

Topological transition from a hopfion to a toron via flexoelectric self-polarization in chiral liquid crystals

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The presence of topological defects in apolar chiral liquid crystals causes orientational distortions, leading to nonuniform strain. This nonuniform strain generates an electric polarization response due to the flexoelectric effect, which induces an internal electric field. Associated with this electric field is an electrostatic self-energy, which has a backreaction on the director field. Calculation of this internal electric field and its resulting backreaction on the director field is complicated. We propose a method to do such, adapting a method recently developed to study the magnetostatic self-interaction effect on skyrmions in chiral ferromagnets. Bloch skyrmions in chiral magnets are solenoidal and are unaffected by the magnetostatic self-interaction. However, Bloch skyrmions in liquid crystals yield nonsolenoidal flexoelectric polarization and, thus, are affected by the electrostatic self-interaction. Additionally, as the flexoelectric coefficients are increased in strength, a transition from a hopfion to a toron is observed in three-dimensional confined systems.

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I. INTRODUCTION

Liquid crystals are unique states of soft matter that exhibit properties between those of conventional liquids and solid crystals. They are characterized by the long-range order of their molecular orientations, described by the director field, allowing for fascinating physical phenomena and applications. The director field encodes spatial variations of the local average molecular alignment direction of the constituent chiral molecules. By anchoring the liquid crystal at solid boundaries, the director can easily be distorted, giving rise to a myriad of topological defects in the liquid crystal [1]. This homeotropic anchoring introduces geometric frustration to the system [2], where the boundary conditions are incompatible with the favored cholesteric twist. It is because of this geometric frustration that topological defects form. Among the multitude of topological defects that can emerge in liquid crystals, skyrmions and hopfions have gained significant interest recently.

Skyrmions were first studied in condensed matter systems, in the context of chiral magnets [3,4]. In those systems, skyrmions are topological spin defects in the magnetization and behave like magnetic quasiparticles. They were later predicted to exist in chiral liquid crystals [5] and nematic liquid crystals [6], where the skyrmions in these systems are defects in the director field that also exhibit particlelike properties.

In addition to skyrmions, the emergence of hopfions in condensed matter systems has gathered great interest [7–10]. Unlike a skyrmion, which can be viewed as a localized vortex-like state with a defined core [3], a hopfion can be interpreted as a twisted skyrmion string, forming a closed loop in real space [11]. They have also been realized experimentally in liquid crystal systems [12,13].

While skyrmions in chiral magnets and chiral liquid crystals are similar in nature, they have distinct manifestations and mechanisms of formation. In chiral magnets, skyrmions are formed due to the competition between exchange interactions and Dzyaloshinskii-Moriya interactions, which arise from the lack of inversion symmetry in the crystal lattice. So, skyrmions in chiral magnets are driven primarily by magnetic interactions, whereas those in liquid crystals arise from molecular orientational order. Essentially, the difference between the two physical systems is spin orientations versus molecular orientations.

Both skyrmions in chiral magnets and in chiral liquid crystals share a common topological nature characterized by a topological homotopy invariant, ensuring stability against perturbations. While their manifestation is unique in both systems, the fundamental stabilization mechanism responsible for skyrmion formation is the same in both systems. This relationship between skyrmions in chiral magnets and chiral liquid crystals was shown by Leonov *et al.* [14].

In magnetic systems, the magnetization induces an internal magnetic field known as the stray, or demagnetizing, field [15], which acts to reduce the total magnetic moment. Associated with this internal demagnetizing field is a magnetostatic self-energy, generated by dipolar self-interactions of the magnetization. Determining the stray field and its backreaction on the magnetization is a difficult task [16], and it can lead to the stabilization/destabilization of topological defect structures [17].

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We study the analog problem, that is, depolarization in liquid crystals. The emergence of topological defects in liquid crystals results in orientational distortions of the director field, creating nonuniform strain in the system. In response to this, an electric polarization is induced, generating an internal electric depolarizing field, similar to the piezoelectric effect [18]. Akin to the stray field energy, there is an electrostatic self-energy corresponding to the depolarizing field [19]. Including this electrostatic self-interaction makes the problem nonlocal. The author recently developed a method to include the demagnetization field and compute its backreaction in chiral ferromagnets [17]. Such methods were also developed recently in the context of knotted string solitons in an extended version of the standard model of particle physics [20], and the backreaction of the Coulomb force in an extension of the Skyrme model [21]. Our aim is to adapt these methods developed originally in models of particle physics and condensed matter systems to soft matter systems.

II. THE CHIRAL LIQUID CRYSTAL MODEL

The system that we wish to model is an apolar chiral liquid crystal, described by a director field $\mathbf{n}(\mathbf{x}) \in \mathbb{R}P^2 \cong S^2/\mathbb{Z}_2$ [22,23]. That is, the director $\mathbf{n}$ is a vector in $\mathbb{R}^3$ of unit length $|\mathbf{n}| = 1$, such that $\mathbf{n}$ and $-\mathbf{n}$ describe the same state, since the director $\mathbf{n}$ is the average molecular alignment direction. Let us define the thin disk Ω_R^d of radius R and thickness d by

$$\Omega_R^d = \left\{ (x, y, z) \in \mathbb{R}^3 : |(x, y)| < R, |z| \leq \frac{d}{2} \right\}. \quad (1)$$

We will restrict our system to the confined geometry

$$\Omega \equiv \lim_{R \rightarrow \infty} \Omega_R^d = \left\{ (x, y, z) \in \mathbb{R}^3 : |z| \leq \frac{d}{2} \right\}, \quad (2)$$

where d is the cell gap under an external electric field with homeotropic anchoring [24]. Let us first begin by considering a nonchiral, apolar nematic liquid crystal. In the simplest approximation with isotropic elasticity, the Frank-Oseen energy for such a liquid crystal takes the simple harmonic form

$$F_{\text{FO}} = \frac{1}{2} K \int_{\mathbb{R}^3} d^3x |\nabla \mathbf{n}|^2. \quad (3)$$

Director configurations that minimize this energy are solutions of the Laplace equation $\Delta \mathbf{n} = \mathbf{0}$. Metastable inhomogeneous solutions of this were found by Belavin and Polyakov in the context of isotropic ferromagnets [25].

More insight can be gained by considering the elastic modes. We can decompose the director gradient tensor $\nabla \mathbf{n}$ into four normal modes $(\mathbf{B}, T, S, \Delta)$, which are distinct irreducible representations (irreps) of the rotation group. The director gradient tensor can then be expressed in terms of these normal modes as [26]

$$\partial_i n_j = -n_i B_j + \frac{1}{2} T \epsilon_{ijk} n_k + \frac{1}{2} S (\delta_{ij} - n_i n_j) + \Delta_{ij}. \quad (4)$$

In liquid-crystal physics terminology, the standard bend vector is

$$\mathbf{B} = -(\mathbf{n} \cdot \nabla) \mathbf{n} = \mathbf{n} \times (\nabla \times \mathbf{n}), \quad (5)$$

the standard pseudoscalar twist is

$$T = \mathbf{n} \cdot (\nabla \times \mathbf{n}), \quad (6)$$

and the standard splay vector is [27]

$$\mathbf{S} = S \mathbf{n}, \quad S = \nabla \cdot \mathbf{n}. \quad (7)$$

The last normal mode is the traceless symmetric tensor Δ , known as the biaxial splay [26]. Note that the biaxial splay Δ is not to be confused with the Laplace operator $\Delta = -\nabla^2$. It is a symmetric traceless tensor such that $\Delta \mathbf{n} = \mathbf{0}$. Let q be the positive eigenvalue of Δ and $\mathbf{n}_1, \mathbf{n}_2$ be the eigenvectors orthogonal to $\mathbf{n}$. Then, the biaxial splay can be given by [28]

$$\Delta = q(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2). \quad (8)$$

Inserting the normal mode decomposition (4) into the Frank-Oseen energy (3) yields

$$F_{\text{FO}} = \frac{1}{2} K \int_{\mathbb{R}^3} d^3x \left\{ \frac{1}{2} |\mathbf{S}|^2 + \frac{1}{2} T^2 + |\mathbf{B}|^2 + \text{Tr}(\Delta^2) \right\}. \quad (9)$$

It is clear that all four modes cost elastic free energy. In particular, the energy cost of splay, twist, and bend deformations is equivalent.

Let us now consider a liquid crystal composed of chiral molecules, with different elastic deformation costs, in the confined geometry (2). Then, the chirality of these molecules is characterized by some pseudoscalar q_0 that couples to the twist T . The associated Frank-Oseen free energy, neglecting the energy cost due to saddle-splay, can be expressed as [29,30]

$$\begin{aligned} F_{\text{FO}} &= \int_{\Omega} d^3x \left\{ \frac{1}{2} K_1 |\mathbf{S}|^2 + \frac{1}{2} K_2 (T + q_0)^2 + \frac{1}{2} K_3 |\mathbf{B}|^2 \right\} \\ &= \int_{\Omega} d^3x \left\{ \frac{K_1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{K_2}{2} \left[\mathbf{n} \cdot (\nabla \times \mathbf{n}) + \frac{2\pi}{p} \right]^2 \right. \\ &\quad \left. + \frac{K_3}{2} [\mathbf{n} \times (\nabla \times \mathbf{n})]^2 \right\}, \end{aligned} \quad (10)$$

where $q_0 = 2\pi/p$ is the cholesteric twist and p is the cholesteric pitch with a defined length at which the director twists by 2π . The cholesteric phase is characterized by the presence of enantiomorphy ($q_0 \neq 0$), and is distinguishable from the nematic phase ($q_0 = 0$). The Frank elastic constants K_1, K_2 , and K_3 determine the energy cost of splay, twist, and bend deformations, respectively. Skyrmion solutions in nematic liquid crystal ($q_0 = 0$) were first theorized in Ref. [6].

Chiral liquid crystals are dielectric materials that respond to external electric fields. This generates a corresponding coupled electric energy of the form [13,31]

$$\mathcal{E}_{\text{elec}} = -\frac{\epsilon_0 \Delta \epsilon}{2} (\mathbf{E}_{\text{ext}} \cdot \mathbf{n})^2, \quad (11)$$

where $\mathbf{E}_{\text{ext}}$ is the external electric field, ϵ_0 is the vacuum permittivity, and $\Delta \epsilon$ is the dielectric anisotropy. We will consider only the applied electric field orthogonal to topological defects in the (x, y) plane, that is, $\mathbf{E}_{\text{ext}} = (0, 0, E_z)$.

In experimental realizations, liquid crystals are placed between parallel plates with a potential difference. This imposes boundary conditions orthogonal to the plates on the liquid-crystal director field. In particular, this can impose strong

homeotropic anchoring conditions [2]

$$\mathbf{n}(x, y, z = \pm d/2) = \mathbf{e}_z = (0, 0, 1). \quad (12)$$

This can be accounted for in two-dimensional systems by including the Rapini-Papoular homeotropic surface anchoring potential [32,33]

$$\mathcal{E}_{\text{anch}} = -\frac{1}{2}W_0n_z^2, \quad (13)$$

where W_0 is the effective surface anchoring strength that favors director alignment in the z direction, that is, $\mathbf{n} = \pm\mathbf{e}_z = (0, 0, \pm 1)$ director configurations. This term acts to mimic homeotropic anchoring conditions at the cell surfaces of a three-dimensional system.

The Frank-Oseen free energy we are interested in, including the electric energy and homeotropic anchoring, is given by the energy functional

$$F_{\text{FO}} = \int_{\Omega} d^3x \left\{ \frac{K_1}{2}(\nabla \cdot \mathbf{n})^2 + \frac{K_2}{2}[\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 + \frac{K_3}{2}[\mathbf{n} \times \nabla \times \mathbf{n}]^2 + K_2q_0[\mathbf{n} \cdot (\nabla \times \mathbf{n})] + V(\mathbf{n}) \right\}, \quad (14)$$

where the potential is $V(\mathbf{n}) = \mathcal{E}_{\text{elec}}$ in three dimensions and $V(\mathbf{n}) = \mathcal{E}_{\text{elec}} + \mathcal{E}_{\text{anch}}$ in two dimensions. While this model does account for an external applied electric field, it does not include the electrostatic self-interaction energy arising from the flexoelectric effect. We will now show how to do this.

