

Certifying macroscopic quantum mechanics via hypothesis testing with finite data

Andreu Riera-Campenya^{1,2,*}, Patrick Maurer², and Oriol Romero-Isart^{1,3}

¹*ICFO—Institut de Ciències Fòniques, The Barcelona Institute of Science and Technology, Castelldefels, Barcelona 08860, Spain*

²*Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, 6020 Innsbruck, Austria*

³*ICREA—Institut Català de Recerca i Estudis Avançats, Barcelona 08010, Spain*



(Received 4 July 2025; accepted 27 March 2026; published 21 May 2026)

We address the challenge of certifying quantum behavior with single macroscopic massive particles, subject to decoherence and finite data. We propose a hypothesis-testing framework that distinguishes between classical and quantum mechanics based on position measurements. While interference pattern visibility in single-particle quantum superposition experiments has been commonly used as a sufficient criterion to falsify classical mechanics, we show that, from a hypothesis-testing perspective, it is neither necessary nor efficient. Focusing on recent proposals to prepare macroscopic superposition states of levitated nanoparticles, we show that the likelihood ratio test—which leverages differences across the entire probability distribution—provides an exponential reduction in measurements needed to reach a given confidence level. These results generalize to a broad class of quantum states, and offer a principled, efficient method to falsify classical mechanics in interference experiments, relaxing the experimental constraints faced by current efforts to test quantum mechanics at the macroscopic scale.

DOI: [10.1103/gttg-mvrq](https://doi.org/10.1103/gttg-mvrq)

I. INTRODUCTION

The gradual emergence of an interference pattern from successive position measurements of single massive particles in a double-slit experiment remains one of the most striking demonstrations of the quantum superposition principle [1,2]. The appearance of such an interference pattern in the position probability distribution, under conditions where classical mechanics cannot account for it, provides a sufficient criterion to certify quantum mechanics [3–8]. Pushing quantum interference experiments to increasingly macroscopic scales, both in mass and in delocalization distance, offers a way to probe the boundaries of quantum mechanics and test potential modifications of the theory [9–12]. However, this endeavor comes at a cost. Larger physical systems are more susceptible to the interaction with their environment, making the visibility of their delicate quantum interference more difficult to observe [13–15]. Reliably certifying quantum mechanics with macroscopic particles therefore requires mitigating environmental decoherence, in-run noise, and run-to-run experimental fluctuations [5]. While recent experimental efforts with levitated particles have focused on reducing noise and decoherence [16–21], the certification process itself has received less attention.

In this work, we focus precisely on this second aspect: the certification process in experiments aiming to prepare macroscopic quantum superposition states of a single massive particle. More specifically, we frame the problem as a binary

hypothesis test [22] where, given a set of N position measurements, we aim to distinguish between two hypotheses: the null hypothesis H_0 , consistent with classical mechanics, and the alternative hypothesis H_1 , which assumes quantum mechanical behavior. Framing the certification problem in this way allows us to compare different statistical tests—such as the likelihood ratio test and the visibility test—and to analyze their performance as a function of the number of measurements N and the level of decoherence.

We apply this framework to recent experimental proposals involving levitated nanoparticles [23,24], a promising platform for testing the superposition principle at unprecedented mass and delocalization scales. Furthermore, we show that our findings extend to a wide range of well-defined macroscopic quantum states. The center-of-mass motion of these systems has been experimentally cooled to the quantum ground state [25–30], and mildly delocalized beyond its zero-point fluctuations [18,31]. For the dynamical protocols in these proposals, we show that, remarkably, the likelihood ratio test—the optimal hypothesis-testing method—could exponentially reduce the number of measurements needed to certify quantum mechanics at a given significance level, outperforming conventional interference visibility tests. As we shall argue, these results can open the door to a decoherence-tolerant certification process that requires fewer measurements, thereby reducing also the impact of run-to-run variability.

Our article is organized as follows. In Sec. II, we define the theoretical model for the classical and quantum dynamics of single macroscopic massive particles in the presence of noise and decoherence. In Sec. III, we introduce the hypothesis-testing framework to certify quantum behavior with finite position measurement data. In Sec. IV, we apply this framework to the specific case of cubic phase state—a standard resource in protocols for creating macroscopic quantum superpositions with levitated nanoparticles—and demonstrate

*Contact author: andreu.riera-campenya@icfo.eu

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

the exponential advantage of the likelihood ratio test in Sec. IV B. In Sec. V A, we provide a physical interpretation of this advantage and formally demonstrate that the contrasting scaling laws are general for a broad class of semiclassical states in Sec. V B. Finally, in Sec. VI we provide concluding remarks.

II. MODEL AND OPEN-SYSTEM DYNAMICS

We consider the one-dimensional center-of-mass motion of a massive particle with mass m , described by position $\hat{x}$ and momentum $\hat{p}$ operators, which satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. At time $t = 0$, the particle is confined in a harmonic potential (e.g., an optical trap) with frequency Ω and is prepared in a thermal state with a mean phonon occupation number $\bar{n}$. We denote by $x_{zp} = \sqrt{\hbar/(2m\Omega)}$ and $p_{zp} = \sqrt{\hbar m\Omega/2}$ the zero-point fluctuations in this potential. For $t > 0$, the system is subjected to a dynamical protocol implemented via a possibly time-dependent potential $V(x, t)$ and evolves according to either classical or quantum equations of motion in the presence of noise and decoherence. For massive particles in highly isolated environments without damping, the dominant sources of decoherence are well captured by a momentum diffusion process [9], whose diffusion coefficient we denote by D . The distinction between classical and quantum dynamics is highlighted in the evolution equation for the Wigner function, which takes the form

$$\begin{aligned} \frac{\partial W_s}{\partial t}(x, p, t) = & \left\{ -\frac{p}{m} \frac{\partial}{\partial x} + \frac{\partial V}{\partial x}(x, t) \frac{\partial}{\partial p} + \frac{D}{2} \frac{\partial^2}{\partial p^2} \right. \\ & \left. + s \sum_{n \geq 1} \frac{(i\hbar/2)^{2n}}{(2n+1)!} \frac{\partial^{2n+1} V}{\partial x^{2n+1}}(x, t) \frac{\partial^{2n+1}}{\partial p^{2n+1}} \right\} \\ & \times W_s(x, p, t), \end{aligned} \quad (1)$$

where $s \in \{0, 1\}$ labels the classical or quantum hypothesis, respectively. In Eq. (1), the first line corresponds to the classical dynamics, including kinetic and potential terms that describe the Hamiltonian flow, along with the diffusive term accounting for noise and decoherence. The second line, proportional to s , corresponds to the quantum term. Note that, in the absence of decoherence, for $D = 0$, Eq. (1) reduces to the Liouville equation for $s = 0$ and to the Schrödinger equation for $s = 1$. The initial condition $W(x, p, 0) = W_i(x, p)$ is independent of the hypothesis and corresponds to the aforementioned thermal state, with variances $v_x = (2\bar{n} + 1)x_{zp}^2$ and $v_p = (2\bar{n} + 1)p_{zp}^2$. After an evolution time t , the classical $p_0(x)$ or quantum $p_1(x)$ position probability distribution can be calculated as $p_s(x) = \int_{-\infty}^{\infty} W_s(x, p, t) dp$. These probability distributions will depend on a set of parameters, labeled as $\theta = (\theta_1, \dots, \theta_K)$, which are determined, for instance, from the properties of the potential $V(x, t)$ defining the protocol.

