolar BEC, where density modulations in the stable supersolid phase arise from the dipolar interactions linked to two gapless Goldstone modes [22]. However, the modified excitation spectrum, influenced by coefficients k_L and Ω , with the 2BI and 3BI terms, underscores the emergence of a stable supersolid in our framework. While a stable supersolid is a hallmark of certain types of quantum matter, including superfluid helium [20,21] and ultracold atomic gases [12], their stability depends on specific conditions, such as interaction strength, trapping potentials, and temperature.

We further examine the static density and spin structure factors, defined as $S_d(k) = N^{-1} |\sum_{\sigma} \sqrt{n_{\sigma}} (u_{\sigma}(k) + v_{\sigma}(k))|^2$ and $S_s(k) = N^{-1} |\sum_{\sigma} \sqrt{n_{\sigma}} \text{sgn}(\sigma) (u_{\sigma}(k) + v_{\sigma}(k))|^2$, with $\text{sgn}(\uparrow) = -\text{sgn}(\downarrow) = 1$ [110], revealing the system's susceptibility to density and spin fluctuations. When $G_{\pm} \neq 0$, the $\omega_{\pm}$ branch predominantly carries spin excitations within the corresponding MI range in k . In contrast, when $G_{\pm} = 0$, the $\omega_{\pm}$ branch denotes only a density mode [124]. Notably, both $S_d^{\pm}$ and S_s vanish as $k \rightarrow 0$, where $S_d^{\pm}$ and $S_s^{\pm}$ indicate the density of the $\omega_{\pm}$ modes and spin static structure factors, respectively; however, for $k \neq 0$, they obey the Feynman relation, $S_d^F(k) = k^2 n_{\sigma} / 2\omega N$, modified for the spin structure factor as $S_s^F(k) = 1 - k^2 n_{\sigma} / 2\omega N$ [126,127].

In Fig. 3(a), we show $S_{d,s}^{\pm}$ for the BBMI in the PW phase. Here, S_d^- increases monotonically across the MI range in k , eventually approaching n_{σ} , while S_d^+ follows the Feynman relation, as shown by the dashed-dotted magenta line. Furthermore, S_s^- reflects spin fluctuations driven by MI, while S_s^+ remains unaffected by perturbations. However total $S_{d,s}$ generates fluctuations within the MI range in the k space. While PBMI also exhibits similar fluctuations in $S_{d,s}$, the MI range of k varies, as illustrated in Fig. 3(b). Compared to BBMI, PBMI

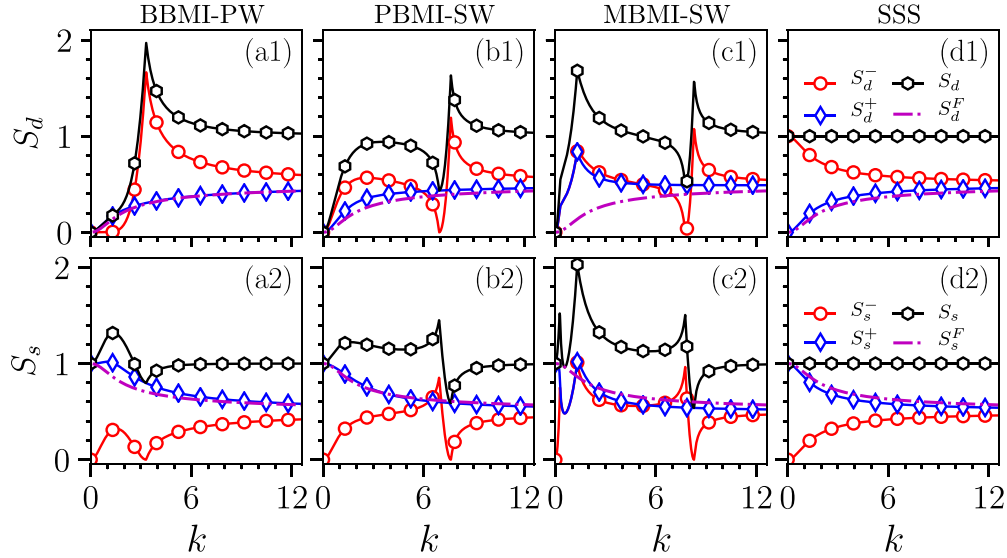


FIG. 3. (Top row) Static density structure factors for the BBMI, PBMI, MBMI, and stable supersolid as depicted in Figs. 1 and 2. (Bottom row) Spin static structure factors corresponding to each respective mode. The dashed-dotted magenta line represents the Feynman criterion, corresponding to the upper branch of the spectrum.

