

Witnessing quantum non-Gaussianity from intensity moments

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Direct measurement of quantum non-Gaussianity requires some variant of a discrete photon-resolving detection, which is feasible only for low mean photon numbers. For a large mean photon number, intensity detection by linear photodiodes provides a continuous signal; therefore, the Fock probabilities of the unknown input state are not directly available. On the other hand, intensity moments can be measured directly, and photon-number moments can be estimated. Therefore, we derive and analyze a quantum non-Gaussianity witness based solely on the photon-number mean and variance (or alternatively, the second-order correlation $g^{(2)}$) of an unknown state. Due to the simplicity of the used photon-number moments, the measurement results are easy to correct for losses and additive noise. We provide examples of simple amplification-based measurement schemes where our witness can be applied directly, thereby opening pathways to proof-of-principle tests and applications.

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I. INTRODUCTION

Quantum non-Gaussian (QNG) states are states that cannot be expressed as statistical mixtures of Gaussian states. Quantum non-Gaussianity rules out any models based on linearized open-system dynamics of a harmonic oscillator, described by a mixture of Gaussian trace-preserving completely positive maps that evolve the system from its ground state [1]. Such dynamics can only produce a classical mixture of Gaussian states in phase space; therefore, it yields only classical non-Gaussianity of the Gaussian states. Gaussian states can always be generated by maximally quadratic Hamiltonians, and they involve arbitrary coherent classical displacement and nonclassical quadrature squeezed states. Therefore, quantum non-Gaussianity and its analysis can reveal genuine nonlinear quantum dynamics beyond such a semiclassical regime through measurable statistics.

Furthermore, many bosonic protocols, such as quantum error correction, require more than just classical Gaussian states or their mixtures [2,3]. These bosonic codes rely fundamentally on quantum non-Gaussian states. In quantum optics, the core components of these codes have already been examined [4,5]. Quantum codes and error correction are vital for fault-tolerant quantum computing and could also play a key role in quantum communication and networking. Moreover, optical non-Clifford gates on bosonic codes demand quantum non-Gaussian magic states as auxiliary resources [6,7]. Additionally, quantum non-Gaussian states of light

are investigated for their potential applications in quantum sensing [8].

In all-optical systems, several experimental breakthroughs have been achieved in recent years in the preparation and manipulation of non-Gaussian states. These include the creation and control of squeezed cat states [9], remote state preparation [10,11], multimode non-Gaussianity by mode-selective photon subtraction [12], the iterative growth of QNG states using a quantum-memory cavity [13], the ultrafast generation of non-Gaussian states using broadband parametric amplifiers [14], the addition of multiple photons to coherent states of light [15], the certification of non-Gaussian qubit coherence [16], and a loop-based optical circuit that can sequentially perform squeezing gates while maintaining states' Wigner negativity [17].

Beyond quantum optics, quantum non-Gaussian states have numerous applications across diverse areas. In trapped ions and superconducting circuits, they are indispensable resources for investigations of bosonic quantum sensors [18–22] and bosonic quantum computing [2,3,23] both for error correction [24–29] and for universality because Gaussian states can be efficiently simulated on a traditional computer [30].

At many platforms, however, the preparation and detection of non-Gaussian states is still far from trivial, and therefore, simple and easily implementable criteria to verify quantum non-Gaussianity are very useful. This situation is especially limiting in quantum optics, as the light is control and probe of many other matter systems. While certainly providing the necessary information, full state tomography does not qualify as either simple to perform or, in the absence of Wigner negativity, simple to evaluate.

Several witnesses of quantum non-Gaussianity have been proposed in the literature, tailored to specific practical scenarios. Generally, they rely on identifying observable statistics that are unattainable by any mixture of Gaussian states. For example, Ref. [31] defines a witness based on the value of the

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Wigner function at the origin of the phase space $W(0, 0)$, and the mean photon number m , and proves that $W(0, 0)$ cannot be lower than a certain function of m for any mixture of Gaussian states. Similarly, the single-photon probability p_1 cannot be lower than a function of the vacuum probability p_0 for any mixtures of Gaussians [32,33]. The latter approach has been thoroughly studied and extended over the years [34–36], including a witness based on the mean photon number beside the vacuum probability p_0 [37]. Other works rely on alternative quasiprobability distributions [38,39], generalized squeezing [40], expectation of the operator $e^{-c\hat{x}^2} + e^{-c\hat{p}^2}$ as a function of c [41]. Most recently, a procedure for detecting non-Gaussianity in cluster states based on non-Gaussian nullifiers has been proposed in Ref. [42].

The notion of quantum non-Gaussianity has been extended to form hierarchies (with the corresponding witnesses) [43–45]: A state possessing n -photon non-Gaussianity is a state that cannot be expressed as a mixture of squeezed displaced lower-order Fock states. In this sense, the notion of quantum non-Gaussianity discussed in this work corresponds to single-photon non-Gaussianity.

In experimental schemes involving large-gain amplification to compensate for subsequent loss and noise, individual Fock probabilities might be inaccessible due to the magnitudes of intensities created. However, photon-number moments after and before amplification can be estimated (including loss and noise correction) from the integrated-intensity moments [46–48], which provides a strong motivation to derive a non-Gaussianity witness based on moments only. Simultaneously, a single-mode pump of the amplifier effectively filters other modes of the signal; therefore, the output state is close to perfect single mode. Regarding nonclassicality, multiple witnesses have been formulated in terms of photon-number and integrated-intensity moments [48]; however, similar criteria for non-Gaussianity are missing from the literature. In this article, therefore, we *ab initio* introduce a quantum non-Gaussianity witness in terms of the photon-number mean and variance. Analogously to the probability-based witness, for a fixed value of the mean photon number $m \equiv \langle \hat{N} \rangle$, the photon-number variance $s^2 \equiv \langle (\hat{N} - m)^2 \rangle$ cannot go below a certain value (dependent on m) if the state is a single-mode mixture of Gaussian states. Due to the fact that it is based on the first two moments, the proposed witness behaves favorably regarding additive noise and loss, which is easy to exploit.

First, we provide the analytic form of this moment-based witness and provide proof of its validity. Then, we suggest a few experimental schemes involving amplification where it is convenient to use. Finally, we analyze the applicability of the proposed witness for lossy Fock states and lossy photon-added thermal states, and discuss how to treat loss and additive noise in general.

II. MOMENT-BASED QNG WITNESS

Mixtures of Gaussian states in a single mode can be represented through their Wigner functions as

$$W(x, p) = \int dP(\lambda) W_\lambda(x, p)$$

for some index variable λ and $P(\lambda)$ denoting a probability measure on the set of possible values of λ . The functions $W_\lambda(x, p)$ are pure Gaussian states obeying the Heisenberg uncertainty principle:

$$W_\lambda(x, p) \propto \exp \left\{ -\frac{1}{2} \begin{pmatrix} x - x_\lambda \\ p - p_\lambda \end{pmatrix}^T C_\lambda^{-1} \begin{pmatrix} x - x_\lambda \\ p - p_\lambda \end{pmatrix} \right\},$$

where x_λ and p_λ denote mean quadratures and C_λ the covariance matrix of the two quadratures, $\sqrt{\det C_\lambda} \geq 1/4$.

For such states, the first two photon-number moments can be calculated as

$$\begin{aligned} \langle \hat{N} \rangle &= \int dx dp \left(x^2 + p^2 - \frac{1}{2} \right) W(x, p) \\ &= \int dP(\lambda) \int dx dp \left(x^2 + p^2 - \frac{1}{2} \right) W_\lambda(x, p) \\ &= \int dP(\lambda) \langle \hat{N}_\lambda \rangle, \end{aligned} \quad (1)$$

$$\begin{aligned} \langle \hat{N}^2 \rangle &= \int dx dp [(x^2 + p^2)^2 - (x^2 + p^2)] W(x, p) \\ &= \int dP(\lambda) \int dx dp (x^2 + p^2)^2 W_\lambda(x, p) \\ &\quad - \int dP(\lambda) \langle \hat{N}_\lambda \rangle - \frac{1}{2} = \int dP(\lambda) \langle \hat{N}_\lambda^2 \rangle, \end{aligned} \quad (2)$$

where for the first equality of Eq. (2), we used the identity that the symmetrically ordered second moment can be given as $(\hat{a}^{\dagger 2} \hat{a}^2)_S = \hat{N}^2 + \hat{N} + 1/2$. That is, for mixtures, noncentered photon-number moments are the appropriate averages of the component moments. Note that the second moment defined in Eq. (2) does not involve ordering, that is, $\langle \hat{N}^2 \rangle = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle$. We will use the notations $m \equiv \langle \hat{N} \rangle$ and $s^2 \equiv \langle \hat{N}^2 \rangle - m^2$ for the photon-number mean and variance, respectively. Note that although the photon-number variance cannot be, in general, calculated as the average of the component variances, given a fixed value of m , minimizing s^2 is equivalent to minimizing the noncentered second moment $\langle \hat{N}^2 \rangle$.

For dim states (states for which the probability of observing three or more photons is negligible), the Hanbury Brown-Twiss coincidence scheme provides a straightforward way to estimate vacuum and single-photon probabilities p_0 and p_1 . For any given value of p_0 , there is a maximal value of p_1 that a mixture of Gaussian states can possibly attain [33]. Therefore, a measurement result where the single-photon probability exceeds this limit indicates quantum non-Gaussianity.

Similarly, we will show that mixtures of Gaussian states in a single mode have a minimal photon-number variance for a fixed mean photon number. This minimum-variance curve can be given by the following parametric equation:

$$m_{\text{NG}}(r) = \frac{e^{6r}}{4} - \frac{1}{2} + \frac{e^{-2r}}{4}, \quad (3)$$

$$s_{\text{NG}}^2(r) = \frac{3e^{4r}}{8} - \frac{1}{2} + \frac{e^{-4r}}{8}, \quad (4)$$

with $r \geq 0$. In other words, there are pairs of values of m and s^2 inaccessible to mixtures of Gaussian states (area below the orange line in Fig. 1), and this can be used to witness the non-

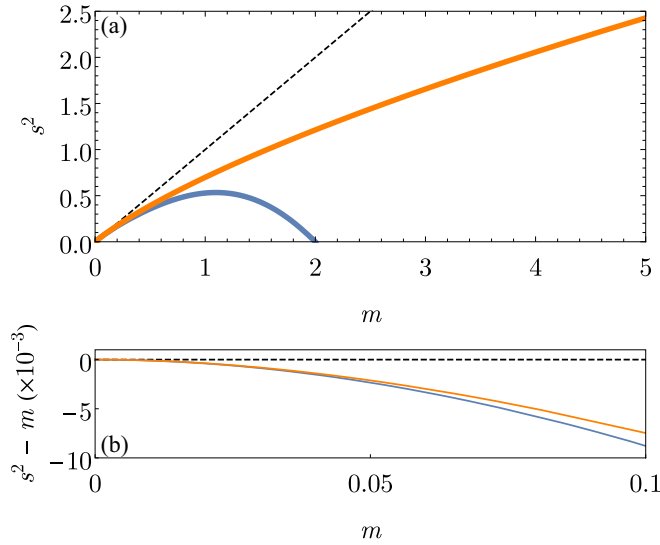


FIG. 1. Comparison of QNG witnesses. (a) In terms of photon-number mean and variance. The blue line shows the border of the converted Fock probability-based witness, and the orange line shows the moment-based witness: The latter has an extended applicability toward bright states. The areas below either line correspond to pairs of values indicating quantum non-Gaussianity. The dashed black line shows the border of the nonclassicality witness, $s^2 < m$. (b) The same lines in terms of $s^2 - m$ and m for a narrower range of mean values, showing that the two witnesses are equivalent for low values of m .

