

Berry-phase-induced canonical structure of quantum fluctuations across the Mott-superfluid phase diagram

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We develop a unified field-theoretical description of the Bose-Hubbard model that captures low-energy dynamics across the insulator-to-superfluid transition. We show that commonly used cumulant expansions about the Mott ground state fail in the superfluid phase not only quantitatively but also fundamentally, because they impose an incorrect canonical structure. Expanding instead about the appropriate mean-field saddle point, we derive an effective action in which the dynamics of fluctuations are governed by a Berry-phase term, giving rise to an emergent magnetic field in the associated oscillator description. This field encodes particle-hole symmetry breaking in both the Mott-insulator phase and the superfluid, and vanishes at the point where a pure Goldstone mode emerges. The resulting effective hydrodynamic theory admits a speed of sound that approaches the Bogoliubov limit in the weakly interacting regime, demonstrating consistency across the entire phase diagram, from the Mott-insulator to the deep superfluid regime. While the total density remains continuous within mean-field theory, we find that quantum fluctuations generate a discontinuous contribution at the quantum phase transition, which vanishes at the tip of the Mott lobe.

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I. INTRODUCTION

Quantum field theory is a cornerstone of many-particle physics, providing powerful tools to describe their collective behavior. Lattice vibrations in solids, for example, are elegantly captured by mapping their quantum field theoretic description to a collection of harmonic oscillators, representing noninteracting phonons. The Bose-Hubbard model (BHM), on the other hand, offers a paradigmatic example of an interacting bosonic system that undergoes a quantum phase transition (QPT) between a Mott insulator (MI) and a superfluid (SF) phase [1–10].

In the MI phase, the low-lying excitations are either particlelike or holelike. In the SF phase, they can further be classified according to whether they couple to amplitude or phase fluctuations of the order parameter (OP). A defining feature is the emergence of a gapless phaselike Goldstone mode, exhibiting linear dispersion at long wavelengths and characterized by a specific speed of sound. The SF also supports a gapped amplitudelike Higgs mode. These excitations and their decay processes have been the subject of detailed investigation using numerical approaches based on the Gutzwiller ansatz [11–16]. In the deep SF regime, Bogoliubov mean-field theory serves as a benchmark for the OP and the sound velocity,

while in the deep MI regime, the model reduces to exactly solvable independent sites, providing another anchor point for a theory of excitations.

Field-theoretic techniques have played a central role in analyzing the BHM allowing for a physically transparent description in terms of an action. The seminal work of Fisher *et al.* [17] introduced such an effective action via a cumulant expansion following a Hubbard-Stratonovich transformation (HST), yielding reliable results in the MI phase and accurately locating the phase boundary. However, the theory quickly loses quantitative accuracy deeper into the SF phase [18,19]. A key development involved applying a second HST [18], which extends the theory further into the SF regime, though it still falls short of recovering the Bogoliubov limit.

Efforts to derive a low-energy effective field theory (EFT) for the SF phase, built upon these HST-based approaches, inherit the same limitations [20–22]. As a result, a consistent low-energy EFT of the BHM that captures the correct excitation spectrum and matches the Bogoliubov benchmark remains elusive.

In this paper, we develop a unified framework to describe excitations across both MI and SF phases using a refined field-theoretic approach. Starting from an HST and a cumulant expansion around the mean-field saddle point, we construct an effective local action that decouples into a free theory for the excitations. Our approach identifies the role of the Berry-phase term in the canonical structure governing the excitations at the Gaussian level, leading to the generation of the Higgs gap at the quantum phase transition. This allows us to systematically characterize both particle-hole modes in the MI and the Goldstone and Higgs modes in the SF. Remarkably, the hydrodynamical speed of sound, written explicitly in terms of single-particle Green's functions, attains the Bogoliubov

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value deep in the SF regime, attesting to the consistency of the theory across the phase diagram. Finally, we explore the role of dimensionality by going beyond mean field and finding the effect of the quantum Gaussian fluctuations on the density, revealing the role of the poles of the Green's matrix in regularizing the theory.