Having defined the theoretical model for the classical and quantum dynamics, we now introduce the statistical framework used to distinguish between the two hypotheses using finite data.

III. CERTIFICATION VIA HYPOTHESIS TESTING

After the protocol terminates at time t , a measurement is performed on the system, yielding an outcome Y_1 . We model the measurement using a standard linear model, $Y_1 = X_1 + R_1$, where X_1 is the true measurement outcome (e.g., the particle position) and R_1 is a zero-mean Gaussian random variable with variance v_R that represents random measurement noise. This description naturally captures the retrodiction of measurement outcomes in continuously monitored systems [32,33]. The state of the massive particle is then reinitialized, and the protocol is repeated N times, generating a dataset $\{Y_1, \dots, Y_N\}$. The measurement outcomes are assumed to be independent and sampled from the distribution $p_s(Y)$, with $s \in \{0, 1\}$ corresponding to either classical or quantum dynamics.

A. Test statistics: Visibility and likelihood ratio

To test our hypothesis, we use a random variable Z , the so-called test statistic, which is a function of the measurement data $Z = f(Y_1, \dots, Y_N)$. In the limit of infinite data, $N \rightarrow \infty$, any difference between the predictions of quantum and classical mechanics can be used to certify quantum behavior with certainty. However, with finite data, a threshold must be selected to decide when a null hypothesis is rejected. The threshold is typically determined by fixing the significance level α , which is defined as the probability of rejecting the null hypothesis H_0 when the dataset $\{Y_1, \dots, Y_N\}$ is actually generated under the null hypothesis, that is, a false positive incorrectly suggesting that classical mechanics has been falsified. Once the significance level is set, the statistical power of the test can be evaluated. It is defined as $1 - \beta$, and corresponds to the probability of correctly rejecting the null hypothesis [34], that is, correctly falsifying classical mechanics. See Fig. 1 for further details. Hence, the test power can be used as a figure of merit to evaluate the performance of a statistical test given a desired significance level α . Let us now discuss two test statistics: the visibility test and the likelihood ratio test.

The visibility test, defined by the random variable $Z = V$, is given by the ratio

$$V = \frac{n_{\max} - n_{\min}}{n_{\max} + n_{\min}}, \quad (2)$$

where $n_{\max}$ and $n_{\min}$ are the counts in the bins corresponding to the maximum $Y_{\max}$ and minimum $Y_{\min}$ of the interference pattern associated with the quantum probability distribution. This statistic provides a measure of the local contrast near the minimum of the interference pattern. The specific procedures used to determine these values, along with further details about this definition, are detailed in Appendix C. With infinite data, the number of counts converges to the true probabilities. Therefore, in the presence of an interference pattern for the quantum distribution—absent under the classical hypothesis—one could certify quantum mechanics with the visibility test by observing $V > 0$.

The (log-) likelihood ratio test $Z = \Lambda$ of a set of independent measurements is defined as the logarithm of the ratio of the likelihood of the data under the alternative hypothesis to

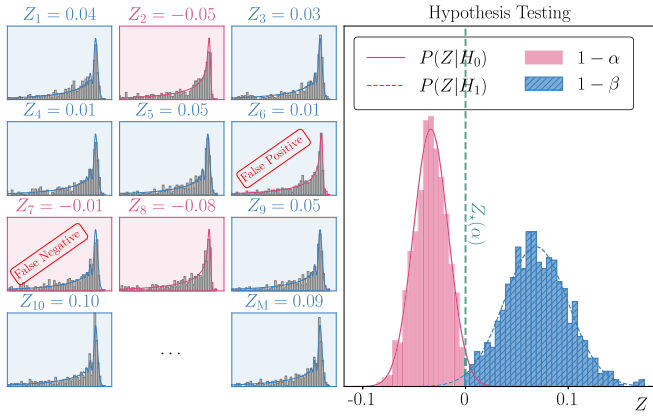


FIG. 1. (Left) Schematic representation of $M = 5000$ classical (red solid) and quantum (blue solid) datasets $\{Y_1, \dots, Y_N\}$ with $N = 500$, from which histograms (gray) are constructed. A test statistic Z_i is computed for each instance and compared against a decision threshold $Z_*(\alpha) = 0$ (for notational convenience). Instances with $Z_i < 0$ (red background) are classified as classical; that is, H_0 is accepted, while those with $Z_i > 0$ (blue background) are classified as quantum, corresponding to acceptance of H_1 . (Right) Empirical distribution of Z values superimposed with theoretical distributions $P(Z|H_0)$ and $P(Z|H_1)$ that correspond to the prediction of classical (red, solid) and quantum (blue, hatched) mechanics, respectively. The area to the left of the threshold and under the $P(Z|H_0)$ curve corresponds to the confidence level $1 - \alpha$. The area to the right of the threshold and under the $P(Z|H_1)$ curve corresponds to the test power $1 - \beta$.

the likelihood of the data under the null hypothesis, namely,

$$\Lambda = \frac{1}{N} \sum_{i=1}^N \log \left[\frac{p_1(Y_i)}{p_0(Y_i)} \right], \quad (3)$$

where the factor $1/N$ is added for normalization. Neyman-Pearson's lemma ensures that the likelihood ratio test Λ is optimal for hypothesis testing under mild assumptions [35]. Since the likelihood ratio test is a sum of independent and identically distributed random variables, the central limit theorem guarantees that it converges to a Gaussian random variable of decaying variance as the number of measurements N goes to infinity. The expectation value of the distribution is given by

$$\langle \Lambda \rangle_s = \begin{cases} D(p_1 \| p_0), & \text{for } s = 1, \\ -D(p_0 \| p_1), & \text{for } s = 0, \end{cases} \quad (4)$$

where $\langle \cdot \rangle_s$ is the average according to $p_s(Y)$, and $D(p \| q) = \int p(x) [\log(p(x)) - \log(q(x))] dx \geq 0$ is the non-negative relative entropy, with equality only if $p = q$. Therefore, given a sufficiently large number of measurements N and that $p_1(Y) \neq p_0(Y)$, quantum mechanics could be always certified using the likelihood ratio test by observing $\Lambda > 0$. We note in passing that hypothesis testing across a continuum of hypotheses has been used to quantify the degree of macroscopicity of quantum superpositions [10,36,37]. This approach, however, should be distinguished from the binary certification problem considered here.

B. Performance metric

To compare the two statistical tests, we compute the number of measurements N_* required to falsify classical mechanics at a significance level $\alpha = 2.86 \times 10^{-7}$ —corresponding to the conventional 5-sigma threshold—and with statistical power $1 - \beta \geq 0.9973$. In other words, we determine the minimal N_* such that the probability of falsely identifying classical data as quantum is less than one in a million, while correctly identifying quantum data exceeds 99.73%. This metric provides a direct and quantitative measure of a test's efficiency in distinguishing classical from quantum dynamics based on the position distribution at a specified level of statistical confidence.