III. FLEXOELECTRIC POLARIZATION

When liquid crystals possess macroscopic electric polarization $\mathbf{P}_f$ (where $\mathbf{P}_f$ is spontaneous or induced by some external, nonelectric field-related, factors), then they induce a linear-in-field energy contribution [34]. One such source of macroscopic electric polarization generation is related to orientational distortions in liquid crystals. The case we consider here is molecules with permanent dipole moments. This leads to piezoelectric effects and generates a dipolar piezoelectric-like polarization. However, piezoelectricity is due to uniform strain, whereas this polarization is caused by the mechanical curvature, or flexion, of the director field $\mathbf{n}$, and is called flexoelectric [22]. If we fix the splay and induce polarization, this generates an additional energy contribution given by

$$F_S = \frac{1}{2}K_1|\mathbf{n}(\nabla \cdot \mathbf{n}) - c_1\mathbf{P}|^2 + \frac{1}{2}\mu|\mathbf{P}|^2. \quad (15)$$

Varying this with respect to the polarization yields [35]

$$\frac{\delta F_S}{\delta \mathbf{P}} = 0 \Rightarrow \mathbf{P} = \left(\frac{c_1K_1}{c_1^2K_1 + \mu} \right) (\nabla \cdot \mathbf{n})\mathbf{n}. \quad (16)$$

Similarly, we can do the same thing for fixed-bend-induced polarization, which generates a polarization

$$\mathbf{P} = \left(\frac{c_3K_3}{c_3^2K_3 + \mu} \right) [\mathbf{n} \times (\nabla \times \mathbf{n})]. \quad (17)$$

A formal theory of these flexoelectric effects was developed by Meyer [18]. This can be expressed as [19,36]

$$\mathbf{P}_f = e_1[(\nabla \cdot \mathbf{n})\mathbf{n}] + e_3[\mathbf{n} \times (\nabla \times \mathbf{n})], \quad (18)$$

where e_1 and e_3 are, respectively, the piezoelectric constants for the splay and bend of the molecules [37]. These coefficients are material dependent and can be measured directly via the electric current produced by the periodic mechanical flexing of the liquid crystals bounding surfaces [38].

Analogous to demagnetization in chiral magnets, the flexoelectric polarization produces internal sources of electric fields [18]; i.e., it induces an electric dipole moment $\mathbf{p}$, where $\mathbf{p} = \mathbf{P}_f$. In fact, it generates a continuous electric dipole moment distribution $\mathbf{P}_f : \Omega \rightarrow \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3$ is the confined geometry (2). The electric potential $\varphi : \Omega \rightarrow \mathbb{R}$ associated with this continuous dipole distribution induces an internal electric field $\mathbf{E} = -\nabla\varphi$. It satisfies a Poisson equation for electrostatics [39,40]

$$\Delta\varphi = -\nabla^2\varphi = \frac{1}{\epsilon_0}\rho, \quad \rho = -(\nabla \cdot \mathbf{P}_f), \quad (19)$$

where ρ_e is the electric charge density and the Laplacian is $\Delta = -\nabla^2$ on $\mathbb{R}^3$.

Using the definition of the electric potential, we see that Gauss' law is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \rho = -(\nabla \cdot \mathbf{P}_f), \quad (20)$$

where ρ is the electric charge density. The electric field induced by the dipole distribution $\mathbf{P}_f$ coincides, therefore, with the electric field induced by the charge distribution $-(\nabla \cdot \mathbf{P}_f)$. Hence, we may think of $-(\nabla \cdot \mathbf{P}_f)$ as an electric charge density. Furthermore, we can write

$$\nabla \cdot (\epsilon_0\mathbf{E} + \mathbf{P}_f) = \nabla \cdot \mathbf{D} = 0, \quad (21)$$

where $\mathbf{D}$ is the electric displacement field. Hence, there is no space charge.

Suppose we have a pair of electric dipole moments $\mathbf{P}_f^{(1)}$ and $\mathbf{P}_f^{(2)}$. Their interaction energy is

$$E_{\text{int}} = -\mathbf{P}_f^{(1)} \cdot \mathbf{E}^{(2)} = -\mathbf{P}_f^{(2)} \cdot \mathbf{E}^{(1)}, \quad (22)$$

where $\mathbf{E}^{(2)}$ is the electric field induced by the polarization $\mathbf{P}_f^{(2)}$ at the position of $\mathbf{P}_f^{(1)}$, and vice versa. Therefore, the flexoelectric energy coincides with the energy of a continuous dipole density distribution, which is [41,42]

$$F_{\text{flexo}} = -\frac{1}{2} \int_{\Omega} d^3x \mathbf{E}(\mathbf{x}) \cdot \mathbf{P}_f(\mathbf{x}), \quad (23)$$

where $\mathbf{E}$ is the induced electric field [Eq. (A2)]. We will later want to compute the variation of the flexoelectric energy with respect to the director field $\mathbf{n}$. For this reason, it proves more useful to express the flexoelectric energy in terms of the scalar electric potential φ ,

$$\begin{aligned} F_{\text{flexo}} &= \frac{1}{2} \int_{\Omega} d^3x \mathbf{P}_f \cdot \nabla\varphi \\ &= -\frac{1}{2} \int_{\Omega} d^3x (\nabla \cdot \mathbf{P}_f)\varphi + \frac{1}{2} \oint_{\partial\Omega} \mathbf{ds} \cdot (\varphi\mathbf{P}_f) \\ &= \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} d^3x \varphi \Delta\varphi + \frac{1}{2} \oint_{\partial\Omega} \mathbf{ds} \cdot (\varphi\mathbf{P}_f), \end{aligned} \quad (24)$$

by the divergence theorem. We will restrict ourselves to situations where the boundary conditions ensure that the boundary

term vanishes. In this case, the flexoelectric energy F_{flexo} coincides with the electrostatic self-energy of the charge distribution $-(\nabla \cdot \mathbf{P}_f)$. To see this, we use the general identity $\varphi \Delta \varphi = \nabla \varphi \cdot \nabla \varphi - \nabla \cdot (\varphi \nabla \varphi)$ and the divergence theorem to express the flexoelectric energy as

$$F_{\text{flexo}} = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} d^3 \mathbf{x} |\nabla \varphi|^2 = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} d^3 \mathbf{x} |\mathbf{E}|^2. \quad (25)$$

We now detail the two cases of interest. The first is the flexoelectric self-interaction energy of a translation-invariant skyrmion in a chiral liquid crystal, and the second is a hopfion in the confined geometry (2).

A. Translation-invariant skyrmion system

We impose the translation invariance to be along the z direction, such that the director field $\mathbf{n}$ is independent of z and, therefore, so is the polarization $\mathbf{P}_f$. We assume that there is an energetically preferential orientation for the director $\mathbf{n} = \mathbf{e}_z$ due to some intrinsic anisotropy or an external applied electric field. Note that we could have equally chosen $\mathbf{n} = -\mathbf{e}_z$ as the energy is invariant under the transformation $\mathbf{n} \mapsto -\mathbf{n}$. That is, the directors $\mathbf{n}$ and $-\mathbf{n}$ describe the same physical state, with energy $F[\mathbf{n}] = F[-\mathbf{n}]$. We assume that our polarization field has compact support such that there exists $R_0 > 0$ whereby $\mathbf{n}(x, y) = \mathbf{e}_z$ and, whence, $\mathbf{P}_f(x, y) = \mathbf{0}$ for all $r := |(x, y)| \geq R_0$.

The total electrostatic self-energy must either vanish or diverge as the polarization field $\mathbf{P}_f$ is translation invariant. However, if we consider the energy per unit length in the z direction, then this may be finite. This coincides with the energy of the thick disk $\Omega_R^1 = \{\mathbf{x} : x^2 + y^2 \leq R^2, |z| \leq 1/2\}$ in the limit $R \rightarrow \infty$ [17]. In this case, the flux of the boundary term $\varphi \mathbf{P}_f$ through the disk wall vanishes since the polarization $\mathbf{P}_f$ vanishes on the boundary. Furthermore, the flux through the top and bottom of the disk cancel out due to the translation invariance of $\varphi \mathbf{P}_f$. Thus, the electrostatic self-energy of the translation-invariant system is

$$F_{\text{flexo}} = \frac{\epsilon_0}{2} \int_{\mathbb{R}^2} d^2 x \varphi \Delta \varphi, \quad (26)$$

where $\Delta = -\partial_1^2 - \partial_2^2$ is the usual flat Laplacian on $\mathbb{R}^2$, and the electric scalar potential φ satisfies

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{\epsilon_0} \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} \right). \quad (27)$$

Consider the asymptotic $r \rightarrow \infty$ behavior of the electric potential $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$. Any solution of the Poisson equation $\Delta \varphi = \rho/\epsilon_0$ on the plane has a multipolar expansion

$$\varphi = -\frac{q}{2\pi\epsilon_0} \ln r + O(r^{-1}), \quad (28)$$

where $q = \int_{\mathbb{R}^2} d^2 x \rho$ is the total charge. Such functions are generally logarithmically unbounded. However, let us define $B_R(\mathbf{y})$ to be the ball of radius R centered at a point $\mathbf{y}$ in $\mathbb{R}^2$, that is,

$$B_R(\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{y}| < R\}. \quad (29)$$

Then, the total charge electric charge in the system must vanish,

$$q = \int_{\mathbb{R}^2} d^2 x \rho = \lim_{R \rightarrow \infty} \oint_{\partial B_R(0)} d\mathbf{s} \cdot \mathbf{P}_f = 0, \quad (30)$$

by the divergence theorem, since the flexoelectric polarization has compact support. Therefore, we can conclude that φ has (at least) a $1/r$ localization.

B. Hopfion system

We now consider the three-dimensional case with the strong homeotropic anchoring (12). This leads to the cell boundary condition for the polarization,

$$\mathbf{P}_f(r, z = \pm d/2) = e_1(\partial_z n_z) \mathbf{e}_z - e_3 \partial_z \mathbf{n}. \quad (31)$$

That is, we compute the energy of a thin disk $\Omega_R^d = \{\mathbf{x} : r^2 = x^2 + y^2 \leq R^2, |z| \leq d/2\}$ in the limit $R \rightarrow \infty$. As in the translation-invariant case, the director, and hence the polarization, has compact support in the (x, y) plane for fixed cell height values z . So, the director field tends to a constant on the disk boundary, that is, $\mathbf{n}(r \rightarrow \infty, z) = \mathbf{e}_z$. This means that the self-induced flexoelectric polarization identically vanishes on the disk boundary $\mathbf{P}_f(r \rightarrow \infty, z) = \mathbf{0}$. Therefore, we must have that the electric scalar potential also vanishes on the disk boundary, $\varphi(r \rightarrow \infty, z) = 0$. In order to compute the electric potential $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$, we need to extend beyond the cell boundary. This leads to the interior/exterior interface problem

$$\begin{cases} \Delta \varphi = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}_f, & \text{in } \Omega, \\ \Delta \varphi = 0, & \text{in } \mathbb{R}^3/\Omega. \end{cases} \quad (32)$$

In three dimensions, we have the multipole expansion

$$\varphi = -\frac{q}{4\pi\epsilon_0 r} + O(r^{-2}) \quad (33)$$

and the regularity condition $\varphi(r, |z| \rightarrow \infty) \rightarrow 0$.

IV. VARIATION OF THE FLEXOELECTRIC ENERGY

So far, we have shown how to include the electrostatic self-energy and compute the electric scalar potential φ by solving Poisson's equation (19) for fixed director field configuration $\mathbf{n}$. However, we need to compute the backreaction of the self-induced electric field $\mathbf{E}$ on the director field $\mathbf{n}$. To do this, we need to calculate the first variation of the flexoelectric energy $F_{\text{flexo}}(\mathbf{n})$ with respect to the director field $\mathbf{n}$.

