

Coarse-grained quantum thermodynamics: Observation-dependent quantities, observation-independent laws

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In both classical and quantum thermodynamics, physical quantities are typically assigned objective values defined independently of our observations. We then refer to the “work performed by a gas,” or the “entropy of the gas,” regardless of how they are evaluated. Here, we question this conception in the context of quantum thermodynamics, estimating how the definition of pivotal thermodynamic quantities is affected by experimental instruments of limited precision. We find that the coarse-grained thermodynamic quantities frequently lead to different conclusions from those drawn in fine-grained scenarios. For instance, the irreversibility of a process, or its work payoff, can significantly vary with the instrument precision. We show nonetheless that coarse-grained thermodynamic quantities satisfy the *same* relations (i.e., the second law inequality, the relation between dissipation and distinguishability of a process from its time reverse, and the quantum work fluctuation theorems) as their fine-grained counterparts. These results highlight the observation independence of relations linking thermodynamic quantities, which are themselves observation dependent.

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I. INTRODUCTION

In traditional formulations of thermodynamics, notions such as thermal equilibrium, thermodynamic entropy, or extractable work are often introduced as objective macroscopic properties of a system, without explicit reference to the information available to an observer. In quantum mechanics, in contrast, the notion of quantum states and their dynamics is more explicitly understood as reflecting the information that an agent is able to acquire about the system. This shift in perspective inspires the question of whether a similar epistemic viewpoint might also be meaningful (or even necessary) in the thermodynamic context. In fact, in real experiments, agents do not dispose of instruments of infinite precision. Consequently, physical properties can only be reconstructed in a coarse-grained manner, which inherently limits the amount of information that can be acquired. This limitation takes on particular importance in thermodynamics, where key quantities

like entropy and work are fundamentally tied to information. Such considerations gave rise to the field of information thermodynamics, which examines how constraints on information processing shape fundamental thermodynamic limits. A paradigmatic example is the modern understanding of Maxwell’s demon, where the extractable work reflects the level of knowledge the agent has about the system [1–3]. From this perspective, there appears to be no inherent meaning in, e.g., the notion of “work performed on or by a gas,” or of the “thermodynamic entropy of the gas” *per se*, without specifying the observer’s level of knowledge and their experimental ability to engineer the state of the gas.

While a redefinition of the main thermodynamic quantities under coarse graining emerges as a natural response to these considerations, it raises a number of related questions such as how these quantities evolve with varying experimental precision, what energetic costs are associated with coarse graining, and whether thermodynamic laws remain invariant under such changes. Providing answers to these questions is the central purpose of this study.

Various different types of coarse-graining procedures have already been explored in the thermodynamic context for small classical and quantum systems, following either Hamiltonian dynamics [4–6] or stochastic evolution [7–12]. Most of these approaches focus on eliminating fast degrees of freedom in the dynamics [13,14] or on marginal observations of selected states or currents [15–19]. In contrast, here we develop a framework for quantum coarse graining that explicitly

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models the finite precision of measurement devices, representing their limited energy resolution through a family of finite-degree projectors that decimate the system's Hilbert space. This coarse graining homogenizes blocks within the density operator according to the chosen energy resolution, providing a quantum analog of classical phase-space partitioning [20,21]. Similar techniques have been recently used to define coarse-grained or “observational” entropies [22–25], with applications in the statistical mechanics description of equilibration in isolated quantum systems [26–28].

In this work, we focus on the role of coarse graining in the thermodynamics of driven dissipative processes as seen from the point of view of modern information thermodynamics [2,3]. Crucially, we show that, while finite measurement resolution can significantly alter the apparent properties of a process (such as work distributions, entropy production, and effective dynamics), the fundamental thermodynamic laws remain invariant under changes in this resolution. As an example, coarse graining may affect the estimation of a process's degree of reversibility, making an irreversible process appear reversible (or vice versa), yet the average coarse-grained dissipative work required to transfer the system between coarse-grained thermal states always exceeds or equals their free-energy difference. This leads to a family of second-law-like relations valid for any level of coarse graining. In what follows, we first establish quantitative relations between work costs, coarse graining, and entropy production in operational terms, linking coarse graining to information thermodynamics. We then introduce a two-point measurement scheme scenario to derive coarse-grained counterparts of nonequilibrium relations, including the connection between dissipation and distinguishability from time-reversed processes [4,5], as well as quantum work fluctuation theorems [29,30]. From this, we conclude that every agent arrives at the same form of the fundamental thermodynamic laws, regardless of observational precision, even though the thermodynamic quantities involved depend on that precision.

The rest of this paper is organized as follows. Section II introduces the coarse-graining procedure employed in this paper and the corresponding thermal states by applying the principle of maximum entropy to the case of a state estimated via limited precision measurements. Sections III–V focus on dissipative work due to driving protocols. In particular, we introduce a two-point measurement scheme based on coarse-grained measurements (Sec. III), and derive second-law-like inequalities (Sec. IV) and work fluctuation theorems (Sec. V) for any resolution of the measurement apparatus. Section VI discusses the relation between the type of coarse graining here considered and that in other previous works (e.g., Refs. [4,5]). Finally, Sec. VII extends the framework to nonorthogonal coarse graining, modeling readout noise and energy values assigned to different measurement outcomes, and discusses the implications for thermodynamic relations.

II. COARSE-GRAINED THERMAL STATES AND COARSE-GRAINING OPERATION

The tight link binding statistical mechanics and information theory has been highlighted in a number of seminal works [31–33]. One of the most relevant evidences of this

interconnection is provided by the “principle of maximum entropy” [34,35]. This principle states that the probability distribution that best represents the state of one's own knowledge is the one that leaves the greatest residual uncertainty (i.e., maximum entropy) consistent with given constraints (usually imposed by general conservation laws of macroscopic observables like energy, momentum, or number of particles). However, in any practical investigation, the constraints entering the description of equilibrium states are themselves available only with finite precision, reflecting the limited resolution of the instruments or preparation procedures used to characterize the system.

Consider a physical system whose mean values of a number of physical observables are assigned based on experimentally accessible information with finite resolution. These observables could be, for instance, its energy, the number of particles composing the system, etc. Let us call $\langle O^{(k)} \rangle$ the mean values of M physical observables of the system, with $k = 1, \dots, M$. The constraints on the mean values of such observables can be written as

$$\langle O^{(k)} \rangle = \sum_{i=1}^N p_i O_i^{(k)}, \quad (1)$$

where $O_i^{(k)}$ are the possible values that each of the set of mutually commuting physical observable $\hat{O}^{(k)}$ can take in a measurement process, with probability p_i , that is, $\sum_{i=1}^N p_i = 1$. Under these constraints, the probability distribution that exhibits maximum entropy (hence, the most likely according to the principle of maximum entropy) has the form

$$p_i = \frac{1}{Z(\beta_1, \dots, \beta_M)} \exp\left(-\sum_{k=1}^M \beta_k O_i^{(k)}\right), \quad (2)$$

where $\beta_1, \dots, \beta_M$ are so-called “Lagrange multipliers,” whose values are determined from the average values $\langle O^{(k)} \rangle = -\partial \ln Z / \partial \beta_k$, and where the normalization constant $Z(\beta_1, \dots, \beta_M) = \sum_{i=1}^N \exp(-\sum_{k=1}^M \beta_k O_i^{(k)})$ is the partition function. (We note that the derivation can be extended to quantum statistical mechanics and to noncommuting observables [36,37].)

One of the prime examples of the application of this principle is the derivation of the “thermal (Gibbs) state”:

$$\tau_\beta \equiv \frac{1}{Z} \sum_j e^{-\beta \epsilon_j} |\epsilon_j\rangle \langle \epsilon_j| = e^{-\beta(H-F)}, \quad (3)$$

with $Z = \sum_j e^{-\beta \epsilon_j} = e^{-\beta F}$ being the partition function of the system (where F is the equilibrium free energy), $\beta = (k_B T)^{-1}$ the inverse temperature, and $\{|\epsilon_j\rangle\}$ the energy eigenstates, with corresponding eigenvalues ϵ_j of the Hamiltonian H . This state maximizes the von Neumann entropy $S(\tau_\beta) = -\text{Tr}[\tau_\beta \ln(\tau_\beta)]$ under the average energy constraint or, equivalently, minimizes the average energy $\langle H \rangle = \text{Tr}[\tau_\beta H]$ for a fixed von Neumann entropy.

As a consequence, whenever one does not have access to the complete microscopic state of a system, the optimal state they shall attribute to it is the generalized Gibbs state in Eq. (2) derived from the known mean values of the system observables [34]. The link with thermodynamics is

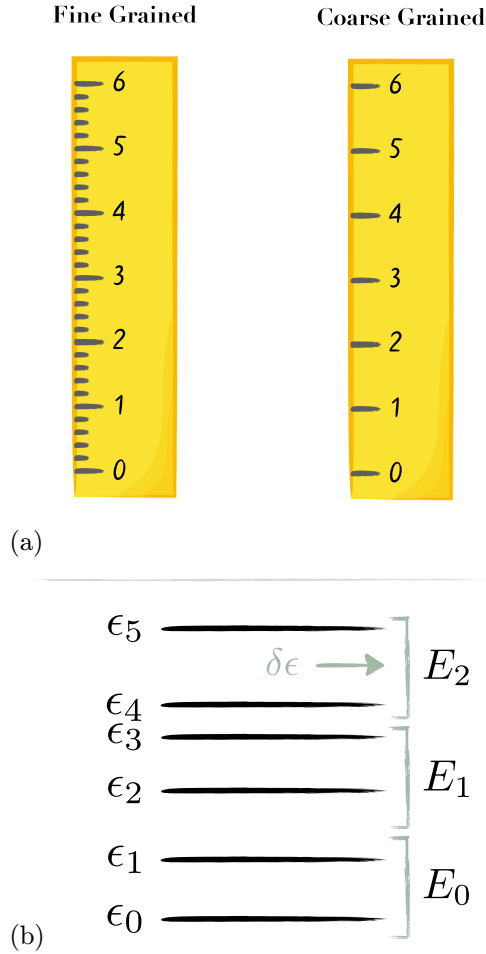


FIG. 1. Coarse graining of a ruler and of the energy levels of a thermodynamic system. (a) A ruler shown without (left) and with (right) coarse graining. Coarse graining reduces the resolution of the instrument. Note: This is a schematic illustration (any instrument appears coarse grained relative to one with higher precision, and only an idealized, infinitely precise instrument would be truly non-coarse-grained). (b) Diagrammatic example illustrating how the levels E_j in Eq. (11) are defined from the fine-grained energy levels ϵ_j , using intervals of width $\delta\epsilon$.

established through the notion of equilibrium [38]. In other words, under interactions that preserve the relevant set of observables appearing in Eq. (2), the interaction between systems prepared in generalized Gibbs states does not produce any observable change in their states, nor any currents in the average values [Eq. (1)]. This is particularly clear in the case of energy: Systems in thermal contact may exchange heat with each other until they reach equilibrium states at the same temperature τ_β .

In practice, however, not only does one not have direct access to the microscopic states of the system, but also the average values that one can evaluate are affected by the accuracy of the measuring instruments. Performing a measurement under coarse graining limits the ability to distinguish microscopic states of the system due to the limited resolution of the measurement device [see pictorial representation in Fig. 1(a)]. By referring to the example in Fig. 1, let us label as j the

analog of the “fine-grained” indentations of the ruler in the subfigure on the left, and as J that of the “coarse-grained” indentations in the subfigure on the right.