IV. CASE STUDY: CUBIC PHASE STATES

To apply and analyze the proposed certification method, we focus on dynamical protocols $V(x, t)$ for creating macroscopic quantum superpositions with a single, initially cooled, levitated nanoparticle in a wide nonharmonic potential [23,24]. These protocols commonly lead to the preparation of cubic phase states [38]. While we use them as a primary example, we show in Sec. VB that the following analysis extends to the entire class of states for which quantum mechanics is imprinted as a high-frequency feature of the wavefunction under a slowly varying envelope. These states arise naturally in the Wentzel-Kramers-Brillouin (WKB) approximation and in related semiclassical constructions [39–41].

A. Key properties

Cubic phase states [42–44] are characterized by an oscillatory behavior on one side of the position distribution's maximum and a rapid decay on the opposite side (see inset of Fig. 3). As shown in Appendix A, the associated characteristic function $\chi_s(k) = \int p_s(Y) \exp(ikY) dY$, in one-to-one correspondence with the probability distribution, is given by

$$\chi_s(k) = \frac{\exp(-is\theta_3 k^3/3 - \theta_2 k^2/2)}{\sqrt{1 + 2i\theta_1 k}}, \quad (5)$$

for $s \in \{0, 1\}$. The parameters θ_1 , θ_2 , and θ_3 are real numbers subject to $\theta_2 > 0$ and $\theta_3/(\theta_2\theta_1) \in [0, 1]$, which ensures the positivity of their associated quantum state. The role of the parameters θ_1 , θ_2 , and θ_3 can be understood from their appearance in the expression of the cumulants $c_{s,j}$ of the distribution $p_s(Y)$ associated with $\chi_s(k)$. We find that $\mu_s = c_{s,1} = -\theta_1$, where μ_s is the average of the distribution, and $v_s = c_{s,2} = \theta_2 + 2\theta_1^2$, where v_s is the variance. Finally, the parameter θ_3 is associated with the skewness (i.e., the third cumulant) of the distribution as $c_{s,3} = 2s\theta_3 - 8\theta_1^3$. Interestingly, the third cumulant is the only one that contains information about the hypothesis s , and the quantum contribution has the effect of reducing the classical skewness; that is, it makes the distribution more symmetric. Higher-order cumulants depend only on the parameter θ_1 as $c_{s,j} = (j-1)!(-2\theta_1)^j/2$, with $j \geq 4$. Finally, we note that the purity of the associated quantum state can be expressed as $\sqrt{\theta_3/(\theta_2\theta_1)}$.

Cubic phase states display different quantum and classical measurement statistics. This distinction is highlighted by the fact that their associated distributions are related through the

Airy transform [45]

$$p_1(Y) = \frac{1}{|\sqrt[3]{\theta_3}|} \int_{-\infty}^{\infty} \text{Ai}\left(\frac{Y - Y'}{\sqrt[3]{\theta_3}}\right) p_0(Y') dY', \quad (6)$$

where $\text{Ai}(x)$ denotes the Airy function. Hence, the quantum and classical distributions remain distinct for any finite θ_3 , coinciding only in the limit $\theta_3 \rightarrow 0$. We are interested in how this difference depends on noise and decoherence. To quantify their impact on position measurements, we express the observed distributions $p_s(Y)$ —both classical and quantum—as convolutions of the ideal noiseless distributions $\tilde{p}_s(X)$, corresponding to $\bar{n} = 0$, $D = 0$, and $v_R = 0$, with a Gaussian kernel. This yields

$$p_s(Y) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} \exp\left[-\frac{(Y - X)^2}{2\sigma^2}\right] \tilde{p}_s(X) dX, \quad (7)$$

where $\sigma^2 = (\theta_2 - \theta_3/\theta_1) + v_R$ quantifies the effect of noise and decoherence on the measured data.

Having established the theoretical description of cubic phase states, we proceed to a performance analysis to quantify the efficiency of the certification frameworks.

B. Scaling analysis of N_*

Focusing on the protocols that generate the cubic phase states discussed above, we now discuss the results of our certification method. In Fig. 2(a), we show the convergence of the likelihood ratio test statistic Λ toward the relative entropy as a function of the number of measurements N , for cubic phase states and using the parameter values specified in Table I, corresponding to Ref. [24]. The rapid convergence with N ensures that the likelihood ratio test can be used to discriminate between the classical and quantum scenarios with a 5-sigma significance level already for $N_* \approx 1500$ measurements. We note that such a N_* will always exist for the likelihood ratio test, since the variance of the likelihood ratio test decays as $1/N$ for sufficiently large N . Indeed, according to Eq. (6), the two probability distributions $p_0(Y)$ and $p_1(Y)$ are different and thus always distinguishable with a sufficiently large dataset.

While the likelihood ratio test can always distinguish between the classical and quantum hypotheses given sufficient data, this task becomes increasingly challenging as the decoherence increases. In Fig. 2(b), we compare the metric N_* for both the likelihood ratio test and the visibility test as a function of σ^2 . As shown, the likelihood ratio test consistently outperforms the visibility test across all plotted decoherence values. The difference between the two tests is highlighted by the scaling behavior of N_* with σ^2 . While the likelihood ratio test has a polynomial scaling with decoherence, the visibility test exhibits exponential growth. Thus, using the likelihood ratio test as a test statistic leads to an exponentially more efficient testing. In turn, this improved efficiency enables certification under less stringent experimental conditions. For example, for the proposal of Ref. [24], the visibility test requires pulse energy precision $p_{\text{err}} \lesssim 10^{-5}$ for $N_* \approx 10^4$, whereas the likelihood ratio test relaxes this limiting constraint by over an order of magnitude. This raises the question of the physical origin of these results and whether they are an artifact of the specific state under consideration. We address this in Sec. V, where

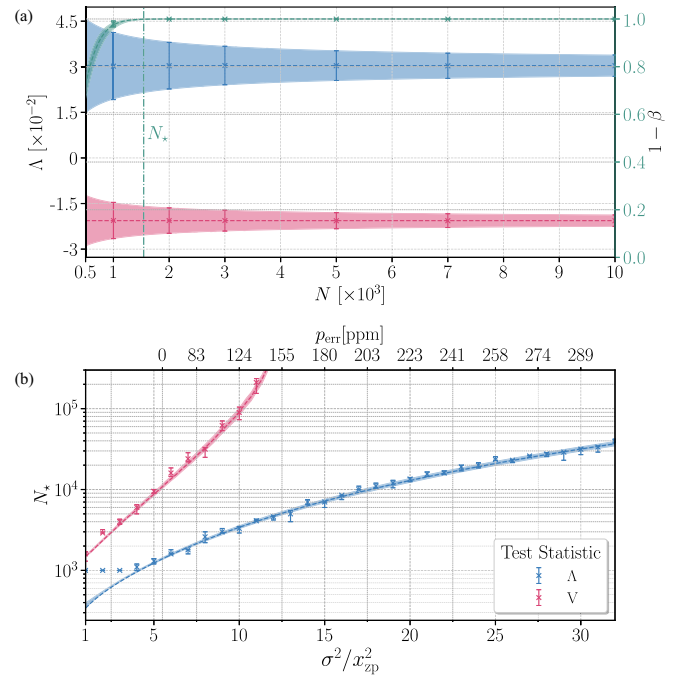


FIG. 2. (a) Mean (symbols) and standard deviation (error bars) of the distribution of Λ simulated over $M = 5000$ runs each for the classical (red) and quantum (blue) hypotheses as a function of the number of measurements N using the parameter values specified in Table I. Dashed lines and shaded regions represent the mean value and standard deviation obtained by integration in the large N limit. Green symbols show the (empirical) statistical power $1 - \beta$ for a significance level $\alpha = 2.86 \times 10^{-7}$, whereas the dashed green line shows the statistical power obtained by integration in the asymptotic limit. Green error bars and shaded regions correspond to a $\pm 5\%$ uncertainty in σ^2 and θ_3 . The vertical dash-dotted line shows N_* , i.e., the number of measurements N for which the power is above 99.73% for all parameters in the $\pm 5\%$ error window. (b) Number of measurements N_* needed to certify quantum mechanics as a function of σ^2 (lower x axis) and the corresponding relative precision error p_{err} in the pulse energy (upper x axis) as defined in Ref. [24]. The symbols and error bars correspond to a statistical power $1 - \beta > 99.73\%$ at a significance level $\alpha = 2.86 \times 10^{-7}$ for the visibility (red) and likelihood ratio (blue) test statistics obtained by $M = 2000$ runs with θ_3 within a $\pm 5\%$ error window of the value specified in Table I. The corresponding dashed lines and shaded regions show the values obtained by integration in the large N limit. Details about numerical methods can be found in Appendix E.