Before proceeding with the variation of the flexoelectric energy, we opt to work in dimensionless units. This will also make numerical simulations more palatable. Let us consider an energy and length rescaling with $E = E_0 \hat{E}$ and $x = L_0 \hat{x}$. We choose to set our length and energy scales as

$$L_0 = \frac{1}{q_0} \frac{K_1}{K_2}, \quad E_0 = \frac{1}{q_0} \frac{K_1^2}{K_2}. \quad (34)$$

Then, the rescaled energy is

$$\begin{aligned} \hat{F}_{\text{FFO}} = \int_{\Omega} d^3x \left\{ \frac{1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{1}{2} \frac{K_2}{K_1} [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 \right. \\ \left. + \frac{1}{2} \frac{K_3}{K_1} (\mathbf{n} \times \nabla \times \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})] \right. \\ \left. + \frac{1}{q_0^2} \frac{K_1}{K_2^2} V(\mathbf{n}) \right\} + \hat{F}_{\text{flexo}}, \end{aligned} \quad (35)$$

where the rescaled flexoelectric energy is determined to be

$$\hat{F}_{\text{flexo}} = \frac{\epsilon}{2} \int_{\Omega} \hat{\varphi} \Delta_{\hat{x}} \hat{\varphi} d^3\hat{x}, \quad \Delta_{\hat{x}} \hat{\varphi} = -\frac{1}{\epsilon} \nabla_{\hat{x}} \cdot \mathbf{P}. \quad (36)$$

Here, we have expressed the polarization $\mathbf{P}_f$ in terms of a rescaled polarization $\mathbf{P}$, where

$$\mathbf{P}_f = \frac{e_1}{L_0} \mathbf{P}, \quad \mathbf{P} = (\nabla_{\hat{x}} \cdot \mathbf{n}) \mathbf{n} + \frac{e_3}{e_1} [\mathbf{n} \times (\nabla_{\hat{x}} \times \mathbf{n})], \quad (37)$$

and introduced the dimensionless vacuum electric permittivity

$$\epsilon = \frac{K_1 \epsilon_0}{e_1^2}. \quad (38)$$

A necessary requirement for topological solitons to exist in a field theory is the successful evasion of the Hobart-Derrick theorem [43]. The Hobart-Derrick theorem is a nonexistence theorem that states if the energy functional $F[\mathbf{n}]$ is not stationary against spatial rescaling, then $\mathbf{n}$ cannot be a solution of the field equations. So, let us consider a coordinate rescaling $\mathbf{x} \mapsto \mathbf{x}' = \mu \mathbf{x}$, for some positive scaling parameter $\mu \in \mathbb{R}_{>0}$. Then, the director field necessarily rescales as $\mathbf{n}_{\mu} = \mathbf{n}(\mu \mathbf{x})$. We find that the polarization $\mathbf{P}$ transforms as $\mathbf{P}_{\mu} = \mu \mathbf{P}(\mu \mathbf{x})$ and the necessary electric scalar potential scaling behavior is $\varphi_{\mu}(\mathbf{x}) = \varphi(\mu \mathbf{x})$.

We now focus on the situation at hand: translation-invariant configurations in the $\mathbf{e}_z$ direction. Recall that we are considering the finite energy of a thick slab $\Omega = \mathbb{R}^2 \times [0, 1]$. Under the energy and length rescalings [Eq. (34)], the thick slab gets mapped to $\Omega' = \mathbb{R}^2 \times [0, t]$, where $t = L_0^{-1}$ is the rescaled slab thickness. We then consider the energy per unit length F/t of this system, which is defined by the two-dimensional adimensional energy functional

$$\begin{aligned} F_{\text{FFO}} = \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{1}{2} \frac{K_2}{K_1} [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 \right. \\ \left. + \frac{1}{2} \frac{K_3}{K_1} (\mathbf{n} \times \nabla \times \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})] \right. \\ \left. + \frac{1}{q_0^2} \frac{K_1}{K_2^2} V(\mathbf{n}) + \frac{\epsilon}{2} \varphi \Delta \varphi \right\}, \end{aligned} \quad (39)$$

where the electric potential satisfies the dimensionless Poisson equation

$$\Delta \varphi = -\frac{1}{\epsilon} \nabla \cdot \mathbf{P}, \quad \mathbf{P} = (\nabla \cdot \mathbf{n}) \mathbf{n} + \frac{e_3}{e_1} [\mathbf{n} \times (\nabla \times \mathbf{n})]. \quad (40)$$

Therefore, under the coordinate rescaling $\mathbf{x} \mapsto \mathbf{x}' = \mu \mathbf{x}$, the dimensionless energy functional becomes

$$F_{\text{FFO}}(\mu) = F_2 + \frac{1}{\mu} F_1 + \frac{1}{\mu^2} F_0 + F_{\text{flexo}}. \quad (41)$$

For skyrmions to exist in this model, we require the energy to be stable against spatial rescalings, which yields the Derrick scaling constraint

$$\left. \frac{dF_{\text{FFO}}}{d\mu} \right|_{\mu=1} = -(F_1 + 2F_0) = 0. \quad (42)$$

While the potential energy F_0 is positive semi-definite, the first-order (in spatial derivatives) term F_1 can be negative, and thus provide stability. Hence, the Hobart-Derrick nonexistence theorem can be evaded.

We note that the flexoelectric self-energy is scale invariant in two dimensions and is thus unable to provide stability against spatial rescalings. However, in comparison with chiral ferromagnets, the magnetostatic self-energy there can stabilize skyrmions as it behaves like a potential under coordinate rescalings [17].

Let $\mathbf{n}_t$ be a smooth variation of $\mathbf{n} = \mathbf{n}_0$ through fields of compact support and define $\delta \mathbf{n} = \partial_t \mathbf{n}_t|_{t=0}$. Denote by φ_t the associated unique solution of Eq. (40) with source $-\frac{1}{\epsilon} \nabla \cdot \mathbf{P}_t$ decaying to 0 at infinity, and $\hat{\varphi} = \partial_t \varphi_t|_{t=0}$. It is important to note that, while $\delta \mathbf{n}$ has compact support, neither $\varphi = \varphi_0$ nor $\hat{\varphi}$ do: as argued above, they are $1/r$ localized. Following Ref. [17], the variation of F_{flexo} induced by $\mathbf{n}_t$ is found to be given by

$$\left. \frac{d}{dt} \right|_{t=0} F_{\text{flexo}}(\mathbf{n}_t) = \int_{\mathbb{R}^2} d^2x (\text{grad}_{\mathbf{n}} F_{\text{flexo}}) \cdot \delta \mathbf{n}, \quad (43)$$

where the corresponding gradient is

$$\begin{aligned} \text{grad}_{\mathbf{n}} F_{\text{flexo}} = \frac{e_3}{e_1} [((\nabla \times \mathbf{n}) \times \nabla \varphi) + (\nabla \times (\nabla \varphi \times \mathbf{n}))] \\ - \nabla (\nabla \varphi \cdot \mathbf{n}) + (\nabla \cdot \mathbf{n}) \nabla \varphi. \end{aligned} \quad (44)$$

We can easily extend the analysis in the previous section to the three-dimensional confined geometry (2). The variation is found to be

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} F_{\text{flexo}}(\mathbf{n}_t) = \int_{\Omega} d^3x \nabla \varphi \cdot \dot{\mathbf{P}} - \oint_{\partial \Omega} \mathbf{ds} \cdot (\varphi \dot{\mathbf{P}}) \\ - \frac{\epsilon}{2} \oint_{\partial \Omega} \mathbf{ds} \cdot (\varphi \nabla \hat{\varphi} - \hat{\varphi} \nabla \varphi). \end{aligned} \quad (45)$$

We can treat this as a thick disk in the limit where the radius of the disk tends to infinity. Then, using the same argument as in the two-dimensional case, the surface contribution term must vanish on the curved part of the disk in the infinite radius limit. We will consider only systems in which this boundary term vanishes, such as the case of hopfions (66) in the confined geometry (2). Therefore, the variation of the flexoelectric energy is determined to be

$$\left. \frac{d}{dt} \right|_{t=0} F_{\text{flexo}}(\mathbf{n}_t) = \int_{\Omega} d^3x (\text{grad}_{\mathbf{n}} F_{\text{flexo}}) \cdot \delta \mathbf{n}. \quad (46)$$

V. RELATION TO CHIRAL MAGNETS

The stability of two-dimensional skyrmions in chiral liquid crystals arises from the same mechanism responsible for the existence of skyrmions in chiral ferromagnetic systems [14]. This is due to the chiral interactions imposed by the handedness of the system. Consider the one-constant approximation where the bend, splay, and twist constants are all

equal ($K_i = K$). This corresponds to an apolar, chiral liquid crystal [23]. For such liquid crystals in an applied electric field $\mathbf{E}_{\text{ext}} = (0, 0, E_z)$, the free energy in the one-constant approximation can be reduced to the following expression:

$$F_{\text{FFO}} = \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2} (\nabla \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})] + \frac{1}{q_0^2} \frac{1}{K} V(\mathbf{n}) + \frac{\epsilon}{2} \varphi \Delta \varphi \right\}, \quad (47)$$

where we have used the identity [15]

$$(\nabla \mathbf{n})^2 = (\nabla \cdot \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 + [\mathbf{n} \times (\nabla \times \mathbf{n})]^2 + \nabla \cdot [(\mathbf{n} \cdot \nabla) \mathbf{n} - (\nabla \cdot \mathbf{n}) \mathbf{n}], \quad (48)$$

which holds for any unit vector $\mathbf{n}$. This is the energy density of a chiral ferromagnet in the absence of an external magnetic field with the Dzyaloshinskii-Moriya interaction (DMI) arising from the Dresselhaus spin-orbit coupling (SOC) [44]. The dielectric anisotropy energy in liquid crystals plays the same role as uniaxial anisotropy in chiral magnets.

VI. NUMERICAL METHOD

Our interests lie in computing the self-induced flexoelectric polarization of topological solitons in the above systems. For simplicity, we will use the one-constant approximation from here on out. We will now detail our method for obtaining solitons in the three-dimensional system, but the method reduces easily to the two-dimensional translation-invariant case. Topological solitons in this model are minimizers of the adimensional flexoelectric Frank-Oseen free energy

$$F_{\text{FFO}} = \int_{\Omega} d^3x \left\{ \frac{1}{2} (\nabla \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})] + \frac{1}{q_0^2} \frac{1}{K} V(\mathbf{n}) + \frac{1}{2} \mathbf{P} \cdot \nabla \varphi \right\}, \quad (49)$$

where the electric scalar potential φ is subject to the constraint

$$\begin{cases} \Delta \varphi = -\frac{1}{\epsilon} \nabla \cdot \mathbf{P}, & \text{in } \Omega, \\ \Delta \varphi = 0, & \text{in } \mathbb{R}^3 / \Omega. \end{cases} \quad (50)$$

The adimensional polarization is

$$\mathbf{P} = (\nabla \cdot \mathbf{n}) \mathbf{n} + \frac{e_3}{e_1} [\mathbf{n} \times (\nabla \times \mathbf{n})]. \quad (51)$$

We develop a method to find director fields $\mathbf{n} \in \mathbb{R}P^2$ that simultaneously minimize the flexoelectric Frank-Oseen energy (49) and solve the electric potential constraint (50). This method is implemented for NVIDIA CUDA architecture and is adapted from a similar nonlocal method developed to determine skyrmion crystals stabilized by ω mesons [45,46].

The inclusion of the flexoelectric polarization self-interaction introduces nonlocality into the minimization problem. Now, the minimization method assumes that Poisson's equation is satisfied during each iteration of the algorithm. We now need to ensure that it indeed is. That is, during every iteration of the minimization algorithm, the electric potential φ must solve Poisson's equation $\Delta \varphi = \frac{1}{\epsilon} \rho$ with the source $\rho = -(\nabla \cdot \mathbf{P})$. This can be approached by reformulating the problem as an unconstrained optimization problem: minimize

the functional

$$F(\varphi) = \frac{1}{2} \int_{\mathbb{R}^3} d^3x |\mathrm{d}\varphi|^2 + \frac{1}{\epsilon} \int_{\Omega} d^3x \varphi (\nabla \cdot \mathbf{P}) \quad (52)$$

with respect to φ , where the director field $\mathbf{n}$ is fixed and, thus, the divergence of the polarization $\mathbf{P}$ is also fixed, with

$$\begin{aligned} \nabla \cdot \mathbf{P} &= \frac{e_3}{e_1} \{ (\nabla \times \mathbf{n})^2 - \mathbf{n} \cdot [\nabla (\nabla \cdot \mathbf{n})] + \mathbf{n} \cdot \nabla^2 \mathbf{n} \\ &\quad + (\nabla \cdot \mathbf{n})^2 + \mathbf{n} \cdot [\nabla (\nabla \cdot \mathbf{n})] \}. \end{aligned} \quad (53)$$

We will use a nonlinear conjugate gradient method with a line search strategy to solve this unconstrained problem. The conjugate stepsize is determined using the Fletcher-Reeves method.