Let $\{\Pi_J\}$, with $\sum_J \Pi_J = \mathbb{1}$, be a set of nonzero projectors, which we call the coarse-graining projectors, and let us consider such a coarse graining to be performed in the energy basis:

$$\Pi_J = \sum_{j \in \mathcal{J}_J} |\epsilon_j\rangle\langle\epsilon_j|, \quad (4)$$

with intervals $\mathcal{J}_J = [\sum_{K=1}^{J-1} \text{Tr}[\Pi_K] + 1, \sum_{K=1}^J \text{Tr}[\Pi_K]]$ that we call “slots,” each of them containing a number of microstates $N_J \equiv \text{Tr}[\Pi_J]$. Under such a coarse graining, the original “fine-grained” Hamiltonian $H = \sum_j \epsilon_j |\epsilon_j\rangle\langle\epsilon_j|$ becomes

$$\check{H} = \sum_{J=1}^{\check{N}} E_J \Pi_J, \quad (5)$$

with degenerate eigenvalues E_J , and where we labeled $\check{N}$ the number of slots. We will refer to this observable as the “coarse-grained energy” (we consider $\check{N} > 1$). Here, the intervals $\mathcal{J}_J$ define the coarse-grained slots. These, in principle, can be chosen arbitrarily. However, in this work we are particularly interested in a physically motivated prescription based on the energy resolution of the measuring instruments. Specifically, we choose the (fine-grained) ground state as a natural reference point to define the intervals [39], and define each slot J to contain all eigenstates $\{|\epsilon_j\rangle\}$ such that $(J-1)\delta\epsilon \leq \epsilon_j < J\delta\epsilon$, where $\delta\epsilon$ denotes the smallest energy difference resolvable by the instruments, and $J = 1, 2, \dots, \check{N}$ [see Fig. 1(b) for an illustrative example] [40].

Given the above constraint, and assuming energy is the only relevant conserved quantity, the maximum entropy principle yields what we will call the “coarse-grained thermal state”:

$$\check{\tau}_\beta \equiv e^{-\beta(\check{H}-\check{F})} = \sum_J \frac{e^{-\beta E_J}}{\check{Z}} \Pi_J, \quad (6)$$

where $\check{Z} \equiv e^{-\beta\check{F}} = \sum_J e^{-\beta E_J} N_J$ is the coarse-grained partition function. This form follows from applying the maximum entropy principle under the constraint that only coarse-grained energy measurements are accessible: Since measurement outcomes correspond to projectors Π_J , the optimal state must be block diagonal in this basis, with weights proportional to $e^{-\beta E_J}$ within each slot. One should note, however, that the specific values E_J in Eq. (6) are not determined by the maximum entropy principle alone, but need to be derived from further assumptions ensuring thermodynamic consistency, as we will shortly see. Coarse-grained thermal states in this form provide the central mathematical tool for the analysis developed in this work.

In order to introduce a link between coarse-grained and fine-grained pictures, we use the coarse-graining mapping $\mathcal{C}_{\Pi_J}(\rho) \rightarrow \check{\rho}$, associated with a set of coarse-graining projectors $\{\Pi_J\}$, which transforms any state in the fine-grained description to the coarse-grained one:

$$\mathcal{C}_{\Pi_J}(\rho) \equiv \sum_J \frac{\text{Tr}[\rho \Pi_J]}{N_J} \Pi_J = \sum_J \check{\rho}_J \frac{\Pi_J}{N_J}, \quad (7)$$

where $\check{\rho}_J$ denotes the coarse-grained eigenvalues of $\check{\rho} \equiv \mathcal{C}_{\Pi_J}(\rho)$, with $\sum_J \check{\rho}_J = 1$ (see also Ref. [24]). Notice also the need to keep the factor N_J in the denominator since $\text{Tr} \Pi_J = N_J$. Loosely speaking, the role of the map $\mathcal{C}_{\Pi_J}$ is essentially to flatten the density operator ρ within the slots defined by the projectors Π_J ; that is, it replaces each block J of the fine-grained state by a flat diagonal block proportional to the maximally mixed state $\mathbb{1}_J$.

The map $\mathcal{C}_{\Pi_J}$ captures the fact that coarse graining leads to a loss of the microscopic information about the state ρ , thereby maximizing entropy within the slots inaccessible to the coarse-grained instruments. In particular, $\mathcal{C}_{\Pi_J}(\rho + \delta\rho) = \mathcal{C}_{\Pi_J}(\rho)$ for any perturbation $\delta\rho$ such that $\text{Tr}[\delta\rho \Pi_J] = 0$ for all J , i.e., for any $\delta\rho$ that does not modify the overall population of the slots J . Consequently, the map $\mathcal{C}_{\Pi_J}$ corresponds to a logically irreversible operation, which transforms fine-grained states with potentially different microscopic structures to the same (block-flattened) density $\check{\rho}$. Importantly, this description does not depend on how the slots are chosen. In particular, we note that $\{\Pi_J\}$ may in principle represent any projective partition of the energy eigenbasis; the derivations in the orthogonal coarse-graining setting rely only on the fact that the associated map $\mathcal{C}_{\Pi_J}$ is completely positive, trace preserving, and unital, and therefore do not depend on whether the sets $\mathcal{J}_J$ correspond to contiguous energy intervals.

From a different but related viewpoint, one may also ask whether the coarse-grained description should be formulated on a reduced Hilbert space. Although the coarse-grained state $\check{\rho}$ can be equivalently represented on a reduced Hilbert space of dimension $\check{N}$, since each block $\Pi_J \check{\rho} \Pi_J$ is proportional to the identity within the corresponding slot, we keep the full Hilbert space throughout. This is convenient because the unitary dynamics implementing the driving protocol is naturally defined on the fine-grained space and may, in general, act nontrivially within each slot. Retaining the full Hilbert space therefore allows us to describe consistently both the coarse-grained state assignment and the underlying physical dynamics within a single framework.

On the more technical side, $\mathcal{C}_{\Pi_J}$ is a completely positive and trace-preserving (CPTP) map, which can be written in terms of Kraus operators as $\mathcal{C}(\rho) = \sum_k M_k \rho M_k^\dagger$ fulfilling $\sum_k M_k^\dagger M_k = \mathbb{1}$. This map preserves any state already flattened within the slots, which are all invariant states of the map. It is also an idempotent operation, i.e., $\mathcal{C}_{\Pi_J}^2(\rho) = \mathcal{C}_{\Pi_J}(\rho)$, and it preserves the identity operator, $\mathcal{C}_{\Pi_J}(\mathbb{1}) = \mathbb{1}$; that is, it is a *unital* map. It hence verifies $S(\mathcal{C}_{\Pi_J}(\rho)) \geq S(\rho)$ for any ρ . Another important property of the map $\mathcal{C}_{\Pi_J}$ is that it preserves the coarse-graining energy $\text{Tr}[\check{H}\rho] = \text{Tr}[\check{H}\mathcal{C}_{\Pi_J}(\rho)]$ for any state ρ . In this sense, the coarse-grained state is an effective description of the system. We also notice that the convolution of the map $\mathcal{C}_{\Pi_J}$ with von Neumann entropy,

$$\begin{aligned} S(\mathcal{C}_{\Pi_J}(\rho)) &= \sum_J \sum_{j \in \mathcal{J}_J} \left(\frac{\check{\rho}_J}{N_J} \right) \ln \left(\frac{\check{\rho}_J}{N_J} \right) \\ &= \sum_J \check{\rho}_J \ln \left(\frac{\check{\rho}_J}{N_J} \right), \end{aligned} \quad (8)$$

naturally leads to the definition of coarse-grained or *observational* entropies [22–24, 26, 27].

We now apply the coarse-graining map $\mathcal{C}_{\Pi_J}$ in Eq. (7) to the thermal state τ_β . The result of this operation must correspond to the coarse-grained thermal state $\check{\tau}_\beta$ in Eq. (6) within the precision of the instruments, that is,

$$\mathcal{C}(\tau_\beta + \delta\tau) = \check{\tau}_\beta, \quad (9)$$

where $\delta\tau$ is a (fine-grained) perturbation such that $\text{Tr}[\delta\tau \Pi_J] = 0$ for all J . This consistency condition is key to determining the coarse-grained energy values E_J in Eq. (5) and, as we see in Appendix A, it is in fact implied by the second law of thermodynamics. Applying the preservation of the coarse-graining Hamiltonian to the thermal state τ_β , and using the condition in Eq. (9), we obtain [41]

$$e^{-\beta E_J} N_J = \sum_{j \in \mathcal{J}_J} e^{-\beta \epsilon_j}, \quad (10)$$

which also implies $e^{-\beta \check{F}} = \sum_J e^{-\beta E_J} N_J = \sum_j e^{-\beta \epsilon_j} = e^{-\beta F}$, or, equivalently, $\check{F} = F$. Based on Eq. (9), each coarse-grained energy level takes on a value

$$E_J = -k_B T \ln \left(\frac{Z_J}{N_J} \right), \quad (11)$$

where by $Z_J \equiv \text{Tr}[e^{-\beta H} \Pi_J]$ we denoted the partition function within the slot J . Notably, the coarse-grained eigenvalues E_J depend not only on the energy eigenvalues of the original Hamiltonian H , but also on the temperature of the reservoir T .

In Appendix B, we show that coarse-grained eigenvalues E_J in Eq. (11) lie in the interval between the minimum eigenvalue in the slot, $E_J^{\min} \equiv \min_{j \in \mathcal{J}_J} \epsilon_j$, and their average:

$$E_J^{\min} \leq E_J \leq \bar{E}_J, \quad (12)$$

where $\bar{E}_J = \sum_{j \in \mathcal{J}_J} \epsilon_j / N_J$ is the slot mean. We notice that these two limiting cases correspond to the low- and high-temperature limits, respectively: For low temperatures, $Z_J / N_J \sim e^{-\beta E_J^{\min}}$, and hence $E_J \sim E_J^{\min}$. On the other hand, for high temperatures ($\beta \epsilon_j \ll 1$), we can series expand the exponential term, thereby obtaining $Z_J \simeq N_J (1 - \beta \bar{E}_J)$, which implies $E_J \simeq \bar{E}_J$ from Eq. (11). The bounds in Eq. (12) guarantee that the energy eigenvalues appear distinguishable to coarse-grained instruments, i.e., $|E_J - E_K| > \delta\epsilon$ for any pair of slots K, J , and that they do not intersect (analogously to the case of the reduction in resolution of a ruler in Fig. 1).

Finally, the prescription introduced above specifies a thermal state for any size of coarse-grained slots, which are determined by the accuracy of the agent's instruments to measure energies. Each such range defines a different coarse-grained energy for a given temperature. In the upcoming sections, the state in Eq. (6) will be adopted as the basis for a formulation of the thermodynamic laws for coarse-grained thermal states.

III. DISSIPATIVE WORK AND FLUCTUATIONS

The information-theoretic measure of the quantum relative entropy between the states subject to a (generally irreversible) thermodynamic process is directly related to

the average dissipated work during the process [4–6,42–44]. Moreover, fluctuations in dissipative work feature universal properties characterized by a set of nonequilibrium equalities conventionally referred to as fluctuation theorems [29,30,45]. Here, we are interested in the dissipative work in processes described *and performed* by both fine-grained and coarse-grained agents. In the following, we first review the prototypical scenario corresponding to a fine-grained agent F , and then introduce the key modifications in the case of the coarse-grained agent C . Importantly, in our setup, the different level of resolution has implications in the way an experiment is prepared and not only observed.

A. Two-point measurement scheme and thermodynamic relations for a fine-grained agent

Consider a quantum system with time-dependent Hamiltonian $H(t)$ that is initially prepared in a thermal state τ_β^0 , to undergo unitary evolution within which the Hamiltonian of the system is changed from $H_0 \equiv H(0)$ to $H_\tau \equiv H(\tau)$ in a time window $t \in (0, \tau]$ by changing some control parameter of the Hamiltonian, $\lambda(t)$, according to a prescribed protocol $\Lambda = \{\lambda(t); 0 \leq t \leq \tau\}$. The dynamical evolution occurs in isolation from the thermal environment [46], so that it can be described by a unitary operator $U(t, 0) \equiv \mathcal{T} e^{-i \int_0^t H[\lambda(s)] ds}$, where $\mathcal{T}$ denotes the time-ordering operator. At the end of the process, the system is generally in a nonequilibrium state, $\rho(\tau) = U(\tau, 0) \tau_\beta^0 U(\tau, 0)^\dagger$, which then interacts with the thermal environment and relaxes to the thermal state corresponding to the final Hamiltonian, τ_β^τ . The dissipated work in the process, $W_{\text{diss}} = W - \Delta F$, is defined as the extra amount of work that has been invested in the transformation, W , on the top of the free-energy difference, ΔF , between the initial and final equilibrium states.