we identify the mechanism as a low-pass filtering effect of decoherence and prove its universality for a broad class of well-defined states.

V. DISCUSSION

A. Decoherence as a low-pass filter

To better understand how decoherence impacts the distinguishability between quantum and classical cubic phase states, in Fig. 3 we examine three indicators: interference visibility, Wigner function negativity, and Jeffreys divergence. The latter, defined as $J(p_1 \| p_0) \equiv D(p_1 \| p_0) + D(p_0 \| p_1)$, quantifies the statistical distance between the quantum and

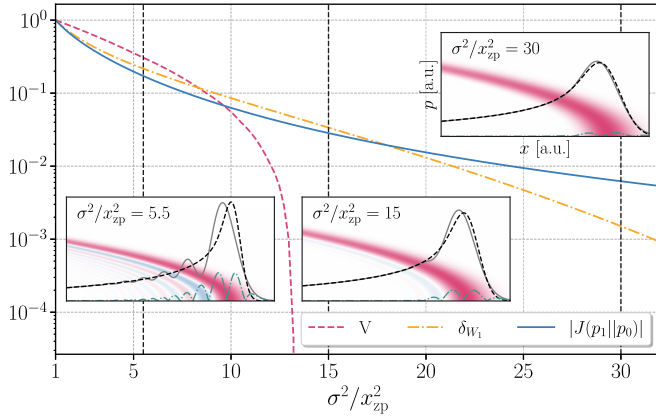


FIG. 3. Visibility V (red, dashed), Wigner function negativity $\delta_{W_1} = \int |W_1(x, p)| dx dp - 1$ (orange, dash-dotted), and Jeffreys divergence $J(p_1 || p_0)$ (blue, solid) as a function of σ^2 for θ_3 as specified in Table I. All functions are normalized to their value at $\sigma^2 = x_{zp}^2$. The insets show the quantum and classical probability distribution $p_1(x)$ (gray, solid), $p_0(x)$ (black, dashed), their absolute difference $|p_1(x) - p_0(x)|$ (green, dash-dotted), and the Wigner function $W_1(x, p)$ (red and blue shaded regions represent positive and negative values of the Wigner function, respectively) as a function of scaled position and momentum at $\sigma^2/x_{zp}^2 = \{5.5, 15, 30\}$ indicated by the three vertical dashed black lines and inside of each inset. Due to the scale invariance of the system (see Appendix B), axis tick values in the inset are omitted.

classical position distributions. For clarity, each measure is normalized to its value at $\sigma^2 = x_{zp}^2$. At low decoherence values (e.g., $\sigma^2 \approx 5.5x_{zp}^2$), all three measures successfully distinguish between the classical and quantum hypotheses. This can be seen in the leftmost inset, which shows interference visibility, Wigner negativity, and a clear difference between the quantum and classical distributions. As decoherence increases, visibility drops to zero around $\sigma^2 \approx 13x_{zp}^2$. In contrast, Wigner function negativity persists to significantly higher values of σ^2 , highlighting the presence of noncontextual quantum features [46]. Ultimately, Jeffreys divergence emerges as the most robust indicator, remaining nonzero up to $\sigma^2 \gtrsim 30x_{zp}^2$. Its resilience to decoherence arises because, as seen in Eq.(7), decoherence acts as a low-pass filter, exponentially suppressing the high-frequency interference fringes that the visibility test relies on. In contrast, the likelihood ratio test leverages information from the entire distribution. Thus, distinguishing quantum from classical dynamics in the presence of decoherence is analogous to inferring the shape of an object concealed beneath a cloth. As decoherence increases, the cloth thickens, making the object harder to discern, but the underlying features remain and can therefore be tested.

B. Polynomial versus exponential scaling of N_*

Since the exponential advantage of the likelihood ratio test is based on the low-pass filtering of high-frequency features, the contrasting behavior observed in Fig. 2 generalizes beyond cubic phase states under two conditions: (1) The probability distribution exhibits a separation of length scales, consisting of a slowly varying envelope and rapidly oscillating interfer-

ence fringes; and (2) noise and decoherence can be modeled as a Gaussian convolution with a kernel of width σ^2 [c.f. Eq. (7)]. We reiterate that states fulfilling condition (1) naturally arise in the WKB approximation and related semiclassical constructions [39–41], whereas condition (2) provides an accurate approximation of decoherence effects for systems evolving in wide nonharmonic potentials [38].

The number of measurements N_* required to falsify classical mechanics at a significance level $\alpha = \Phi(-a)$ and statistical power $1 - \beta = \Phi(b)$ is determined by the distance between the mean values of the test statistic relative to their variances (see Appendix E for details), that is,

$$N_* = \left[\frac{b\sqrt{v_1(Z)} + a\sqrt{v_0(Z)}}{\langle Z \rangle_1 - \langle Z \rangle_0} \right]^2. \quad (8)$$

Here, $\langle Z \rangle_s$ and $v_s(Z) = N \times \text{Var}_s(Z)$ denote the expectation value and the N -independent part of the variance in the large N limit under hypothesis $s \in \{0, 1\}$, respectively, and $\Phi(x)$ is the cumulative distribution function of the normal distribution.

Evaluating Eq. (8) for both test statistics Z and the class of states defined by conditions (1) and (2) reveals the contrasting *exponential* versus *polynomial* scaling. Here, we sketch the main steps of the derivation in Appendix F.