II. CONSISTENT FIELD THEORY

The BHM, introduced in Ref. [24], is governed by the Hamiltonian $\hat{H} = \hat{H}_J + \hat{H}_{\text{loc}}$:

$$\hat{H}_J = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j, \quad \hat{H}_{\text{loc}} = \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i. \quad (1)$$

Here, i, j label the lattice sites and $\langle i, j \rangle$ the nearest-neighbor bonds; $\hat{a}_i^\dagger$ ($\hat{a}_i$) is the bosonic creation (annihilation) operator and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ the site occupation number; $J > 0$ denotes the tunneling amplitude, $U > 0$ the on-site interaction, and μ the chemical potential.

The prevalent field-theoretical description (see Appendix A) employs a cumulant expansion around the Mott insulator phase that leads to a nonphysical description of the superfluid regime. This includes an order parameter that decays rather than increases with J/U (see Appendix B) and a non-normalized spectral function [18].

We now turn to describe a consistent field-theoretical formulation that resolves these issues. It is equivalent at the mean-field level to the thermal averaging with respect to the decoupled site Hamiltonian with an on-site drive of strength $\psi^{(0)}$,

$$\hat{H}_{\text{MF}} = \sum_i \left[\frac{U}{2} \hat{n}_i(\hat{n}_i - 1) - \mu \hat{n}_i - \psi^{(0)}(\hat{a}_i^\dagger + \hat{a}_i) \right]. \quad (2)$$

Here, $\psi^{(0)}$, calculated self-consistently using $\langle \hat{a}_i \rangle_{\text{MF}} = (zJ)^{-1} \psi^{(0)}$ (with z the coordination number), is the mean-field OP, acquiring a finite value in the SF phase. We employ a truncation of the Fock space such that the maximal occupation number n_{max} retained for each site [1] increases until convergence is achieved. This saddle-point approximation attains the same OP as the Gutzwiller mean-field (GMF) approach [25].

Proceeding beyond mean field (see Appendix C), taking $\psi = \psi^{(0)} + \delta\psi$,

$$S_E[\delta\psi, \delta\psi^*] = \frac{1}{2} \sum_{\mathbf{k}, \omega} \delta\Psi_{\mathbf{k}}^\dagger(i\omega) \mathcal{G}(\mathbf{k}, i\omega) \delta\Psi_{\mathbf{k}}(i\omega) \quad (3)$$

is the action, with $\delta\Psi_{\mathbf{k}}^\dagger(i\omega) = (\delta\psi_{\mathbf{k}}^*(i\omega), \delta\psi_{-\mathbf{k}}(-i\omega))$ the particle-hole spinor and $\mathcal{G}$ the Green's matrix, given by

$$\mathcal{G}(\mathbf{k}, i\omega) = \begin{pmatrix} \mathcal{J}_{\mathbf{k}}^{-1} + G(i\omega) & G'(i\omega) \\ G'(-i\omega) & \mathcal{J}_{-\mathbf{k}}^{-1} + G(-i\omega) \end{pmatrix}. \quad (4)$$

Here, $\mathcal{J}_{\mathbf{k}}$ is the spatial Fourier transform of the tunneling matrix $\mathcal{J}_{ij} \equiv J\delta_{\langle i,j \rangle}$; $G(i\omega)$ and $G'(i\omega)$ are the Fourier transformed Green's function $G_{i,j}(\tau - \tau') = -\langle \mathcal{T} \hat{a}_i(\tau) \delta \hat{a}_j^\dagger(\tau') \rangle_{\text{MF}}$ and the anomalous Green's function $G'_{i,j}(\tau - \tau') = -\langle \mathcal{T} \hat{a}_i(\tau) \delta \hat{a}_j(\tau') \rangle_{\text{MF}} = -\langle \mathcal{T} \delta \hat{a}_i^\dagger(\tau) \delta \hat{a}_j^\dagger(\tau') \rangle_{\text{MF}}$, respectively; and $\delta \hat{a}_i(\tau) = \hat{a}_i(\tau) - \langle \hat{a}_i(\tau) \rangle_{\text{MF}}$. Retaining

the time dependence of the anomalous Green's function is necessary for attaining the correct excitation spectrum in the SF phase [26].