maintains stability at small values of k , leading to steady $S_{d,s}^\pm$, whereas BBMI shows significant fluctuations. Notably, PBMI induces a significant change in $S_{d,s}^-$ in the course of the onset of the roton instability ($k \approx 8$), indicating the dynamical instability of the supersolid phase. In contrast, $S_{d,s}^+$ remains fluctuation-free, showing no MI in the ω_+ mode. However, Fig. 3(c) reveals unexpected fluctuations in $S_{s,d}^+$, confirming the emergence of MI in ω_+ and the presence of IAC, where $S_{s,d}^\pm$ overlap within the MI range. In stable phases, $S_{s,d}(k) = 1$ with $S_{s,d}^\pm$ obeying the Feynman relations [see Fig. 3(d)]. In all scenarios, the structure factors exhibit the inversion symmetry, $S(-k) = S(k)$, and increase with momentum k , asymptotically approaching the limit value $S^\pm(k \rightarrow \infty) = n_\sigma$. Hence, the total structure factor reaches $S(k \rightarrow \infty) = 2n_\sigma = 1$, as expected.

We have explored the various types of MIs concerning the coupling parameters, specifically PW and SW quantum phases. Additionally, we identified conditions for obtaining stable bosenets and supersolids and investigated the role of Rabi coupling. We have analyzed the sensitivity of the density and spin structure factors to fluctuations and their characteristics. In the next section, we will further examine the role of nonlinear interactions on MI.

IV. EFFECT OF INTERACTIONS ON MI

In this section, we examine MIs, considering nonlinear interactions with fixed coupling strengths. This is a crucial step of the analysis because, in coupled BECs, intra- and interspecies interactions play a significant role in determining the stability of the nonlinear matter-wave dynamics [3,5,128].

A. Role of two-body and intraspecies interactions

To understand how interactions affect MI, we examine changes in the MI magnitude ($A_L^\pm$) and bandwidths ($B_w^\pm$, where $\pm$ refers to the corresponding $\omega_\pm$ modes), which drive the

system's nonlinear dynamical behavior [33]. Initially, we set $\chi = \chi_{\uparrow\downarrow} = -1$ for the PW case ($\Omega = 1, k_L = 0.5$). The MI magnitude A_L^- decreases linearly for ω_- as the two-body interactions g , and $g_{\uparrow\downarrow}$ vary simultaneously, as shown in Fig. 4(a). When $g = g_{\uparrow\downarrow} \geq 2$, MI disappears ($A_L^- = 0$), leading to the phonon-mode softening, while bandwidth B_w^- follows the same linear trend, as shown in Fig. 4(a). In the PW case, the ω_+ mode is not involved in MI ($A_L^+ = B_w^+ = 0$). Next, setting $\chi = \chi_{\uparrow\downarrow} = 1$, PW does not exhibit MI in the considered range of the cubic nonlinearity, $[-2, +2]$. Instead, it exhibits the phonon-mode softening in the ω_- branch, leading to the emergence of stable bosenets. However, for $g < -2$, MI emerges, and both scenarios display a constant gapped mode, $\Delta_g = 2\Omega$, unaffected by the interactions. We define the gap between the $\omega_\pm$ modes as Δ_g .