For the likelihood ratio test ($Z = \Lambda$), we assume the probability distribution $p_s(x)$ admits a *Gram-Charlier A* series, valid for distributions that decay at least exponentially in x^2 as $|x| \rightarrow \infty$ [47]. This series expresses the distribution in terms of Hermite polynomials $H_n(x)$ with coefficients $c_s(n)$ defined in terms of Bell polynomials of the hypothesis-dependent cumulants. In the limit of large σ^2 , one arrives at $N_* \approx (a + b)^2 \sigma^2 / (\mu_1 - \mu_0)^2$, when the hypotheses differ in the mean values μ_s , while $N_* \approx 2(a + b)^2 \sigma^4 / (\sigma_1^2 - \sigma_0^2)^2$ for equal mean values but different variances σ_s . When the first two cumulants are hypothesis independent, a case often encountered in the systems of interest [38], orthogonality of the Hermite polynomials in the expansion leads to

$$N_* \approx (a + b)^2 \left[\sum_{n \geq 3} \frac{[c_1(n) - c_0(n)]^2}{n! \sigma^{2n}} \right]^{-1}, \quad (9)$$

giving polynomial scaling of leading order $(\sigma^2)^d$ and $d = \min\{n \geq 3 | c_1(n) - c_0(n) \neq 0\}$. Conversely, for the visibility test ($Z = V$), condition (1) ensures that the interference fringes oscillate rapidly, with a length scale $1/k_0$, as compared to the width of the Gaussian noise kernel. Therefore, the convolution in condition (2) suppresses the fringe contrast exponentially by a factor $\exp(-k_0^2 \sigma^2)$. Given that the visibility is defined in terms of bin counts that follow a multinomial distribution on bins of size Δx , one can show that

$$N_* \approx \frac{\exp(k_0^2 \sigma^2)}{2v^2} \left[\frac{b}{\sqrt{A_1(x_0)\Delta x}} + \frac{a}{\sqrt{A_0(x_0)\Delta x}} \right]^2. \quad (10)$$

Here, v is the fringe visibility, and $A_s(x_0)$ is the value of the hypothesis-dependent envelope of $p_s(x)$ evaluated in the vicinity of the fringe position $x \approx x_0$. The comparison between the polynomial scaling of Eq. (9) and the exponential scaling of Eq. (10) formally demonstrates that the likelihood ratio test yields an exponential reduction in N_* relative to the

visibility test, for all dynamical protocols satisfying conditions (1) and (2).

VI. CONCLUSION

In summary, we have introduced a likelihood ratio-based hypothesis-testing framework that uses position measurements to certify quantum mechanics with finite data. Our analysis shows that this method can outperform the standard visibility test under realistic conditions relevant to levitated nanoparticle experiments. Specifically, for equivalent statistical significance and power, it distinguishes classical from quantum dynamics using exponentially fewer measurements and, moreover, can succeed even when decoherence fully suppresses quantum interference visibility. That is, we have shown that while finite visibility is sufficient and aesthetically pleasing for certifying quantum mechanics in interference experiments, it is, from a hypothesis-testing perspective, not necessary. These results pave the way for certifying quantum superpositions of macroscopic particles under less restrictive experimental conditions.

ACKNOWLEDGMENTS

We thank the Q-Xtreme synergy group for fruitful discussions. This research has been supported by the European Research Council (ERC) under the Grant Agreement No. 951234 (Q-Xtreme ERC-2020-SyG). A.R.-C. acknowledges funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 101103589 (DecoXtreme, HORIZON-MSCA-2022-PF-01-01).

A.R.-C. and P.M. developed the statistical certification framework and derived the analytical results. P.M. performed the numerical analysis. O.R.-I. originated the idea and supervised the project. All authors discussed the results and contributed to the writing and editing of the manuscript.

DATA AVAILABILITY

The data that support the findings of this article are openly available [48]; embargo periods may apply.

APPENDIX A: CHARACTERISTIC FUNCTION OF CUBIC PHASE STATES

In this Appendix, we derive the analytic expression for the characteristic function of the cubic phase states.

To obtain the characteristic function of a (position) cubic phase state, we evolve the initial Wigner function $W_i(x, p)$ —that of an uncorrelated Gaussian state with variances v_x and v_p —from just before $t_i = t_p^-$ to just after $t_f = t_p^+$ a cubic pulse. More precisely, the evolution is driven by a time-dependent potential of the form $V_{\text{cubic}}(x, t) = -\hbar(\gamma/2)(x/x_{zp})^3\delta(t - t_p)$, where γ is a dimensionless parameter that sets the strength of the pulse. The dynamics follow Eq. (1) with $s = 0$ or $s = 1$. Note that for the classical case, the $\hbar$ in $V_{\text{cubic}}(x, t)$ is only a conventional choice, and does not imply that the dynamical laws are classical or quantum. The equation can be solved analytically for both the classical and the quantum hypotheses.

One obtains the characteristic function

$$\begin{aligned} \chi_s(k_x, k_p) = \exp \left\{ is \frac{\gamma}{3} (p_{zp} k_p)^3 - \frac{(v_p/p_{zp}^2)}{2} (p_{zp} k_p)^2 \right. \\ \left. - (v_x/x_{zp}^2) \frac{(x_{zp} k_x)^2}{2} \frac{1}{1 - 2i\gamma(v_x/x_{zp}^2)p_{zp} k_p} \right\} \\ \times 1/\sqrt{1 - 2i\gamma(v_x/x_{zp}^2)p_{zp} k_p}, \end{aligned} \quad (\text{A1})$$

corresponding to the Fourier transform of $W_s(x, p, t_f)$. Setting $k_x = 0$ and $k_p \mapsto k$, one obtains the characteristic function in the form displayed in Eq. (5). Note that under Gaussian dynamics, the shape of the state will be preserved while the position and momentum variables can be mixed by symplectic transformations [49].

APPENDIX B: SCALE INVARIANCE

In this Appendix, we discuss the scale invariance properties of the position probability distributions for cubic phase states.

The probability distribution associated with Eq. (5) exhibits the scale invariance $p_s(X; \theta_1, \theta_2, \theta_3) = |\lambda| p_s(\lambda X; \lambda \theta_1, \lambda^2 \theta_2, \lambda^3 \theta_3)$, for $\lambda \neq 0$. Since the random measurement noise v_R is unaffected by this transformation, only the parameters θ_k scale, the relative impact of measurement noise can be reduced by increasing λ . Specifically, the effective variance of the Gaussian convolution above is transformed as $\sigma^2 \mapsto \lambda^2(\theta_2 - \theta_3/\theta_1) + v_R$, showing that the contribution of v_R becomes negligible for large λ . Physically, such scaling corresponds to a squeezing transformation, which can be implemented by evolving the particle under an inverted harmonic potential [24,50]. This property is used to produce the Figures displayed in the main text.

APPENDIX C: DEFINITION OF THE VISIBILITY TEST STATISTIC

In this Appendix, we provide a rigorous definition of the interference visibility statistic used in our analysis, including the procedure for determining the integration bins corresponding to the maximum and minimum of the interference pattern.

In Eq. (2), we have introduced the definition of the visibility V as a function of the counts n_{max} and n_{min} . To be precise, we define it as follows. Let Y_{max} and Y_{min} be the values of the second maximum and first minimum of the quantum position distribution $p_1(Y)$, which are at a distance $\Delta \equiv |Y_{\text{max}} - Y_{\text{min}}|$, and which we assume exist. Then, we define two intervals $I_{\text{max}} = [Y_{\text{max}} - \Delta/2, Y_{\text{max}} + \Delta/2]$ and $I_{\text{min}} = (Y_{\text{min}} - \Delta/2, Y_{\text{min}} + \Delta/2]$, centered around Y_{max} and Y_{min} . Given the dataset $\{Y_1, \dots, Y_N\}$, we define $n_{\text{max}} = \sum_{i=1}^N \mathbf{1}(Y_i \in I_{\text{max}})$ and $n_{\text{min}} = \sum_{i=1}^N \mathbf{1}(Y_i \in I_{\text{min}})$ as the number of counts on the intervals I_{max} and I_{min} , respectively. Here, $\mathbf{1}(c)$ is the indicator function that gives one if the condition c is true and zero otherwise. The number of counts in the I_{max} and I_{min} bins are part of a multinomial distribution, whose statistical properties—for instance, the mean values, variances, and covariances—can be evaluated. To conclude, we note that in the definition of Eq. (2), we omit the conventional absolute value, since the classical distribution could have a negative visibility. In such

TABLE I. Relation between the physical parameters discussed in the proposal of Ref. [24] and the parameters of the characteristic function in Eq. (5). Numerical values are obtained by inserting the values in Table I of Ref. [24], including the assumption of negligible measurement noise $v_R = 0$.