The average dissipated work in the process is proportional to the quantum relative entropy between the states of the system in forward and time-reversal processes at any intermediate instant of time [4,5]:

$$\langle W \rangle - \Delta F = k_B T S(\rho(t) || \Theta^\dagger \tilde{\rho}(\tau - t) \Theta), \quad (13)$$

where $\langle W \rangle = \text{Tr}[H_\tau \rho(\tau)] - \text{Tr}[H_0 \tau_\beta^0]$ is the average work performed during the implementation of protocol Λ , and we introduced the system states $\rho(t) = U(t, 0) \tau_\beta^0 U(t, 0)^\dagger$ and $\tilde{\rho}(\tau - t) = \tilde{U}(\tau - t, 0) \Theta \tau_\beta^\tau \Theta^\dagger \tilde{U}(\tau - t, 0)^\dagger$ at intermediate times $t \in [0, \tau]$ in forward and time-reversal process, respectively. The time-reversed process is defined by the application of the (reversed) driving protocol $\tilde{\Lambda} = \{\lambda(\tau - t); 0 \leq t \leq \tau\}$ over the final thermal state τ_β^τ , as described by the unitary $\tilde{U}(t, 0) = \mathcal{T} e^{-i \int_0^t \Theta H[\lambda(\tau - s)] \Theta^\dagger ds}$. In the above equations, Θ denotes the time-reversal operator in quantum mechanics, which reverses the sign of observables with odd parity under time reversal. We also emphasize that the two density operators appearing in Eq. (13) are evaluated at the same point in time of their respective evolution [47]. As a result, the dissipative work can be interpreted as quantifying the distinguishability between the forward and time-reversed processes. In this sense, it serves as a measure of how effectively one can infer the arrow of time from the observed physical process.

The positivity of the quantum relative entropy in Eq. (13) is a manifestation of the second law of thermodynamics; that is, the average external work required to perform protocol Λ cannot be smaller than the system's free-energy change, $\langle W \rangle \geq \Delta F$. Stronger thermodynamic constraints also appear in the setup by taking into account work fluctuations. These fluctuations can be accessed via projective measurements of the system Hamiltonian at the beginning and at the end of the unitary evolution, a framework that has been dubbed two-point measurement scheme [29,30].

In the TPM scheme, a first projective measurement of H_0 is performed on the initial state τ_β^0 , obtaining an outcome n with probability $p_n^{(0)} = e^{-\beta \epsilon_n^{(0)}} / Z_0$ and corresponding eigenvalue $\epsilon_n^{(0)}$. Then, after the implementation of the protocol Λ through the unitary $U(\tau, 0)$, a second projective measurement of H_τ is performed, giving outcome m with eigenvalue $\epsilon_m^{(\tau)}$. The very same procedure can also be applied to the time-reversed process by exchanging the order of the measurements and replacing $U(\tau, 0)$ by $\tilde{U}(\tau, 0)$.

Combining the statistics of measurements outcomes in many realizations of the TPM scheme, one can construct the work probability distribution associated with the forward protocol Λ as $P(W) = \sum_{nm} P_{n,m} \delta(W - W_{n,m})$, where $P_{n,m} = p_n^{(0)} |\langle \epsilon_m^{(\tau)} | U(\tau, 0) | \epsilon_n^{(0)} \rangle|^2$ is the joint probability to obtain outcomes n in the first measurement and m in the second one, and $\delta(x)$ denotes the Dirac delta function with $W_{n,m} \equiv \epsilon_m^{(\tau)} - \epsilon_n^{(0)}$. Analogously, the work probability distribution in the time-reversed process reads $\tilde{P}(W) = \sum_{nm} \tilde{P}_{nm} \delta(W + W_{n,m})$, with $\tilde{P}_{nm} = p_m^{(\tau)} |\langle \epsilon_n^{(0)} | \Theta^\dagger \tilde{U}(\tau, 0) \Theta | \epsilon_m^{(\tau)} \rangle|^2$ being the joint probability to obtain, correspondingly, outcomes m and n before and after the application of protocol $\tilde{\Lambda}$. These two probability distributions verify the detailed fluctuation theorem [29,48]:

$$\frac{P(W)}{\tilde{P}(-W)} = e^{\beta(W - \Delta F)}, \quad (14)$$

which provides a link between irreversibility (through the probabilities of work in forward and time-reversed processes) and the dissipative work, in every realization of the TPM. The fluctuation theorem in Eq. (14), together with the integral version $\langle e^{-\beta(W - \Delta F)} \rangle = 1$, can be regarded as deeper manifestations of the second law, putting constraints over the structure of fluctuations of thermodynamic quantities [45].

B. Coarse-grained TPM scheme

We now extend the validity of Eqs. (13) and (14) to coarse-grained quantities. Let us first specify how exactly the lack of resolution of the instruments of the coarse-graining agent C affects the standard TPM scheme (see Fig. 2). Our starting point is again the (fine-grained) thermal state $\tau_\beta^{(0)}$, over which a first projective energy measurement of $\tilde{H}_0$ is performed. In this case, agent C obtains outcome I , corresponding to eigenvalue $E_I^{(0)}$, with probability

$$\tilde{p}_I^{(0)} = \text{Tr}[\tau_\beta^{(0)} \Pi_I^{(0)}] = \frac{e^{-\beta E_I^{(0)}}}{Z_0} N_I^{(0)}, \quad (15)$$

where we used Eq. (10). After the measurement, agent C *reprepares* the state $\Pi_I^{(0)} / N_I^{(0)}$ and applies the unitary evolution $U(\tau, 0)$ associated with protocol Λ , with $U(\tau, 0)$

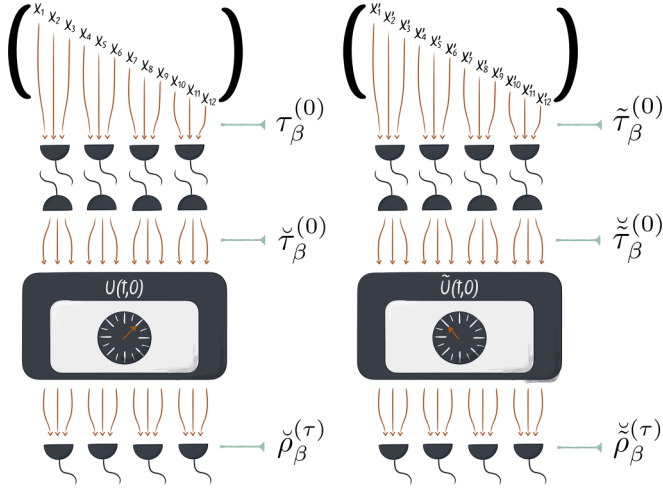


FIG. 2. Schematic of the operations performed on the system in the forward (left) and time-reversed (right) processes. A thermodynamic system is initially prepared in the (fine-grained) state $\tau_\beta^{(0)}$ in the forward process, or in $\check{\tau}_\beta^{(0)} = \Theta \tau_\beta^{(\tau)} \Theta^\dagger$ for the time-reversed process. It is then measured and reprepared using coarse-grained instruments, yielding the state $\check{\tau}_\beta^{(0)}$ (or $\check{\tau}_\beta^{(0)} = \Theta \tau_\beta^{(\tau)} \Theta^\dagger$ for the time reversal) at time $t = 0$. Subsequently, the system evolves under a unitary evolution $U(t, 0)$ (or $\tilde{U}(t, 0)$ in the time-reversed case) during the interval $t \in (0, \tau]$, reaching the final state $\check{\rho}_\beta^{(\tau)} = U(\tau, 0) \check{\tau}_\beta^{(0)} U^\dagger(\tau, 0)$ at $t = \tau$ ($\check{\rho}_\beta^{(\tau)} = \tilde{U}(\tau, 0) \check{\tau}_\beta^{(0)} \tilde{U}^\dagger(\tau, 0)$ at $t = \tau$ for the time-reversed process).

potentially acting on fine-grained degrees of freedom that are not accessible to the coarse-graining agent C [49]. At the end of the protocol, a second measurement of $\tilde{H}_\tau$ is performed, obtaining the outcome J associated with the eigenvalue $E_J^{(\tau)}$. Finally, the system is put again in contact with the thermal reservoir until it equilibrates with the fine-grained state $\tau_\beta^{(\tau)}$ corresponding to the final Hamiltonian H_τ .

The joint probability of obtaining outcomes I and J in the coarse-graining process then reads

$$\check{P}_{I,J} = \check{p}_I^{(0)} \text{Tr} \left[\Pi_J^{(\tau)} U(\tau, 0) \frac{\Pi_I^{(0)}}{N_I^{(0)}} U^\dagger(\tau, 0) \right]. \quad (16)$$

It is worth noting that the TPM procedure with coarse-grained instruments effectively involves a reparation of the initial fine-grained thermal state into the corresponding coarse-grained thermal state. Specifically, the ensemble average over the postmeasurement states yields

$$\check{\tau}_\beta^{(0)} = \sum_I \text{Tr}[\tau_\beta^{(0)} \Pi_I^{(0)}] \frac{\Pi_I^{(0)}}{N_I^{(0)}} = \mathcal{C}_{\Pi_I^{(0)}}(\tau_\beta^{(0)}), \quad (17)$$

which is precisely the result of applying the coarse-graining operation defined in Eq. (7). This stands in contrast to a procedure in which the initial state $\tau_\beta^{(0)}$ is merely *projected* onto the I th subspace, yielding $\Pi_I^{(0)} \tau_\beta^{(0)} \Pi_I^{(0)} / \check{p}_I^{(0)}$, which retains the fine-grained structure of the probability distribution within the slot I . We observe that, while these two procedures appear to have the same effect from the perspective of a coarse-graining agent C , they affect the underlying fine-grained degrees of

freedom and subsequent dynamics (i.e., unitary evolution), as observed by any agent.

The time-reversed protocol is defined as follows. It begins with the (fine-grained) thermal state $\tau_\beta^{(\tau)}$, upon which a measurement of the coarse-grained Hamiltonian $\tilde{H}_\tau$ is performed, yielding outcome J . As in the forward process, this measurement is accompanied by the reparation of the state $\Pi_J^{(\tau)} / N_J^{(\tau)}$. At the ensemble level, this is equivalent to applying the coarse-graining operation to the initial state, resulting in

$$\check{\tau}_\beta^{(\tau)} = \sum_J \text{Tr}[\tau_\beta^{(\tau)} \Pi_J^{(\tau)}] \frac{\Pi_J^{(\tau)}}{N_J^{(\tau)}} = \mathcal{C}_{\Pi_J^{(\tau)}}(\tau_\beta^{(\tau)}). \quad (18)$$

Subsequently, the unitary evolution $\tilde{U}$ associated with protocol $\tilde{\Lambda}$ is applied, after which the final measurement of the coarse-grained Hamiltonian $\tilde{H}_0$ is performed, obtaining outcome I . Therefore, the joint probability of obtaining outcomes J and I in the time-reversed (coarse-grained) process reads

$$\check{P}_{I,J} = \check{p}_J^{(\tau)} \text{Tr} \left[\Pi_I^{(0)} \tilde{U}(\tau, 0) \frac{\Pi_J^{(\tau)}}{N_J^{(\tau)}} \tilde{U}^\dagger(\tau, 0) \right], \quad (19)$$

where $\check{p}_J^{(\tau)} = \text{Tr}[\tau_\beta^{(\tau)} \Pi_J^{(\tau)}] = \frac{e^{-\beta E_J^{(\tau)}}}{Z_\tau} N_J^{(\tau)}$.

The coarse-grained TPM scheme introduced here provides a systematic way to describe thermodynamic processes under limited measurement resolution, making explicit how finite precision affects both the statistics of observed work and the reconstruction of the system's state. In the following, we illustrate these concepts through a concrete example involving a driven 12-level system.