The spectrum comprises the on-shell frequencies of the quadratic action [Eq. (3)] following an analytic continuation $i\omega \rightarrow \omega$. They are obtained by requiring a vanishing determinant of the Green's matrix $\mathcal{G}$, yielding $\omega_{\lambda, \mathbf{k}}$ with λ indexing the excitation bands, while the corresponding kernel, $(\mathcal{U}_{\lambda, \mathbf{k}}, \mathcal{V}_{\lambda, \mathbf{k}}^*)^T$, comprises the quasiparticle and hole amplitudes of the excitations [11] up to a normalization factor. Equivalently, the excitation spectrum is extracted from the poles of the one-particle correlator, $\langle \delta\psi_{\mathbf{k}}(\omega) \delta\psi_{\mathbf{k}}^*(\omega) \rangle$ (see Appendix D).

The resulting spectrum comprises n_{max} bands [Fig. 1(a)] identical to the spectrum found in the GMF approximation [1]. Each band, due to the anomalous Green's function, includes excitations that are a superposition of a particle and a hole, reflected by simultaneously finite $\mathcal{U}_{\mathbf{k}}$ and $\mathcal{V}_{\mathbf{k}}$ [11], with the ratio of these two amplitudes characterizing this superposition [Fig. 1(a)].

III. CANONICAL STRUCTURE OF THE LOW-ENERGY THEORY

The low-energy theory can serve to clarify the physical content of the theory of excitations. At the Gaussian level, a theory of harmonic oscillators is expected to govern the physics, but it is not *a priori* clear which details contribute to the spectrum [Fig. 1(a)]. To elucidate this, we perform the low-energy-momentum expansion of Eq. (3), to get (see Appendix E)

$$\mathcal{L}_E = \frac{1}{2} \delta\Psi^\dagger \left[(M - K_s \nabla^2 - K_\tau \partial_\tau^2) \sigma_0 + Z \sigma_3 \partial_\tau \right. \\ \left. + (\Delta - \tilde{K}_\tau \partial_\tau^2) \sigma_1 \right] \delta\Psi, \quad (5)$$

where σ_i are the Pauli matrices, $M = G + (zJ)^{-1}$ is the tuning parameter across the QPT, $2K_s = \partial_{\mathbf{k}}^2 \mathcal{J}_{\mathbf{k}}^{-1}$ is the spatial stiffness, $2K_\tau = -\partial_{i\omega}^2 G$ is the temporal stiffness, $Z = \partial_{i\omega} G$ is the Berry-phase term, $\Delta = G'$ is the particle-hole mixing amplitude, and $2\tilde{K}_\tau = -\partial_{i\omega}^2 G'$ is the anomalous temporal stiffness. All coefficients are evaluated at $(\mathbf{k}, i\omega) = 0$ and are real. The Berry-phase coefficient, Z , extends the expression given in Ref. [17] into the SF phase, allowing us to find when there is particle-hole symmetry, particularly the existence of a pure Goldstone excitation in the phase diagram. We can compare the vanishing of the Berry-phase term with the exact condition of particle-hole symmetry, $|\mathcal{U}_{\mathbf{k}}| = |\mathcal{V}_{\mathbf{k}}|$ [11] [Fig. 1(b)]. In addition, in the MI phase the Berry-phase term differentiates between a relativistic ($Z = 0$) and a nonrelativistic ($Z \neq 0$) action. In contrast, in the SF phase, the effective action is always nonrelativistic due to the anomalous temporal terms.

For a real OP, the pure amplitude and the pure phase excitations are along the real (h) and imaginary (g) parts of the OP perturbation, $\delta\psi(\mathbf{r}, \tau) = h(\mathbf{r}, \tau) + ig(\mathbf{r}, \tau)$. They can also be considered as a superposition of particle and hole excitations or, alternatively, of upper- and lower-band excitations. To study the dynamics in a familiar form, we write the Hamiltonian given by the Legendre transformation of the real-time