To summarize, the full numerical algorithm is implemented as follows:

- (1) Perform a step of the arrested Newton flow method for the director field $\mathbf{n}$ using a fourth-order Runge-Kutta method.
- (2) Solve Poisson's equation for the electric potential φ using nonlinear conjugate gradient descent with the Fletcher-Reeves method.
- (3) Compute the total energy of the configuration $(\mathbf{n}_i, \varphi_i)$ and compare it to the energy of the previous configuration $(\mathbf{n}_{i-1}, \varphi_{i-1})$. If the energy has increased, arrest the flow.
- (4) Check the convergence criteria: $\|E_{\text{dis}}(\mathbf{n})\|_{\infty} < \epsilon$. If the convergence criteria have been satisfied, then stop the algorithm.
- (5) Repeat the process (return to step 1).

VII. LIQUID-CRYSTAL SKYRMIONS

In our model, liquid-crystal skyrmions are smooth maps $\mathbf{n} : \mathbb{R}^2 \rightarrow \mathbb{R}P^2$ that minimize the flexoelectric Frank-Oseen energy (49), and are subject to the vacuum boundary condition $\mathbf{n}(r \rightarrow \infty) = \mathbf{e}_z$. This boundary condition compactifies the domain $\mathbb{R}^2 \cup \{\infty\} \cong S^2$, such that skyrmions belong to the homotopy group $\pi_2(\mathbb{R}P^2) = \pi_2(S^2) = \mathbb{Z}$. That is, skyrmions have an associated integer-valued topological degree, given explicitly by

$$Q_{\text{Sk}} = \frac{1}{4\pi} \epsilon^{ijk} \int_{\mathbb{R}^2} d^2x \left(n_i \frac{\partial n_j}{\partial x} \frac{\partial n_k}{\partial y} \right). \quad (54)$$

Numerical relaxation of the flexoelectric Frank-Oseen energy (49) is carried out using the arrested Newton flow and nonlinear conjugate gradient descent methods for the director field $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$ and electric potential $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$, respectively. We impose vacuum boundary conditions, $\mathbf{n} \rightarrow \mathbf{e}_z$ and $\varphi \rightarrow 0$, as $r \rightarrow \infty$.

A. Twist favored Bloch skyrmions

In the one-constant approximation, the chiral liquid-crystal model becomes equivalent to that of a chiral ferromagnetic model with the Dresselhaus DMI term, favoring Bloch modulations. The Bloch skyrmion ansatz is given by

$$\mathbf{n}_{\text{Bloch}}(r, \theta) = \sin f(r) \mathbf{e}_{\theta} + \cos f(r) \mathbf{e}_z, \quad (55)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a monotonically decreasing radial profile function with boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. Within this ansatz, the flexoelectric polarization is determined

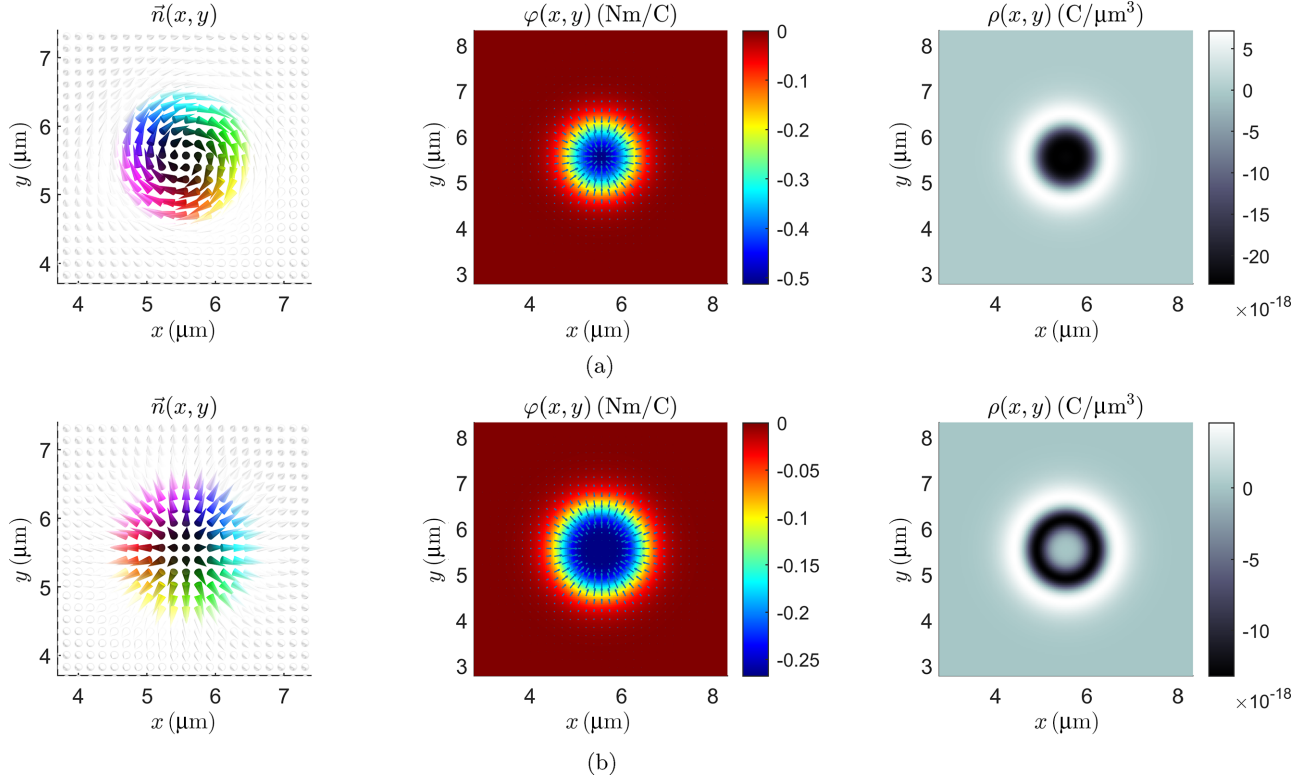


FIG. 1. Flexoelectric polarization effect on a skyrmion in a liquid crystal in the one-constant approximation. A twist favored Bloch skyrmion is shown in panel (a), and a splay-bend favored Néel skyrmion in panel (b). In each subfigure is the director $\mathbf{n}(x, y)$, and the flexoelectric potential $\varphi(x, y)$ is shown alongside the electric field vector $\mathbf{E} = -\nabla\varphi$. Also shown is the associated self-induced electric charge density $\rho = -\nabla \cdot \mathbf{P}_f$. It can be seen that the Bloch skyrmion has a core of negative charge, surrounded by an outer ring of positive charge. However, the Néel skyrmions have a neutral core surrounded by an internal negatively charged ring and a positively charged outer ring. For material properties, we use the elastic coefficient $K = 10$ pN and the cholesteric pitch is chosen to be $p = 7$ μm . The flexoelectric coefficients are chosen to be $e_1 = 2$ pC m $^{-1}$ and $e_3 = 4$ pC m $^{-1}$, with an applied electric field of strength $E_z = 1$ V μm^{-1} and dielectric anisotropy $\Delta\epsilon = 3.7$.

to be purely radial,

$$\mathbf{P}_{\text{Bloch}} = \frac{e_3}{e_1} \frac{1}{r} \sin^2 f(r) \mathbf{e}_r. \quad (56)$$

While the Bloch ansatz (55) is solenoidal, its associated polarization is not,

$$\nabla \cdot \mathbf{P}_{\text{Bloch}} = \frac{e_3}{e_1} \frac{1}{r} \frac{df}{dr} \sin 2f(r) \neq 0. \quad (57)$$

Bloch skyrmions in chiral magnets are solenoidal and are unaffected by the magnetostatic self-interaction. However, Bloch skyrmions in liquid crystals yield nonsolenoidal flexoelectric polarization and thus are affected by the electrostatic self-interaction. This can be seen in Fig. 1(a).

B. Splay and bend favored Néel skyrmions

What happens if we now consider liquid crystals, which prefer splay and bend, opposed to twist. Let us remain in the one-constant approximation. Then, the Frank-Oseen free

energy takes the form

$$\begin{aligned} F &= \frac{K}{2} \int_{\Omega} d^3x \{ (\mathbf{S} + \mathbf{S}_0)^2 + T^2 + (\mathbf{B} + \mathbf{B}_0)^2 \} \\ &= \frac{K}{2} \int_{\Omega} d^3x \{ (\nabla \cdot \mathbf{n})^2 + 2\mathbf{S}_0 \cdot \mathbf{n}(\nabla \cdot \mathbf{n}) + [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 \\ &\quad + [\mathbf{n} \times (\nabla \times \mathbf{n})]^2 - 2\mathbf{B}_0 \cdot [(\mathbf{n} \cdot \nabla)\mathbf{n}] + \text{const.} \}. \end{aligned} \quad (58)$$

If we choose $\mathbf{S}_0 = \mathbf{B}_0 = q_0 \mathbf{e}_z$, then the model reduces to that of the chiral magnet with the DMI term arising from the Rashba SOC. That is, the free energy becomes

$$F = \int_{\Omega} d^3x \left\{ \frac{K}{2} (\nabla \mathbf{n})^2 + Kq_0 [n_z (\nabla \cdot \mathbf{n}) - \mathbf{n} \cdot \nabla n_z] + V(\mathbf{n}) \right\}, \quad (59)$$

where we have included the potential term and used the identity (48) again. The factor of q_0 was chosen for convenience as we can pick the same energy and length scales as the twist favored model. It also allows us to compare splay-bend favored Néel skyrmions with the twist favored Bloch skyrmions. For more insight into the various DMI types and their associated skyrmions, see e.g., [47].

Let us now include the electrostatic self-energy, and employ the same length $L_0 = 1/q_0$ and energy $E_0 = K/q_0$ scales

as before. Then, in the translation-invariant case, the normalized free energy of this splay-bend favored liquid-crystal model becomes

$$F = \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2} (\nabla \mathbf{n})^2 + [n_z (\nabla \cdot \mathbf{n}) - \mathbf{n} \cdot \nabla n_z] + \frac{1}{q_0^2} \frac{1}{K} V(\mathbf{n}) + \frac{\epsilon}{2} \varphi \Delta \varphi \right\}. \quad (60)$$

It is well-known that the Rashba DMI term prefers Néel hedgehog skyrmions, given by the ansatz

$$\mathbf{n}_{\text{Néel}}(r, \theta) = \sin f(r) \mathbf{e}_r + \cos f(r) \mathbf{e}_z. \quad (61)$$

The self-induced polarization, coming from the Néel ansatz (61), is

$$\begin{aligned} \mathbf{P}_{\text{Néel}} = & \left[\frac{1}{r} \sin^2 f(r) + \left(1 - \frac{e_3}{e_1} \right) \frac{1}{2} \sin 2f(r) \frac{df}{dr} \right] \mathbf{e}_r \\ & + \left[\frac{1}{2r} \sin 2f(r) + \left(\cos^2 f(r) + \frac{e_3}{e_1} \sin^2 f(r) \right) \frac{df}{dr} \right] \mathbf{e}_z. \end{aligned} \quad (62)$$

Unlike the Bloch polarization, the Néel polarization picks up an out-of-plane component. The divergence of this polarization is also nonzero. We note that if the flexoelectric coefficients are equal, $e_1 = e_3$, then the divergence of the polarization for both Bloch and Néel ansätze is the same,

$$\nabla \cdot \mathbf{P}_{\text{Bloch}} = \nabla \cdot \mathbf{P}_{\text{Néel}} = \frac{1}{r} \frac{df}{dr} \sin 2f(r). \quad (63)$$

So, they will generate the same electrostatic potential and, thus, they will be energy degenerate for equal flexoelectric coefficients. For unequal flexoelectric coefficients, $e_1 \neq e_3$, the associated self-induced polarizations yield different electric scalar potentials and, hence, distinct skyrmions. This can be seen in Fig. 1(b).