Gaussianity of quantum states of light even if their Wigner function is strictly positive. While a bit cumbersome, we can also express the analytic witness of quantum non-Gaussianity in a nonparametric form as

$$m > \frac{(4s^2 + \sqrt{16s^2(s^2 + 1) + 1} + 2)^{3/2}}{12\sqrt{3}} + \frac{\sqrt{3}}{4\sqrt{4s^2 + \sqrt{16s^2(s^2 + 1) + 1} + 2}} - \frac{1}{2}. \quad (5)$$

As detailed in Sec. I, many witnesses of quantum non-Gaussianity have been developed, each applicable in different scenarios. Since neither of these is based solely on photon-number moments, and also because the use cases are different, providing a fair comparison to the proposed witness is not straightforward. Nevertheless, for the sake of simplicity, we transformed the vacuum- and single-photon-probability-based witness from Ref. [33] to photon-number mean m and variance s^2 (see the Appendix for derivation). Figure 1 shows this transformed witness with a blue line. From Fig. 1(b), we can see that the two witnesses become equivalent as $m \ll 1$: Using the right-hand side of Eq. (5), we obtain that

$$m_{\text{NG}} = s_{\text{NG}}^2 + s_{\text{NG}}^4 + \mathcal{O}(s_{\text{NG}}^6),$$

which is the same as the low-mean behavior of the probability-based witness [33]. However, for larger values of the mean photon number m , the moment-based witness considerably relaxes the condition for identifying QNG. Note that this is

not necessarily a fair comparison (the transformation of the probability-based witness is only exact if the probability of observing three or more photons, p_{3+} , is zero); you will find a better comparison in terms of non-Gaussianity depth in Sec. IV. Also, the areas of applicability are different: The main advantage of the moment-based witness is that it is suitable in scenarios involving amplification.

In the following, we first show the step-by-step proof for the proposed moment-based witness [Eqs. (3) and (4)]. We will also discuss the extension of the witness in the multimode setting for the two extremes: independent modes and identical modes. We furthermore provide equivalent formulations of the witness where the variance in the witness Eq. (4) is replaced by the appropriate values of the noncentered second moment $\langle \hat{N}^2 \rangle$, the second integrated-intensity moment $\langle W^2 \rangle$, and the second-order correlation function $g^{(2)}(0)$.

A. Proof

The proof is structured as follows: we will take an arbitrary mixture and show that each component's contribution to the second moment of the mixture can be reduced by replacing it with a specific optimal displaced squeezed vacuum without changing the contribution to the mean photon number. That is, we can reduce the photon-number variance of any Gaussian mixture by replacing each component with the appropriate displaced squeezed vacuum. Then, we will show that the second moment of the photon number for such optimal displaced squeezed vacua is a convex function, and therefore, the minimum-variance Gaussian mixture for a fixed value of the mean photon number consists of a single displaced squeezed vacuum state, the photon numbers of which can be described by Eqs. (3) and (4). Note that throughout this article, we use the convention $\hat{x} = (\hat{a} + \hat{a}^\dagger)/2$, $\hat{p} = (\hat{a} - \hat{a}^\dagger)/2i$ for the quadrature operators; however, this choice does not have any bearing on the functional form of the witness [Eqs. (3) and (4)] itself since that only involves photon-number moments.

1. Parametrization of Gaussian states

Any single Gaussian state can be constructed from a thermal state with mean photon number $\bar{n}$ by first squeezing it along the x axis with squeezing parameter r , then rotating it about the origin by an angle φ , and finally displacing it by (d_x, d_p) . To make the construction unique, without the loss of generality, we will restrict ourselves to $r \geq 0$ and $\varphi \in [0, \pi)$. In this parametrization, the Wigner function of the resulting state is a two-dimensional Gaussian distribution centered at (d_x, d_p) with the covariance matrix

$$C = \sigma^2 \begin{bmatrix} e^{-2r} \cos^2 \varphi + e^{2r} \sin^2 \varphi & -\sinh 2r \sin 2\varphi \\ -\sinh 2r \sin 2\varphi & e^{-2r} \sin^2 \varphi + e^{2r} \cos^2 \varphi \end{bmatrix},$$

where we introduced the notation $\sigma^2 \equiv (2\bar{n} + 1)/4$ for the variance of the initial thermal state. We will further parametrize any Gaussian state through the tuple $(\sigma, r, \varphi, d_x, d_p)$. Plugging this state into Eqs. (1) and (2), we

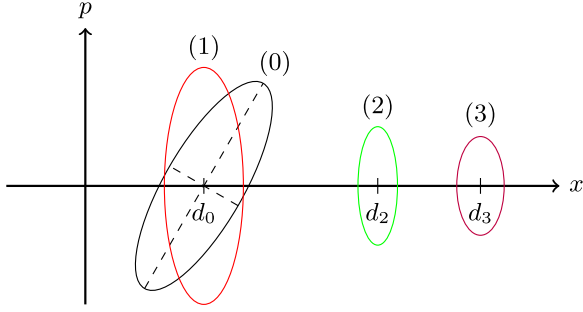


FIG. 2. Steps of optimizing the photon-number variance of a Gaussian state without changing the mean photon number. We start off with an arbitrary Gaussian state represented by ellipse (0), and first rotate it about its center ($d_0, 0$) (1), and then increase its distance from the origin while keeping its squeezing parameter unchanged until we arrive at the displaced squeezed vacuum marked (2). State (3) represents the minimal photon-number variance displaced squeezed vacuum given the mean photon number and has the moments described in Eqs. (3) and (4).

obtain for the mean and the variance of the photon number of this state:

$$m = -\frac{1}{2} + \text{Tr } C + d_x^2 + d_p^2, \quad (6)$$

$$s^2 = -\frac{1}{4} - 4\sigma^4 + 2(\text{Tr } C)^2 + 4\sigma^2 e^{-2r} (d_x \cos \varphi + d_p \sin \varphi)^2 + 4\sigma^2 e^{2r} (-d_x \sin \varphi + d_p \cos \varphi)^2, \quad (7)$$

where $\text{Tr } C = 2\sigma^2 \cosh 2r$.

2. Orientation of the state

Let us look at a single component of the mixture and choose a frame of reference in which its center lies on the x axis, in $x = d_0$ [see ellipse (0) in Fig. 2]. We will denote its parameters by $(\sigma_0, r_0, \varphi_0, d_0, 0)$. The corresponding photon-number mean m_0 and variance s_0^2 from Eqs. (6) and (7) are

$$m_0 = -\frac{1}{2} + d_0^2 + \text{Tr } C_0, \quad (8)$$

$$s_0^2 = -\frac{1}{4} - 4\sigma_0^4 + 2(\text{Tr } C_0)^2 + 4d_0^2 \sigma_0^2 e^{-2r} + 2d_0^2 \sigma_0^2 \cosh 2r_0 \sin^2 \varphi_0. \quad (9)$$

Since the trace of C_0 does not depend on the value of the rotation angle φ_0 , neither does m_0 . Therefore, the first goal is to find the value of φ that minimizes s_0^2 while leaving r_0 and d_0 (and, as a consequence, m_0) unchanged. Looking at Eq. (9) and keeping in mind that we assumed $r_0 \geq 0$ and $\varphi_0 \in [0, \pi)$, this is trivially achieved by setting the rotation angle to zero. In other words, reorienting the state such that the eigendirection of its covariance matrix corresponding to the smaller eigenvalue is aligned with the x axis results in a lower photon-number variance while leaving the mean photon number unchanged.

This improved state is represented by the ellipse (1) in Fig. 2 and has the parameters $(\sigma_1 = \sigma_0, r_1 = r_0, \varphi_1 = 0, d_1 = d_0, 0)$. The corresponding covariance matrix and

photon-number moments are

$$C_1 = \sigma_0^2 \times \begin{bmatrix} e^{-2r_0} & 0 \\ 0 & e^{2r_0} \end{bmatrix} \quad (10)$$

and

$$m_1 = m_0 = -\frac{1}{2} + d_0^2 + 2\sigma_0^2 \cosh 2r_0, \\ s_1^2 = -\frac{1}{4} - 4\sigma_0^4 + 2(\text{Tr } C_1)^2 + 4d_0^2 \sigma_0^2 e^{-2r_0} \\ = -\frac{1}{4} - 4\sigma_0^4 + 8\sigma_0^4 \cosh^2 2r_0 + 4d_0^2 \sigma_0^2 e^{-2r_0}. \quad (11)$$

3. Lower-variance displaced squeezed vacuum

Now that we have arrived at state (1), let us compare it with the displaced squeezed vacuum (2) in Fig. 2 that has the same mean photon number and the same value of the squeezing parameter. That is, state (2) is represented by the tuple $(\sigma_2 = 1/2, r_2 = r_0, \varphi_2 = 0, d_2 \neq d_0, 0)$, and therefore has the covariance matrix

$$C_2 = \frac{1}{4} \begin{bmatrix} e^{-2r_0} & 0 \\ 0 & e^{2r_0} \end{bmatrix}.$$

Using that the mean photon numbers are equal in states (1) and (2), we obtain

$$d_0^2 + 2\sigma_0^2 \cosh 2r_0 = d_2^2 + \frac{1}{2} \cosh 2r_0. \quad (12)$$

Turning to variances, we have

$$s_2^2 = -\frac{1}{2} + \frac{1}{2} \cosh^2 2r_0 + d_2^2 e^{-2r_0}. \quad (13)$$

Therefore, using Eqs. (11)–(13), we obtain

$$s_1^2 - s_2^2 = e^{-2r_0} \left(\sigma_0^2 - \frac{1}{4} \right) \\ \times [4d_0^2 + e^{2r_0} (-1 + 4\sigma_0^2 \cosh 4r_0 + \sinh 4r_0)].$$

Each term in this equation is non-negative: $\sigma_0^2 - \frac{1}{4} \geq 0$, $d_0^2 \geq 0$, $-1 + 4\sigma_0^2 \cosh 4r_0 \geq -1 + 4\sigma_0^2 \geq 0$, and $\sinh 4r_0 \geq 0$, since we use non-negative values only for the squeezing parameter. As a consequence, $s_2^2 \leq s_1^2$, so we achieved a decrease in the variance without changing the mean photon number ($m_2 = m_1 = m_0$). The moments of state (2) are then

$$m_2 = -\frac{1}{2} + d_2^2 + \frac{1}{2} \cosh 2r_0 = d_2^2 + \sinh^2 r_0, \\ s_2^2 = -\frac{1}{2} + \frac{1}{2} \cosh^2 2r_0 + d_2^2 e^{-2r_0} = \frac{\sinh^2 2r_0}{2} + d_2^2 e^{-2r_0}.$$

4. Optimal displaced squeezed vacuum

Now, we will find the displaced squeezed vacuum with the minimal variance for a fixed value of the mean photon number m . This optimal squeezed vacuum is represented by ellipse (3) in Fig. 2 and has the parameters $(\sigma_3 = 1/2, r_3 = r \neq r_0, \varphi_3 = 0, d_3 \neq d_2, 0)$. The optimization problem is then

$$m_3 = d_3^2 + \sinh^2 r = m_0 = \text{fixed}, \\ s_3^2 = \frac{\sinh^2 2r}{2} + e^{-2r} d_3^2 \\ = \frac{\sinh^2 2r}{2} + e^{-2r} (m_0 - \sinh^2 r) = \text{min.}$$

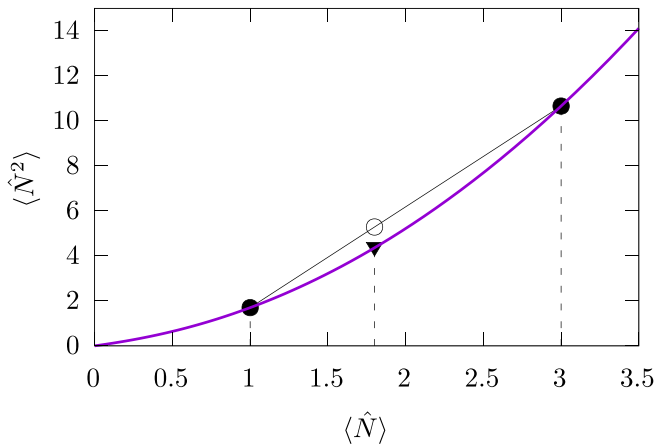


FIG. 3. Behavior of the QNG witness [Eqs. (3) and (4)] for a mixture of two optimal displaced squeezed vacua. The purple line shows the noncentered second moment of an optimal displaced squeezed vacuum as a function of its mean photon number, that is, $s_{\text{NG}}^2(r) + m_{\text{NG}}^2(r)$ as a function of $m_{\text{NG}}(r)$ ($r \geq 0$). The mixture components are shown with black disks, with mean photon numbers one and three. Any state corresponding to their mixture lies in this representation on the thin black line connecting the two black disks. The black circle corresponds to a 60%–40% mixture of the initial states, which has a mean photon number of 1.8. However, since the purple line represents a convex function, we obtain a lower second moment for a single optimal displaced squeezed vacuum with the same mean, 1.8 (black triangle).