For the SW case ($\Omega = 1, k_L = 4$) with $\chi = \chi_{\uparrow\downarrow} = -1$, Fig. 4(b) shows $A_L^\pm$ and $B_w^\pm$, revealing MI in both $\omega_\pm$ modes. Initially, the ω_- mode exhibits MBMI under the A2BI, while the ω_+ mode displays PBMI. As the 2BI changes from attractive to repulsive, the MBMI in ω_- transforms to PBMI and eventually stabilizes. Concurrently, the ω_+ mode evolves from PBMI to the stability, resulting in a stable supersolid phase. The MBMI mode features multiple instability bands. To calculate the magnitude of instability ($A_L^\pm$) and bandwidth ($B_w^\pm$) for MBMI, we average the values based on the number of bands appear in the respective mode. In this context, the MBMI mode reveals a linear decrease in A_L^- during the transition from the attractive to repulsive cubic nonlinearity. Without the cubic nonlinearity ($g = g_{\uparrow\downarrow} = 0$), the system remains unstable ($A_L^- \neq 0$) due to the action of the attractive quintic nonlinearity with $\chi = \chi_{\uparrow\downarrow} = -1$, indicating that the attractive 3BI alone induces the instability [see Fig. 4(b)]. Furthermore, A_L^- continues to decrease with the increase of g , stabilizing at $g \geq 2$. A_L^+ also decreases, reaching $A_L^+ = 0$ at $g = 0.6$, marking the disappearance of the gapless IAC mode

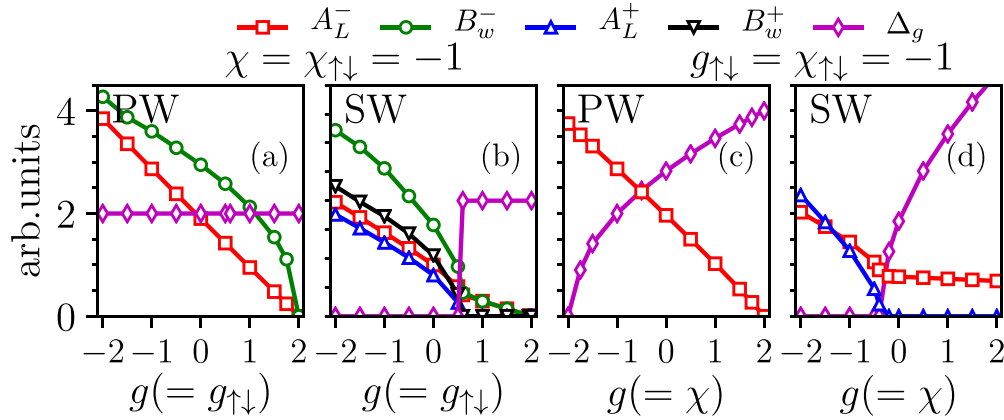


FIG. 4. The variation in the loss of the MI magnitude ($A_L^\pm$) and bandwidth ($B_w^\pm$), along with the gap between $\omega_\pm$ modes (Δ_g), is shown as a function of the interaction strengths for the PW [(a) and (c)] and SW [(b) and (d)] phases, respectively. In the PW phase, the coupling parameters are fixed to $k_L = 0.5$ and $\Omega = 1$, while for the SW phase, the parameters are $k_L = 4$ and $\Omega = 1$.

and transitioning to a gapped mode between the $\omega_\pm$ ones. The bandwidth $B_w^\pm$ shows a similar behavior. Conversely, at $\chi = \chi_{\uparrow\downarrow} = 1$, the stabilization point shifts to $g \geq -2$ for SW, while ω_+ is not involved.

The transition from gapped to gapless modes, based on varying the interaction strength, is crucially important for comprehending their behavior. With fixed two- or three-body interactions, all modes remain gapped except for the gapless IAC mode. In the gapped mode, ω_+ maintains a constant minimum across the interaction range, as depicted in Figs. 4(a) and 4(b). The interplay between the intra- and interspecies two- and three-body interactions leads to diverse behaviors for the gapped modes. Fixing the interspecies interactions ($g_{\uparrow\downarrow} = \chi_{\uparrow\downarrow} = -1$) and simultaneously varying intraspecies interactions g and χ , we observe the following trends. In the PW phase, Δ_g remains zero for $g < -1.81$. Beyond this threshold, Δ_g increases exponentially with $g = \chi$. As g increases further, A_L^- decreases linearly up to $g < 2$, beyond which no MI occurs, and $A_L^- = 0$ [see Fig. 4(c)]. In contrast, in the SW regime, $A_L^\pm$ decrease gradually; specifically, $A_L^+ = 0$ and $A_L^- \approx 1.55$ at $g = \chi = -0.4$, while Δ_g remains zero. Beyond this point, no MI is observed in the ω_+ mode, while $A_L^- \neq 0$, indicating consistent instability in the SW region in the present case. Furthermore, Δ_g exhibits exponential growth, as shown in Fig. 4(d).