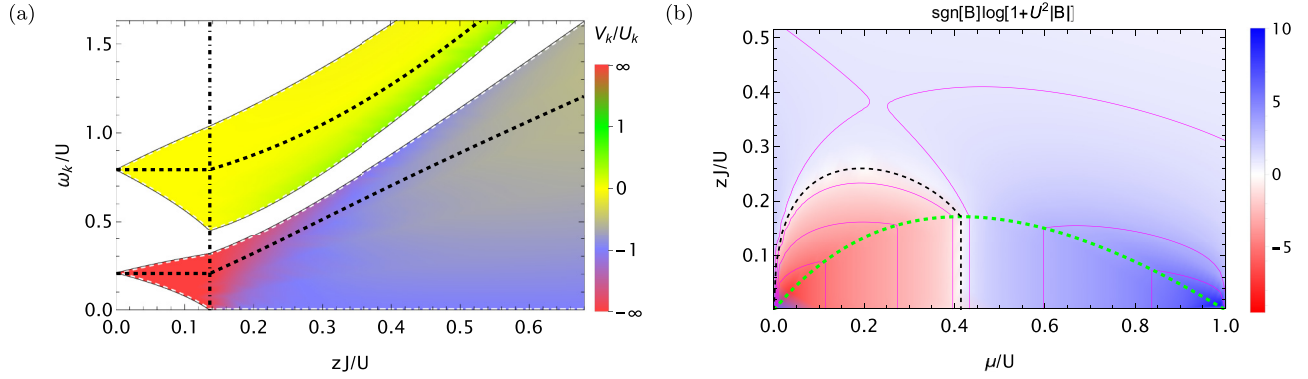


FIG. 1. Role of the Berry phase in the formation of the excitation spectrum. (a) The excitation spectrum depicting $\omega_{\lambda,k}$ as a function of J/U , showing the two lowest bands ($\lambda = 1, 2$), corresponding singularities, $\omega_{\lambda,\pi/2}$ (black dashed lines), and phase transition point (vertical black dot-dashed). Color indicates the ratio between particle and hole amplitudes of the excitation (V_k/U_k): pure hole (red), pure particle (yellow), pure phase (blue), and pure amplitude excitation (green). In the MI phase the excitations are pure particle/hole, and this nature is approximately maintained into the SF near the phase transition. In the SF regime, the $k \approx 0$ excitations are mostly phase and amplitude for the gapless and gapped band, respectively. For larger J/U , the gapless band becomes particlelike, as predicted by the Bogoliubov spectrum [23]. Here, $\mu/U = (\mu/U)_c/2$, where $(\mu/U)_c$ is the value for the tip of the first Mott lobe, for the one-dimensional (1D) chain. (b) The dimensionless magnetic field B , or Berry-phase coefficient Z , in the low-energy effective two-dimensional (2D) anharmonic oscillator description representing the strength of the Berry-phase term; $B = 0$ corresponds to a particle-hole symmetry, highlighted by the $|\mathcal{U}_k| = |\mathcal{V}_k|$ black curve of the gapless band, calculated with the exact quadratic theory [11], leading to a pure phase (Goldstone) mode.

Lagrangian ($\mathcal{L}_E(\tau) \rightarrow \mathcal{L}(t) = -\mathcal{L}_E(it)$) (see Appendix F),

$$\mathcal{H}_{\mathbf{k}}^+ = \frac{|p_{h,\mathbf{k}} + \mathcal{B}g_{\mathbf{k}}/2|^2}{2\mathcal{M}_h} + \frac{\mathcal{K}_h(\mathbf{k})|h_{\mathbf{k}}|^2}{2} + \frac{|p_{g,\mathbf{k}} - \mathcal{B}h_{\mathbf{k}}/2|^2}{2\mathcal{M}_g} + \frac{\mathcal{K}_g(\mathbf{k})|g_{\mathbf{k}}|^2}{2}, \quad (6)$$