This problem is straightforward to solve analytically by requiring $ds_3^2/dr = 0$; this yields $0 = 1 + e^{8r} - 2e^{2r} - 4e^{2r}m_0$. While the latter equation does not allow us to express the optimal r as a function of m_0 in a closed form, it clearly works the other way around, yielding the parametric form in Eqs. (3) and (4). We can also express the optimal displacement as $d_3^2 = e^{2r}(e^{4r} - 1)/4$, which is non-negative for $r \leq 0$, so there are no issues with nonphysicality.

5. Convexity of the second moment of the optimal displaced squeezed vacuum

So far, we have shown that any mixture of Gaussian states can be replaced by a mixture of optimal displaced squeezed vacua that have the same mean m but a smaller variance s^2 , or equivalently, a smaller noncentered second moment $\langle \hat{N}^2 \rangle$. Here, we will show that the second moment $\langle \hat{N}^2 \rangle$ of optimal displaced squeezed vacua is a convex function of the mean photon number m . From the parametric form in Eqs. (3) and (4), we obtain

$$\frac{d^2}{dm_{\text{NG}}^2}(s_{\text{NG}}^2 + m_{\text{NG}}^2) = 2 - \frac{4}{3e^{8r} - 1},$$

which is non-negative for any $r \geq 0$, implying convexity. Using this convexity, we can again apply Jensen's inequality to show that the second moment of the photon number of a single optimal displaced squeezed vacuum with a given mean m is smaller than that of the mixture of multiple optimal displaced squeezed vacua with the same mean photon number. Figure 3 demonstrates Jensen's inequality for two components in the mixture. For a multicomponent mixture, one can replace

any pair of components by the optimal displaced squeezed vacuum having the same mean photon number as the average of the two components and continue on with this procedure until only a single component remains.

That is, we have shown that for a fixed value m of the mean photon number, the mixture of Gaussian states that has the smallest value of $\langle \hat{N}^2 \rangle$ consists of a single Gaussian state, namely, an optimal displaced squeezed vacuum. Note that, of course, a phase-randomized version of this state produces the same photon-number moments.

B. Multimode case

Previously, we assumed that the amplification process with a single-mode pump efficiently filters out a single output mode before the detection. However, we still discuss here some aspects for multimode light. While tackling the general case of arbitrarily interdependent modes is not feasible, the two extremes of either all modes being independent or all modes being identical are instructive.

1. Independent modes

Considering M independent (uncorrelated) modes, measuring the total number of photons ($\hat{N} = \sum_{i=1}^M \hat{a}_i^\dagger \hat{a}_i$) corresponds to the sum of independent random variables, and therefore both the means and variances corresponding to the individual modes add up. The mean and the variance per mode are then

$$\bar{m} = \frac{1}{M} \sum_{i=1}^M m_i, \quad \bar{s}^2 = \frac{1}{M} \sum_{i=1}^M s_i^2.$$

Clearly, with m_i fixed, the minimal variance in each mode is attained by the appropriate displaced squeezed vacuum. Taking two of these modes (with means m_i and variances s_i^2 , with $i \in \{1, 2\}$), it can be shown that replacing them by optimal displaced squeezed vacua with $m'_1 = m_1 + m_2$ and $m'_2 = 0$, the resulting variance contribution will be smaller than the original, $s_1^2 + s_2^2 (=0) < s_1^2 + s_2^2$ (see Fig. 4, diamond versus triangle). Continuing with this process pairwise, it follows that the lowest variance is attained by a single mode containing all the photons, and the remaining modes empty. Therefore, the previously attained single-mode QNG witness is applicable in the multimode case as-is if the modes are independent.

Importantly, since beam splitters do not change the sum of the number of photons in a pair of modes, interdependent modes that are only dependent due to mixing on a network of beam splitters are, for our purposes, equivalent to the case of all modes being independent. Therefore, again, the non-Gaussianity witness is applicable in its single-mode form in such cases.

2. Identical modes

As the other extreme, we may also consider M ($M \geq 1$) identical modes. Clearly, in this case we just multiply the photon numbers of a single mode by the number of modes, and therefore the moments become

$$m_M = M m_0, \quad s_M^2 = M^2 s_0^2,$$

with m_0 and s_0^2 denoting the mean and variance of the photon number of a single copy of the mode. Due to this sim-

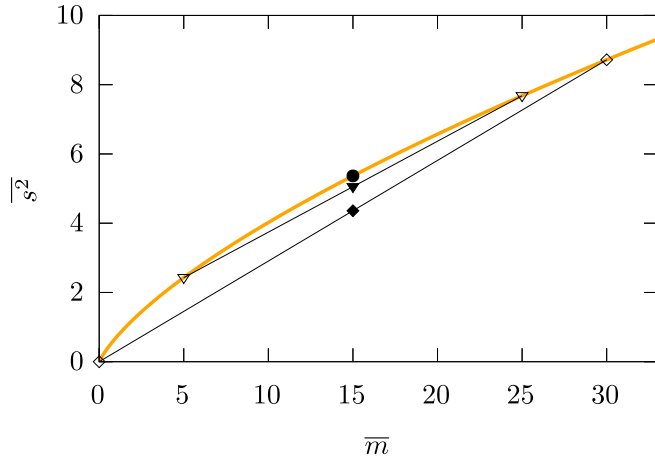


FIG. 4. Behavior of the QNG witness [Eqs. (3) and (4)] for two independent modes: number variance per mode and mean photon number per mode. The black disk corresponds to two identical optimally squeezed modes with $m_1 = m_2 = 15$. The black filled triangle corresponds to the average from the two states denoted by the empty triangles ($m_1 = 5$, $m_2 = 25$), the average photon number per mode is again 15; however, the variance is lower than in the identical case. The black filled diamond shows the average from $m_1 = 0$ and $m_2 = 30$ (empty diamonds): Due to the convexity of the optimal variance curve, this is the setup that provides the lowest variance for $\bar{m} = 15$.

ple form (both m_M and s_M^2 strictly increasing with m_0 and s_0^2 , respectively), the state that minimizes the variance is, again, an optimal displaced squeezed vacuum. Therefore, the minimum-variance curve can be given by

$$m_M = M m_{\text{NG}}(r), \quad s_M^2 = M^2 s_{\text{NG}}^2(r).$$

Figure 5 shows that if the number of modes M is unknown, assuming $M = 1$ is the worst-case scenario. That is, the single-mode witness provides a stricter criterion for non-Gaussianity than any higher-mode witness.

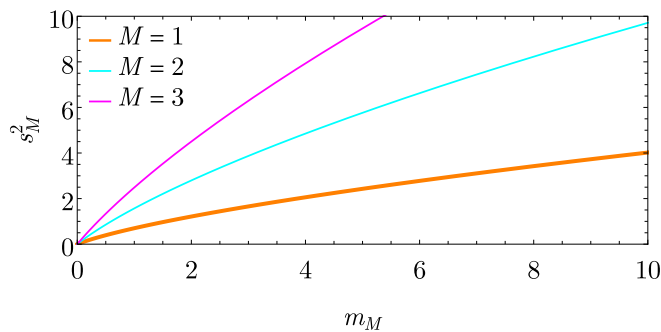


FIG. 5. Mean and variance of photon numbers for optimal displaced squeezed vacua in the case of M identical modes. This illustrates that assuming $M = 1$ provides the worst-case scenario in the identical multimode case; that is, if such a state can be represented by a point below the $M = 1$ orange line, it also qualifies as QNG for any $M > 1$.

C. Equivalent formulations

So far, for mathematical convenience, we worked mainly with the photon-number mean and variance. In the following, we provide a few equivalent formulations of witness that may be handier depending on the scenario.

1. Formulation in terms of the noncentered second moment

For any fixed mean photon number,

$$\langle \hat{N} \rangle = m_{\text{NG}}(r) = \frac{e^{6r}}{4} - \frac{1}{2} + \frac{e^{-2r}}{4},$$

parametrized by $r \geq 0$, the noncentered second moment of the photon number is lower bounded for any mixture of Gaussian states (see Fig. 3, purple line):

$$\begin{aligned} \langle \hat{N}^2 \rangle &\geq m_{\text{NG}}^2(r) + s_{\text{NG}}^2(r) \\ &= \frac{e^{12r}}{16} - \frac{e^{6r}}{4} + \frac{e^{4r}}{2} - \frac{1}{4} - \frac{e^{-2r}}{4} + \frac{3e^{-4r}}{16}. \end{aligned}$$

Importantly, note that for mixtures of arbitrary states, it is the noncentered second moment that transforms linearly; that is, it is the convexity of this quantity that is relevant in the proof.

2. Formulation in terms of integrated-intensity moments

Advantageously, for the experiments with classical integrated-intensity detectors, one can also describe the photon number distribution as a Poissonian random variable with a random intensity parameter W , the integrated intensity [48,49], which corresponds to the detector's response over a finite time window. In mathematical terms, this is a mixed Poisson distribution, and we can express the relationship between the characteristic functions as $C_N(\xi) = C_W[(e^{i\xi} - 1)/i]$ (see Ref. [49], Sec. 14.9). For example, a thermal number distribution can be obtained as a Poissonian with an exponentially distributed mean. The relationship between the moments of the integrated-intensity and the photon-number moments is simply $\langle W \rangle = \langle \hat{N} \rangle$ and $\langle W^2 \rangle = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle$. That is, for a fixed mean photon number m , the second moment of the integrated intensity is a strictly increasing function of the second moment of the photon number, so the second integrated-intensity moment is minimal exactly when the second moment of the photon number is minimal. The proposed QNG witness is therefore straightforward to express in terms of integrated-intensity moments. For any fixed mean intensity,

$$\langle W \rangle = m_{\text{NG}}(r) = \frac{e^{6r}}{4} - \frac{1}{2} + \frac{e^{-2r}}{4},$$

parametrized by $r \geq 0$, the second moment of the integrated intensity is lower bounded for any mixture of Gaussian states:

$$\begin{aligned} \langle W^2 \rangle &\geq m_{\text{NG}}^2(r) + s_{\text{NG}}^2(r) - m_{\text{NG}}(r) \\ &= \frac{e^{12r}}{16} - \frac{e^{6r}}{2} + \frac{e^{4r}}{2} + \frac{1}{4} - \frac{e^{-2r}}{2} + \frac{3e^{-4r}}{16} \\ &\equiv W_{\text{NG}}^2(r). \end{aligned}$$

That is, if we have a pair of experimental averages $\bar{W}$ and $\bar{W}^2$, then $\bar{W}^2 < W_{\text{NG}}^2[m_{\text{NG}}^{-1}(\bar{W})]$ is a witness of quantum non-Gaussianity (with m_{NG}^{-1} denoting the inverse function of m_{NG}).

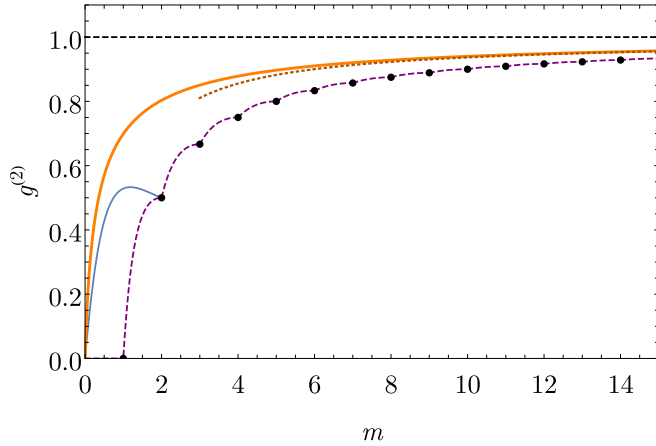


FIG. 6. Formulation of the QNG witness in terms of second-order correlation function and mean photon number. The black dashed line represents the witness for nonclassicality, the solid orange represents line the witness of non-Gaussianity, whereas the blue line represents the transformed witness of quantum non-Gaussianity from Eq. (5) in Ref. [33]. The dotted orange line shows the asymptotic approximation of the orange line, $1 - (m + \frac{1}{2})^{-1} + 3 \cdot 2^{-5/3} (m + \frac{1}{2})^{-4/3} - (m + \frac{1}{2})^{-2}$. The black disks show the $g^{(2)}$ values of Fock states $n = 1, \dots, 14$, $1 - n^{-1}$. The points below the purple dashed line are nonphysical.