B. Effect of the three-body and interspecies interactions

Next, we analyze the effects of the three-body interactions, with $g = g_{\uparrow\downarrow} = -1$. By varying 3BI strength, we attain a stable mode for PWs at $\chi = 0.5$. In comparison, A_L^- and B_w^- are relatively large for $\chi < -2$, with A_L^- diminishing rapidly at $\chi = 0.5$ and entering the stable regime thereafter, as illustrated in Fig. 5(a). This fact indicates that the three-body interaction can be employed to design dynamical patterns. For the SW ($\Omega = 1, k_L = 4$), A_L^- and B_w^- initially decrease, attaining $A_L^- = 0$ at $\chi = 0.5$. However, $A_L^- \neq 0$ holds for $\chi > 0.75$, with A_L^- increasing linearly and indicating that, with a fixed attractive two-body interaction strengths, SWs get stabilized only for $0.5 < \chi < 0.75$, as shown in Fig. 5(b). The ω_+ mode gets stabilized and the gapless IAC disappears at $\chi = 0$. It

does not reappear at $\chi > 0$, suggesting the existence of a constant gapped mode.

With $g = g_{\uparrow\downarrow} = 1$, the behavior of A_L^- and B_w^- for PW and SW phases is similar to that shown in Figs. 5(a) and 5(b). However, the PW stabilization point shifts to $\chi = -0.5$, and SW stabilizes in interval $\chi = [-0.55, -0.5]$, with A_L^+ and B_w^+ vanishing at $\chi = -1.2$. Compared to the prior case, the system gets stabilized with very weak repulsive two-body and attractive three-body interactions.

For the interaction parameters $g = \chi = 1$, we observe a significant transition in the PW and SW phases, from a gapped state to a gapless one, as $g_{\uparrow\downarrow} = \chi_{\uparrow\downarrow}$ varies, see Figs. 5(c) and 5(d). In this interaction regime, the mode A_L^+ becomes zero, indicating a transition from gapless to a gapped state in the SW phase [see Figs. 4(d) and 5(d)]. The stability of the PW phase is confined to $-0.5 < g_{\uparrow\downarrow} < 1.8$. Outside this range, the PW phase is unstable, as shown in Fig. 5(c). The SW phase is always unstable, confirming the persistent instability under conditions $A_L^- \neq 0$, as depicted in Fig. 5(d).

V. PROPOSAL FOR THE EXPERIMENT

We further propose an experimental realization for BEC in the ^{39}K atomic gas [73–76], with $N \sim 10^4$ atoms. To create a quasi-1D cigar-shaped BEC, we consider weak axial and strong transverse trapping frequencies: $(\omega_x, \omega_\perp)/2\pi = (6, 300)$ Hz, leading to a transverse length scale of $a_\perp \sim 2.33 \mu\text{m}$. The attractive two-body scattering lengths are $a_{\uparrow\uparrow} = a_{\downarrow\downarrow} = a_{\uparrow\downarrow} = -4.4062a_0$, where a_0 is the Bohr radius, yielding dimensionless interaction strengths of $g = g_{\uparrow\downarrow} \approx -2$. Such interactions can be tuned using the Feshbach resonance [66–68] under the action of the magnetic field below 507 G [129]. The two-body loss rates vanish for symmetric two-body interactions [76]. For three-body interactions, we estimate the coupling constant for R3BI as $\chi = \{0.5 - 2.0\}$, corresponding to $\lambda_3 = \{0.71 - 2.84\} \times 10^{-38} \text{ m}^6 \text{ s}^{-1}$, which is about $\simeq 100$ times larger than the dominant three-body loss coefficient, $K_3/6 \sim 3 \times 10^{-40} \text{ m}^6 \text{ s}^{-1}$ [74,130]. The three-body interactions are characterized by the scattering hypervolume D , the above-mentioned complex quantity whose real and imaginary components correspond to energy