Parameter	Physical parameters	Value
$\theta_1/(\lambda x_{zp})$	$(2\bar{n} + 1)(kx_{zp})(\Omega t_1)^3$	69.04
$\theta_2/(\lambda x_{zp})^2$	$(2\bar{n} + 1 + \bar{a})$	6.001
$\theta_3/(\lambda x_{zp})^3$	$(kx_{zp})(\Omega t_1)^3$	34.52
λ	$-\cosh(\Omega_4 t_4)(t_3/t_1) - \sinh(\Omega_4 t_4)/(\Omega_4 t_1)$	-59.67

cases, including the absolute value would artificially degrade the performance of the test. Also, we remark that for large decoherence it is not guaranteed that $Y_{\max}$ and $Y_{\min}$ exist. In that case, it is customary to set the visibility to zero. This makes the visibility of the quantum distribution decay continuously to zero.

APPENDIX D: PHYSICAL PARAMETERS OF THE PROPOSAL

In this Appendix, we show how the physical parameters discussed in the proposal of Ref. [24] relate to the parameters of the characteristic function in Eq. (5).

The proposal consists of a five-step protocol implemented with a single charged dielectric silica nanoparticle of mass m . Each step $0 \leq i \leq 4$ is fully characterized by its duration t_i , decoherence rate Γ_i , and potential $V_i(x) = -m\Omega_i^2/(2k^2)[\cos^2(kx - \phi_i) - kx \sin(2\phi_i)]$, with frequency Ω_i and phase ϕ_i . The potential can be realized by optical standing wave pulses with wave number k and a linear electric potential. At step 0, the particle is initialized by cooling it to a thermal state of mean phonon occupation $\bar{n}$ in a harmonic potential with frequency $\Omega_0 = \Omega$. After that, the protocol follows a sequence of free evolutions and evolutions in cubic and inverted potentials with fine-tuned parameters to generate a cubic phase state as in Eq. (5). Following the derivation of Ref. [24], it can then be shown that the relation between the physical parameters and the parameters of the characteristic function are given by Table I, where

$$\frac{\bar{a}}{\Omega^2} = 4\Gamma_2 t_1^2 + \frac{4\Gamma_1 t_1^3}{3} + \frac{\exp(2\Omega_4 t_4)(4\Gamma_3 t_3^3/3 + 2\Gamma_4/\Omega_4^3)}{4\lambda^2}.$$

APPENDIX E: DETAILS ON NUMERICAL EVALUATION

In this Appendix, we show how we determine the statistical power $1 - \beta$ and number of measurements N_* required to falsify classical mechanics for M runs per hypothesis.

First, we evaluate the mean value μ and standard deviation $\sqrt{\text{Var}_0(Z)}$ of the test statistic Z for the null hypothesis. The threshold is then set to $Z_* = \mu + 5\sqrt{\text{Var}_0(Z)}$, which corresponds to a significance level $\alpha \approx 2.86 \times 10^{-7}$. This allows us to evaluate $M_> \equiv \sum_{i=1}^M \mathbf{1}(Z_i > Z_*)$, which is the total number of runs with a test statistic Z_i that lies above the threshold.

Using $M_>$, we evaluate the Wilson score interval, which guarantees that the statistical power $1 - \beta$ is with a probability

$1 - \varepsilon$ within the interval $[w_-(\varepsilon), w_+(\varepsilon)]$, where

$$w_{\pm}(\varepsilon) = \frac{M_> + z_{\varepsilon}^2/2}{M + z_{\varepsilon}^2} \pm \frac{z_{\varepsilon}/2}{M + z_{\varepsilon}^2} \sqrt{\frac{4(M - M_>)M_>}{M} + z_{\varepsilon}^2}, \quad (\text{E1})$$

with $z_{\varepsilon} = \Phi^{-1}(1 - \varepsilon/2)$, where $\Phi(x) = \int_{-\infty}^x \exp(-s^2/2)/\sqrt{2\pi} ds$ denotes the cumulative distribution function of the normal distribution. For a conservative estimate, we evaluate the power at the lower interval and $\varepsilon = 0.05$, i.e., $1 - \beta = w_-(0.05)$. Note that $w_-(\varepsilon) \leq 1/(1 + z_{\varepsilon}^2/M)$, which requires a minimum number of measurements to ensure $1 - \beta \geq 99.73\%$, which is why in Fig. 2(b) the empirical N_* does not drop below 1000.

In order to determine N_* , we consider a $\pm 5\%$ error margin for the parameters. Specifically, we fix the parameters θ_i used to evaluate the likelihood ratio but sample the data from a probability distribution with parameters within the interval $[0.95\theta_i, 1.05\theta_i]$. This accounts for potential uncertainties and ensures robustness in the estimation of N_* .

Finally, the shaded regions can be obtained by first evaluating the mean value $\langle Z \rangle_s$ and variance $\text{Var}_s(Z)$ under hypothesis $s \in \{0, 1\}$ in the large N limit. We reject the null hypothesis when $Z_i > \langle Z \rangle_0 + a\sqrt{\text{Var}_0(Z)}$, which for Gaussian distributed test statistics corresponds to a significance level $\alpha = \Phi(-a)$. The statistical power reads

$$1 - \beta = 1 - \Phi\left(\frac{\langle Z \rangle_0 + a\sqrt{\text{Var}_0(Z)} - \langle Z \rangle_1}{\sqrt{\text{Var}_1(Z)}}\right), \quad (\text{E2})$$

and exceeds a value $1 - \beta \geq 1 - \Phi(-b) = \Phi(b)$ if $\langle Z \rangle_1 - \langle Z \rangle_0 \geq a\sqrt{\text{Var}_0(Z)} + b\sqrt{\text{Var}_1(Z)}$. Using $\text{Var}_s(Z) = v_s(Z)/N$ allows us to derive the number of measurements N_* required to falsify classical mechanics in the large N limit, namely,

$$N_* = \left[\frac{b\sqrt{v_1(Z)} + a\sqrt{v_0(Z)}}{\langle Z \rangle_1 - \langle Z \rangle_0} \right]^2. \quad (\text{E3})$$

APPENDIX F: DERIVATION OF SCALING LAWS

Here, we demonstrate that the contrasting *exponential* versus *polynomial* scaling of the visibility and likelihood ratio test with noise generalizes beyond cubic phase states under the following conditions: (1) The probability distribution exhibits a separation of length scales, consisting of a slowly varying envelope and rapidly oscillating interference fringes; and (2) noise and decoherence can be modeled as a Gaussian convolution.