3. Formulation in terms of the second-order correlation $g^{(2)}$

The second-order correlation function can be expressed from the mean and the variance of photon numbers as

$$g^{(2)} \equiv \frac{\langle :\hat{N}^2: \rangle}{\langle \hat{N} \rangle^2} = \frac{\langle W^2 \rangle}{\langle W \rangle^2} \equiv 1 + \frac{s^2}{m^2} - \frac{1}{m},$$

with the pair of colons denoting normal ordering. Clearly, for a fixed value of m , $g^{(2)}$ is linearly increasing with s^2 . As a consequence, for a fixed value of the mean photon number $\langle \hat{N} \rangle = \langle W \rangle = m_{\text{NG}}(r)$, the second-order correlation function is lower bounded for any mixture of Gaussian states:

$$\begin{aligned} g^{(2)}(r) &\geq 1 + \frac{s_{\text{NG}}^2(r)}{m_{\text{NG}}^2(r)} - \frac{1}{m_{\text{NG}}(r)} \\ &= \frac{e^{12r} + 2e^{10r} + 3e^{8r} - 4e^{6r} - 3e^{4r} - 2e^{2r} + 3}{(e^{6r} + e^{4r} + e^{2r} - 1)^2} \\ &\equiv g_{\text{NG}}^{(2)}(r). \end{aligned}$$

In other words, similarly to the previous instance, if the experimental value of $g^{(2)}$ is lower than $g_{\text{NG}}^{(2)}[m_{\text{NG}}^{-1}(\bar{W})]$ (a point below the orange line in Fig. 6), it is a witness of the quantum non-Gaussianity of the state.

Figure 6 shows the transformation of the probability-based witness with blue and the moment-based witness with orange. The usual nonclassicality witness of $g^{(2)} < 1$ is shown with the dashed black line. For large values of r , we can use the approximation $m_{\text{NG}}(r) \approx e^{6r}/4 - 1/2$, and the QNG border in terms of $g^{(2)}$ can be approximated from below

$$g_{\text{NG}}^{(2)} > 1 - \frac{1}{m + \frac{1}{2}} + \frac{3}{2^{5/3} (m + \frac{1}{2})^{4/3}} - \frac{1}{(m + \frac{1}{2})^2}. \quad (14)$$

This approximation is shown with the orange dotted line. The purple dashed line shows the boundary of physical states (for any state with mean photon number m , the physical state that achieves minimal variance is a mixture of Fock states $|[m]\rangle$ and $|[m] + 1\rangle$). The distance between this boundary of physical states and the QNG boundary has its maximum in $m = 1$ and decreases as $m^{-4/3}$, which shows that, practically, the required experimental precision for QNG witnessing increases with the mean photon number m .

The black disks, on the other hand, correspond to the $g^{(2)}$ values of pure Fock states, $1 - n^{-1}$, showing that for not too large values of the mean photon number, there is enough distance between the orange line and the disks to certify the non-Gaussianity of states other than Fock states.

Note that similar results can be obtained for the Fano factor

$$\mathcal{F} \equiv \frac{\text{Var}(\hat{N})}{\langle \hat{N} \rangle} = \frac{s^2}{m}.$$

Namely, $\mathcal{F} < 1$ is a known nonclassicality witness. While using the Eqs. (3) and (4), one can get a similar witness for QNG using Fano factor since if only mixtures of Gaussian states are considered,

$$\mathcal{F} \geq \frac{s_{\text{NG}}^2(r)}{m_{\text{NG}}(r)} \equiv \mathcal{F}_{\text{NG}}(r).$$

III. EXAMPLES OF DIFFERENT MEASUREMENT SCHEMES

The moment-based non-Gaussianity witness is applicable in any situation where there is a reliable way to infer the values of m and s^2 (or its alternatives). In what follows, we will show a few straightforward examples of doing that. Note that our aim here is not to solve a realistic experimental problem, but rather to show some examples where our witness could be applied. Therefore, our solution details only the ideal case, while for the noisy cases, a similar analysis could be applied as detailed in Sec. IV C.

These examples rely on the fact that we can express the photon-number moments $m \equiv \langle \hat{a}^\dagger \hat{a} \rangle$ and $s^2 \equiv \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - m^2$ by expanding the integral expressions at the beginning of Sec. II to

$$m = \langle \hat{x}^2 \rangle + \langle \hat{p}^2 \rangle - \frac{1}{2}, \quad (15)$$

$$s^2 = \text{Var} \hat{x}^2 + \text{Var} \hat{p}^2 + 2\text{Cov}(\hat{x}^2, \hat{p}^2) - \frac{1}{4}, \quad (16)$$

where $\text{Var} \hat{O} \equiv \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$ and

$$\begin{aligned} \text{Cov}(\hat{x}^2, \hat{p}^2) &\equiv \int x^2 p^2 W(x, p) dx dp - \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle \\ &= \left\langle \frac{\hat{x}^2 \hat{p}^2 + \hat{p}^2 \hat{x}^2}{2} + \frac{1}{8} \right\rangle - \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle. \end{aligned}$$

A. Witness from homodyning

Performing a homodyne measurement is the most straightforward approach to estimating quadrature moments. Looking at Eqs. (15) and (16), we need the moments $\langle \hat{x}^2 \rangle$, $\langle \hat{p}^2 \rangle$, $\langle \hat{x}^4 \rangle$, $\langle \hat{p}^4 \rangle$, and $\int x^2 p^2 W(x, p) dx dp$. While the first four of these are straightforward to obtain, $\langle \hat{x}^2 \hat{p}^2 \rangle$ is not. However,

by measuring in four directions: $0, \pi/4, \pi/2$, and $3\pi/4$, this cross term can also be obtained by showing that $\langle \hat{q}_{\pi/4}^4 \rangle + \langle \hat{q}_{3\pi/4}^4 \rangle = (\langle \hat{q}_0^4 \rangle + \langle \hat{q}_{\pi/2}^4 \rangle)/2 + 3 \int x^2 p^2 W(x, p) dx dp$, with $\hat{q}_\varphi \equiv \hat{x} \cos \varphi + \hat{p} \sin \varphi$ denoting the quadrature in direction φ . After straightforward manipulations, this leads to the following form for the photon-number moments:

$$m = \frac{1}{2} [\langle \hat{q}_0^2 \rangle + \langle \hat{q}_{\pi/4}^2 \rangle + \langle \hat{q}_{\pi/2}^2 \rangle + \langle \hat{q}_{3\pi/4}^2 \rangle - 1], \quad (17)$$

$$s^2 = \frac{2}{3} [\langle \hat{q}_0^4 \rangle + \langle \hat{q}_{\pi/4}^4 \rangle + \langle \hat{q}_{\pi/2}^4 \rangle + \langle \hat{q}_{3\pi/4}^4 \rangle] - \frac{1}{4} [\langle \hat{q}_0^2 \rangle + \langle \hat{q}_{\pi/4}^2 \rangle + \langle \hat{q}_{\pi/2}^2 \rangle + \langle \hat{q}_{3\pi/4}^2 \rangle]^2 - \frac{1}{4}. \quad (18)$$

So after measuring the quadrature moments in these four directions, one can estimate the photon-number moments and check if they are a witness of quantum non-Gaussianity, that is, if the corresponding point (m, s^2) lies below the orange line in Fig. 1. Alternatively, we can convert the criterion from m and s^2 to the average quadrature moments $Q^2 \equiv \frac{1}{4} \sum_{k=0}^3 \langle \hat{q}_{k\pi/4}^2 \rangle$ and $Q^4 \equiv \frac{1}{4} \sum_{k=0}^3 \langle \hat{q}_{k\pi/4}^4 \rangle$ as

$$Q_{\text{NG}}^2(r) = \frac{e^{6r} + e^{-2r}}{8}, \quad (19)$$

$$Q_{\text{NG}}^4(r) = \frac{3}{128} [e^{12r} + 8e^{4r} - 4 + 3e^{-4r}]. \quad (20)$$

That is, we have witnessed QNG if for $Q^2 = Q_{\text{NG}}^2(r)$, the average of the fourth quadrature moments $Q^4 < Q_{\text{NG}}^4(r)$.

Phase-random local oscillator

A broadly available detection scheme that involves amplification using a strong single-mode local oscillator is a homodyne measurement with a phase-random local oscillator. Essentially, this approach assumes that the local oscillator is not phase locked to the signal, so it randomizes the phase of the state and subsequently measures a single quadrature. By changing variables, we obtain the mean photon number as

$$\begin{aligned} \langle \hat{N} \rangle &= \int dx dp \left(x^2 + p^2 - \frac{1}{2} \right) W(x, p) \\ &= \int dr d\varphi \left(r^2 - \frac{1}{2} \right) \times r W(r \cos \varphi, r \sin \varphi) \\ &= 2\pi \int_0^\infty dr \left(r^2 - \frac{1}{2} \right) \times r \int_0^{2\pi} \frac{d\varphi}{2\pi} W(r \cos \varphi, r \sin \varphi) \\ &= \int_0^\infty dr \left(r^2 - \frac{1}{2} \right) \tilde{W}(r), \end{aligned}$$

where

$$\tilde{W}(r) \equiv r \int_0^{2\pi} d\varphi W(r \cos \varphi, r \sin \varphi)$$

is exactly the Wigner function of the state after phase randomization in polar coordinates, including the Jacobi determinant

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial r} & \frac{\partial p}{\partial \varphi} \end{vmatrix} = r.$$

Similarly,

$$\begin{aligned} \langle \hat{N}^2 \rangle &= \int dx dp (x^2 + p^2)(x^2 + p^2 - 1) W(x, p) \\ &= \int_0^\infty dr r^2 (r^2 - 1) \tilde{W}(r). \end{aligned}$$

Clearly, this also means that the phase-randomization step does not affect the photon-number mean or variance. Using the symmetry of the Wigner function of this randomized state, Eqs. (17) and (18) simplify to

$$m = 2\langle \hat{q}^2 \rangle - \frac{1}{2}, \quad (21)$$

$$s^2 = \frac{8}{3}\langle \hat{q}^4 \rangle - 4\langle \hat{q}^2 \rangle^2 - \frac{1}{4}, \quad (22)$$

where $\hat{q}$ denotes an arbitrary quadrature of the phase-randomized state. As a result, the non-Gaussianity witness derived in terms of m and s^2 is easy to express via $\langle \hat{q}^2 \rangle$ and $\langle \hat{q}^4 \rangle$: We have a witness of quantum non-Gaussianity if for $\langle \hat{q}^2 \rangle = Q_{\text{NG}}^2(r)$, the fourth moment $\langle \hat{q}^4 \rangle < Q_{\text{NG}}^4(r)$. The quantities $\langle \hat{q}^2 \rangle$ and $\langle \hat{q}^4 \rangle$ are, of course, straightforward to estimate from a finite sample of realizations of $\hat{q}$ measured in the homodyne setup with a phase-random local oscillator. So all that remains is to check if the point determined by these empirical values lies below the parametric curve defined by Eqs. (19) and (20). This witness represents an alternative to the approach used in Ref. [34].

B. Phase-sensitive amplification

An alternative approach uses optical preamplification followed by integrated-intensity detection and analyzing photon-number moments estimated from them. This approach was first proposed and experimentally verified [50] to show the quantum non-Gaussianity of states without phase dependence, specifically heralded single photons. In what follows, we extend that technique to arbitrary single-mode states: Using the amplified moments, we estimate the original values of m and s^2 before amplification in order to be able to apply the witness given in Eqs. (3) and (4).

The action of a phase-sensitive amplifier, assuming that it amplifies (antisqueezes) the x quadrature, can be expressed as

$$W'(x, p) = W(xe^{-r}, pe^r)$$

for some value $r > 0$. This amounts to multiplying the x quadrature by e^r and the p quadrature by e^{-r} . For high values of the parametric gain r , the output intensity is then approximately proportional to $\hat{x}^2$. Since Eqs. (17) and (18) only contain even moments, parametric homodyning as described in Refs. [51,52] is also applicable to verify quantum non-Gaussianity. This would require amplification in four directions, similarly to the standard homodyne case (Sec. III A).