VIII. LIQUID-CRYSTAL HOPFIONS

A hopfion can be interpreted as a twisted skyrmion string, forming a closed loop in real space. Cross-sectional views [top panel of Fig. 2(a)] show the skyrmion twisting as it winds around the hopfion core, changing from an in-plane skyrmion ($Q_{\text{Sk}} = -1$) to an out-of-plane antiskyrmion ($Q_{\text{Sk}} = +1$). A cross-section view through the center of the disk ($z = 0$) gives a different perspective of the hopfion, showing the structure of Bloch skyrmionium ($Q_{\text{Sk}} = 0$), or a 2π vortex [48] [bottom panel of Fig. 2(a)].

Hopfions comprise interlinked closed-loop preimages of constant $\mathbf{n}(x, y, z)$ [7]. The linking of closed-loop preimages of antipodal points in $S^2/\mathbb{Z}_2 \cong \mathbb{R}P^2$ defines a homotopy invariant. This homotopy invariant is the Hopf index $Q_{\text{Hopf}} \in \pi_3(\mathbb{R}P^2) = \pi_3(S^2) = \mathbb{Z}$, associated with the fibration $\mathbb{Z}_2 \rightarrow S^2 \rightarrow \mathbb{R}P^2$. The Hopf index of a given vector field $\mathbf{F}$ can be computed by constructing a gauge potential $\mathbf{A}$, such that $\mathbf{F} = \nabla \times \mathbf{A}$ and [49,50]

$$F_i = \frac{1}{8\pi^2} \epsilon_{ijk} \mathbf{n} \cdot (\partial_j \mathbf{n} \times \partial_k \mathbf{n}). \quad (64)$$

Then, the Hopf index is given by [51]

$$Q_{\text{Hopf}} = - \int_{\Omega} d^3x \mathbf{F} \cdot \mathbf{A}. \quad (65)$$

We will consider a hopfion with Hopf index $Q_{\text{Hopf}} = 1$, defined by the ansatz [52]

$$\mathbf{n}_{\text{Hopf}} = \begin{pmatrix} \frac{4\Sigma r(\Theta \cos \theta - (\Lambda - 1) \sin \theta)}{(1 + \Lambda)^2} \\ \frac{4\Sigma r(\Theta \sin \theta + (\Lambda - 1) \cos \theta)}{(1 + \Lambda)^2} \\ 1 - \frac{8\Sigma^2 r^2}{(1 + \Lambda)^2} \end{pmatrix}, \quad (66)$$

where we have introduced the three functions [11]

$$\Theta(z) = \tan\left(\frac{\pi z}{d}\right), \quad (67a)$$

$$\Sigma(r, z) = \frac{1}{d} \left[1 + \left(\frac{2z}{d} \right)^2 \right] \sec\left(\frac{\pi r}{2d}\right), \quad (67b)$$

$$\Lambda(r, z) = \Sigma^2 r^2 + \frac{\Theta^2}{4}. \quad (67c)$$

Using the hopfion ansatz (66) as an initial configuration for our numerical minimization algorithm, the relaxed director solution is plotted in Fig. 2(a). Consider the closed-loop preimages of $\mathbf{n} = (1, 0, 0)$ and $\mathbf{n} = (-1, 0, 0)$ by letting us construct two isosurface tubes, each corresponding to closed-loop preimages about $\mathbf{n} = (\pm 1, 0, 0)$. Such a construction is shown in Fig. 3(a). By considering the preimages of $\mathbf{n} = (\pm 1, 0, 0)$, it is clear to see that they do indeed link only once.

In our simulations, we use a grid with lattice points $n_x = n_y = 128$ and $n_z = 192$ to model $\mathbb{R}^3$. This corresponds to a discretized model of the disk Ω with lattice points $128 \times 128 \times 64$. We add a layer of vacuum above and below the disk, with the same thickness as the disk, d . Different vacuum layer thicknesses were tried, but a vacuum layer with thickness of $O(d)$ was sufficiently accurate. For the electric scalar potential $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$, we employ a central finite difference method throughout $\mathbb{R}^3$. However, for the director field $\mathbf{n} : \Omega \rightarrow \mathbb{R}P^2$, we use a central finite difference method in the bulk of the disk Ω and a backward/forward finite difference method on the disk boundary $\partial\Omega$.

The resulting electrostatic properties of the minimal energy $Q_{\text{Hopf}} = 1$ hopfion, associated with the flexoelectric Frank-Oseen energy (49), are plotted in Fig. 4 for the hopfion. We use similar material properties as in the two-dimensional skyrmion case. This is carried out within the one-constant approximation with $K = 10$ pN. We set the distance between the cell plates to be $d = 7 \mu\text{m}$, with an applied potential difference of $U = 2$ V between the plates. This generates an electric field of strength $E_z = U/d = 0.29 \text{ V } \mu\text{m}^{-1}$, and the dielectric anisotropy constant is chosen to be $\Delta\epsilon = 4.8$. These parameters are similar to those employed in Ref. [13]. The flexoelectric constants are normally estimated to be of the order of magnitude $\sim 3 \text{ pC m}^{-1}$ [34] and for nematic liquid crystals in the range $1\text{--}10 \text{ pC m}^{-1}$ [38]. So we choose to set them to initially be equal with $e_1 = e_3 = 4 \text{ pC m}^{-1}$. We set our box size for $\mathbb{R}^3$ to be $l_x = l_y = l_z = 3d$ and choose the cholesteric pitch to be $p/d = 1$. Hence, the lattice spacing is $\Delta l_x = \Delta l_y = 0.165 \mu\text{m}$ and $\Delta l_z = 0.110 \mu\text{m}$.

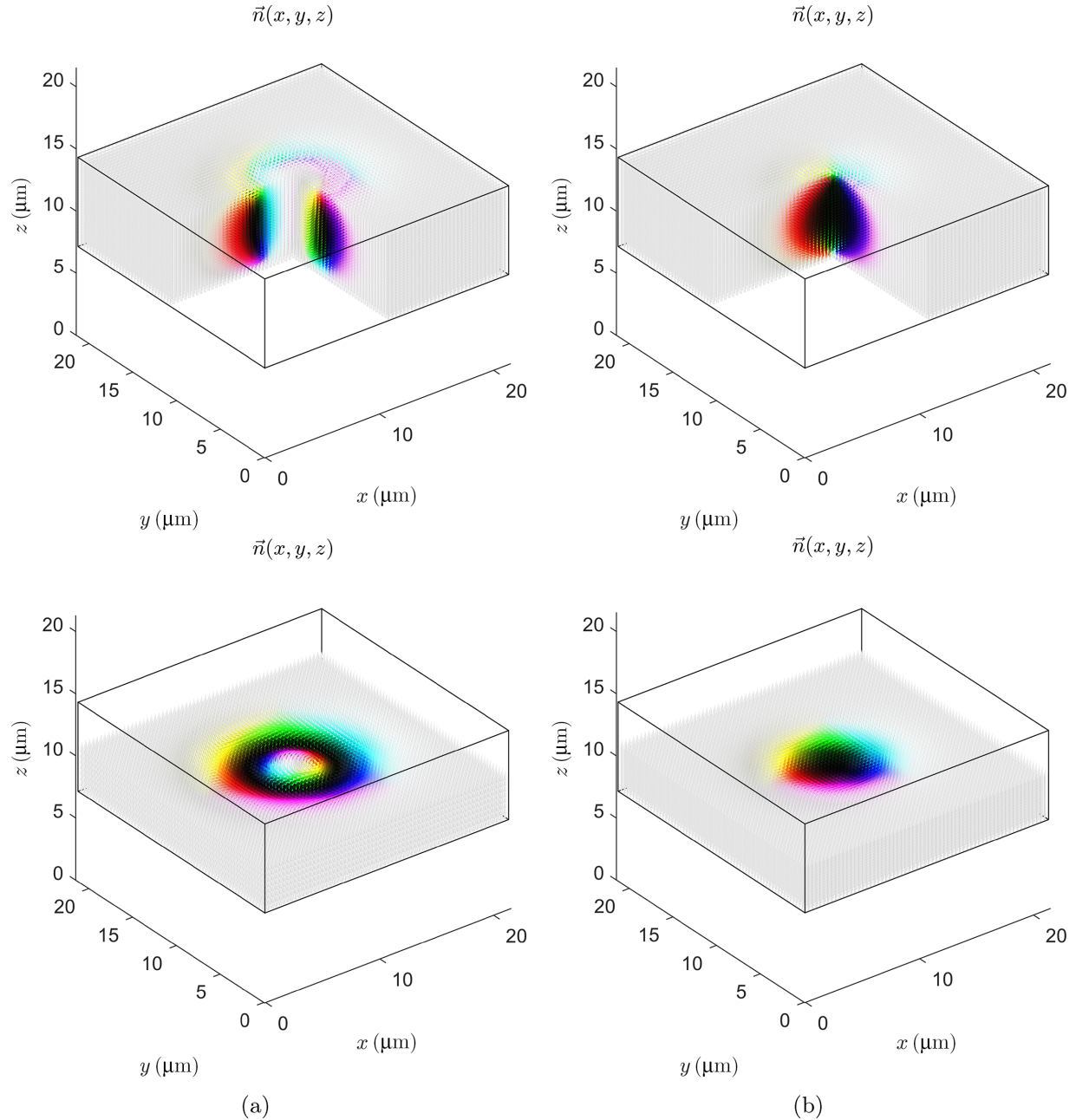


FIG. 2. Flexoelectric polarization effect on a hopfion in a liquid crystal in the one-constant approximation $K_i = K = 10$ pN. Shown is the relaxed director field $\mathbf{n}(x, y, z)$ solution of the flexoelectric Frank-Oseen energy (49), starting from the hopfion ansatz (66) as an initial configuration. Plots of the director field $\mathbf{n}$ are shown for (a) a hopfion and (b) a toron. The distance between the cell plates is taken to be $d = 7 \mu\text{m}$, with an applied voltage of $U = 2$ V. This gives an applied electric field strength of $E_z = U/d = 0.29 \text{ V } \mu\text{m}^{-1}$ and we set the dielectric anisotropy to be $\Delta\epsilon = 4.8$. The cholesteric pitch p is taken to be the same length as the distance between the cell plates, $p = d$. We set the flexoelectric coefficients to be equal $e_1 = e_3$. There is a phase transition from a hopfion to a toron as the flexoelectric coefficients are increased from $e_i = 4 \text{ pC m}^{-1}$ to $e_i = 8 \text{ pC m}^{-1}$. (a) Hopfion ($e_1 = e_3 = 4 \text{ pC m}^{-1}$) and (b) toron ($e_1 = e_3 = 8 \text{ pC m}^{-1}$).

The electric scalar potential $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ and the electric charge density $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}$ are shown in Fig. 4. The left column shows a cross-section view on the xy plane at the center of the disk Ω . (In our numerical simulations this corresponds to $z = 3d/2$.) This view shows the skyrmionium structure and it is clear to see that both the electric potential and charge density (in this cross-section of the hopfion) behave similar to the two-dimensional skyrmion. A cross-section view on the xz plane at $y = 3d/2$ is displayed in the right column of

Fig. 4, showing the axial symmetry of the electric potential and charge density.