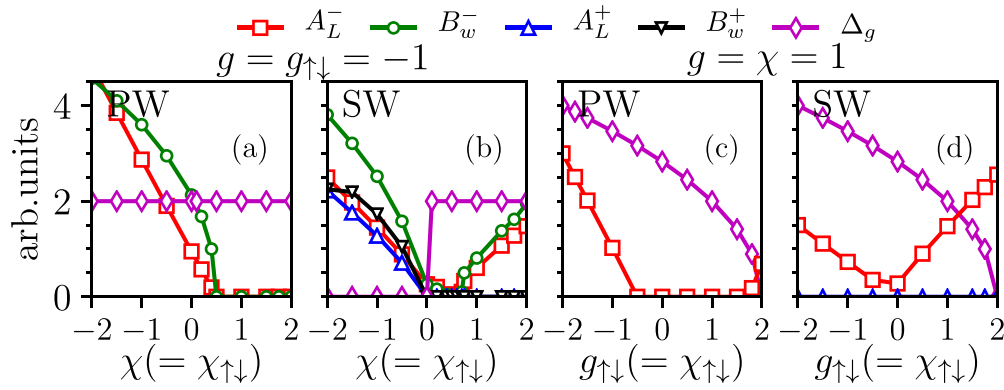


FIG. 5. The variation in the MI magnitude ($A_L^\pm$) and bandwidth ($B_w^\pm$), along with the gap between $\omega_\pm$ modes (Δ_g), is shown against different interaction strengths for the PW [(a) and (c)] and SW [(b) and (d)] phases, with coupling parameters same as in Fig. 4.

shifts and three-body losses, respectively. Our results indicate that $|\text{Re}(D)| \gg |\text{Im}(D)|$, placing the system near the resonance [86–88,90]. Parameters Ω and k_L are readily tunable by adjusting the Raman laser frequency, wavelength, and geometry, allowing precise control over the system's properties. In ultracold gases, the excitation spectrum can be probed using two-photon Bragg spectroscopy [131–136]. Therefore, our predictions are relevant for the experimental realization.

VI. CONCLUSIONS AND PERSPECTIVES

We have investigated the stability of quantum phases in spin-orbit-coupled Bose-Einstein condensates with two- and three-body interactions, following the Bogoliubov-de-Gennes approach. Firstly, we have found that attractive two-body interactions alone lead to various MI (modulation instability) scenarios, including the baseband and zero-wavenumber MIs in the PW (plane-wave) phase, which lead to the formation of deterministic rogue waves, as seen in both scalar and vector BEC models. Additionally, we have also identified new types of MI in the nontrivial stripe-wave (SW) phase, resulting in passband and mixed-band MIs that give rise to the emergence of nonlinear oscillatory matter waves and complex matter-wave patterns. Notably, the passband MI displays the roton instability alone, while the mixed-band MI drives phonon and roton instabilities.

Secondly, we aimed to suppress the instabilities and achieve stable quantum phases. In this regard, our results demonstrate that the introduction of the three-body repulsive interactions can transform the destabilized PW phase into a stable superfluid boselet (bosonic-droplet) phase. Additionally, we show that the unstable SW phase can become a stable supersolid under certain conditions. Thus, our findings reveal

the emergence of supersolidity and boselets in SW and PW phases, respectively.

Further, we have found that the density and spin structure factors, $S_{d,s}^\pm$, are sensitive to the fluctuations and characterize the respective MI phases. As MI is present in the ω_- mode and absent in the ω_+ one, $S_{d,s}^-$ exhibits fluctuations, while $S_{d,s}^+$ remains in the fluctuation-free state. However, the total structure factors $S_{d,s}$ indicate overall instability through fluctuations. Specifically, in the MBMI phase, fluctuations in $S_{d,s}^+$ occur solely when an instability-avoided crossing occurs between the $\omega_\pm$ modes. When a stable phase emerges, the structure factors stay constant.

Finally, we have demonstrated the crucial role of nonlinear interactions in achieving stable and unstable states which, further, plays an important role in understanding the gapped nature of the spectrum. We are currently extending the analysis to explore how repulsive three-body interactions can lead to stable supersolid stripe phases in two-dimensional spin-orbit-coupled BECs.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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