where $\mathcal{H}_{\mathbf{k}}^+$ is the Hamiltonian for real time t and for half of the k space. The fields $h_{\mathbf{k}}(t)$ and $g_{\mathbf{k}}(t)$ are the spatial Fourier transformed complex fields of h and g , respectively, and $p_{h,\mathbf{k}}(t)$ and $p_{g,\mathbf{k}}(t)$ are the corresponding canonical momenta. The physical parameters are $\mathcal{B} = 2Z$, $\mathcal{M}_h = 2(K_\tau + \tilde{K}_\tau)$, $\mathcal{M}_g = 2(K_\tau - \tilde{K}_\tau)$, $\mathcal{K}_h = 2(M + \Delta + k^2 K_s)$, and $\mathcal{K}_g = 2(M - \Delta + k^2 K_s)$. In the MI phase, where the anomalous Green's function vanishes, the masses are equal, $\mathcal{M}_h = \mathcal{M}_g$, as well as the spring constants, $\mathcal{K}_h = \mathcal{K}_g$; hence, Eq. (6) describes an isotropic two-dimensional harmonic oscillator with a k -dependent spring constant and a constant magnetic field. In the SF phase, Eq. (6) describes an anisotropic harmonic oscillator in a constant magnetic field. The expressions for the parameters as functions of the Green's functions are provided in Appendix F.

The low-energy approximation for the excitation spectrum is given by the frequencies of the normal modes of the Hamiltonian, written in the form $\mathcal{H}_{\mathbf{k}}^+ = \omega_{+, \mathbf{k}} a_{+, \mathbf{k}}^* a_{+, \mathbf{k}} + \omega_{-, \mathbf{k}} a_{-, \mathbf{k}}^* a_{-, \mathbf{k}}$ [27]. These frequencies are functions of the oscillator and cyclotron frequencies expressed by the Hamiltonian parameters. The spring constants at $\mathbf{k} = 0$, $\mathcal{K}_{h,g} = 2[G + (zJ)^{-1} \pm G']$, vanish at the QPT, where the lower band gap closes. The magnetic field $\mathcal{B}$ (and the Berry-phase term) is responsible for two effects: First, it sets the gap between the lower and upper bands in the MI phase, and second, it breaks the time-reversal symmetry in the oscillator picture that relates to broken particle-hole symmetry in Eq. (5). While in dual vortex descriptions of the transition the Berry-phase term couples vortices to the background density,

leading to an effective Magnus force (see, e.g., Ref. [28] and references within), here we show that in the bosonic description, at the level of Gaussian fluctuations, it generates the Higgs gap at the QPT.

IV. THE HYDRODYNAMICAL SPEED OF SOUND

In the SF regime, after a suitable basis change, Eq. (5) represents a Klein-Gordon (KG) action, with KG mass $m_{\pm}^{(\text{SF})} \propto \sqrt{G + (zJ)^{-1} \pm G'}$. The spectrum becomes gapless in the SF phase, as expected by the Goldstone theorem. The lower mode φ_- acquires a zero mass ($G + (zJ)^{-1} = G'$) throughout the SF phase, as also is numerically verified. The low-energy band is thus governed by the hydrodynamical action (see Appendix G)

$$S_{\text{HD}}[\varphi_-] = \int d\tau d\mathbf{r} \frac{\rho_s^{(1)}}{2} [(\nabla \varphi_-)^2 + c_s^{-2} (\partial_\tau \varphi_-)^2], \quad (7)$$

where the coefficient $\rho_s^{(1)} = K_s$ characterizes the stiffness of the low-energy single-particle excitation rather than the full superfluid density, which is determined by the response of the action to a local twist field $\theta(\mathbf{r}, \tau)$ [29]. The speed of sound c_s characterizing the dispersion relation $\omega_- = c_s k$ for low momenta is

$$c_s^2 = \frac{2K_s \Delta}{Z^2 - 2(\tilde{K}_\tau - K_\tau) \Delta}. \quad (8)$$

It is remarkable that this expression, although based on a low-energy theory, is demonstrated in Fig. 2(a) to be numerically identical to the GMF results [1]. The mode φ_+ , described by a massive (gapped) KG action, admits a quadratic dispersion relation for small k , $\omega_+ = \epsilon_+^{(\text{SF})} + k^2/2m_+^{(\text{SF})}$ (other parameters are given explicitly in Appendix G).