Alternatively, note that that all terms in Eqs. (15) and (16) ($\langle \hat{x}^2 \rangle$, $\langle \hat{p}^2 \rangle$, $\text{Cov}(\hat{x}^2, \hat{p}^2)$) are directly available through a double homodyne setup, at the cost of additional noise. This noise can, however, be corrected for. For example, in the horizontal

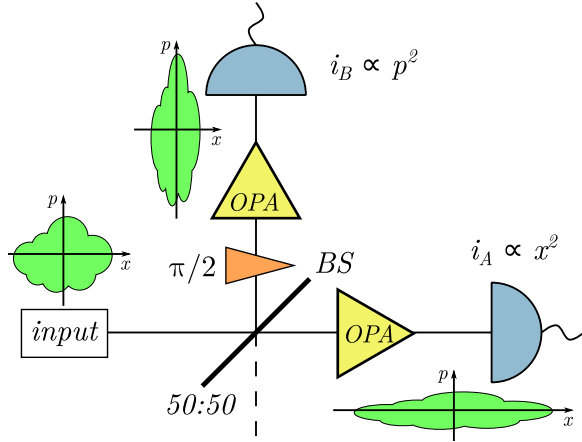


FIG. 7. Double parametric homodyne scheme. The input state is first split on a balanced beam splitter (BS). Then the state in the horizontal arm is strongly parametrically amplified (OPA) in the x quadrature direction, and the state in the vertical arm is first phase shifted by $\pi/2$ and then parametrically amplified by an identical amplifier (OPA). The integrated intensities are then measured in either arm, and are proportional to x^2 and p^2 , respectively.

arm of Fig. 7, we have

$$\hat{x} = \frac{1}{\sqrt{2}}\hat{x}_0 + \frac{1}{\sqrt{2}}\hat{x}_{\text{vac}}, \quad \Downarrow \quad \langle \hat{x}^2 \rangle = \frac{1}{2}\langle \hat{x}_0^2 \rangle + \frac{1}{8},$$

$$\langle \hat{x}^4 \rangle = \frac{1}{4} \left[\langle \hat{x}_0^4 \rangle + \frac{3}{2}\langle \hat{x}_0^2 \rangle + \frac{1}{8} \right],$$

where $\hat{x}_{\text{vac}}$ denotes the x quadrature of vacuum ($\langle \hat{x}_{\text{vac}} \rangle = \langle \hat{x}_{\text{vac}}^3 \rangle = 0$, $\langle \hat{x}_{\text{vac}}^2 \rangle = 1/4$, $\langle \hat{x}_{\text{vac}}^4 \rangle = 3/16$). The effect is identical when looking at the moments of $\hat{p}$. The covariance of $\hat{x}_0^2$ and $\hat{p}_0^2$ is unaffected by the addition of vacuum; there is only a factor of $1/4$ coming from the beam splitter. We can then plug in the quadrature moments that have been corrected for noise into Eqs. (15) and (16) to estimate m and s^2 , and finally check if the estimated point lies below the orange line in Fig. 1.

C. Phase-insensitive amplification

A phase-insensitive amplifier amplifies the signal at the cost of adding noise from the idler mode. In the simplest form, its action can be described as [53]

$$\hat{x} = \sqrt{G}\hat{x}_s + \sqrt{G-1}\hat{x}_i,$$

$$\hat{p} = \sqrt{G}\hat{p}_s - \sqrt{G-1}\hat{p}_i,$$

for some gain $G > 1$ with $(\hat{x}_{s/i}, \hat{p}_{s/i})$ denoting the signal/idler quadratures and $(\hat{x}, \hat{p})$ the output quadratures. Knowing the various quadrature moments of the idler mode, the amplified quadrature moments are straightforward to express and correct for.

As the simplest example, let us consider the idler to be vacuum ($\Rightarrow \langle \hat{x}_i \rangle = \langle \hat{p}_i \rangle = 0$, $\langle \hat{x}_i^2 \rangle = \langle \hat{p}_i^2 \rangle = 1/4$, $\langle \hat{x}_i^4 \rangle = \langle \hat{p}_i^4 \rangle = 3/16$, $\text{Cov}(\hat{x}_i^2, \hat{x}_i) = \text{Cov}(\hat{p}_i^2, \hat{p}_i) = 0$). In this case, we can express the amplified photon-number moments as

$$M = G(\langle \hat{x}_s^2 \rangle + \langle \hat{p}_s^2 \rangle) + \frac{G-1}{2} - \frac{1}{2} = mG + G - 1, \quad (23)$$

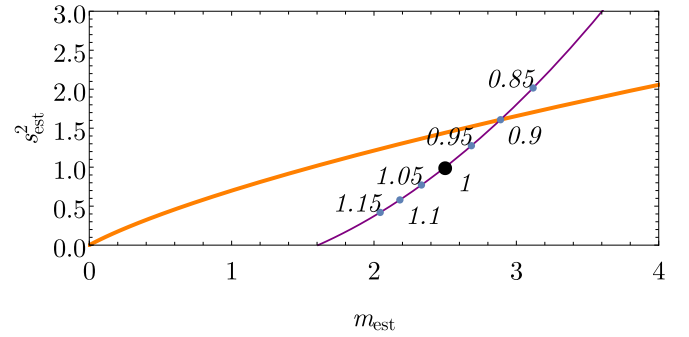


FIG. 8. Sensitivity of the corrected value of the photon-number mean and variance to the estimated gain G_{est} . Assuming that the true values are $m = 2.5$, $s^2 = 1$ (indicated with a black disk), and $G = 100$, we calculated from Eqs. (23) and (24) the values $M = 349$ and $S^2 = 42\,200$. Then, we inverted Eqs. (23) and (24) using different values of estimated gain G_{est} (purple line). Specific values of G_{est}/G are indicated with labeled blue disks along the purple line. The orange line shows the border of the moment-based QNG witness.

$$S^2 = G^2[\text{Var}\hat{x}^2 + \text{Var}\hat{p}^2 + 2\text{Cov}(\hat{x}^2, \hat{p}^2)]$$

$$+ G(G-1)(\langle \hat{x}_s^2 \rangle + \langle \hat{p}_s^2 \rangle) - \frac{1}{4}$$

$$= G^2 s^2 + G(G-1)m + (3G^2 - 2G - 1)/4, \quad (24)$$

with m and s^2 denoting the initial photon-number mean and variance of the signal. Using Eqs. (23) and (24) and G known, we can estimate m and s^2 from M and S^2 and check whether these estimates meet the QNG witness in Eqs. (3) and (4). Of course, the same can be done based on integrated-intensity moments, which amounts to replacing Eq. (24) by

$$W^2 = S^2 + M^2 - M = G^2(s^2 + m^2 - m)$$

$$+ \frac{G-1}{4}[7(G-1) + 16Gm]. \quad (25)$$

Alternatively, we can substitute m and s^2 in Eqs. (23) and (24) [or in Eqs. (23) and (25)] with $m_{\text{NG}}(r)$ and $s_{\text{NG}}^2(r)$ from Eqs. (3) and (4) to obtain a witness in terms of the amplified moments.

In experiments, we know the value of G only up to a finite precision quantified by the error bars around G_{est} . We should therefore consider this finite precision in the estimation of the preamplification moments. Figure 8 shows how the estimated preamplification moments change when assuming different values for G_{est}/G . Clearly, by assuming a larger value for estimated gain G_{est} , both the variance and the mean decrease, moving deeper into the non-Gaussian region (possibly providing even nonphysical values). Therefore, if we have a confidence interval $(G_{\text{min}}, G_{\text{max}})$, the conservative choice is to invert Eqs. (23) and (24) using G_{min} .

IV. RESILIENCE OF QUANTUM NON-GAUSSIANITY TO LOSS AND NOISE

In the previous section, we looked at a few possible schemes where m and s^2 may be estimated without photon-number resolving detectors. In the current section, we

look at how the moment-based witness [Eqs. (3) and (4)] behaves assuming that these estimates are exact. This allows us to explore the limits of the applicability of the moment-based QNG witness independently of the particulars of the experiment from which the estimates m_{est} and s_{est}^2 were obtained. We can also look at it as the infinite sample size limit of the measurement, where $m_{\text{est}} = m$ and $s_{\text{est}}^2 = s^2$.

Quantum non-Gaussianity depth [35] measures the maximum loss (or equivalently, the minimal transmittance $\eta_{\min}$) a given measured state can suffer and still be recognized as QNG by the witness under consideration. Note that this is a different concept from QNG measures, which are most often defined by a distance between the density operator and the density operator of the Gaussian state with the same mean quadratures and covariance matrix [54]. The calculation of such measures does require full state tomography. Quantum non-Gaussianity depth, in contrast, can be determined without state tomography by using a calibrated attenuator in front of the detector. We will use QNG depth as a figure of merit to compare our moment-based witness to the witness based on Fock-0 and Fock-1 probabilities [33] for a few types of states.

In the following, we first look at the quantum non-Gaussianity depth of Fock states and photon-added thermal states as examples. We will further examine how loss and additive noise affect the witness and how it may be corrected for.

A. QNG depth of Fock states

As the simplest demonstration of the applicability of the witness [Eqs. (3) and (4)], we calculate how much signal loss a Fock state can tolerate in order for the witness to still recognize its non-Gaussianity. It is straightforward to show that if a Fock state $|n\rangle$ passes through a beam splitter with transmissivity η , its photon-number mean and variance become

$$m = \eta n, \quad s^2 = \eta(1 - \eta)n. \quad (26)$$

For any set of values n and η , it is easy to check whether (m, s^2) lies below or above the border of the non-Gaussianity witness given by Eqs. (3) and (4). Figure 9(a) shows how the moments of Fock states 1–5 change as they pass through a beam splitter with different values of the transmissivity $\eta \in [0, 1]$. For $\eta = 1$, each curve starts at $(n, 0)$ (black disks) and arrives in the origin when η is decreased to zero. For $n > 1$, each curve intersects with the QNG border for some value $\eta^* > 0$ (which defines the QNG depth), and this η^* increases with n . This means that, expectedly, higher Fock states can tolerate less loss, that is, their QNG depth is smaller.

Table I compares the QNG depths using the probability-based witness [using that $p_0 = (1 - \eta)^n$ and $p_1 = n\eta(1 - \eta)^{n-1}$ for lossy Fock states] and the moment-based witness. For the Fock-1 case, we have a mixture of vacuum and single photon, which is always non-Gaussian. For Fock-2 and higher, the new witness Eqs. (3) and (4) provides a considerably larger depth (smaller $\eta_{\min}$) for all cases without a need to specify photon-number probabilities. Note that for the probability-based witness, this assumes that we can estimate both p_0 and p_1 quite accurately, even if they are both close to zero, which might be difficult in practice.

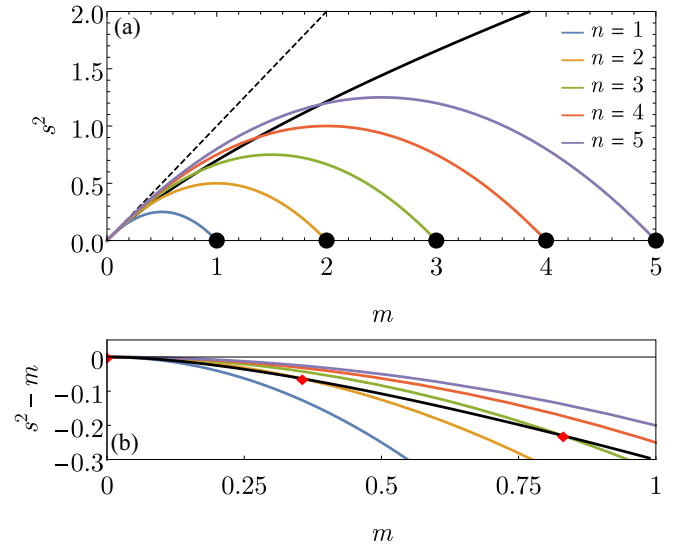


FIG. 9. QNG depth of Fock states without added noise. (a) The photon-number mean and variance of lossy Fock states 1–5, the proposed QNG witness is shown with the black solid line. The QNG depths are given by the intersections of the colored curves with the black line. The black disks indicate the lossless, $\eta = 1$ states. The dashed black line shows the nonclassicality border $s^2 = m$. (b) The same lines depicted via $s^2 - m$ as a function of the mean. The numerically calculated intersection points for Fock states 1–3 are indicated with red diamonds.