C. Case study: A driven 12-level system

As an illustrative example, let us consider a 12-level harmonic oscillator of Hamiltonian $H_0 = \hbar\omega (a^\dagger a + \mathbb{1}/2)$, with a and $a^\dagger$ being the annihilation and creation operators such that $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and $\omega = 2$, $\hbar = 1$. Imagine that a coarse-grained measurement is carried out on such a system, and that this measurement is only able to resolve a minimum of, say, three energy levels [50], i.e., $\mathcal{C} = \{P_1, P_2, P_3, P_4\}$ with $P_1 = \sum_{j=1}^3 |\epsilon_j\rangle\langle\epsilon_j|$, $P_2 = \sum_{j=4}^6 |\epsilon_j\rangle\langle\epsilon_j|$, $P_3 = \sum_{j=7}^9 |\epsilon_j\rangle\langle\epsilon_j|$, and $P_4 = \sum_{j=10}^{12} |\epsilon_j\rangle\langle\epsilon_j|$. Figure 3 shows the comparison between the fine-grained density matrix of the system $\tau_\beta^{(0)}$ and its coarse-grained counterpart $\check{\tau}_\beta^{(0)}$.

Suppose that, immediately after having measured and reprepared the system at $t = 0$, an agent turns on a time-independent driving force f acting on the system in the state $\check{\tau}_\beta^{(0)}$. The Hamiltonian, for $t \in (0, \tau]$, takes the form $H_\tau = \hbar\omega (a^\dagger a + \mathbb{1}/2) + f(a^\dagger + a)/\sqrt{2}$, which does not commute with the initial Hamiltonian and therefore induces transitions between energy eigenstates [51].

Figure 4 reports the trends of fine-grained and coarse-grained dissipated work (relative entropy) and average work as the driving force varies in the range $f = [0, 5]$, maintaining $\tau = 1$ and $\beta = 1$. The level of coarse graining is set by the energy resolution $\delta\epsilon = \alpha|\epsilon_1 - \epsilon_0|$, where α controls the granularity of the measurement. The figure shows that the dissipated work measured on the coarse-grained states, $\check{\rho}_\tau$ and $\check{\tau}_\beta^{(\tau)}$, is higher than that measured on fine-grained states. How-

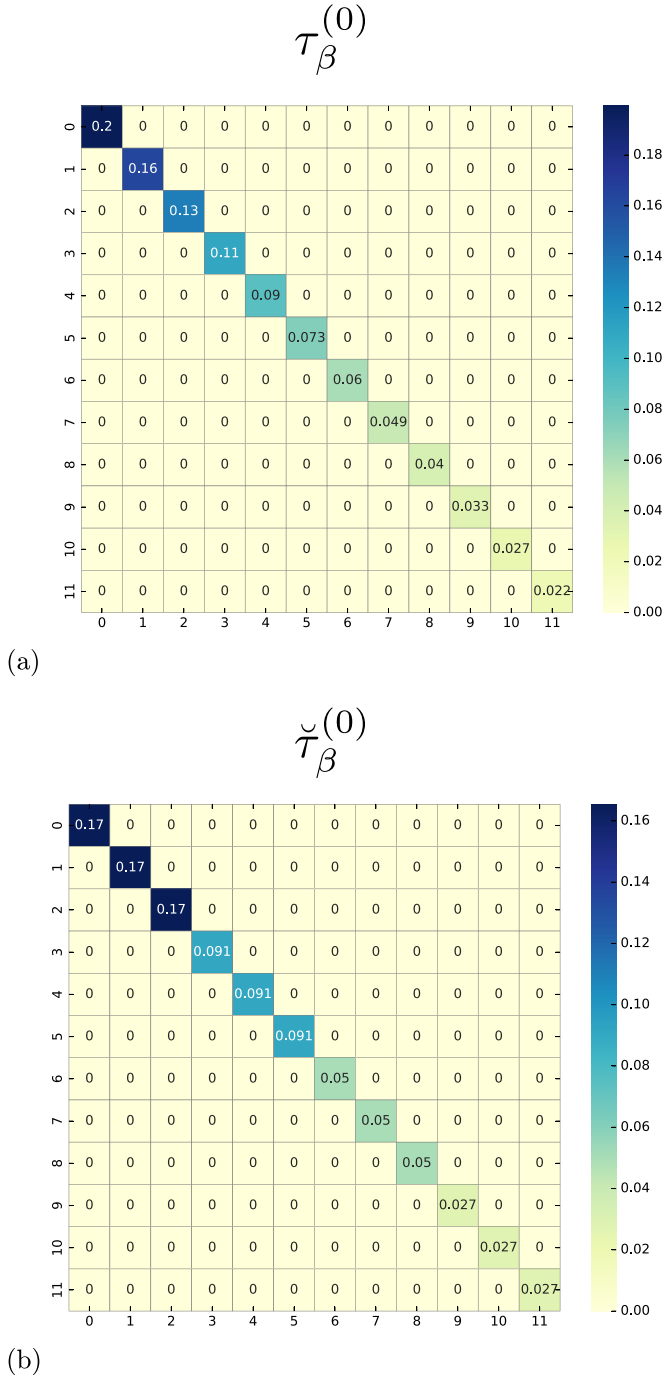


FIG. 3. Matrix elements of fine- and coarse-grained density matrices of a harmonic oscillator. Fine-grained [panel (a)] and coarse-grained [panel (b)] density matrices of the thermal states $\tau_\beta^{(0)}$ and $\tilde{\tau}_\beta^{(0)}$, respectively, corresponding to a 12-level harmonic oscillator. The system's Hamiltonian is $H_0 = \hbar\omega(a^\dagger a + \mathbb{1}/2)$, having set $\omega = 2$ and $\hbar = 1$. The thermal state $\tilde{\tau}_\beta^{(0)}$ is defined as in Eq. (3), with $\epsilon_j^{(0)} = \hbar\omega(j + 1/2)$ being the energy levels of the harmonic oscillator, and $\beta = (k_B T)^{-1}$, with $k_B = 1$, $T = 10$. Here, each coarse-grained interval is composed of a 3×3 fine-grained submatrices (i.e., $\text{Tr}[P_j] = 3$).

ever, starting from different initial conditions, the opposite can occur, indicating that no general rule applies. The same outcome is also observable in the case of the dissipative work.

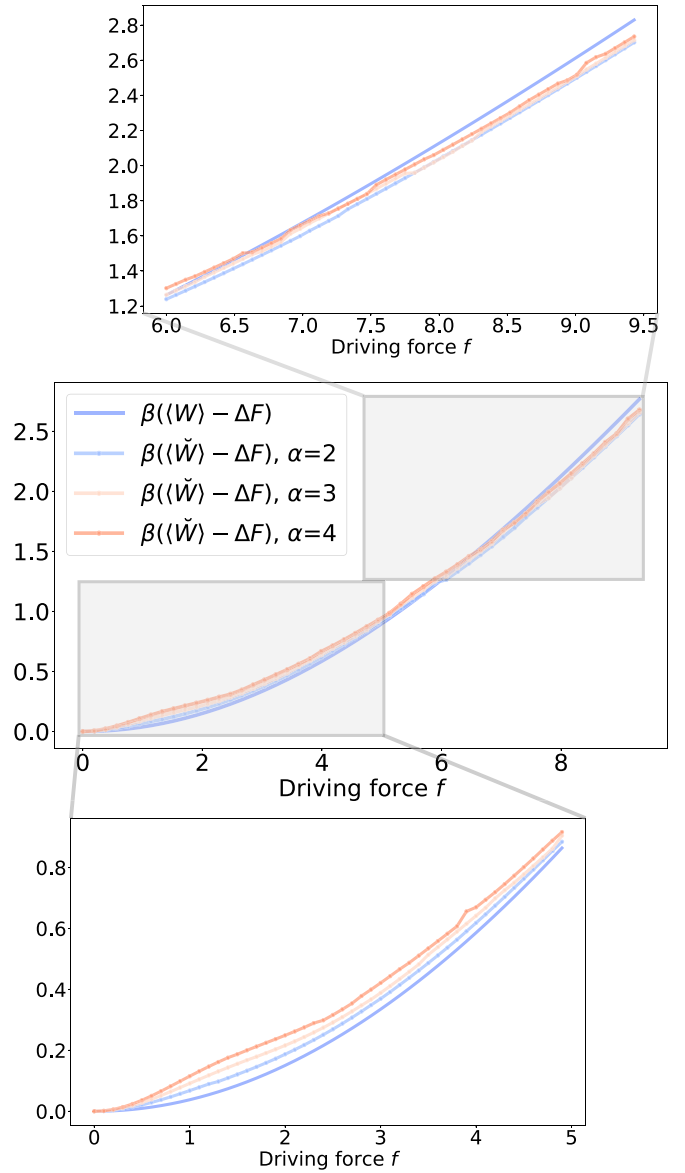


FIG. 4. Trends of fine- and coarse-grained dissipative work for a harmonic oscillator under a time-independent driving force $f \in [0, 9.5]$. The initial state is as in Fig. 3, and the final state is prepared by applying a constant force f within the interval $t \in (0, \tau]$, with Hamiltonian $H_t = \hbar\omega(a^\dagger a + 1/2) + f(a^\dagger + a)/\sqrt{2}$ (with f expressed in energy units, and $\hbar = 1$). The plots show fine-grained and coarse-grained dissipated work (relative entropy) and average work as functions of f , for fixed $\tau = 1$, $\beta = 1$, and $\omega = 2$. (Full details on the case study considered are provided in Sec. III C.) Coarse graining is performed by grouping eigenstates of the initial and final Hamiltonians into intervals of width $\delta\epsilon = \alpha|\epsilon_1 - \epsilon_0|$, where α is a coarse-graining factor that dictates the size of the width of the coarse graining and $\{\epsilon_j\}$ are the eigenstates of the initial Hamiltonian. The insets illustrate that coarse-grained dissipative work $\tilde{W}_{\text{diss}}$ can be either smaller or larger than the fine-grained value W_{diss} , with $\tilde{W}_{\text{diss}} \geq W_{\text{diss}}$ for $f \in [0, 6]$ and $\tilde{W}_{\text{diss}} < W_{\text{diss}}$ for $f > 7$.

IV. SECOND-LAW-LIKE INEQUALITIES UNDER COARSE GRADING

Under the conditions of the coarse-grained TPM protocol, the relative entropy (dissipated work) $S(\tilde{\rho}(t)||\Theta^\dagger$

$\check{\rho}(\tau - t)\Theta$) among the density operators in the forward and time-reversal processes in the coarse-grained TPM, $\check{\rho}(t) = U(t, 0)\check{\tau}_\beta^{(0)}U^\dagger(t, 0)$ and $\check{\rho}(\tau - t) = \check{U}(\tau - t, 0)\check{\tau}_\beta^{(\tau)}\check{U}^\dagger(\tau - t, 0)$, respectively, reads

$$\begin{aligned} S(\check{\rho}(t)||\Theta^\dagger\check{\rho}(\tau - t)\Theta) \\ = \text{Tr}[\check{\rho}(t)\ln\check{\rho}(t)] - \text{Tr}[\check{\rho}(t)\ln\Theta^\dagger\check{\rho}(\tau - t)\Theta] \\ = \text{Tr}[\check{\tau}_\beta^{(0)}\ln\check{\tau}_\beta^{(0)}] - \text{Tr}[\check{\rho}(\tau)\ln\check{\tau}_\beta^{(\tau)}]. \end{aligned} \quad (20)$$

In the last equality, we used the fact that unitary evolution preserves the von Neumann entropy, along with the microreversibility principle for nonautonomous systems [29], ensuring that $\Theta^\dagger\check{U}(\tau - t, 0)\Theta = U^\dagger(\tau, t)$. Expanding the two terms in Eq. (20), we obtain

$$\text{Tr}[\check{\tau}_\beta^{(0)}\ln\check{\tau}_\beta^{(0)}] = -\beta(\text{Tr}[\check{H}_0\check{\tau}_\beta^{(0)}] - F_0), \quad (21a)$$

$$\text{Tr}[\check{\rho}(\tau)\ln\check{\tau}_\beta^{(\tau)}] = -\beta(\text{Tr}[\check{H}_\tau\check{\rho}(\tau)] - F_\tau). \quad (21b)$$

We note that, because the coarse-grained energies depend on the temperature, $\text{Tr}[\check{H}_0\check{\tau}_\beta^{(0)}] \neq -\partial \ln(\check{Z}_0)/\partial\beta$, with $\check{Z}_0 = \sum_J e^{-\beta E_J^{(0)}} \text{Tr}[\Pi_J]$, in contrast to the fine-grained case.