As the flexoelectric coefficient strength is increased from $e_i = 4 \text{ pC m}^{-1}$ to $e_i = 8 \text{ pC m}^{-1}$, the hopfion collapses and transitions to a toron state. This toron terminates at point defects due to the boundary conditions enforced by the strong homeotropic anchoring [Eq. (12)] [7], and can be seen in Fig. 2(b). It is clear from Fig. 3(b) that the knotted preimages of $\mathbf{n} = \pm \mathbf{e}_1$ become unlinked. The resulting electric scalar

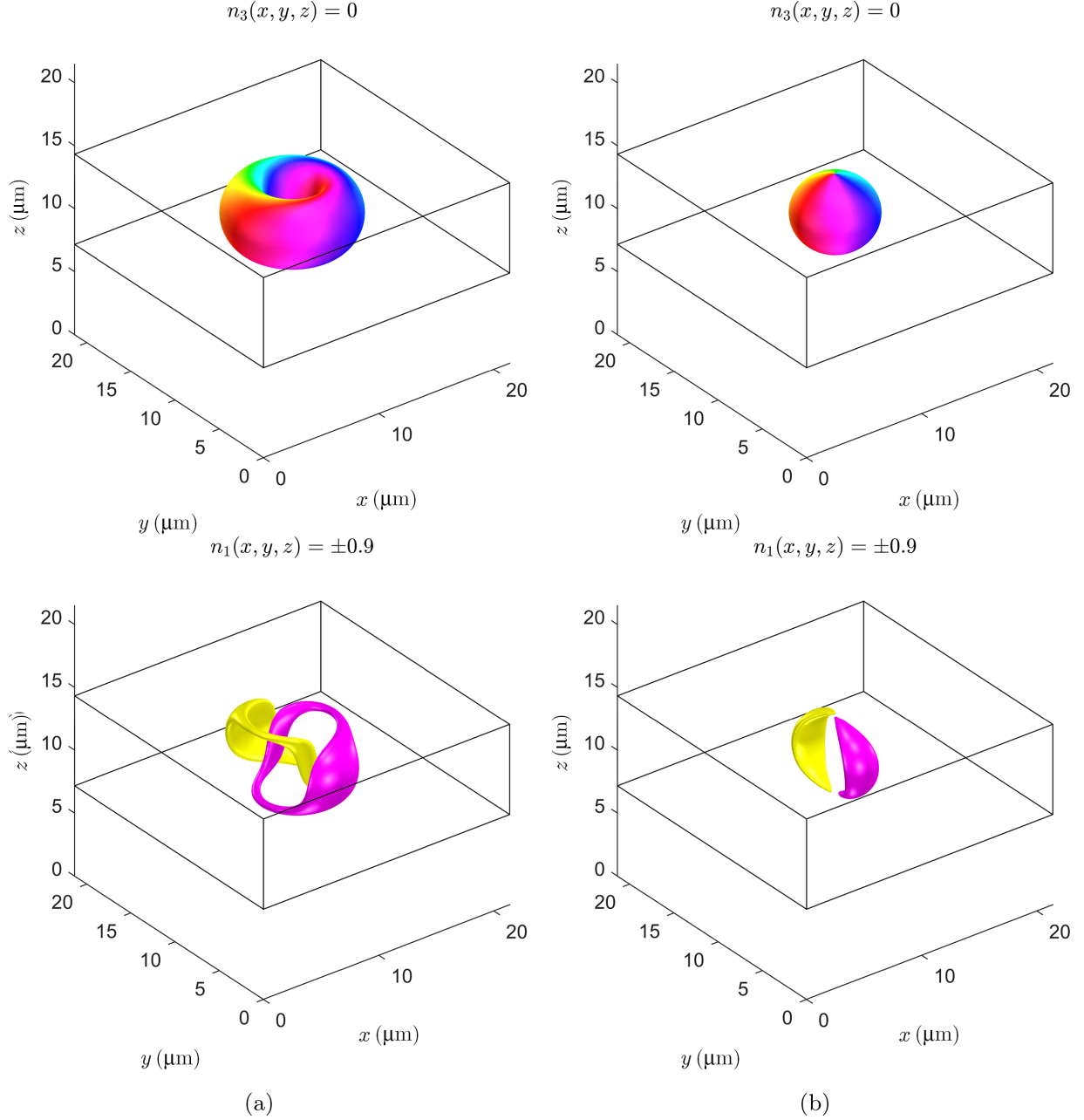


FIG. 3. Isosurface plots of the director field: The top row is the field $n_3 = 0$ and the bottom row is the linking of the field $n_1 = \pm 0.9$. The hopfion is shown in panel (a) and the toron in panel (b). The parameter set is detailed in Fig. 2. As the flexoelectric coefficients are increased from $e_i = 4 \text{ pC m}^{-1}$ to $e_i = 8 \text{ pC m}^{-1}$, the hopfion transitions to a toron. It is clear to see from the bottom row that the knots become unlinked. (a) Hopfion ($e_1 = e_3 = 4 \text{ pC m}^{-1}$) and (b) toron ($e_1 = e_3 = 8 \text{ pC m}^{-1}$).

potential and electric charge density for the toron are plotted in Fig. 5.

IX. CONCLUSION

In this paper, we have studied the backreaction of the self-induced flexoelectric polarization on topological defects in chiral liquid crystals. Due to flexoelectricity, topological defects induce spatially-varying strain into the liquid crystal, which generates an internal polarization (18). From the polarization emerges an electric field (A2) with an associated electrostatic self-energy (23). We show how to compute this

self-energy and its corresponding backreaction on the director field by introducing the electric scalar potential (A1) and solving the Poisson equation (19).

The problem we have investigated here is the analog problem of demagnetization in chiral magnetic systems. While the two physical systems are similar, the manifestation of topological defects in each system is unique. Further, the electrostatic self-interaction behaves differently in both systems. In chiral magnets, the electrostatic scalar potential depends on the divergence of the order parameter, whereas it depends on the divergence of the polarization induced by the order parameter in chiral liquid crystals. We showed

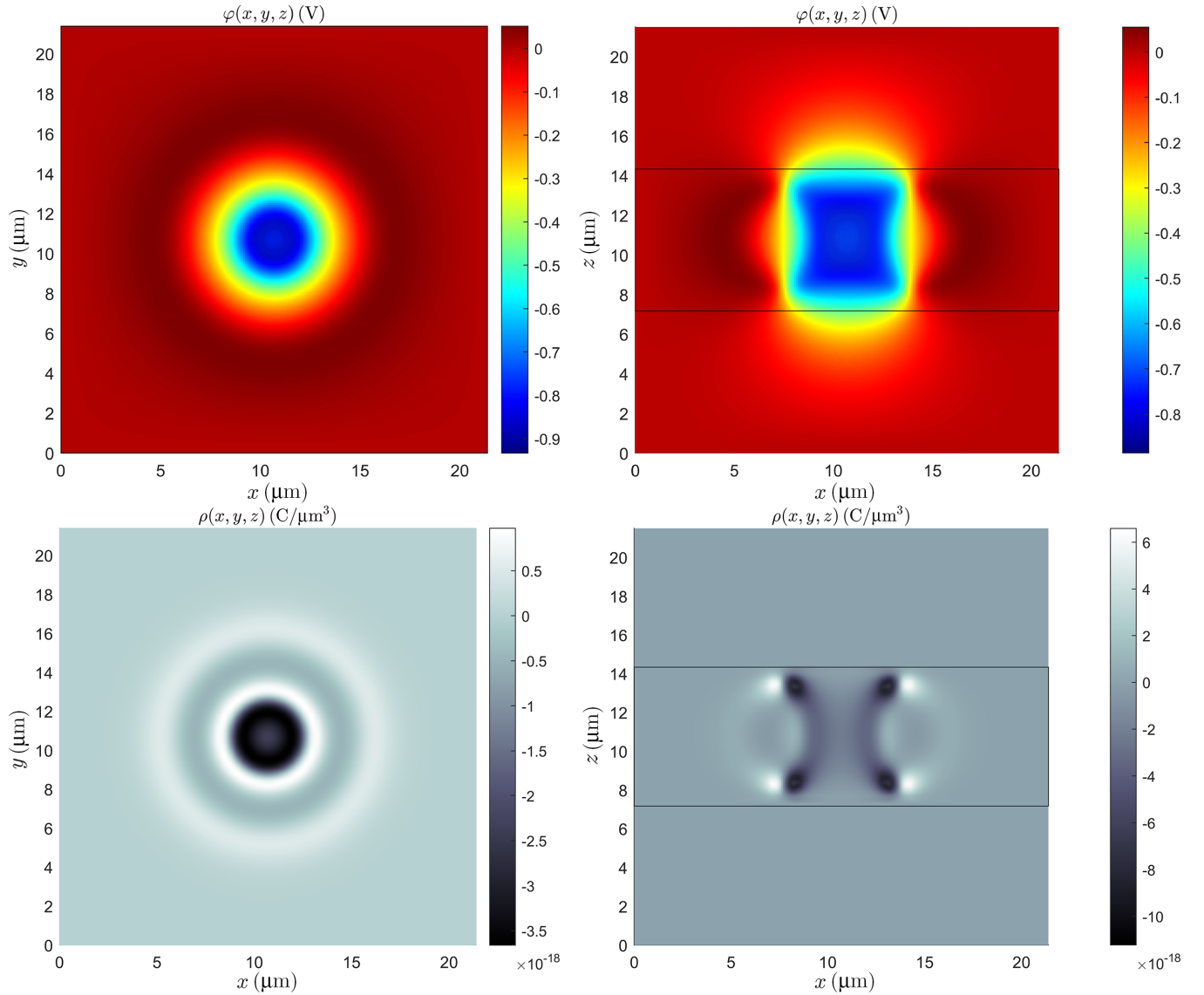


FIG. 4. Flexoelectric polarization effect on a hopfion in a liquid crystal in the one-constant approximation. The top row shows the self-induced electrostatic scalar potential ϕ , which is a solution of the Poisson equation $\Delta\phi = \rho/\epsilon_0$, and the bottom row displays the electric charge density $\rho = -\nabla \cdot \mathbf{P}_f$. It is clear to see that the electric charge density is confined within the confined geometry Ω , whereas the electric scalar potential extends into $\mathbb{R}^3/\Omega$. The parameter set is detailed in Fig. 2 with flexoelectric coefficients $e_1 = e_3 = 4 \text{ pC m}^{-1}$.

that, in two-dimensional chiral liquid-crystal systems, the electrostatic self-energy is conformally invariant. However, in two-dimensional chiral magnetic systems, the demagnetizing self-energy behaves like a potential and can provide scale stability to magnetic skyrmions and aid in evading the Hobart-Derrick Theorem. Finally, magnetic Bloch skyrmions are unaffected by the electrostatic self-interaction due to the Bloch ansatz being solenoidal. Although the Bloch skyrmion is solenoidal, its self-induced polarization is not. Therefore, liquid-crystal Bloch skyrmions are affected by the electrostatic self-interaction.