Let us note that our witness extends the applicability range from microscopic states ($m \ll 1$) to mesoscopic states (m could be well above 1), but in principle it can be applied to any state. For large values of the squeezing parameter r , Eqs. (3) and (4) simplify to

$$m_{\text{NG}}(r) \approx \frac{e^{6r}}{4}, \quad s_{\text{NG}}^2(r) \approx \frac{3e^{4r}}{8} = \frac{3}{8}(4m)^{2/3}, \quad (27)$$

and therefore the equation for QNG depth can be solved asymptotically and yields $1 - \eta_{\min} = \frac{3}{8}4^{2/3}n^{-1/3}$ (see Fig. 10, black dashed line indicating the asymptotic slope). We can see that this analytic approximation is close to the numerically calculated values for large Fock states (orange points). Note that the compared (p_0, p_1) -based witness is most suited for low mean photon numbers. A similar extension of range for mesoscopic states could be achieved also to the probability-based witness, but that needs a more sophisticated setup with multiple single-photon detectors [36,55]

TABLE I. QNG depth $1 - \eta_{\min}$ of different Fock states via the probability-based witness and the moment-based witness.

n	$1 - \eta_{\min}$	
	Prob.-based	Moment-based
1	1	1
2	0.63	0.82
3	0.51	0.72
4	0.46	0.66
5	0.42	0.61

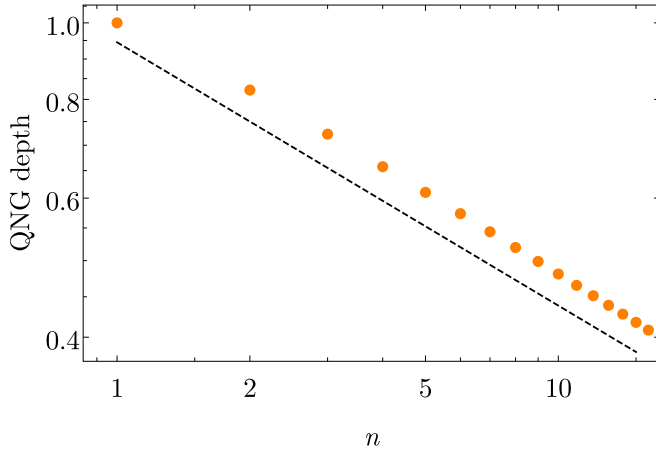


FIG. 10. The numerically calculated QNG depths [defined by the witness Eqs. (3) and (4)] of Fock states 1–15 as a function of n (orange points) on a doubly logarithmic scale and the asymptotic slope of $-1/3$ indicated for reference.

and/or a more complex witness involving only numerically attainable critical values [45].

B. QNG depth of photon-added thermal states

The photon-number distribution of a k -photon-added thermal state with (initial) mean photon number $\bar{n}$ can be given as [56]

$$p_{+k}(n) = \frac{\bar{n}^{n-k}}{(1+\bar{n})^{n+1}} \binom{n}{k}, \quad n \geq k, \quad (28)$$

which, in mathematical terms, is a random variable obtained by adding k to a negative binomial distribution with parameters $k+1$ and $(1+\bar{n})^{-1}$ [i.e., it is distributed as $\text{NB}(k+1, 1/(1+\bar{n})) + k$]. Note that this describes the case of adding k photons realistically to a thermal state, meaning that $\bar{n} = 0$ is equivalent to our previous case of Fock states.

The photon-number mean and variance are therefore

$$m_{+k}(\bar{n}) = k + (k+1)\bar{n}, \quad s_{+k}^2(\bar{n}) = (k+1)\bar{n}(\bar{n}+1).$$

After passing through a beam splitter with transmittance η , these moments change to

$$\begin{aligned} m_{+k}(\bar{n}, \eta) &= \eta[k + (k+1)\bar{n}], \\ s_{+k}^2(\bar{n}, \eta) &= \eta^2(k+1)\bar{n}(\bar{n}+1) \\ &\quad + \eta(1-\eta)[k + (k+1)\bar{n}]. \end{aligned} \quad (29)$$

Figure 11 shows the photon-number mean and variance of single-photon-added thermal states as $\bar{n}$ is increased from 0 to 0.4. We can see that the $\bar{n} = 0$ case coincides with a lossy single photon, which is always in the QNG regime for $\eta > 0$. If we increase the size of the initial thermal state, the variance of the lossless state ($\eta = 1$, black disks) increases, until at $\bar{n} \approx 0.4$, it reaches the QNG border. This means that the moment-based witness cannot show the non-Gaussianity of single-photon-added thermal states whose initial mean photon number $\bar{n}$ is over 0.4.

The numerically calculated values of QNG depth for both the probability-based and the moment-based witnesses are

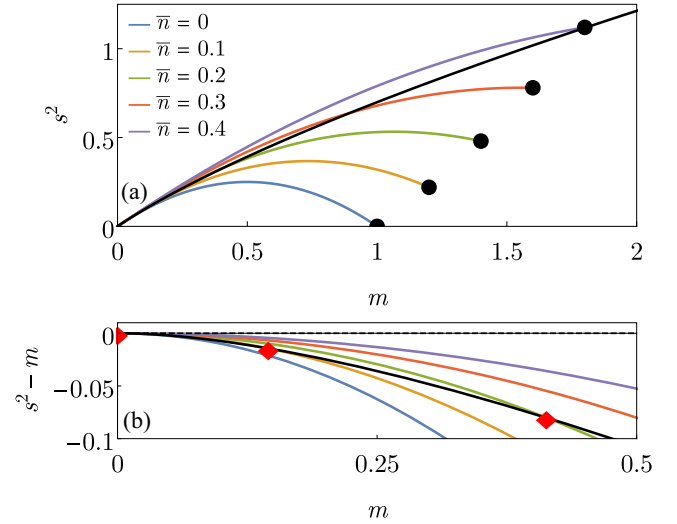


FIG. 11. QNG of single photon-added thermal states. (a) Photon-number mean and variance of single-photon-added thermal states with initial mean photon number $\bar{n}$ between 0 and 0.4. The black disks correspond to the lossless, $\eta = 1$ case. (b) $s^2 - m$ for a narrower range of mean values. The red diamonds indicate where the curves corresponding to $\bar{n} = 0, 0.1$, and 0.2 intersect the border of the moment-based QNG witness.

collected in Table II. We can see that if we add more and more photons, then the QNG depth mostly decreases. That is, by adding more photons, we do not obtain a more non-Gaussian state. If that is our main goal, then adding only a single photon is more efficient.

On the other hand, if we increase the initial mean photon number $\bar{n}$ of the thermal state, the QNG depth also decreases. This is again expected behavior, as the contribution of the classical thermal state will be higher compared to the added Fock state. For low values of $\bar{n}$, the moment-based method shows again a higher QNG depth compared to the probability-based witness. This, however, changes for larger values of $\bar{n}$, where the moment-based method fails to work around $\bar{n} \approx 0.4$ as the same moments can be increasingly well approximated with Gaussian distributions. In contrast, the probability-based method works even for larger thermal states. This is because for $\eta = 1$, the probability of zero photons is always zero for any photon-added thermal state, which automatically implies non-Gaussianity. However, in a realistic experiment with dif-

TABLE II. QNG depths $1 - \eta_{\min}$ of k -photon-added thermal states with initial mean photon number $\bar{n}$ using the probability- (P) and the moment-based (M) witnesses.

$\bar{n}$	$k = 1$		$k = 2$		$k = 3$	
	P	M	P	M	P	M
0	1	1	0.63	0.82	0.51	0.72
0.1	0.76	0.88	0.54	0.71	0.45	0.61
0.2	0.62	0.71	0.47	0.56	0.41	0.47
0.3	0.52	0.44	0.42	0.36	0.38	0.29
0.4	0.45	0	0.38	0.06	0.35	0.02

ferent noises and imperfect detection, the property of $p_0 = 0$ is difficult to achieve.

C. Loss and noise correction

As an extension to the previously investigated cases of Fock states [Eq. (26)] and photon-added thermal states [Eq. (29)], let us look at a general state with initial photon-number moments m_0 and s_0^2 and assume that it suffers loss that can be described with the aggregate transmissivity η_{agg} and an (aggregated) independent additive noise with mean m_{noise} and variance s_{noise}^2 . The altered moments can be given as

$$\begin{aligned} m &= \eta_{\text{agg}} \cdot m_0 + m_{\text{noise}}, \\ s^2 &= \eta_{\text{agg}}(1 - \eta_{\text{agg}})m_0 + \eta^2 s_0^2 + s_{\text{noise}}^2. \end{aligned} \quad (30)$$

The added loss η_{loss} has the same effect as the attenuation for QNG depth η , the two of them will have a combined attenuation of $\eta_{\text{agg}} = \eta_{\text{loss}} \times \eta$. We have two choices on how to handle this situation:

(1) Accepting loss: We check the non-Gaussianity of the lossy state, in which case the QNG depth will be simply smaller ($\eta'_{\text{min}} = \eta_{\text{min}}/\eta_{\text{loss}}$).

(2) Removing loss: If we know the value of loss η_{loss} , we may simply correct the moments of our state for the given loss. In this case, the QNG depth does not change.

Which case we decide on depends on the actual situation. In general, removing the loss is applicable if we have a precise understanding of the loss (e.g., correcting for detection inefficiency), so that we do not overcompensate. Note that if we want to be precise, we should obtain a confidence interval for the loss and take the lower bound to remain on the safe side.

The situation for the noises is very similar: If one knows both m_{noise} and s_{noise}^2 exactly (by measuring the properties of the noise separately), they can be subtracted from the empirical values of m and s^2 to remove their bias. Our method of using moments is extremely useful here, as for additive noises, the mean and the variance simply add up, and so they can be very easily corrected for noise. In contrast, if we are interested in the probability distribution, then an additive noise means a convolution, the inversion of which is numerically very difficult. So again, if we can characterize the imperfection precisely, we can remove its effect entirely. Note that in practice, different noises can have a different effect in Eq. (30), e.g., in the case of amplification and a Gaussian additive noise, the bias m_{noise} will come from the variance for preamplification noise, and will come from the mean for postamplification noise.

For the sake of completeness, let us also check what happens if we cannot remove the noise. In this case, the QNG depth decreases, but not as trivially as for losses. Let us look at a numerical example of Fock states with two different types of noises (Fig. 12). This means that we have the following values in Eq. (30): $m_0 = n$ and $s_0 = 0$. In the first case (solid lines), the noise is Poissonian with mean $\bar{n}$, so we have $m_{\text{noise}} = s_{\text{noise}}^2 = \bar{n}$. In the second case (dashed line), the noise is thermal with mean $\bar{n}$, so $m_{\text{noise}} = \bar{n}$ and $s_{\text{noise}}^2 = \bar{n}(\bar{n} + 1)$.

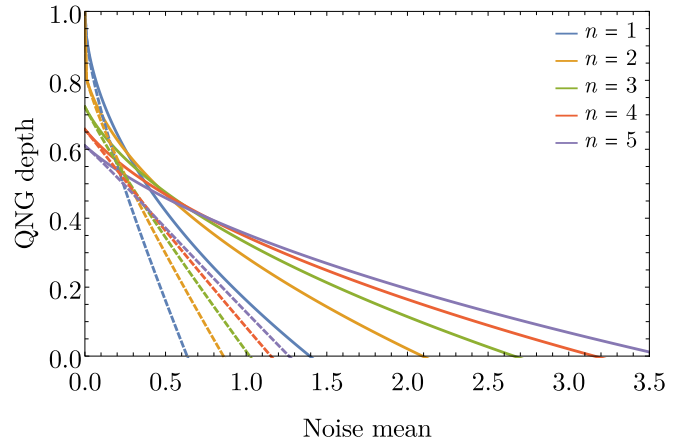


FIG. 12. QNG depth of Fock states one to five assuming an independent noise added during detection. Solid lines: Poissonian noise; dashed lines: thermal noise. The dashed and solid lines of the same color correspond to the same Fock state (see legend).

The behavior of the QNG depth is very similar in both cases: More added noise decreases the QNG depth of Fock states, and the higher the Fock state, the less sensitive its QNG depth is to this bias. The difference is that for the same noise mean, the Poissonian noise gives a significantly higher QNG depth, as it has a smaller variance than the thermal noise for the same mean. Note that the tolerable noise in either case has the same magnitude as the signal, which gives a good applicability for practical scenarios. If we have an even worse signal-to-noise ratio, the earlier discussed option of subtracting the estimated values of m_{noise} and s_{noise}^2 helps to reduce the disruptive effect of the added noise.

V. CONCLUSION AND DISCUSSION

Non-Gaussian states are important for different quantum applications. One may characterize such states by performing a complete tomography, but if we are interested only in their non-Gaussianity, providing a non-Gaussianity witness is a sufficient and, in general, simpler task. The standard method in the literature to achieve this is to perform a coincidence measurement, which can characterize the probability distribution of the unknown state at low photon numbers. A witness based on zero- and single-photon probabilities [33] can then be applied to establish the quantum non-Gaussianity of a state: For a fixed value of zero-photon probability p_0 , there is a maximal value of the single-photon probability p_1 that any mixture of Gaussian states might reach. If the experimental value of p_1 exceeds this limit, one can certify quantum non-Gaussianity. This approach works well for specific tasks but requires single-photon detectors, which limits its applicability.