Recombining the two expressions (21a) and (21b) within Eq. (20), we achieve

$$S(\check{\rho}(t)||\Theta^\dagger\check{\rho}(\tau - t)\Theta) = \beta(\langle\check{W}\rangle - \Delta F) \geq 0, \quad (22)$$

where we introduced the mean “effective work” under coarse graining $\langle\check{W}\rangle = \text{Tr}[\check{H}_\tau\check{\tau}_\beta^{(\tau)}] - \text{Tr}[\check{H}_0\check{\tau}_\beta^{(0)}]$. Equation (22) establishes a fundamental link between information theory and thermodynamics at every level of the instrument’s precision, linking the relative entropy between density operators in forward and time-reversed processes to the work and free energy changes of the system. Indeed, they provide a family of second-law-like relations, $\langle\check{W}\rangle \geq \Delta F$, one for each possible level of coarse graining, as directly implied by the positivity of the quantum relative entropy.

For each level of coarse-grained resolution, the quantity $\langle\check{W}_{\text{diss}}\rangle = \langle\check{W}\rangle - \Delta F$ can be interpreted as the average amount of work, beyond the free-energy difference, required to implement the protocol Λ , as perceived by the corresponding coarse-grained agent. Importantly, because the measurement procedure itself alters the dynamics (via measurement and reparation), there is, in general, no strict relationship between $\check{W}_{\text{diss}}$ and the fine-grained dissipative work W_{diss} . That is, depending on the resolution, one may find either $\check{W}_{\text{diss}} \geq W_{\text{diss}}$ or $\check{W}_{\text{diss}} < W_{\text{diss}}$.

There is, however, a special situation in which the inequalities (22) (and, consequently, the family of second-law-like relations) are hierarchically ordered with respect to the level of resolution: The coarser the resolution, the smaller the relative entropy in Eq. (22), and hence the smaller the amount of dissipative work detected by the coarse-graining agent. That happens when the coarse-graining operation C_{Π_J} corresponding to an agent’s resolution and the unitary evolution implementing the driving protocol commute, i.e.,

$$C_{\Pi_J^{(0)}}[U(t, 0)\sigma U^\dagger(t, 0)] = U(t, 0)C_{\Pi_J^{(0)}}(\sigma)U^\dagger(t, 0), \quad (23)$$

for any fine-grained density operator σ . This condition requires that the unitary evolution does not involve fine-grained

degrees of freedom. In that case, we have

$$\begin{aligned} \beta(\langle W \rangle - \Delta F) &= S(\rho(t)||\Theta^\dagger\check{\rho}(\tau - t)\Theta) \\ &\geq S(C_{\Pi_J^{(0)}}[\rho(t)]||\Theta^\dagger C_{\Pi_J^{(0)}}[\check{\rho}(\tau - t)]\Theta) \\ &= S(\check{\rho}(t)||\Theta^\dagger\check{\rho}(\tau - t)\Theta) = \beta(\langle\check{W}\rangle - \Delta F), \end{aligned} \quad (24)$$

where the inequality follows from the data processing inequality for the quantum relative entropy and the fact that C_{Π_J} is a CPTP map.

We conclude that, when agent C can fully distinguish the changes in the system’s state due to the protocol Λ (meaning those changes affect only the degrees of freedom accessible to them), the limitations in the TPM scheme’s energy measurements can only lead to an underestimation of the work required for the process. On the contrary, driving protocols exceeding the resolution of the agent may result in work payoffs either larger or smaller than $\langle W \rangle$.

V. FLUCTUATION THEOREMS UNDER COARSE GRAINING

We now focus on the work probability distribution in the coarse-grained TPM protocol in both forward and time-reversed processes. They are defined as

$$P(\check{W}) = \sum_{I,J} \check{P}_{I,J} \delta(\check{W} - \check{W}_{I,J}), \quad (25a)$$

$$\bar{P}(\check{W}) = \sum_{I,J} \check{\bar{P}}_{I,J} \delta(\check{W} + \check{W}_{I,J}), \quad (25b)$$

with $\check{W}_{I,J} = E_J^{(\tau)} - E_I^{(0)}$ being the work measured by the coarse-graining agent C in every realization of the TPM and where $\check{P}_{I,J}$ are defined in Eq. (16). Microreversibility implies the following relation between the joint probability distributions in forward and time-reversal processes:

$$\frac{\check{P}_{I,J}}{\check{\bar{P}}_{I,J}} = \left(\frac{\check{p}_J^{(\tau)} N_I^{(0)}}{\check{p}_I^{(0)} N_J^{(\tau)}} \right) = e^{-\beta(\check{W}_{I,J} - \Delta F)}. \quad (26)$$

Introducing Eq. (26) into expressions (25a) and (25b), we obtain the detailed fluctuation theorem for the coarse-grained work probability distributions:

$$\frac{\check{P}(\check{W})}{\check{\bar{P}}(-\check{W})} = e^{\beta(\check{W} - \Delta F)}. \quad (27)$$

As in the fine-grained case, the detailed theorem (27) implies the integral version:

$$\begin{aligned} \langle e^{-\beta(\check{W} - \Delta F)} \rangle &= \int dW \check{P}(\check{W}) e^{-\beta(\check{W} - \Delta F)} \\ &= \int dW \check{\bar{P}}(-\check{W}) = 1. \end{aligned} \quad (28)$$

Equations (27) and (28) extend, respectively, Crooks’ work fluctuation relation [48] and the Jarzynski inequality [52] for the dissipative work to coarse-grained scenarios as introduced here.

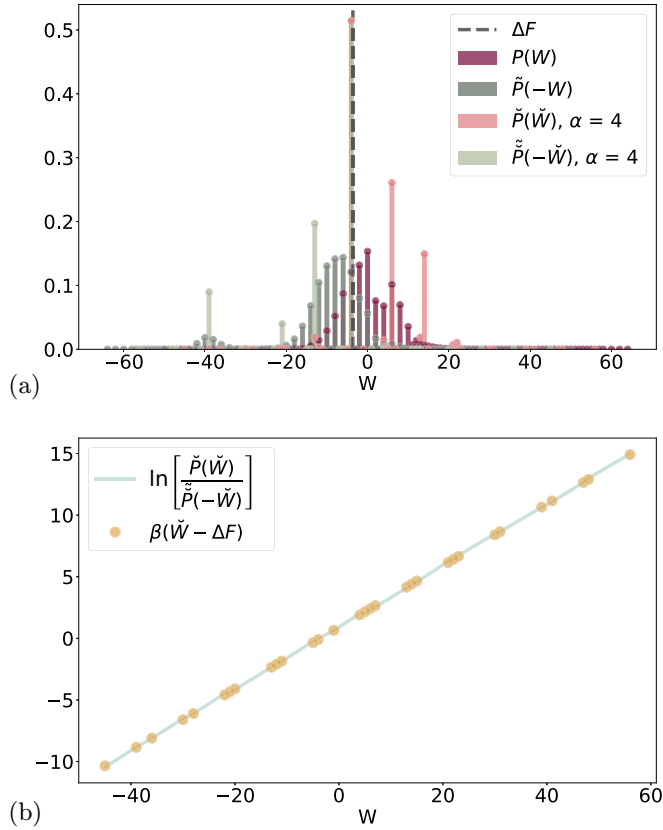


FIG. 5. (a) Probability distributions for forward and time-reversed trajectories in fine- and coarse-grained scenarios. Fine- and coarse-grained probability distributions for a 24-level harmonic oscillator subjected to a time-independent driving force $f = 3.8$ at inverse temperature $\beta = 1/4$. The setup is the same as in Sec. III C, except that a 24-level system is considered here instead of 12 levels. In the fine-grained scenario, the forward and backward distributions are smooth and approximately bell-shaped, intersecting at the free-energy difference ΔF . In the coarse-grained case, the number of data points is significantly reduced, and the distributions appear less regular, though the intersection still occurs at the same ΔF . The horizontal axis corresponds to the work W , expressed in units of energy. (b) Illustration of the fluctuation theorem for forward and time-reversed trajectories in coarse-grained scenarios. The left- and right-hand sides of Eq. (27) (up to a logarithm) are plotted and shown to coincide exactly, confirming that the detailed fluctuation theorem remains valid under coarse graining.

An important application of the detailed fluctuation theorem in Eq. (14) and its integral counterpart has been the determination of equilibrium free-energy differences—a genuine equilibrium property—from the measurement of the work performed in (arbitrary) nonequilibrium conditions [53,54]. The practical relevance of Eqs. (27) and (28) is that they allow to determine the difference in equilibrium free energy from the measurements performed on the system under arbitrary resolution of the coarse graining, as long as $\tilde{N} > 1$. Importantly, this result holds regardless of the precision with which the average dissipative work is estimated, and independently of whether the underlying unitary evolution involves fine-grained degrees of freedom. This feature is illustrated in Fig. 5.

VI. NONINVASIVE COARSE-GRAINING TPM SCHEME

One may wish to compare the coarse-grained TPM scheme introduced in Sec. III with a simpler variant in which the system is not reprepared following the energy measurements with coarse-grained projectors $\Pi_I^{(0)}$ and $\Pi_J^{(\tau)}$. This alternative may seem less invasive, as it avoids modifying the postmeasurement state through reparation. We refer to this protocol as the “noninvasive TPM scheme,” which consists of the following steps: (1) The fine-grained thermal state is subjected to a coarse-grained energy measurement at $t = 0$; (2) the projected state $\Pi_I^{(0)} \tau_\beta^{(0)} \Pi_I^{(0)} / N_I$, corresponding to measurement outcome I , evolves unitarily under a unitary evolution until $t = \tau$; (3) a second coarse-grained energy measurement is performed, yielding outcome J . In this case, the joint probability of obtaining outcomes I and J in the initial and final measurements, respectively, is given by $\tilde{P}_{I,J} = \text{Tr}[\Pi_J^{(\tau)} U(\tau, 0) \Pi_I^{(0)} \tau_\beta^{(0)} \Pi_I^{(0)} U^\dagger(\tau, 0)]$, with an analogous expression holding for the time-reversed process.

Importantly, in the noninvasive TPM scheme, the key results derived in Secs. IV and V [namely, the nonequilibrium relation for dissipative work in Eq. (22) and the fluctuation theorem in Eq. (27)] no longer hold. Although this scheme may appear less invasive at first glance, the absence of the reparation step alters the structure of the process in a way that invalidates those results. In particular, for the average dissipative work, we find

$$\begin{aligned} \beta(\langle W \rangle - \Delta F) &= S(\rho(t) || \Theta^\dagger \tilde{\rho}(\tau - t) \Theta) \\ &\geq S(\bar{\rho}(t) || \Theta^\dagger \tilde{\bar{\rho}}(\tau - t) \Theta) \\ &\neq \beta(\langle \tilde{W} \rangle - \Delta F), \end{aligned} \quad (29)$$

where $\bar{\rho}(t) = U(t, 0) \sum_I \Pi_I^{(0)} \tau_\beta^{(0)} \Pi_I^{(0)} U^\dagger(t, 0)$ is the system state at an arbitrary intermediate time t of the forward process, $\tilde{\bar{\rho}}(t) = \tilde{U}(\tau - t, 0) \sum_J \Pi_J^{(\tau)} \tau_\beta^{(\tau)} \Pi_J^{(\tau)} \tilde{U}^\dagger(\tau - t, 0)$ its time-reversal counterpart, and $\langle \tilde{W} \rangle = \text{Tr}[\bar{\rho}(\tau) \tilde{H}_\tau] - \text{Tr}[\tau_\beta^{(0)} \tilde{H}_0]$ the average work measured by the coarse-graining agent.