To be precise, condition (1) implies that we can write down the probability distribution of the two hypotheses as

$$p_s(x) = A_s(x)[1 + sv \cos(k_0 x)], \quad (\text{F1})$$

where $A_s(x)$ is the slowly varying envelope and s distinguishes the quantum ($s = 1$) from the classical ($s = 0$) hypothesis. The quantum distribution contains interference fringes with visibility v and spatial frequency k_0 . On the other hand, condition (2) implies that after noise and decoherence the probability distribution in position is given by the Gaussian

convolution

$$(p_s * g_\sigma)(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-x')^2}{2\sigma^2}\right] p_s(x') dx', \quad (\text{F2})$$

where σ quantifies the noise strength.

Our goal is to quantify how the number of measurements N_* required to certify quantum mechanics scales with the decoherence strength σ , for both the visibility $Z = V$ and the likelihood ratio $Z = \Lambda$. To this end, we use Eq. (E3) and compute the expectation $\langle Z \rangle_s$ and standard deviation $v_s(Z)$ of the test statistic Z under the two hypotheses $s = 0$ and $s = 1$.

We start with the likelihood ratio $Z = \Lambda$. First, we assume that the distribution $p_s(x)$ admits a *Gram-Charlier A* series, valid for distributions that decay at least exponentially in x^2 as $|x| \rightarrow \infty$ [47], namely,

$$p_s(x) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(x-\mu_s)^2}{2\sigma_s^2}\right] \sum_{n \geq 0} \frac{c_s(n)}{n! \sigma_s^n} H_n\left(\frac{x-\mu_s}{\sigma_s}\right), \quad (\text{F3})$$

where $c_s(n) \equiv B_n(0, 0, \kappa_s(3), \dots, \kappa_s(n))$ denote the complete Bell polynomials, $\kappa_s(n)$ is the n th cumulant of the distribution $p_s(x)$, and $H_n(x)$ denote the Hermite polynomials. Due to condition (2), noise modifies the variance as $\sigma_s^2 \rightarrow (\sigma_s^2 + \sigma^2)$. To obtain the scaling, we evaluate $\langle Z \rangle_s$ and $v_s(Z)$ by inserting the Gram-Charlier A series, expanding $\log(1+x) \simeq x$ and approximating the denominator of the resulting

expression by a Gaussian, valid for large enough σ . When the hypotheses differ in the mean values, $N_* \simeq (a+b)^2 \sigma^2 / (\mu_1 - \mu_0)^2$, while for equal mean values but different variances $N_* \simeq 2(a+b)^2 \sigma^4 / (\sigma_1^2 - \sigma_0^2)^2$. When the first two cumulants are identical, orthogonality of Hermite polynomials leads to

$$N_* \simeq (a+b)^2 \left[\sum_{n \geq 3} \frac{[c_1(n) - c_0(n)]^2}{n! \sigma^{2n}} \right]^{-1}, \quad (\text{F4})$$

giving polynomial scaling with $(\sigma^2)^d$ and $d = \min\{n \geq 3 \mid c_1(n) - c_0(n) \neq 0\}$.

For the visibility test, we return to Eq. (F1). Condition (1) implies that $A_s(x)$ is slowly varying on the scale of $1/k_0$ and σ , and therefore $(p_s * g_\sigma)(x) \simeq A_s(x)[1 + sv \exp(-k_0^2 \sigma^2 / 2) \cos(k_0 x)]$, showing exponential suppression of fringe contrast v . The visibility is defined in terms of bin counts that follow a multinomial distribution on bins of size Δx . Applying the delta method [51], we evaluate $\langle Z \rangle_s$ and $v_s(Z)$ near the fringe position $x \simeq x_0$, obtaining

$$N_* \simeq \frac{\exp(k_0^2 \sigma^2)}{2v^2} \left[\frac{b}{\sqrt{A_1(x_0) \Delta x}} + \frac{a}{\sqrt{A_0(x_0) \Delta x}} \right]^2. \quad (\text{F5})$$

Finally, comparing Eqs. (F4) and (F5) demonstrates that the likelihood ratio test yields an exponential reduction in N_* relative to the visibility test, for all dynamical protocols satisfying conditions (1) and (2).

- [1] A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa, Demonstration of single-electron buildup of an interference pattern, *Am. J. Phys.* **57**, 117 (1989).
- [2] T. Juffmann, A. Milic, M. Müllneritsch, P. Asenbaum, A. Tsukernik, J. Tüxen, M. Mayor, O. Cheshnovsky, and M. Arndt, Real-time single-molecule imaging of quantum interference, *Nat. Nanotechnol.* **7**, 297 (2012).
- [3] M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. van der Zouw, and A. Zeilinger, Wave-particle duality of C_{60} molecules, *Nature (London)* **401**, 680 (1999).
- [4] B. Brezger, L. Hackermüller, S. Uttenthaler, J. Petschinka, M. Arndt, and A. Zeilinger, Matter-wave interferometer for large molecules, *Phys. Rev. Lett.* **88**, 100404 (2002).
- [5] K. Hornberger, S. Gerlich, P. Haslinger, S. Nimmrichter, and M. Arndt, Colloquium: Quantum interference of clusters and molecules, *Rev. Mod. Phys.* **84**, 157 (2012).
- [6] S. Eibenberger, S. Gerlich, M. Arndt, M. Mayor, and J. Tüxen, Matter-wave interference of particles selected from a molecular library with masses exceeding 10 000 amu, *Phys. Chem. Chem. Phys.* **15**, 14696 (2013).
- [7] T. Kovachy, P. Asenbaum, C. Overstreet, C. A. Donnelly, S. M. Dickerson, A. Sugarbaker, J. M. Hogan, and M. A. Kasevich, Quantum superposition at the half-metre scale, *Nature (London)* **528**, 530 (2015).
- [8] Y. Y. Fein, P. Geyer, P. Zwick, F. Kiałka, S. Pedalino, M. Mayor, S. Gerlich, and M. Arndt, Quantum superposition of molecules beyond 25 kDa, *Nat. Phys.* **15**, 1242 (2019).
- [9] O. Romero-Isart, Quantum superposition of massive objects and collapse models, *Phys. Rev. A* **84**, 052121 (2011).
- [10] S. Nimmrichter and K. Hornberger, Macroscopicity of mechanical quantum superposition states, *Phys. Rev. Lett.* **110**, 160403 (2013).
- [11] A. Bassi, K. Lochan, S. Satin, T. P. Singh, and H. Ulbricht, Models of wave-function collapse, underlying theories, and experimental tests, *Rev. Mod. Phys.* **85**, 471 (2013).
- [12] M. Arndt and K. Hornberger, Testing the limits of quantum mechanical superpositions, *Nat. Phys.* **10**, 271 (2014).
- [13] E. Joos and H. D. Zeh, The emergence of classical properties through interaction with the environment, *Z. Phys. B* **59**, 223 (1985).
- [14] W. H. Zurek, Decoherence and the transition from quantum to classical—revisited, in *Quantum Decoherence: Poincaré Seminar 2005*, edited by B. Duplantier, J.-M. Raimond, and V. Rivasseau (Birkhäuser, Basel, 2007), pp. 1–31.
- [15] M. Schlosshauer, *Decoherence and the Quantum-To-Classical Transition* (Springer, Berlin, 2007).
- [16] L. Dania, D. S. Bykov, F. Goschin, M. Teller, A. Kassid, and T. E. Northup, Ultrahigh quality factor of a levitated nanomechanical oscillator, *Phys. Rev. Lett.* **132**, 133602 (2024).
- [17] E. Bonvin, L. Devaud, M. Rossi, A. Militaru, L. Dania, D. S. Bykov, O. Romero-Isart, T. E. Northup, L. Novotny, and M. Frimmer, State expansion of a levitated nanoparticle in a dark harmonic potential, *Phys. Rev. Lett.* **132**, 253602 (2024).
- [18] M. Rossi, A. Militaru, N. Carlon Zambon, A. Riera-Campeny, O. Romero-Isart, M. Frimmer, and L. Novotny, Quantum delocalization of a levitated nanoparticle, *Phys. Rev. Lett.* **135**, 083601 (2025).