We note that the low-frequency expansion accurately approximates the speed of sound, but the band gap ϵ_+ agrees only qualitatively with the excitation gap as determined by

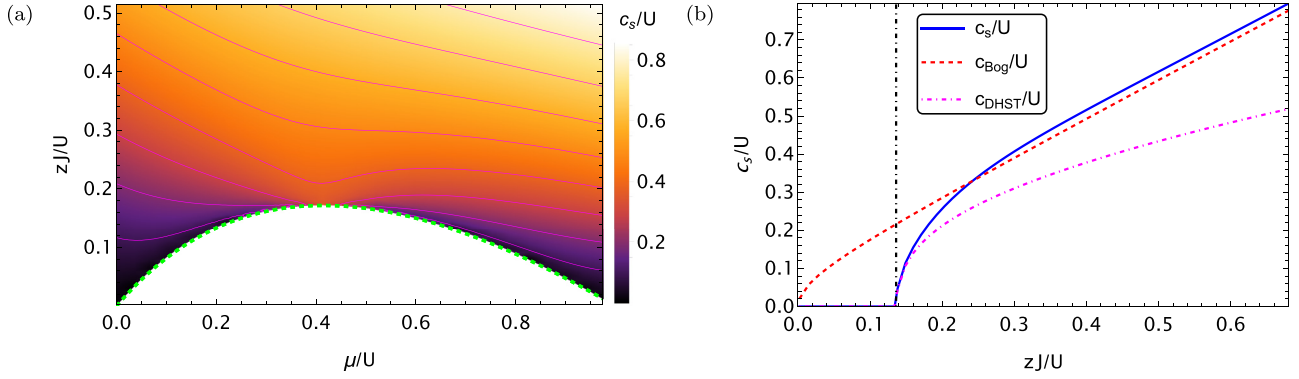


FIG. 2. Hydrodynamical description of the Goldstone mode. (a) The dimensionless sound velocity c_s/U as a function of μ and J within the SF phase (with contour lines in magenta). The dashed green line indicates the boundary of the SF with the $n_0 = 1$ Mott lobe; within the latter, the sound velocity is ill-defined (white). We take the lattice constant $a = 1$, for the 1D chain. (b) Comparison of c_s/U derived from Bogoliubov approximation (red, dashed) [23], the double HST (magenta, dot-dashed) [18], and the field theory presented here (blue, solid). The latter is consistent with the Bogoliubov approximation in the deep SF regime. The slight discrepancy is due to the weakly interacting nature of the Bogoliubov approximation that amounts to a low-order expansion in J/U . Vertical black dot-dashed line indicates the phase transition point. Here, $\mu/U = (\mu/U)_c/2$.

the full theory (see Appendix G and Fig. 5). The reason behind this loss of accuracy is in the expansion of the singular Green's functions in ω . This expansion has a finite radius of convergence, given by the nearest pole to the point of expansion, $\omega = 0$. The poles $\omega_\lambda(\mathbf{k}_r)$ of the Green's matrix determinant are evaluated when $\mathcal{J}_{\mathbf{k}_r} = 0$ for each band [see Appendix H and Fig. 1(a)]. The value of the lowest pole separates the lower band from the first gapped band; hence, the latter is not accurately described by the low-energy theory.

V. THE GREEN'S MATRIX POLES AND FLUCTUATIONS

We now demonstrate that the poles of the Green's matrix determinant in the exact theory carry physical significance for the calculation of thermodynamic quantities when going beyond mean field. Integrating the quadratic excitation action gives a contribution to the grand-canonical potential, $\Omega \approx -T \ln Z_{\text{mf}} + \Omega_{\text{fluc}}$, with

$$\begin{aligned} \Omega_{\text{fluc}} &= T \ln \prod_{\mathbf{k} \in \text{BZ}^+, \omega_n} \det \mathcal{G}(\mathbf{k}, i\omega_n) \\ &= \frac{1}{\beta} \sum_{\mathbf{k}, \omega_n} \ln \left(\frac{1}{\mathcal{J}_{\mathbf{k}}^2} \prod_{\lambda > 0} \frac{i\omega_n - \omega_{\lambda, \mathbf{k}}}{i\omega_n - \omega_{\lambda, \mathbf{k}_r}} \right). \end{aligned} \quad (9)$$

The Matsubara sum gives a thermal excitation energy expectation value that vanishes at zero temperature and a zero-point energy term $\sum_{\lambda > 0, \mathbf{k}} (\omega_{\lambda, \mathbf{k}} - \omega_{\lambda, \mathbf{k}_r})/2$. In the Bogoliubov limit, the diverging sum requires some regularization via dimensional, momentum-cutoff, or convergence factor [30,31]. In our approach, the regulator appears naturally via the poles of the Green determinant.