In our work, we introduced a witness for the non-Gaussianity of quantum states that relies only on measuring the mean and the variance of photon numbers. Similarly to its Fock probability-based counterpart, we obtained a parametric curve [Eqs. (3) and (4)] (Fig. 1, orange line), which provides a lower limit on the photon-number variance achievable by mixtures of Gaussian states given a fixed value of the mean. Thus,

any state whose photon number variance lies below this limit has to be non-Gaussian. By converting the probability-based witness to photon-number mean and variance, we can compare its behavior with the proposed witness: Their boundaries coincide for low values of the mean photon number, but the moment-based witness significantly relaxes the QNG requirement for larger values. We provided alternative formulations of the witness, among others, we expressed it in terms of the mean and the second-order correlation $g^{(2)}$ (Fig. 6, orange line), which provides a simple QNG witness similar to the $g^{(2)} < 1$ witness of nonclassicality (dashed line). Note that a current, independent approach came to our attention, which uses correlation functions to characterize the non-Gaussianity of states [57]. Similarly to the standard approach, they also use single-photon detectors, importantly, our witness does not need single-photon detectors, so it is applicable in any scenario that allows one to infer the first two photon-number moments of the state. We also examined the multimode case for the two extremes: independent modes and identical modes. In either case, the single-mode witness is applicable.

The proposed witness is applicable in any context where we can reliably estimate the photon-number mean and variance; we provided a few such experimental scenarios. In general, estimating the mean for a dim state is not straightforward, but multiple methods can obtain it with high precision. One method is to apply a local oscillator and perform a homodyne measurement, which can fully characterize the state. We showed that for our witness, a non-phase-locked local oscillator is enough, which provides a homodyning in a phase-randomized way. Another common method to measure the mean is to amplify the signal and measure the intensity of the amplified state. We showed that in both the phase-insensitive and phase-sensitive cases, we may apply our QNG witness with minimal modifications. Note that we provided our examples for an ideal case as a proof-of-principle demonstration. However, since we showed that practical losses and noises are straightforward to deal with, we expect our witness to be applicable to real (mesoscopic and/or multimode) experimental setups (such as Refs. [50,58]), extending the characterization of non-Gaussianity well above the currently achievable limits.

Finally, assuming that we have obtained accurate estimates of the photon-number mean and variance of the input state, we analyzed this witness for a few simple states as an example. For this purpose, we used the notion of QNG depth, which is defined as the maximum value of loss $(1 - \eta_{\min})$ a given state can suffer and still be witnessed as non-Gaussian. We investigated Fock states and photon-added thermal states, and in both cases, we achieved a decent QNG depth, which is better than the standard (p_0, p_1) witness [33] in practical scenarios.

In general, the proposed moment-based witness [Eqs. (3) and (4)] performs best in regimes where the probability of observing three or more photons is non-negligible, but the overall mean photon number remains small (~ 1 – 10). For very low photon numbers, the moment-based witness performs similarly to the standard probability-based witness. For very high mean photon numbers, the moment-based witness becomes increasingly strict (Fig. 10), as even for perfect Fock- n states, the QNG depth decreases as $n^{-1/3}$. Note that even though the witness is focused to Fock states and their im-

perfect versions, it applies to arbitrary quantum states. As is typical for witnesses, it is difficult to fully assess its practicality in general. For example, we can use it to provide a QNG witness for small odd cat states, but even cat states display super-Poissonian photon statistics ($g^{(2)}(0) > 1$), and therefore cannot be identified as QNG by the proposed witness.

As a further advantage, due to the favorable properties of the statistical mean and variance, any loss and additive intensity noise can be very easily dealt with. To obtain the probability distribution after such imperfections, we need to deconvolve the noise, which is numerically a difficult task. In contrast, noise only modifies the moments linearly, so by accurately characterizing the magnitude of loss and noise precisely, we can easily minimize their effect. Note that knowing the full Wigner function or, at least the photon-number probabilities for higher values of the photon number, provides, of course, more information than merely (m, s^2) or the probabilities (p_0, p_1) from the compared witness. However, estimating the Wigner function or higher photon-number probabilities is a more complex task (technically and/or witness wise) than the estimation of the first two photon-number moments. So another advantage of the moment-based witness is exactly that it is applicable to brighter states without added complexity.

In summary, we introduced a witness for quantum non-Gaussianity based on the photon-number mean and variance, which is particularly suited for experiments involving high-gain amplification. Other than providing an extension of the applicability of QNG witnesses to brighter states, as a further advantage, it also provides straightforward ways to correct for loss and noise. As an outlook, this approach also conveniently lends itself to an extension toward higher-order quantum non-Gaussianity as a counterpart of Refs. [36,43].

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DATA AVAILABILITY

No datasets were generated or analyzed during the current study. All theoretical curves presented in the figures can be reproduced using the analytical expressions derived in the text.

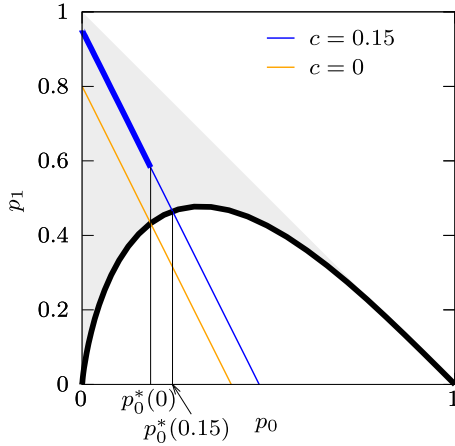


FIG. 13. Illustration of converting the Fock probability-based QNG witness [Eq. (A1)] to a witness in terms of m and s^2 . The probability-based witness identifies states as non-Gaussian if they lie above the thick black line, in the gray area (above the gray area, $p_0 + p_1 > 1$, the states are nonphysical). For a fixed value of the mean photon number m and $c = 0.15$, the potential states lie on the blue line [see Eq. (A4)]. Therefore, in this example, we have a non-Gaussianity witness for $p_0 < p_0^*(0.15)$. However, we do not actually know the value of c . Nevertheless, assuming $c = 0$ (orange line) and check the non-Gaussianity criteria for that, $p_0 < p_0^*(0)$, makes sure that we are in the gray area, since then $p_0 < p_0^*(0) \leq p_0^*(0.15)$ (i.e., the state will be in thicker part of the blue line).

APPENDIX: CONVERTING THE FOCK PROBABILITY-BASED WITNESS TO PHOTON-NUMBER MEAN AND VARIANCE

The non-Gaussianity witness in terms of zero- and single-photon probabilities from Ref. [33] is

$$\begin{aligned}\tilde{p}_0(r) &= \frac{\exp\left\{-\frac{1}{4}(e^{4r}-1)(1-\tanh r)\right\}}{\cosh r}, \\ \tilde{p}_1(r) &= \frac{(e^{4r}-1)\exp\left\{-\frac{1}{4}(e^{4r}-1)(1-\tanh r)\right\}}{4\cosh^3 r}.\end{aligned}\quad (\text{A1})$$

with $r \geq 0$ (see black line in Fig. 13). This means that for any fixed value of vacuum probability $p_0 = \tilde{p}_0(r)$, the value of the single-photon probability cannot exceed $\tilde{p}_1(r)$ for any mixture of Gaussian states. Therefore, given an experimental pair of values (p_0, p_1) that lies above the black line in Fig. 13 (in the grey area), one can certify non-Gaussianity. The witness does not say anything about the non-Gaussianity of states that lie below this black line.

After straightforward manipulations of the definitions, we can calculate the photon-number mean and variance as

$$m \equiv \sum_{n=0}^{\infty} n p_n = 2 - 2p_0 - p_1 + \sum_{n=2}^{\infty} p_n(n-2), \quad (\text{A2})$$

$$\begin{aligned}s^2 &\equiv \sum_{n=0}^{\infty} n^2 p_n - m^2 = 2p_0 - (m-2)(m-1) \\ &\quad + \sum_{n=2}^{\infty} p_n(n-2)(n-1).\end{aligned}\quad (\text{A3})$$

Introducing the notation $c \equiv \sum_{n=2}^{\infty} p_n(n-2)$ and suppose it is a fixed value along with the mean m , we obtain the equation

$$p_1 = 2 - m + c - 2p_0 \quad (\text{A4})$$

for the line representing possible states in the $p_1 - p_0$ graph (see Fig. 13, orange and blue lines). Quantum non-Gaussianity under these assumptions is equivalent to p_0 being smaller than $p_0^*(c)$, the abscissa of the intersection point of this line and the black curve prescribed by Eq. (A1) (which defines the section of the line within the gray, QNG area). It is straightforward to show that $c' < c \Leftrightarrow p_0^*(c') < p_0^*(c)$. Therefore, if p_0 is smaller than $p_0^*(0)$, then $p_0 < p_0^*(0) \leq p_0^*(c)$ implies non-Gaussianity for all values of c . That is, we established that assuming $c = 0 \Leftrightarrow p_{3+} = 0$ is the worst-case scenario if we want to show quantum non-Gaussianity for a fixed value of the mean photon number.

In order to obtain a bound for the moments instead of probabilities, we can also express p_0 from Eq. (A3). We get that the non-Gaussianity witness $p_0 < p_0^*(0)$ is equivalent to $s^2 < 2p_0^*(0) - (m-2)(m-1) + \sum_{n=2}^{\infty} p_n(n-2)(n-1)$. Since $\sum_{n=2}^{\infty} p_n(n-2)(n-1) \geq 0$ (with equality for $p_{3+} = 0$), the state is certainly QNG if

$$s^2 < 2p_0^*(0) - (m-2)(m-1). \quad (\text{A5})$$

As a reminder, we calculate $p_0^*(0)$ by taking the intersection point for $c = 0$ (i.e., $p_{3+} = 0$), and the right-hand side of Eq. (A5) corresponds to exactly same case.

In summary, we can convert the Fock probability-based witness to photon-number mean and variance by assuming $p_{3+} = 0$, which yields

$$\tilde{m}(r) = \tilde{p}_1(r) + 2 \times (1 - \tilde{p}_1(r) - \tilde{p}_0(r)). \quad (\text{A6})$$

$$\tilde{s}^2(r) = \tilde{p}_1(r) + 4 \times (1 - \tilde{p}_1(r) - \tilde{p}_0(r)) - \tilde{m}^2(r). \quad (\text{A7})$$

Importantly, the conversion is only exact if $p_{3+} = 0$, for $p_{3+} > 0$, the transformed witness Eqs. (A6) and (A7) is stricter than the original, (A1). Furthermore, the transformed witness's applicability is restricted to $m \leq 2$. However, note that the original witness is also inconclusive if $p_0 = p_1 = 0$, which is not restrictive for the moment-based witness.

[1] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, *Rev. Mod. Phys.* **84**, 621 (2012).

[2] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, *Phys. Rev. A* **64**, 012310 (2001).