- [19] S. Lindner, P. Juschitz, J. Rieser, Y. Y. Fein, M. Debiassac, M. A. Ciampini, M. Aspelmeyer, and N. Kiesel, Hollow-core fiber loading of nanoparticles into ultra-high vacuum, *Appl. Phys. Lett.* **124**, 143501 (2024).
- [20] S. Dago, J. Rieser, M. A. Ciampini, V. Mlynář, A. Kugi, M. Aspelmeyer, A. Deutschmann-Olek, and N. Kiesel, Stabilizing nanoparticles in the intensity minimum: Feedback levitation on an inverted potential, *Opt. Express* **32**, 45133 (2024).
- [21] G. F. M. Tomassi, D. Veldhuizen, B. Melo, D. Candoli, A. Riera-Campeny, O. Romero-Isart, N. Meyer, and R. Quidant, Accelerated state expansion of a nanoparticle in a dark inverted potential, *Phys. Rev. Res.* **8**, L012026 (2026).
- [22] M. Tsang, Testing quantum mechanics: A statistical approach, *Quantum Meas. Quantum Metrol.* **1**, 84 (2013).
- [23] M. Roda-Llodes, A. Riera-Campeny, D. Candoli, P. T. Grochowski, and O. Romero-Isart, Macroscopic quantum superpositions via dynamics in a wide double-well potential, *Phys. Rev. Lett.* **132**, 023601 (2024).
- [24] L. Neumeier, M. A. Ciampini, O. Romero-Isart, M. Aspelmeyer, and N. Kiesel, Fast quantum interference of a nanoparticle via optical potential control, *Proc. Natl. Acad. Sci. USA* **121**, e2306953121 (2024).
- [25] U. Delić, M. Reisenbauer, K. Dare, D. Grass, V. Vuletić, N. Kiesel, and M. Aspelmeyer, Cooling of a levitated nanoparticle to the motional quantum ground state, *Science* **367**, 892 (2020).
- [26] L. Magrini, P. Rosenzweig, C. Bach, A. Deutschmann-Olek, S. G. Hofer, S. Hong, N. Kiesel, A. Kugi, and M. Aspelmeyer, Real-time optimal quantum control of mechanical motion at room temperature, *Nature (London)* **595**, 373 (2021).
- [27] F. Tebbenjohanns, M. L. Mattana, M. Rossi, M. Frimmer, and L. Novotny, Quantum control of a nanoparticle optically levitated in cryogenic free space, *Nature (London)* **595**, 378 (2021).
- [28] A. Ranfagni, K. Børkje, F. Marino, and F. Marin, Two-dimensional quantum motion of a levitated nanosphere, *Phys. Rev. Res.* **4**, 033051 (2022).
- [29] M. Kamba, R. Shimizu, and K. Aikawa, Optical cold damping of neutral nanoparticles near the ground state in an optical lattice, *Opt. Express* **30**, 26716 (2022).
- [30] J. Piotrowski, D. Windey, J. Vijayan, C. Gonzalez-Ballester, A. de los Ríos Sommer, N. Meyer, R. Quidant, O. Romero-Isart, R. Reimann, and L. Novotny, Simultaneous ground-state cooling of two mechanical modes of a levitated nanoparticle, *Nat. Phys.* **19**, 1009 (2023).
- [31] M. Kamba, N. Hara, and K. Aikawa, Quantum squeezing of a levitated nanomechanical oscillator, *Science* **389**, 1225 (2025).
- [32] J. Zhang and K. Mølmer, Prediction and retrodiction with continuously monitored Gaussian states, *Phys. Rev. A* **96**, 062131 (2017).
- [33] J. Lammers and K. Hammerer, Quantum retrodiction in Gaussian systems and applications in optomechanics, *Front. Quantum Sci. Technol.* **2**, 1294905 (2024).
- [34] L. Wasserman, *All of Statistics: A Concise Course in Statistical Inference*, *Springer Texts in Statistics* (Springer, New York, 2004).
- [35] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (John Wiley & Sons, Hoboken, NJ, 2005).
- [36] B. Schirnski, S. Nimmrichter, B. A. Stickler, and K. Hornberger, Macroscopicity of quantum mechanical superposition tests via hypothesis falsification, *Phys. Rev. A* **100**, 032111 (2019).
- [37] B. Schirnski, S. Nimmrichter, and K. Hornberger, Quantum-classical hypothesis tests in macroscopic matter-wave interferometry, *Phys. Rev. Res.* **2**, 033034 (2020).
- [38] A. Riera-Campeny, M. Roda-Llodes, P. T. Grochowski, and O. Romero-Isart, Wigner analysis of particle dynamics and decoherence in wide nonharmonic potentials, *Quantum* **8**, 1393 (2024).
- [39] M. Berry and K. E. Mount, Semiclassical approximations in wave mechanics, *Rep. Prog. Phys.* **35**, 315 (1972).
- [40] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, *Course of Theoretical Physics* (Pergamon Press, Oxford, 1977), Vol. 3, 3rd ed.
- [41] E. J. Heller, *The Semiclassical Way to Dynamics and Spectroscopy* (Princeton University Press, Princeton, NJ, 2018).
- [42] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, *Phys. Rev. A* **64**, 012310 (2001).
- [43] V. Kala, R. Filip, and P. Marek, Cubic nonlinear squeezing and its decoherence, *Opt. Express* **30**, 31456 (2022).
- [44] D. W. Moore and R. Filip, Nonlinear phase gates as Airy transforms of the Wigner function, *npj Quantum Inf.* **11**, 159 (2025).
- [45] O. Vallée and M. Soares, *Airy Functions and Applications to Physics* (World Scientific, Hackensack, NJ, 2010), 2nd ed.
- [46] N. Delfosse, P. Allard Guerin, J. Bian, and R. Raussendorf, Wigner function negativity and contextuality in quantum computation on rebits, *Phys. Rev. X* **5**, 021003 (2015).
- [47] D. L. Wallace, Asymptotic approximations to distributions, *Ann. Math. Stat.* **29**, 635 (1958).
- [48] A. Riera-Campeny, P. Maurer, and O. Romero-Isart, Software to reproduce figures in “Certifying macroscopic quantum mechanics via hypothesis testing with finite data” (2025), <https://doi.org/10.5281/zenodo.15805134>.
- [49] A. Serafini, *Quantum Continuous Variables: A Primer of Theoretical Methods* (CRC Press, Taylor & Francis Group, Boca Raton, FL, 2017), Chap. 3.
- [50] O. Romero-Isart, Coherent inflation for large quantum superpositions of levitated microspheres, *New J. Phys.* **19**, 123029 (2017).
- [51] G. Casella and R. L. Berger, *Statistical Inference* (Duxbury, Pacific Grove, CA, 2002), 2nd ed.