In addition, we have derived a liquid-crystal model that favors splay and bend deformations, and related it to a chiral magnetic system with a DMI term coming from Rashba SOC. A comparison of the effect of the electrostatic self-interaction on skyrmions in the two liquid crystals models was detailed. In particular, for unequal flexoelectric coefficients the two

skyrmion types yield distinct electric scalar potentials and are not degenerate in energy. We also showed that increasing the strength of the flexoelectric coefficients can destabilize hopfions into torons in three-dimensional confined systems.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this

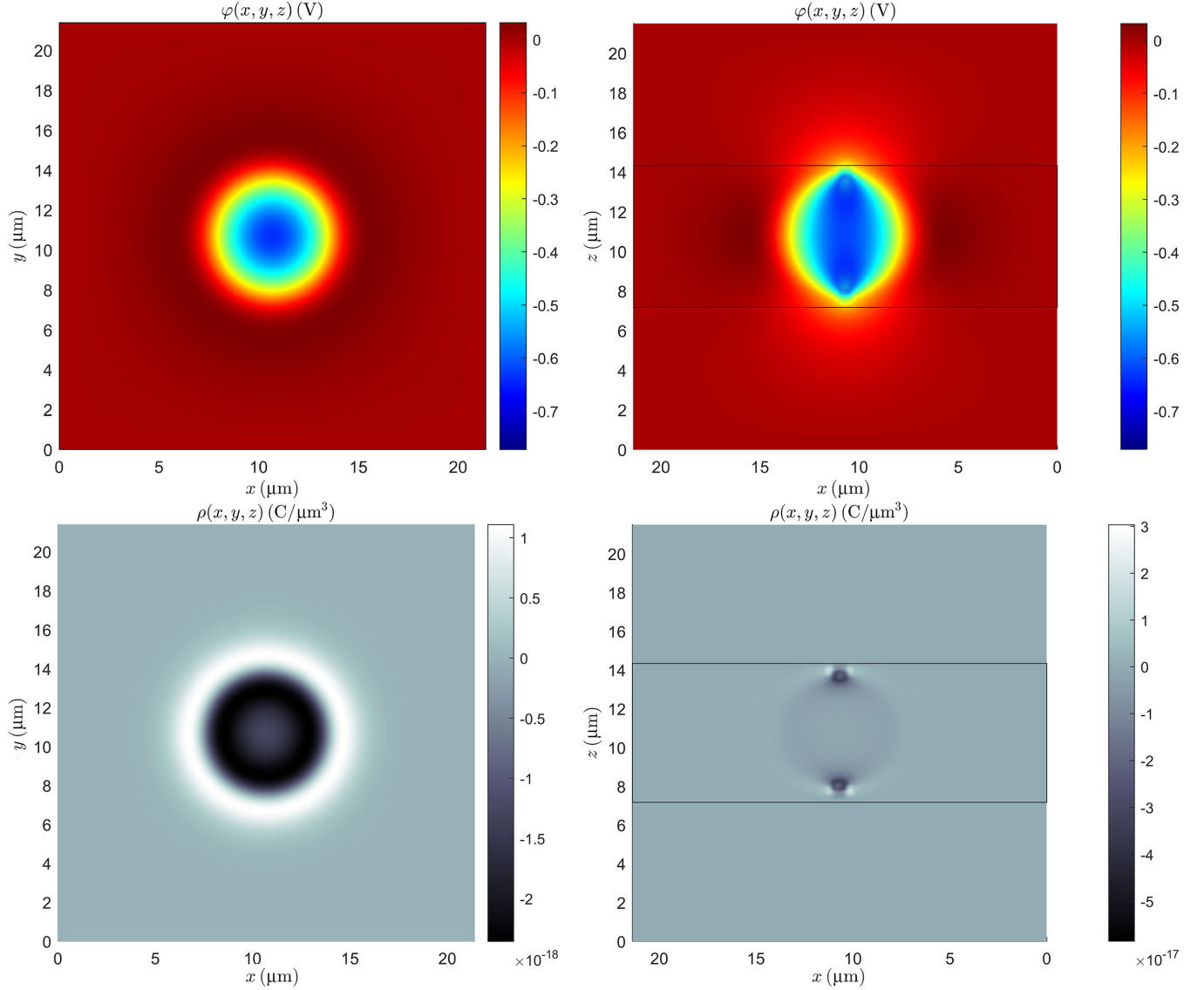


FIG. 5. Transition from a hopfion to a toron due to the flexoelectric self-polarization effect in a liquid crystal in the one-constant approximation. The top row shows the self-induced electrostatic scalar potential φ , which is a solution of the Poisson equation $\Delta\varphi = \rho/\epsilon_0$, and the bottom row displays the electric charge density $\rho = -\nabla \cdot \mathbf{P}_f$. It is clear to see that the electric charge density is confined within the confined geometry Ω , whereas the electric scalar potential extends into $\mathbb{R}^3/\Omega$. The parameter set is detailed in Fig. 2 with flexoelectric coefficients $e_1 = e_3 = 8 \text{ pC m}^{-1}$.

research project. The data are available from the authors upon reasonable request.

APPENDIX A: DEPOLARIZATION

Consider the continuous electric dipole moment distribution $\mathbf{P}_f : \Omega \rightarrow \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3$ is the confined geometry (2). The electric potential $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ associated with this continuous dipole distribution, at a point $\mathbf{x} \in \mathbb{R}^3$, is given by

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{\Omega} d^3\mathbf{y} \left\{ \frac{\mathbf{P}_f(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \right\}, \quad (\text{A1})$$

where ϵ_0 is the permittivity of free space. The corresponding induced electric field $\mathbf{E} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the gradient of this electric potential,

$$\mathbf{E}(\mathbf{x}) = -\nabla_{\mathbf{x}}\varphi(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \int_{\Omega} d^3\mathbf{y} \frac{1}{|\mathbf{x} - \mathbf{y}|^3} \left\{ \mathbf{P}_f(\mathbf{y}) - 3 \frac{\mathbf{P}_f(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} (\mathbf{x} - \mathbf{y}) \right\}. \quad (\text{A2})$$

Let us note that the Green's function for the Laplacian $\Delta = -\nabla^2$ on $\mathbb{R}^3$ is

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}, \quad \Delta_{\mathbf{x}}G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \quad (\text{A3})$$

Then, using the identity

$$\nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} \right) = -\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3}, \quad (\text{A4})$$

we can express the electric potential φ in terms of the Green's function

$$\varphi(\mathbf{x}) = -\frac{1}{\epsilon_0} \int_{\Omega} d^3\mathbf{y} \{ \mathbf{P}_f(\mathbf{y}) \cdot \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \}. \quad (\text{A5})$$

Now, noting that the gradient and Laplacian commute on $\mathbb{R}^3$, the Laplacian of the electric potential is

$$\begin{aligned} \Delta_{\mathbf{x}} \varphi(\mathbf{x}) &= -\frac{1}{\epsilon_0} \int_{\Omega} d^3\mathbf{y} \mathbf{P}_f(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \\ &= -\frac{1}{\epsilon_0} \int_{\Omega} d^3\mathbf{y} \mathbf{P}_f(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \delta(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\epsilon_0} \int_{\Omega} d^3\mathbf{y} \mathbf{P}_f(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \delta(\mathbf{x} - \mathbf{y}) \\ &= -\frac{1}{\epsilon_0} \int_{\Omega} d^3\mathbf{y} \nabla_{\mathbf{y}} \cdot \mathbf{P}_f(\mathbf{y}) \delta(\mathbf{x} - \mathbf{y}) \\ &= -\frac{1}{\epsilon_0} \nabla_{\mathbf{x}} \cdot \mathbf{P}_f(\mathbf{x}). \end{aligned} \quad (\text{A6})$$

Therefore, we see that the electric potential φ satisfies Poisson's equation for electrostatics [39,40]

$$\Delta \varphi = -\nabla^2 \varphi = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}_f. \quad (\text{A7})$$

APPENDIX B: EVADING THE HOBART-DERRICK THEOREM

1. Rescaling the flexoelectric energy

The flexoelectric energy is intrinsically nonlocal. However, we have shown how to overcome this problem by introducing a scalar electric potential, associated with the self-induced polarization. Now, we want to understand how the flexoelectric self-energy transforms under a coordinate rescaling, to see the role it plays in evading the Hobart-Derrick theorem. Recall that the energy of a chiral liquid crystal including the electrostatic self-energy is given by the flexoelectric Frank-Oseen (FFO) energy,

$$\begin{aligned} F_{\text{FFO}} &= F_{\text{FO}} + F_{\text{flexo}} \\ &= \int_{\Omega} d^3x \left\{ \frac{K_1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{K_2}{2} [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 \right. \\ &\quad + \frac{K_3}{2} (\mathbf{n} \times \nabla \times \mathbf{n})^2 + K_2 q_0 [\mathbf{n} \cdot (\nabla \times \mathbf{n})] \\ &\quad \left. + V(\mathbf{n}) + \frac{1}{2} \mathbf{P}_f \cdot \nabla \varphi \right\}, \end{aligned} \quad (\text{B1})$$

subject to the constraint

$$\Delta \varphi = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}_f, \quad \mathbf{P}_f = e_1 [(\nabla \cdot \mathbf{n}) \mathbf{n}] + e_3 (\mathbf{n} \times \nabla \times \mathbf{n}). \quad (\text{B2})$$

The numerical method developed to study the effect of dipole-dipole interactions on skyrmions in chiral magnets can be used to address this constrained minimization problem [17].

First of all, we choose to renormalize energy and length scales such that the energy is dimensionless. This will also make numerical simulations more palatable. Let us consider an energy and length rescaling with $E = E_0 \hat{E}$ and $x = L_0 \hat{x}$. We choose to set our length and energy scales as

$$L_0 = \frac{1}{q_0} \frac{K_1}{K_2}, \quad E_0 = \frac{1}{q_0} \frac{K_1^2}{K_2}. \quad (\text{B3})$$

Then, the rescaled energy is

$$\begin{aligned} \hat{F}_{\text{FFO}} &= \int_{\Omega} d^3x \left\{ \frac{1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{1}{2} \frac{K_2}{K_1} [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 \right. \\ &\quad + \frac{1}{2} \frac{K_3}{K_1} (\mathbf{n} \times \nabla \times \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})] \\ &\quad \left. + \frac{1}{q_0^2} \frac{K_1}{K_2} V(\mathbf{n}) \right\} + \hat{F}_{\text{flexo}}, \end{aligned} \quad (\text{B4})$$

where we still need to determine the dimensionless flexoelectric energy $\hat{F}_{\text{flexo}}$.

Let us consider the rescaling of the electric potential $\varphi = \lambda \hat{\varphi}$, where λ has the units $[\lambda] = \text{Nm C}^{-1}$. We find that the rescaled flexoelectric energy is

$$\begin{aligned} \hat{F}_{\text{flexo}} &= \frac{1\epsilon_0}{2E_0} \int_{\Omega} \lambda^2 \hat{\varphi} \frac{1}{L_0^2} \Delta_{\hat{x}} \hat{\varphi} L_0^3 d^3\hat{x} \\ &= \frac{1}{2} \frac{L_0 \lambda^2 \epsilon_0}{E_0} \int_{\Omega} \hat{\varphi} \Delta_{\hat{x}} \hat{\varphi} d^3\hat{x}. \end{aligned} \quad (\text{B5})$$

It will be convenient to express the polarization $\mathbf{P}_f$ in terms of a rescaled polarization $\mathbf{P}$, where

$$\mathbf{P}_f = \frac{e_1}{L_0} \mathbf{P}, \quad \mathbf{P} = (\nabla_{\hat{x}} \cdot \mathbf{n}) \mathbf{n} + \frac{e_3}{e_1} [\mathbf{n} \times (\nabla_{\hat{x}} \times \mathbf{n})]. \quad (\text{B6})$$

Then, the rescaled Poisson equation can be given in the form

$$\Delta_{\hat{x}} \hat{\varphi} = -\frac{e_1}{\epsilon_0 \lambda} \nabla_{\hat{x}} \cdot \mathbf{P}. \quad (\text{B7})$$

Furthermore, let us introduce the dimensionless vacuum electric permittivity

$$\epsilon = \frac{L_0 \lambda^2 \epsilon_0}{E_0} = \left(\frac{e_1}{\epsilon_0 \lambda} \right)^{-1}. \quad (\text{B8})$$

It can be seen that the necessary electric potential rescaling is given by $\lambda = K_1/e_1$ and, hence, the dimensionless vacuum electric permittivity is found to be

$$\epsilon = \frac{K_1 \epsilon_0}{e_1^2}. \quad (\text{B9})$$

Therefore, the rescaled flexoelectric energy is determined to be

$$\hat{F}_{\text{flexo}} = \frac{\epsilon}{2} \int_{\Omega} \hat{\varphi} \Delta_{\hat{x}} \hat{\varphi} d^3\hat{x}, \quad \Delta_{\hat{x}} \hat{\varphi} = -\frac{1}{\epsilon} \nabla_{\hat{x}} \cdot \mathbf{P}. \quad (\text{B10})$$

2. Scale invariance of the flexoelectric energy in two dimensions

We now focus on the situation at hand: translation-invariant configurations in the $\mathbf{e}_z$ direction. Recall that we are considering the finite energy of a thick slab $\Omega = \mathbb{R}^2 \times [0, 1]$. Under the energy and length rescalings [Eq. (34)], the thick slab gets

mapped to $\Omega' = \mathbb{R}^2 \times [0, t]$, where $t = L_0^{-1}$ is the rescaled slab thickness. We then consider the energy per unit length F/t of this system, which is defined by the two-dimensional adimensional energy functional

$$F_{\text{FFO}} = \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2} (\nabla \cdot \mathbf{n})^2 + \frac{1}{2} \frac{K_2}{K_1} [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 + \frac{1}{2} \frac{K_3}{K_1} (\mathbf{n} \times \nabla \times \mathbf{n})^2 + [\mathbf{n} \cdot (\nabla \times \mathbf{n})] + \frac{1}{q_0^2} \frac{K_1}{K_2} V(\mathbf{n}) + \frac{\epsilon}{2} \varphi \Delta \varphi \right\}, \quad (\text{B11})$$

where the electric potential satisfies the dimensionless Poisson equation

$$\Delta \varphi = -\frac{1}{\epsilon} \nabla \cdot \mathbf{P}, \quad \mathbf{P} = (\nabla \cdot \mathbf{n}) \mathbf{n} + \frac{e_3}{e_1} [\mathbf{n} \times (\nabla \times \mathbf{n})]. \quad (\text{B12})$$