In Eq. (29), the inequality is always warranted by means of the contractivity in the relative entropy, which ensures a useful lower bound for the estimation of the fine-grained dissipative work [4,5]. However, the correspondence between the coarse-grained relative entropy and the work measured by the agent performing the energy measurements in the TPM scheme is not universal. This is due to the fact that the projected thermal states $\sum_I \Pi_I^{(0)} \tau_\beta^{(0)} \Pi_I^{(0)}$ and $\sum_J \Pi_J^{(\tau)} \tau_\beta^{(\tau)} \Pi_J^{(\tau)}$ retain the fine-grained structure of the original fine-grained states within each block and hence cannot be rewritten, in general, as thermal states with Gibbs form. Similarly, this prevents the corresponding work probability distributions $\tilde{P}(\tilde{W}) = \sum_{I,J} \tilde{P}_{I,J} \delta(\tilde{W} - \tilde{W}_{I,J})$ [and the analogous expression for the time-reversed protocol $\tilde{\tilde{P}}(\tilde{W})$], to verify the detailed work fluctuation theorems in Eqs. (27) and (28). This deviation from the standard scheme is illustrated in Fig. 6, which compares dissipative work and relative entropy under the noninvasive protocol, highlighting the divergence between these two quantities despite their similar overall trends.

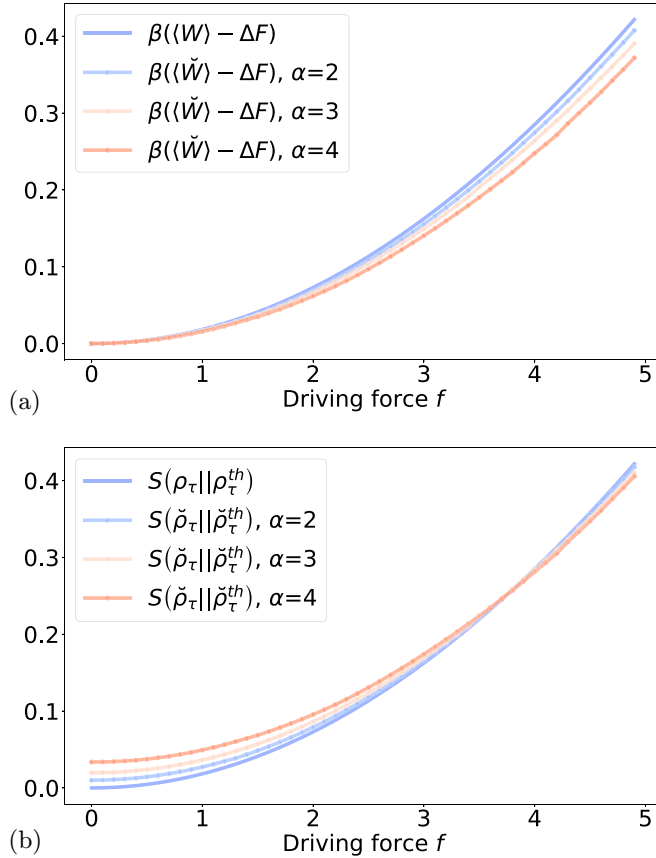


FIG. 6. Comparison between dissipative work [panel (a)] and relative entropy [panel (b)] in fine- and coarse-grained scenarios for the “noninvasive coarse-graining” protocol discussed in Sec. VI. The figure presents the same case study as Fig. 4, but applies the alternative “noninvasive” protocol, where the reparation step is performed using fine-grained instruments, in contrast to the standard protocol used throughout the main text. While in Fig. 4 dissipative work and relative entropy coincide (hence only one is plotted), here they differ, as seen from the mismatch between panels (a) and (b). Nonetheless, in the noninvasive case, the coarse-grained curves retain the overall shape of their fine-grained counterparts, unlike in Fig. 4.

VII. GENERALIZATION TO NONORTHOGONAL (POVM-BASED) COARSE GRADING

Throughout this work, we have modeled finite measurement resolution via an orthogonal coarse graining in the energy basis, represented by a set of projectors $\{\Pi_J\}$ forming a projective-valued measure (PVM), $\sum_J \Pi_J = \mathbb{1}$. This corresponds to a situation in which each microscopic energy eigenstate is assigned to exactly one macroscopic outcome (slot). While this captures finite resolution in the sense of limited distinguishability, it excludes scenarios in which the measurement device may also misidentify outcomes, so that a given microscopic energy may contribute to multiple macroscopic readouts. Such situations naturally arise in experiments where the reported energy is obtained by binning a continuous, noisy measurement signal, or by inferring the energy from measurements performed through an auxiliary degree of freedom (e.g., via population measurements or fluorescence

signals) [55–57], so that the same microscopic energy can lead to different macroscopic readouts with nonzero probability.

A consistent mathematical description of overlapping coarse graining is provided by replacing the PVM by a positive-operator valued measure (POVM) $\{F_J\}$, where each outcome J is associated with a positive operator $F_J \geq 0$ satisfying $\sum_J F_J = \mathbb{1}$. In the present thermodynamic setting, it is natural to restrict to POVMs that are diagonal in the energy eigenbasis $\{|\epsilon_j\rangle\}$, so that

$$F_J = \sum_j w_{J|j} |\epsilon_j\rangle\langle\epsilon_j|, \quad (30)$$

with $w_{J|j} \in [0, 1]$, $\sum_J w_{J|j} = 1 \forall j$. The coefficients $w_{J|j}$ admit a direct operational interpretation as conditional probabilities: $w_{J|j}$ is the probability that the measurement apparatus reports outcome J when the system is in the energy eigenstate $|\epsilon_j\rangle$. Overlapping coarse graining corresponds to the case in which, for a fixed j , more than one coefficient $w_{J|j}$ is nonzero. A physically motivated choice of such weights can be constructed by introducing nominal energy values $\{E_J\}$ associated with the measurement outcomes and a response function g describing the finite resolution of the detector, for instance,

$$w_{J|j} = \frac{e^{-(\epsilon_j - E_J)^2 / (2\sigma^2)}}{\sum_K e^{-(\epsilon_j - E_K)^2 / (2\sigma^2)}}, \quad (31)$$

where σ quantifies the energy uncertainty of the readout. In the limit $\sigma \rightarrow 0$, the POVM effects approach orthogonal projectors onto sharp energy intervals, recovering the PVM coarse graining employed in previous sections.

Given a POVM $\{F_J\}$, the natural analog of the coarse-graining operation introduced in Eq. (7) is a measure-and-reprepare channel of the form

$$\mathcal{C}_F(\rho) = \sum_J \text{Tr}[F_J \rho] \sigma_J, \quad (32)$$

where σ_J is the postmeasurement state associated with outcome J . A particularly natural and unbiased choice (which generalizes the flat reparation within each slot used in the orthogonal case) is $\sigma_J = F_J / \text{Tr}[F_J]$. With this choice, the channel is completely positive, trace preserving, and unital, since

$$\mathcal{C}_F(\mathbb{1}) = \sum_J \text{Tr}[F_J] \frac{F_J}{\text{Tr}[F_J]} = \sum_J F_J = \mathbb{1}. \quad (33)$$

In the special case $F_J = \Pi_J$, this construction reduces exactly to the coarse-graining map $\mathcal{C}_{\Pi_J}$ used throughout the paper. The observable effectively measured by the coarse-grained apparatus can be defined as

$$\check{H} = \sum_J E_J F_J, \quad (34)$$

which, for diagonal POVMs, takes the form $\check{H} = \sum_j \check{\epsilon}_j |\epsilon_j\rangle\langle\epsilon_j|$, with $\check{\epsilon}_j := \sum_J E_J w_{J|j}$. Hence, overlapping coarse graining induces a smooth deformation of the microscopic energy spectrum into effective energies $\check{\epsilon}_j$ reflecting both limited resolution and readout errors. Expectation values of the measured energy are then given by $\langle \check{H} \rangle_\rho = \text{Tr}[\rho \check{H}]$.

While this POVM-based framework provides a consistent and experimentally motivated model of overlapping coarse graining, it leads to qualitative differences with respect to the orthogonal case considered in this work. In particular, POVM outcomes do not define a decomposition of Hilbert space into disjoint energy sectors, and the associated coarse-graining channel cannot be interpreted as flattening independent subspaces. As a consequence, repeated coarse graining generally continues to modify the state, and the channel $\mathcal{C}_F$ is not idempotent, $\mathcal{C}_F^2 \neq \mathcal{C}_F$.

From a thermodynamic perspective, however, the more crucial structural property is unitality. For the choice $\sigma_J = F_J/\text{Tr}[F_J]$, the coarse-graining map $\mathcal{C}_F$ remains unital, which is the key condition ensuring the validity of quantum Jarzynski-type equalities for general quantum channels [58]. In contrast, if the coarse-graining instrument is not unital, additional correction terms appear in fluctuation relations [59,60], reflecting entropy production associated with the measurement backaction itself [61].

Despite unitality, the lack of idempotence and of a sharp sector structure has important consequences. In particular, coarse-grained thermal states $\mathcal{C}_F(\tau_\beta)$ do not generally retain an exact Gibbs form with respect to the effective Hamiltonian $\tilde{H}$, and the forward and backward coarse-grained dynamics are no longer related by a simple time-reversal symmetry. As a result, similarly to the case of TPMs with nonideal measurements [62,63], Crooks-type fluctuation relations are not generically satisfied in the presence of nonorthogonal (POVM-based) coarse graining.

Consequently, while Jarzynski relations may still be recovered for suitably chosen unital POVM-based coarse-graining maps, the invariance of the full set of thermodynamic relations derived in this work (most notably Crooks relations) relies on the stronger structure provided by orthogonal, projective coarse graining. Investigating the precise conditions under which Crooks-type relations and second-law-like inequalities persist for nonorthogonal coarse graining constitutes an interesting direction for future research.

VIII. DISCUSSION

In this work, we have explored the thermodynamic implications of coarse graining in the energy basis from an operational point of view. We have seen that coarse graining as a quantum operation has energetic implications for information thermodynamics, and that, consequently, to be thermodynamically consistent, coarse-grained energies must be selected in a very precise way. Such a choice of energies leads to the definition of coarse-graining thermal states, which are the basis for obtaining several known nonequilibrium relations by only using the information retrieved by an agent who does not dispose of measuring instruments of infinite precision. This focus on energy-diagonal coarse graining reflects a physical modeling choice tailored to finite energy resolution. More general forms of coarse graining, including restrictions on other degrees of freedom or coarse-grained observables not commuting with the Hamiltonian, may be described within a similar mathematical framework but can lead to qualitatively different thermodynamic behavior (a discussion of these cases and their limitations is provided in Appendix D).

We found that the fundamental thermodynamic relations remain formally identical to those obtained using instruments of infinite precision. However, the physical conclusions drawn from coarse-grained quantities can differ significantly from those based on their fine-grained counterparts. For example, when the underlying dynamics involve degrees of freedom that are inaccessible to the coarse-graining agent, the work consumed or produced during a driving protocol may deviate substantially from the value obtained using fine-grained instruments. Importantly, our assessment of work in this study focuses on the system dynamics and does not include additional thermodynamic costs associated with the implementation of the TPM scheme itself. Such costs (arising, for instance, from the realization of projective measurements [64,65]) can be incorporated by extending the TPM framework to account for nonideal measurement processes [62].

We illustrated our findings using a 12-level harmonic oscillator subjected to a thermodynamic evolution driven by a time-independent external force. While in the ideal (fine-grained) case the evolution neither requires nor produces work, we observed that increasing the driving strength leads to a growing amount of effective work production from the perspective of a coarse-grained agent. We also examined how the work probability distributions associated with the forward and time-reversed protocols change with the level of coarse graining, and showed that equilibrium free energies can still be accurately estimated from coarse-grained measurements of nonequilibrium work, regardless of the measurement precision.