- [3] M. H. Michael, M. Silveri, R. T. Brierley, V. V. Albert, J. Salmilehto, L. Jiang, and S. M. Girvin, New class of quantum error-correcting codes for a bosonic mode, *Phys. Rev. X* **6**, 031006 (2016).
- [4] S. Konno, W. Asavanant, F. Hanamura, H. Nagayoshi, K. Fukui, A. Sakaguchi, R. Ide, F. China, M. Yabuno, S. Miki, H. Terai, K. Takase, M. Endo, P. Marek, R. Filip, P. van Loock, and A. Furusawa, Logical states for fault-tolerant quantum computation with propagating light, *Science* **383**, 289 (2024).
- [5] M. V. Larsen, J. E. Bourassa, S. Kocsis, J. F. Tasker, R. S. Chadwick, C. González-Arciniegas, J. Hastrup, C. E. Lopetegui-González, F. M. Miatto, A. Motamedi, *et al.*, Integrated photonic source of Gottesman–Kitaev–Preskill qubits, *Nature (London)* **642**, 587 (2025).
- [6] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, *Phys. Rev. A* **71**, 022316 (2005).
- [7] S. Konno, W. Asavanant, K. Fukui, A. Sakaguchi, F. Hanamura, P. Marek, R. Filip, J.-i. Yoshikawa, and A. Furusawa, Non-Clifford gate on optical qubits by nonlinear feedforward, *Phys. Rev. Res.* **3**, 043026 (2021).
- [8] F. Hanamura, W. Asavanant, S. Kikura, M. Mishima, S. Miki, H. Terai, M. Yabuno, F. China, K. Fukui, M. Endo, and A. Furusawa, Single-shot single-mode optical two-parameter displacement estimation beyond classical limit, *Phys. Rev. Lett.* **131**, 230801 (2023).
- [9] M. Wang, M. Zhang, Z. Qin, Q. Zhang, L. Zeng, X. Su, C. Xie, and K. Peng, Experimental preparation and manipulation of squeezed cat states via an all-optical in-line squeezer, *Laser Photon. Rev.* **16**, 2200336 (2022).
- [10] S. Liu, D. Han, N. Wang, Y. Xiang, F. Sun, M. Wang, Z. Qin, Q. Gong, X. Su, and Q. He, Experimental demonstration of remotely creating Wigner negativity via quantum steering, *Phys. Rev. Lett.* **128**, 200401 (2022).
- [11] D. Han, F. Sun, N. Wang, Y. Xiang, M. Wang, M. Tian, Q. He, and X. Su, Remote preparation of optical cat states based on Gaussian entanglement, *Laser Photon. Rev.* **17**, 2300103 (2023).
- [12] Y.-S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, Non-Gaussian quantum states of a multimode light field, *Nat. Phys.* **16**, 144 (2020).
- [13] H. Simon, L. Caron, R. Journet, V. Cotte, and R. Tualle-Broui, Experimental demonstration of a versatile and scalable scheme for iterative generation of non-Gaussian states of light, *Phys. Rev. Lett.* **133**, 173603 (2024).
- [14] A. Kawasaki, R. Ide, H. Brunel, T. Suzuki, R. Nehra, K. Nakashima, T. Kashiwazaki, A. Inoue, T. Umeki, F. China, M. Yabuno, S. Miki, H. Terai, T. Yamashita, A. Sakaguchi, K. Takase, M. Endo, W. Asavanant, and A. Furusawa, Broadband generation and tomography of non-Gaussian states for ultra-fast optical quantum processors, *Nat. Commun.* **15**, 9075 (2024).
- [15] J. Fadrný, M. Neset, M. Bielak, M. Ježek, J. Bílek, and J. Fiurášek, Experimental preparation of multiphoton-added coherent states of light, *npj Quantum Inf.* **10**, 89 (2024).
- [16] B. E. Asenbeck, L. Lachman, A. Boyer, P. Giri, A. Urvoy, R. Filip, and J. Laurat, Hierarchical verification of non-Gaussian coherence in bosonic quantum states, *Phys. Rev. Lett.* **134**, 233604 (2025).
- [17] T. Yoshida, D. Okuno, T. Kashiwazaki, T. Umeki, S. Miki, F. China, M. Yabuno, H. Terai, and S. Takeda, Sequential and programmable squeezing gates for optical non-Gaussian input states, *PRX Quantum* **6**, 010311 (2025).
- [18] F. Wolf, C. Shi, J. C. Heip, M. Gessner, L. Pezzè, A. Smerzi, M. Schulte, K. Hammerer, and P. O. Schmidt, Motional Fock states for quantum-enhanced amplitude and phase measurements with trapped ions, *Nat. Commun.* **10**, 2929 (2019).
- [19] K. C. McCormick, J. Keller, S. C. Burd, D. J. Wineland, A. C. Wilson, and D. Leibfried, Quantum-enhanced sensing of a single-ion mechanical oscillator, *Nature (London)* **572**, 86 (2019).
- [20] W. Wang, Y. Wu, Y. Ma, W. Cai, L. Hu, X. Mu, Y. Xu, Z.-J. Chen, H. Wang, Y. P. Song, H. Yuan, C.-L. Zou, L.-M. Duan, and L. Sun, Heisenberg-limited single-mode quantum metrology in a superconducting circuit, *Nat. Commun.* **10**, 4382 (2019).
- [21] X. Pan, T. Krisnanda, A. Duina, K. Park, P. Song, C. Y. Fontaine, A. Copetudo, R. Filip, and Y. Y. Gao, Realization of versatile and effective quantum metrology using a single bosonic mode, *PRX Quantum* **6**, 010304 (2025).
- [22] Q. R. Rahman, I. Kladarić, M.-E. Kern, L. Lachman, Y. Chu, R. Filip, and M. Fadel, Genuine quantum non-Gaussianity and metrological sensitivity of Fock states prepared in a mechanical resonator, *Phys. Rev. Lett.* **134**, 180801 (2025).
- [23] S. Lloyd and S. L. Braunstein, Quantum computation over continuous variables, *Phys. Rev. Lett.* **82**, 1784 (1999).
- [24] N. Ofek, A. Petrenko, R. Heeres, P. Reinhold, Z. Leghtas, B. Vlastakis, Y. Liu, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Extending the lifetime of a quantum bit with error correction in superconducting circuits, *Nature (London)* **536**, 441 (2016).
- [25] L. Hu, Y. Ma, W. Cai, X. Mu, Y. Xu, W. Wang, Y. Wu, H. Wang, Y. P. Song, C.-L. Zou, S. M. Girvin, L.-M. Duan, and L. Sun, Quantum error correction and universal gate set operation on a binomial bosonic logical qubit, *Nat. Phys.* **15**, 503 (2019).
- [26] C. Flühmann, T. L. Nguyen, M. Marinelli, V. Negnevitsky, K. Mehta, and J. P. Home, Encoding a qubit in a trapped-ion mechanical oscillator, *Nature (London)* **566**, 513 (2019).
- [27] B. de Neeve, T.-L. Nguyen, T. Behrle, and J. P. Home, Error correction of a logical grid state qubit by dissipative pumping, *Nat. Phys.* **18**, 296 (2022).
- [28] V. V. Sivak, A. Eickbusch, B. Royer, S. Singh, I. Tsioutsios, S. Ganjam, A. Miano, B. L. Brock, A. Z. Ding, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Real-time quantum error correction beyond break-even, *Nature (London)* **616**, 50 (2023).
- [29] B. L. Brock, S. Singh, A. Eickbusch, V. V. Sivak, A. Z. Ding, L. Frunzio, S. M. Girvin, and M. H. Devoret, Quantum error correction of qudits beyond break-even, *Nature (London)* **641**, 612 (2025).
- [30] A. Mari and J. Eisert, Positive Wigner functions render classical simulation of quantum computation efficient, *Phys. Rev. Lett.* **109**, 230503 (2012).
- [31] M. G. Genoni, M. L. Palma, T. Tufarelli, S. Olivares, M. S. Kim, and M. G. A. Paris, Detecting quantum non-Gaussianity via the Wigner function, *Phys. Rev. A* **87**, 062104 (2013).
- [32] R. Filip and L. Mišta, Detecting quantum states with a positive Wigner function beyond mixtures of Gaussian states, *Phys. Rev. Lett.* **106**, 200401 (2011).
- [33] M. Ježek, I. Straka, M. Mičuda, M. Dušek, J. Fiurášek, and R. Filip, Experimental test of the quantum non-Gaussian character

- of a heralded single-photon state, *Phys. Rev. Lett.* **107**, 213602 (2011).
- [34] M. Ježek, A. Tipsmark, R. Dong, J. Fiurášek, L. Mišta, R. Filip, and U. L. Andersen, Experimental test of the strongly nonclassical character of a noisy squeezed single-photon state, *Phys. Rev. A* **86**, 043813 (2012).
- [35] I. Straka, A. Predojević, T. Huber, L. Lachman, L. Butschek, M. Miková, M. Mičuda, G. S. Solomon, G. Weihs, M. Ježek, and R. Filip, Quantum non-Gaussian depth of single-photon states, *Phys. Rev. Lett.* **113**, 223603 (2014).
- [36] I. Straka, L. Lachman, J. Hloušek, M. Miková, M. Mičuda, M. Ježek, and R. Filip, Quantum non-Gaussian multiphoton light, *npj Quantum Inf.* **4**, 4 (2018).
- [37] J. Fiurášek, L. Lachman, and R. Filip, Quantum non-Gaussianity criteria based on vacuum probabilities of original and attenuated state, *New J. Phys.* **23**, 073005 (2021).
- [38] C. Hughes, M. G. Genoni, T. Tufarelli, M. G. A. Paris, and M. S. Kim, Quantum non-Gaussianity witnesses in phase space, *Phys. Rev. A* **90**, 013810 (2014).
- [39] B. Kühn and W. Vogel, Quantum non-Gaussianity and quantification of nonclassicality, *Phys. Rev. A* **97**, 053823 (2018).
- [40] Š. Bräuer, T. Opatrný, and P. Marek, Generalized squeezing as a witness of various quantum properties, *Phys. Rev. Res.* **7**, 033176 (2025).
- [41] L. Happ, M. A. Efremov, H. Nha, and W. P. Schleich, Sufficient condition for a quantum state to be genuinely quantum non-Gaussian, *New J. Phys.* **20**, 023046 (2018).
- [42] V. Kala, R. Filip, and P. Marek, Nullifiers of non-Gaussian cluster states through homodyne measurement, *arXiv:2505.21066*.
- [43] L. Lachman, I. Straka, J. Hloušek, M. Ježek, and R. Filip, Faithful hierarchy of genuine n -photon quantum non-Gaussian light, *Phys. Rev. Lett.* **123**, 043601 (2019).
- [44] U. Chabaud, G. Roeland, M. Walschaers, F. Grosshans, V. Parigi, D. Markham, and N. Treps, Certification of non-Gaussian states with operational measurements, *PRX Quantum* **2**, 020333 (2021).
- [45] J. Fiurášek, Certification of stellar ranks of quantum states of light with a pair of click detectors, *Phys. Rev. A* **112**, 063712 (2025).
- [46] J. Peřina, Normal and antinormal ordering of field operators in quantum optics, *Czech. J. Phys.* **18**, 197 (1968).
- [47] P. Jan, Quantum theory of coherence and nonlinear optics, *J. Eur. Opt. Soc.-Rapid Publ.* **5**, 10048s (2010).
- [48] J. Peřina, V. Michálek, and O. Haderka, Higher-order sub-Poissonian-like nonclassical fields: Theoretical and experimental comparison, *Phys. Rev. A* **96**, 033852 (2017).
- [49] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, EBL-Schweitzer (Cambridge University Press, Cambridge, 1995).
- [50] M. Kalash, M. H. M. Passos, É. Rácz, L. Ruppert, R. Filip, and M. V. Chekhova, Certifying non-classicality and non-Gaussianity through optical parametric amplification, *PRX Quantum* **7**, 033020 (2026).
- [51] M. Kalash and M. V. Chekhova, Wigner function tomography via optical parametric amplification, *Optica* **10**, 1142 (2023).
- [52] I. Barakat, M. Kalash, D. Scharwald, P. Sharapova, N. Lindlein, and M. Chekhova, Simultaneous measurement of multimode squeezing through multimode phase-sensitive amplification, *Optica Quantum* **3**, 36 (2025).
- [53] G. J. Milburn, M. L. Steyn-Ross, and D. F. Walls, Linear amplifiers with phase-sensitive noise, *Phys. Rev. A* **35**, 4443 (1987).
- [54] M. G. Genoni and M. G. A. Paris, Quantifying non-Gaussianity for quantum information, *Phys. Rev. A* **82**, 052341 (2010).
- [55] L. Lachman and R. Filip, Quantum non-Gaussianity from a large ensemble of single photon emitters, *Opt. Express* **24**, 27352 (2016).
- [56] S. M. Barnett, G. Ferenczi, C. R. Gilson, and F. C. Speirits, Statistics of photon-subtracted and photon-added states, *Phys. Rev. A* **98**, 013809 (2018).
- [57] C. Hotter, C. Henke, C. J. van Diepen, P. Lodahl, and A. S. Sørensen, Quantum non-Gaussianity criterion based on photon correlations $g^{(2)}$ and $g^{(3)}$, *Phys. Rev. Lett.* **136**, 243601 (2026).
- [58] N. Nechushtan, H. Zhang, Y. London, M. Meller, H. Amichai, E. Cohen, and A. Pe'er, Ideal detection of entanglement through stimulated disentanglement, in *CLEO 2025* (Optica Publishing Group, Washington, D.C., 2025), p. FF100_5.