A necessary requirement for topological solitons to exist in a field theory is the successful evasion of the Hobart-Derrick theorem [43]. The Hobart-Derrick theorem is a nonexistence theorem that states if the energy functional $F[\mathbf{n}]$ is not stationary against spatial rescaling, then $\mathbf{n}$ cannot be a solution of the field equations. So, let us consider a coordinate rescaling $\mathbf{x} \mapsto \mathbf{x}' = \mu \mathbf{x}$, for some positive scaling parameter $\mu \in \mathbb{R}_{>0}$. Then, the director field necessarily rescales as $\mathbf{n}_\mu = \mathbf{n}(\mu \mathbf{x})$. Obviously, the gradient transforms as $\nabla \mapsto \nabla' = \frac{1}{\mu} \nabla$ under the coordinate rescaling. To determine how the electric potential φ transforms under the coordinate rescaling, we first consider how the polarization $\mathbf{P}$ transforms, which is

$$\begin{aligned} \mathbf{P}_\mu &= (\mu \nabla' \cdot \mathbf{n}_\mu) \mathbf{n}_\mu + \frac{e_3}{e_1} (\mathbf{n}_\mu \times \mu \nabla' \times \mathbf{n}_\mu) \\ &= \mu (\nabla' \cdot \mathbf{n}(\mu \mathbf{x})) \mathbf{n}(\mu \mathbf{x}) + \mu \frac{e_3}{e_1} (\mathbf{n}(\mu \mathbf{x}) \times \nabla' \times \mathbf{n}(\mu \mathbf{x})) \\ &= \mu \mathbf{P}(\mu \mathbf{x}). \end{aligned} \quad (\text{B13})$$

Now, consider the Poisson equation (40), which has the scaling behavior

$$\Delta' \varphi_\mu = -\frac{1}{\mu \epsilon} \nabla' \cdot \mathbf{P}_\mu = -\frac{1}{\epsilon} \nabla' \cdot \mathbf{P}(\mu \mathbf{x}) = \Delta' \varphi(\mu \mathbf{x}). \quad (\text{B14})$$

It is clear to see that the electric scalar potential scaling behavior is $\varphi_\mu(\mathbf{x}) = \varphi(\mu \mathbf{x})$. Therefore, under the coordinate rescaling $\mathbf{x} \mapsto \mathbf{x}' = \mu \mathbf{x}$, the dimensionless energy functional becomes

$$\begin{aligned} F_{\text{FFO}}(\mu) &= \int_{\mathbb{R}^2} d^2x' \left\{ \frac{1}{2} (\nabla' \cdot \mathbf{n})^2 + \frac{1}{2} \frac{K_2}{K_1} [\mathbf{n} \cdot (\nabla' \times \mathbf{n})]^2 + \frac{1}{2} \frac{K_3}{K_1} (\mathbf{n} \times \nabla' \times \mathbf{n})^2 + \frac{1}{\mu} [\mathbf{n} \cdot (\nabla' \times \mathbf{n})] \right. \\ &\quad \left. + \frac{1}{\mu^2} \frac{K_1}{q_0^2 K_2} V(\mathbf{n}) + \frac{1}{2} \mathbf{P} \cdot \nabla' \varphi \right\} \\ &= F_2 + \frac{1}{\mu} F_1 + \frac{1}{\mu^2} F_0 + F_{\text{flexo}}. \end{aligned} \quad (\text{B15})$$

For skyrmions to exist in this model, we require the energy to be stable against spatial rescalings, which yields the Derrick

scaling constraint

$$\left. \frac{dF_{\text{FFO}}}{d\mu} \right|_{\mu=1} = -(F_1 + 2F_0) = 0. \quad (\text{B16})$$

While the potential energy F_0 is positive semi-definite, the first-order (in spatial derivatives) term F_1 can be negative, and thus provide stability. Hence, the Hobart-Derrick nonexistence theorem can be evaded.

We note that the flexoelectric self-energy is scale invariant in two dimensions and is thus unable to provide stability against spatial rescalings. However, in comparison with chiral ferromagnets, the magnetostatic self-energy there can stabilize skyrmions as it behaves like a potential under coordinate rescalings [17].

APPENDIX C: VARIATION OF THE FLEXOELECTRIC ENERGY

So far, we have shown how to include the electrostatic self-energy and compute the electric scalar potential φ by solving Poisson's equation (40) for fixed director field configuration $\mathbf{n}$. Additionally, we have determined that the flexoelectric energy is conformally invariant in two dimensions. However, we need to compute the backreaction of the self-induced electric field $\mathbf{E}$ on the director field $\mathbf{n}$. To do this, we need to calculate the first variation of the flexoelectric energy $F_{\text{flexo}}(\mathbf{n})$ with respect to the director field $\mathbf{n}$.

Two-dimensional system

Let $\mathbf{n}_t$ be a smooth variation of $\mathbf{n} = \mathbf{n}_0$ through fields of compact support and define $\delta \mathbf{n} = \partial_t \mathbf{n}_t|_{t=0}$. Denote by φ_t the associated unique solution of Eq. (40) with source $-\frac{1}{\epsilon} \nabla \cdot \mathbf{P}_t$ decaying to 0 at infinity, and $\dot{\varphi} = \partial_t \varphi_t|_{t=0}$. It is important to note that, while $\delta \mathbf{n}$ has compact support, neither $\varphi = \varphi_0$ nor $\dot{\varphi}$ do: as argued above, they are $1/r$ localized. Following Ref. [17], the variation of F_{flexo} induced by $\mathbf{n}_t$ is

$$\left. \frac{d}{dt} \right|_{t=0} F_{\text{flexo}}(\mathbf{n}_t) = \frac{\epsilon}{2} \int_{\mathbb{R}^2} d^2x (\dot{\varphi} \Delta \varphi + \varphi \Delta \dot{\varphi}). \quad (\text{C1})$$

Let us consider the term

$$\begin{aligned} \int_{\mathbb{R}^2} d^2x \dot{\varphi} \Delta \varphi &= - \int_{\mathbb{R}^2} d^2x \dot{\varphi} (\nabla \cdot \nabla \varphi) \\ &= \int_{\mathbb{R}^2} d^2x \langle \nabla \varphi, \nabla \dot{\varphi} \rangle - \int_{\mathbb{R}^2} d^2x \nabla \cdot (\dot{\varphi} \nabla \varphi) \\ &= \int_{\mathbb{R}^2} d^2x \varphi \Delta \dot{\varphi} + \int_{\mathbb{R}^2} d^2x \nabla \cdot (\varphi \nabla \dot{\varphi} - \dot{\varphi} \nabla \varphi) \\ &= \int_{\mathbb{R}^2} d^2x \varphi \Delta \dot{\varphi} \\ &\quad - \oint_{\partial B_\infty(0)} ds \cdot (\varphi \nabla \dot{\varphi} - \dot{\varphi} \nabla \varphi). \end{aligned} \quad (\text{C2})$$

So, the variation of the electrostatic self-energy is found to be given by

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} F_{\text{flexo}}(\mathbf{n}_t) &= \epsilon \int_{\Omega} d^3x \varphi \Delta \dot{\varphi} \\ &\quad - \frac{\epsilon}{2} \oint_{\partial B_\infty(0)} ds \cdot (\varphi \nabla \dot{\varphi} - \dot{\varphi} \nabla \varphi). \end{aligned} \quad (\text{C3})$$

The surface term vanishes,

$$\begin{aligned}\dot{F}_{\text{flexo}}^{\text{surf}} &= -\frac{\epsilon}{2} \oint_{\partial B_{\infty}(0)} \mathbf{ds} \cdot (\varphi \nabla \dot{\varphi} - \dot{\varphi} \nabla \varphi) \\ &= \lim_{R \rightarrow \infty} \frac{\epsilon}{2} \int_{\partial B_R(0)} (\dot{\varphi} \star \mathbf{d}\varphi - \varphi \star \mathbf{d}\dot{\varphi}) \\ &= \lim_{R \rightarrow \infty} \frac{\epsilon R}{2} \int_0^{2\pi} (\dot{\varphi} \varphi_r - \varphi \dot{\varphi}_r) d\theta \\ &= 0,\end{aligned}\quad (\text{C4})$$

since $\varphi, \dot{\varphi} = O(r^{-1})$ and $\varphi_r, \dot{\varphi}_r = O(r^{-2})$. Varying the Poisson equation (40) with respect to t , we deduce that

$$\Delta \dot{\varphi} = -\frac{1}{\epsilon} \nabla \cdot \left(\frac{\mathbf{d}}{\mathbf{d}t} \bigg|_{t=0} \mathbf{P}(\mathbf{n}_t) \right), \quad (\text{C5})$$

where the variation in the flexoelectric polarization is

$$\begin{aligned}\frac{\mathbf{d}}{\mathbf{d}t} \bigg|_{t=0} \mathbf{P}(\mathbf{n}_t) &= \frac{e_3}{e_1} [\delta \mathbf{n} \times (\nabla \times \mathbf{n}) + \mathbf{n} \times (\nabla \times \delta \mathbf{n})] \\ &\quad + (\nabla \cdot \delta \mathbf{n}) \mathbf{n} + (\nabla \cdot \mathbf{n}) \delta \mathbf{n}.\end{aligned}\quad (\text{C6})$$

Then, the variation of the flexoelectric energy is

$$\begin{aligned}\frac{\mathbf{d}}{\mathbf{d}t} \bigg|_{t=0} F_{\text{flexo}}(\mathbf{n}_t) &= - \int_{\mathbb{R}^2} d^2x \varphi \nabla \cdot \left(\frac{\mathbf{d}}{\mathbf{d}t} \bigg|_{t=0} \mathbf{P}(\mathbf{n}_t) \right) \\ &= \int_{\mathbb{R}^2} d^2x \nabla \varphi \cdot \left(\frac{\mathbf{d}}{\mathbf{d}t} \bigg|_{t=0} \mathbf{P}(\mathbf{n}_t) \right) \\ &= \int_{\mathbb{R}^2} d^2x (\text{grad}_{\mathbf{n}} F_{\text{flexo}}) \cdot \delta \mathbf{n},\end{aligned}\quad (\text{C7})$$

where the corresponding gradient is

$$\begin{aligned}\text{grad}_{\mathbf{n}} F_{\text{flexo}} &= \frac{e_3}{e_1} [((\nabla \times \mathbf{n}) \times \nabla \varphi) + (\nabla \times (\nabla \varphi \times \mathbf{n}))] \\ &\quad - \nabla (\nabla \varphi \cdot \mathbf{n}) + (\nabla \cdot \mathbf{n}) \nabla \varphi.\end{aligned}\quad (\text{C8})$$

To arrive at this, we have used the divergence theorem and the vector calculus identities

$$(\nabla \varphi \cdot \mathbf{n})(\nabla \cdot \delta \mathbf{n}) = \nabla \cdot [(\nabla \varphi \cdot \mathbf{n}) \delta \mathbf{n}] - \nabla (\nabla \varphi \cdot \mathbf{n}) \cdot \delta \mathbf{n} \quad (\text{C9})$$

and

$$\begin{aligned}(\nabla \varphi \times \mathbf{n}) \cdot (\nabla \times \delta \mathbf{n}) &= [\nabla \times (\nabla \varphi \times \mathbf{n})] \cdot \delta \mathbf{n} \\ &\quad - \nabla \cdot [(\nabla \varphi \times \mathbf{n}) \times \delta \mathbf{n}].\end{aligned}\quad (\text{C10})$$

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