In conclusion, our results establish a consistent thermodynamic framework for scenarios in which only partial information (limited by the finite energy resolution of experimental instruments) is available. We examined the impact of limited-precision measurements on the estimation of key quantum thermodynamic quantities and showed that, although the individual quantities depend on the measurement resolution, the fundamental relations among them remain agent independent. These findings open new perspectives in the investigation of the theory of quantum measurements in quantum thermodynamics and, in particular, in its relevance in any experimental scheme seeking to address the concepts of reversibility and the quantum thermodynamic time's arrow.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [66].

APPENDIX A: COARSE GRAINING AND INFORMATION THERMODYNAMICS

In modern approaches to thermodynamics, the second law is considered as a consequence of the fundamental link between energetics and information theory [2,3]. One of the most prominent examples of this strong connection was noticed by R. Landauer. Under the popular lemma “information is physical,” Landauer developed the idea that logically irreversible operations, such as the erasure of a memory, have an associated thermodynamic cost [67], hence providing a route for exorcizing Maxwell’s demon paradox [68]. Here, we explore the thermodynamic implications of having different levels of information as determined by the access to coarse-grained or fine-grained instruments with different levels of energy resolution.

1. Extracting work using fine-grained information

Let us now consider two agents, C and F , equipped with instruments of differing energy resolution. The coarse-grained agent C , whose instruments have limited precision, models the system using the coarse-grained Hamiltonian $\tilde{H}$ defined in Eq. (5). When the system is brought into contact with a thermal reservoir at inverse temperature β , C describes its equilibrium state as $\tilde{\tau}_\beta$ [Eq. (6)]. In contrast, the fine-grained agent F has access to higher-resolution instruments and assigns to the same system the fine-grained Hamiltonian $H = \sum_j \epsilon_j |\epsilon_j\rangle\langle\epsilon_j|$. Accordingly, from F ’s perspective, the equilibrium state of the system is τ_β [Eq. (3)] [69]. The origin of this discrepancy lies in the fact that agent C cannot distinguish τ_β from $\tilde{\tau}_\beta$: From their perspective, the two states are operationally equivalent. However, from agent F ’s viewpoint, the states are markedly different. In particular, F regards $\tilde{\tau}_\beta$ as a nonequilibrium state, and thus as a potential thermodynamic resource [3].

Let us now consider the situation in which the state $\tilde{\tau}_\beta$ is *precisely* prepared by some means and handed to agent F . With access to high-resolution instruments, F may then extract work from $\tilde{\tau}_\beta$ by implementing a thermodynamic protocol that cyclically manipulates the system’s Hamiltonian H , thereby transforming $\tilde{\tau}_\beta \rightarrow \tau_\beta$ while the system remains in contact with the environment [2]. The maximum amount of work extractable in such a process is bounded by the free-energy difference between the initial and final states [2,70,71]:

$$W_{\text{ext}} \leq \mathcal{F}(\tilde{\tau}_\beta) - \mathcal{F}(\tau_\beta) = k_B T S(\tilde{\tau}_\beta || \tau_\beta), \quad (\text{A1})$$

where $\mathcal{F}(\rho) \equiv \text{Tr}[H\rho] - k_B T S(\rho)$ is the nonequilibrium free energy of a generic state ρ with respect to the Hamiltonian H , which reduces to the equilibrium free energy F for thermal

states, and we have $\mathcal{F}(\rho) \geq \mathcal{F}(\tau_\beta) = F$ for any nonequilibrium state ρ . Above, in the right-hand side, we introduced the quantum relative entropy (or Kullback-Leibler divergence), $S(\rho || \sigma) \equiv \text{Tr}[\rho(\ln \rho - \ln \sigma)]$, a non-negative information-theoretic measure of the distinguishability between quantum states [72].

The bound in Eq. (A1) can be saturated (i.e., maximum work can be extracted) in the reversible limit (zero entropy production) [2]. The latter can be achieved, for example, by performing a sudden quench $H \rightarrow H' \equiv -k_B T \ln \tilde{\tau}_\beta$, followed by an isothermal quasistatic modulation of the Hamiltonian back to its initial form, $H' \rightarrow H$, while remaining in equilibrium with the thermal environment [74,75]. We remark that such work extraction protocol requires control over the fine-grained energy levels $|\epsilon_i\rangle$ of H within the slots $\mathcal{J}_j$, and therefore can only be done by using fine-grained instruments. Equation (A1) highlights the fact that a greater ability to detect and manipulate fine-grained degrees of freedom comes with an increased ability to extract work, in the same spirit as in Maxwell’s demon.

Now consider the reverse situation, in which the fine-grained thermal state τ_β is prepared and handed to the coarse-grained agent C . Suppose that C , attempting to replicate the protocol used by agent F in the previous scenario, aims to extract work from τ_β using their coarse-grained instruments. Due to the limited resolution of these instruments, however, agent C can only access the state after the coarse-graining operation has been applied, namely, $\mathcal{C}_{\Pi_j}(\tau_\beta)$. Applying an analogous protocol to the one described above, agent C would then extract an amount of work given by

$$\tilde{W}_{\text{ext}} = k_B T S(\mathcal{C}_{\Pi_j}(\tau_\beta) || \tilde{\tau}_\beta) = 0, \quad (\text{A2})$$

where the last equality is verified whenever $\mathcal{C}_{\Pi_j}(\tau_\beta) = \tilde{\tau}_\beta$, as required in the previous section [cf. Eq. (9)]. It is worth remarking the importance of such a condition to ensure thermodynamic consistency: If $\mathcal{C}_{\Pi_j}(\tau_\beta) \neq \tilde{\tau}_\beta$, the work extractable by the coarse-grain agent C in Eq. (A2) would be strictly positive. If that was possible, then a cycle could be constructed where agent F first prepares τ_β by thermalizing the system in contact with the thermal bath and then agent C extracts work from it, giving the system back to F for a new thermalization. Repeating such a cycle would lead to extracting an infinite amount of work from a single thermal bath, hence violating the second law of thermodynamics.

2. Free energy loss from coarse graining and work cost

In the above, we have seen that access to fine-grained degrees of freedom can increase one’s ability to extract work. In particular, we have seen that a positive amount of work can be extracted by agent F by transforming $\tilde{\tau}_\beta$ into τ_β . We now turn the situation around and focus on the opposite transformation, $\tau_\beta \rightarrow \tilde{\tau}_\beta$, that is, the preparation of state $\tilde{\tau}_\beta$. Interestingly, such an operation is quite different from the perspective of agent F or agent C . For example, if agent F prepares state τ_β and gives it to agent C , the transformation $\tau_\beta \rightarrow \mathcal{C}_{\Pi_j}(\tau_\beta) = \tilde{\tau}_\beta$ occurs spontaneously from the loss of precision entailed by the coarse-grained instruments, leading to a drop in the free energy from agent F ’s perspective:

$$\tilde{\mathcal{F}}(\tilde{\tau}_\beta) - \tilde{\mathcal{F}}(\tau_\beta) = S(\tau_\beta) - S(\tilde{\tau}_\beta) \leq 0, \quad (\text{A3})$$

where we denoted by $\check{\mathcal{F}}(\rho)$ the free energy with respect to $\check{H}$. The equality follows from energy conservation, $\text{Tr}[\check{H}\check{\tau}_\beta] = \text{Tr}[\check{H}\mathcal{C}_{\Pi_j}(\tau_\beta)] = \text{Tr}[\check{H}\tau_\beta]$, and the inequality from the fact that unital channels are entropy nondecreasing, hence $S(\check{\tau}_\beta) = S(\mathcal{C}_{\Pi_j}(\tau_\beta)) \geq S(\tau_\beta)$.

In contrast, a *precise* preparation of state $\check{\tau}_\beta$ from agent F 's perspective is a logically irreversible operation with thermodynamic consequences and an associated work cost, analogous to what happens in information erasure scenarios [2,68]. In particular, if the system is put in contact with a thermal bath at inverse temperature β , agent F will assign state τ_β to it, not $\check{\tau}_\beta$. To prepare $\check{\tau}_\beta$ instead, F must apply additional manipulations to transform τ_β into the desired state. Assuming control over the Hamiltonian H and access to the bath, that transformation can be done in an optimal way by combining again sudden quenches of H with quasistatic modulations (see Appendix B for details), which leads to a positive work cost lower bounded by

$$W_{\text{cost}} \geq \mathcal{F}(\check{\tau}_\beta) - \mathcal{F}(\tau_\beta) = k_B T S(\check{\tau}_\beta || \tau_\beta). \quad (\text{A4})$$

Notice that in opposition to agent C , from agent F 's perspective, the coarse-grained state $\check{\tau}_\beta$ has greater free energy than τ_β , which follows from the non-negativity of the quantum relative entropy $S(\check{\tau}_\beta || \tau_\beta) \geq 0$. This is indeed the reason why F is able to extract work from $\check{\tau}_\beta$ as captured by Eq. (A1). However, now we see that preparing the state $\check{\tau}_\beta$ *precisely* using fine-grained instruments has an equivalent work cost.

The right-hand side of Eq. (A4) provides the minimal work cost for preparing $\check{\tau}_\beta$ in a fine-grained perspective from the true thermal state τ_β , $W_{\text{min}} := k_B T S(\check{\tau}_\beta || \tau_\beta)$. An upper bound to that minimal cost can be obtained by using again the unitality of the map $\mathcal{C}_{\Pi_j}$, which implies that the change in nonequilibrium free energy is upper bounded by the average energy change under coarse graining:

$$W_{\text{min}} \leq \text{Tr}[H\check{\tau}_\beta] - \text{Tr}[H\tau_\beta] \leq \delta\epsilon. \quad (\text{A5})$$

The last inequality above is proven in Appendix B and shows that the (minimal) work cost remains below the precision of the coarse-grained instruments, $\delta\epsilon$, and therefore cannot be detected by agent C . Nevertheless, we stress that nonoptimal irreversible preparations of $\check{\tau}_\beta$ may lead to higher work costs that might also be detectable with coarse-grained instruments.

Finally, we emphasize that the work cost in Eq. (A4) needs to be paid if the state of the system is transformed from the point of view of agent F . Another way to view why there should be a cost in that situation is to consider that the transformation would require forgetting the information about the fine-grained degrees of freedom. This is in contrast to the viewpoint of agent C , which would ascribe the same (macro)state $\check{\tau}_\beta$ to many different fine-grained states, including τ_β , and therefore would not be able to detect any physical transition $\tau_\beta \rightarrow \check{\tau}_\beta$. However, such ignorance of the system's state may lead to *real* consequences when manipulating the system, as we shall see in the following sections.

APPENDIX B: WORK COST FOR THE FINE-GRAINED PREPARATION OF COARSE-GRAINED STATES

In this Appendix, we detail the construction of an optimal thermodynamic protocol for preparing the coarse-grained

thermal state $\check{\rho}_\beta$ from an arbitrary initial state ρ , using fine-grained instruments.

The protocol follows the approach used in Refs. [74,76,77], adapted to the present setting. It consists of the following three steps: (1) A first sudden quench of the Hamiltonian, $H \rightarrow H_0 \equiv -k_B T \ln \rho$, which induces no change in the system state ρ . (2) A quasistatic modulation of the Hamiltonian from $H_0 \rightarrow H_1 \equiv -k_B T \ln \mathcal{C}(\rho)$, while the system is maintained in thermal contact with the environment. (3) A second sudden quench transforming back $H_1 \rightarrow H$ without affecting the system state. In the above sequence, steps (1) and (3) occur on a timescale much shorter than the system's typical relaxation time into the reservoir, while step (2) is assumed to be a quasistatic evolution, slow compared to the relaxation timescale. Consequently, the action of the environment can be neglected during steps (2) and (3), and the system state remains unchanged, resulting in zero entropy change $\Delta S_1 = \Delta S_3 = 0$. The energy changes in these sudden quenches can thus be interpreted as work, $W_1 = \text{Tr}[(H_0 - H)\rho]$ and $W_3 = \text{Tr}[(H - H_1)\mathcal{C}(\rho)]$. In step (2), in contrast, both heat and work are exchanged, with the second law imposing that $Q_2 \leq k_B T \Delta S_2$, where ΔS_2 is the system's entropy change during the quasistatic evolution. When the modulation is sufficiently slow, the system remains approximately in thermal equilibrium at each moment, following a sequence of thermal states of the instantaneous Hamiltonian at inverse temperature β . In this quasistatic limit, the process becomes reversible, and the heat exchanged with the environment equals the entropy change times the temperature, $Q_2 = k_B T [S(\mathcal{C}(\rho)) - S(\rho)]$, where we used the fact that $\mathcal{C}(\rho) = e^{-\beta H_1}$. The work performed in this step is then $W_2 = \Delta E_2 - Q_2$, with $\Delta E_2 = \text{Tr}[H_1 \mathcal{C}(\rho)] - \text{Tr}[H_0 \rho]$. Summing up the three contributions, the total work invested in the protocol reads

$$\begin{aligned} W_{\text{cost}} &\geq W_{\text{min}} = F(\mathcal{C}(\rho)) - F(\rho) \\ &= k_B T [S(\mathcal{C}(\rho) || \tau_\beta) - S(\rho || \tau_\beta)], \end{aligned} \quad (\text{B1})$$

where the inequality in the first line follows from the fact that, in general, finite-time protocols absorb heat $Q_2 \leq \Delta S_2$, and hence $W_2 \geq \Delta E_2 - k_B T \Delta S_2$ (see also below). In the second line of Eq. (B1), we used the fact that $\text{Tr}[H\sigma] = -k_B T \text{Tr}[\sigma \ln \tau_\beta] - k_B T \ln Z$ for $\sigma = \rho, \mathcal{C}_{\Pi_j}(\rho)$.

Equation (A4) implies that, in order to prepare $\check{\tau}_\beta$ at the fine-grained level, a nonzero input work is needed whenever the state $\mathcal{C}_{\Pi_j}(\rho)$ is farther from equilibrium than the initial state ρ . This difference is accounted for by the distinguishabilities of $\mathcal{C}_{\Pi_j}(\rho)$ and ρ from the thermal state τ_β through the quantum relative entropy. In particular, when the initial state is already thermal, i.e., $\rho = \tau_\beta$, the second term on the right-hand side of Eq. (A4) vanishes, and we recover the (strictly positive) cost $W_{\text{cost}} \geq k_B T S(\check{\tau}_\beta || \tau_\beta)$ as reported in Eq. (A4), which matches the extractable work in the reverse process [cf. Eq. (A1)].

The minimal work cost in Eq. (B1) is achieved when the protocol described above is implemented with a clear separation of timescales between the system's relaxation into the thermal bath and the modulation of the Hamiltonian. This corresponds to a reversible process, characterized by zero entropy production. In all other cases, the coarse-graining operation

results in nonzero entropy production:

$$\begin{aligned}\Sigma &= S[\mathcal{C}_{\Pi_J}(\rho)] - S(\rho) - \beta Q \\ &= \beta(W_{\text{cost}} - F[\mathcal{C}_{\text{cost}}(\rho)] + F(\rho)) \geq 0,\end{aligned}\quad (\text{B2})$$

where Q is the total heat absorbed by the system during the process. This additional entropy production comes with an extra work cost, leading to the inequality $W_{\text{cost}} \geq W_{\text{min}}$ in Eq. (B1).

Finally, we prove that the energy change in the system due to the coarse-graining operation, $\Delta E_C := \text{Tr}[H\mathcal{C}_{\Pi_J}(\rho)] - \text{Tr}[H\rho]$, lies below the minimum energy resolution of their experimental instruments:

$$\begin{aligned}\Delta E_C &= \sum_J \left(\bar{E}_J \text{Tr}[\rho \Pi_J] - \sum_{j \in \mathcal{J}_J} \epsilon_j \langle \epsilon_j | \rho | \epsilon_j \rangle \right) \\ &\leq \sum_J (\bar{E}_J - E_J^{\min}) \text{Tr}[\rho \Pi_J] \leq \delta \epsilon \sum_J \text{Tr}[\rho \Pi_J] \\ &= \delta \epsilon.\end{aligned}\quad (\text{B3})$$

The above inequality implies that, from the perspective of agent C , the associated energy change is negligible. Consequently, by the left-hand inequality in Eq. (A5), $W_{\text{min}} \leq \Delta E_C$, the minimum work cost of coarse graining is also negligible for agent C .

APPENDIX C: EFFECT OF COMMUTING DRIVING ON COARSE-GRAINED WORK STATISTICS

One might wonder whether different behavior would arise if the driving and initial Hamiltonians commuted at all times, meaning that no transitions between energy eigenstates would be induced by the dynamics. In this case, the evolution would preserve the energy eigenbasis, and the dynamics could be considered effectively classical as populations would not be redistributed by the unitary evolution.

To explore this scenario, we examine a modified driving protocol where the driven Hamiltonian is expressed as $H_\tau = \hbar\omega(a^\dagger a + 1/2) + \hbar\chi N^2$, where $N = a^\dagger a$ and χ is the third-order susceptibility [78]. This term is formally analogous to the Kerr nonlinearity found in quantum optics, where an intensity-dependent refractive index gives rise to an energy shift proportional to the square of the photon number [79]. We therefore refer to this as a Kerr-type commuting drive. Since $[H_0, H_\tau] = 0$, the time evolution operator is diagonal in the energy eigenbasis and transitions between different energy levels do not occur.

The resulting coarse-grained dissipative work, shown in Fig. 7, exhibits trends that are highly similar to those obtained for the noncommuting drive discussed in Sec. III C. In particular, the dependence on driving strength and coarse-graining resolution follows the same qualitative behavior. This suggests that the main features observed in the coarse-grained thermodynamic quantities are not primarily influenced by the occurrence of quantum transitions between energy eigenstates.

This similarity can be explained by the fact that the effects reported in the main text are primarily governed by the coarse graining of the energy spectrum and the thermal weighting within each coarse-grained energy slot rather than

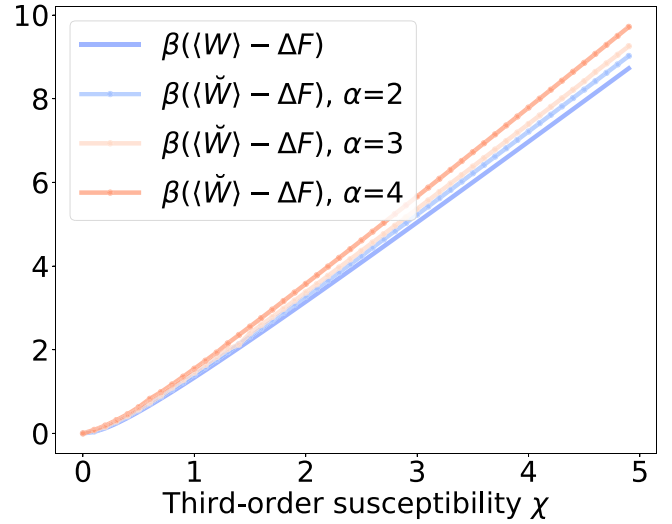


FIG. 7. Trends of fine- and coarse-grained dissipative work for a harmonic oscillator under a Kerr-type commuting drive. The initial state is the thermal state of the harmonic oscillator Hamiltonian $H_0 = \hbar\omega(a^\dagger a + 1/2)$, and the final state is generated by time-independent, number-conserving driving applied for $t \in (0, \tau]$, such that $H_\tau = \hbar\omega(a^\dagger a + 1/2) + \hbar\chi N^2$, satisfying $[H_0, H_\tau] = 0$. The quantities plotted and all parameters are the same as in Fig. 4, with coarse graining performed analogously using the same resolution parameter α . Despite the absence of quantum transitions between energy eigenstates, the qualitative dependence of dissipative work on χ and on the coarse-graining resolution closely mirrors that of the noncommuting driving case. This indicates that the observed coarse-graining effects are largely insensitive to whether the driving induces transitions or only reshapes the energy spectrum.

by the presence or absence of quantum transitions between fine-grained energy eigenstates. Consequently, coarse-grained thermodynamic quantities, such as average work and free-energy differences, are largely insensitive to whether the underlying dynamics generate coherences, provided the same coarse-graining prescription is used. Nevertheless, the observed similarity in coarse-grained thermodynamic quantities does not imply that the underlying dynamics are equivalent. In particular, quantities that probe the structure of the transition probabilities, such as the presence of off-diagonal contributions or measures of transition-induced mixing, vanish identically in the commuting case, yet are generally nonzero when the driving does not commute with the Hamiltonian. While these differences are washed out by coarse graining at the level of average thermodynamic quantities, they highlight the genuinely quantum nature of noncommuting dynamics and clarify in what sense coarse graining can obscure distinctions between classical and quantum driving scenarios.

APPENDIX D: COARSE GRAINING AS LIMITED ENERGY RESOLUTION

In this work, the coarse-graining procedure is restricted to being diagonal in the energy eigenbasis. This is not just a technical simplification, but a physical modeling assumption tailored to the thermodynamic scenario of interest. The framework is designed to capture how thermodynamic quantities and relations are modified when the system's energy can only

be resolved up to finite experimental precision, rather than to describe arbitrary forms of information loss on the quantum state.

More generally, one could also consider coarse-graining procedures acting on other microscopic degrees of freedom, implemented through projective measurements that are not functions of the Hamiltonian. This type of coarse graining can be described using the same mathematical framework adopted here: Projectors can be used to define a partition of Hilbert space, along with an associated coarse-graining map. However, whether thermodynamic relations remain valid in this case depends crucially on the commutation properties between the coarse-grained observables and the Hamiltonian. When coarse graining does not commute with energy, it generally mixes energy populations and coherences, introducing additional physical disturbance rather than merely reflecting limited measurement resolution. This type of mixing is typical of dynamics generated by interactions that do not commute with the Hamiltonian and has observable consequences for work statistics and entropy production [80–82]. As a result, such coarse graining requires a separate thermodynamic analysis and cannot be interpreted simply as a restriction on the observer's access to information.

In contrast, one may consider more general coarse-graining schemes that still probe the energy but with finite accuracy, rather than restricting to sharp projective measurements. A first natural extension is to allow for unsharp measurements

that remain diagonal in the energy basis, described by POVMs with overlapping effects, which model readout noise or imperfect calibration. As discussed in Sec. VII, the associated coarse-graining map is then no longer idempotent and coarse-grained thermal states do not generally retain an exact Gibbs form, so several of the exact thermodynamic identities derived here no longer hold without further assumptions.

Another class of generalizations consists of channels that preserve energy populations, but mix coherences within degenerate subspaces or between nearby energies. This may arise from uncontrolled, yet energy-preserving, interactions with an environment. Although these processes leave the average energy unchanged, they modify work statistics and generally alter fluctuation relations. This requires a reformulation of the thermodynamic framework, as evidenced by studies showing that quantum coherence contributes additional terms to work distributions [81,83] and affects fluctuation theorems for general quantum channels [82,84].

Finally, fully general coarse-graining operations that do not respect the energy eigenspaces correspond to open-system dynamics rather than to limited measurement resolution. In this regime, neither thermal fixed points nor work definitions based on the system Hamiltonian are preserved [30,85], and the resulting entropy production cannot be attributed solely to coarse graining. Investigating thermodynamic relations under such general dynamical coarse graining is therefore a distinct problem, beyond the scope of the present work.

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- [50] Here, we describe the resolution in terms of energy levels rather than an absolute energy scale $\delta\epsilon$ because the harmonic oscillator has equally spaced levels. In this case, an inability to resolve fewer than three energy levels automatically corresponds to a well-defined $\delta\epsilon$.
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