In Fig. 3(a), we show the effects of fluctuations by comparing the average densities calculated using $\langle n \rangle = -\partial \Omega / \partial \mu$ for the Bogoliubov theory and its beyond mean-field contribution (the exact expression for 1D appears in Appendix I), and the field-theoretical approach presented here. Remarkably, the inclusion of fluctuations further improves the agreement between the two approaches. In the MI, the field-theoretical fluctuation contribution to density vanishes, and upon crossing

into the SF phase, we find a jump discontinuity, except for two dimension-dependent points for each Mott lobe. This indicates a change in the order of the QPT from second to first. The density jump is present in all dimensions, although it decreases in size with increasing dimension [Fig. 3(b)].

VI. CONCLUSIONS

We have constructed a unified field-theoretical framework that faithfully describes the mean-field thermodynamics, as well as the excitation dynamics, particularly the low-lying excitations across the MI-SF transition. In each phase, it yields a local effective action: number-conserving in the MI phase and nonconserving in the SF phase, where spontaneous symmetry breaking sets in. The effective action retains the lowest quadratic time-dependent contribution of the anomalous Green's function, which is required to reproduce a consistent spectrum in the two phases.

Compared to GMF approaches, our approach admits a transparent physical picture. While the action successfully decouples into a set of independent two-dimensional harmonic oscillators, with anisotropic potentials that depend on the lattice momentum, these oscillators, strikingly, experience an effective perpendicular “faux” magnetic field arising from the Berry-phase term. At the quantum critical point, the oscillator potential at zero momentum vanishes, yielding the expected gapless phonon mode, while the gap to the amplitude mode is set by the cyclotron frequency of the faux field. This highlights the role of the Berry-phase term in the generation of the apparent gaps in the excitation spectrum. This mechanism applies generally to any field theory that combines relativistic and nonrelativistic time dependence.

Compared with previous field-theoretical descriptions, our unified approach treats the two phases on an equal theoretical footing, delivering numerically precise results even through criticality. Importantly, deep in the SF regime, the order parameter and speed of sound converge to the prediction of Bogoliubov. Furthermore, this framework allows to go beyond mean field and examine the contribution of fluctuations to thermodynamical quantities such as the average density. We

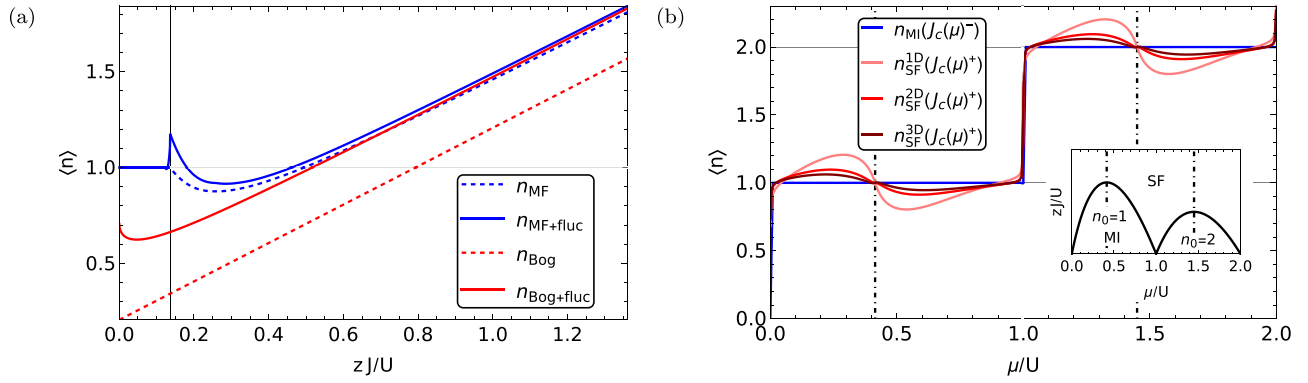


FIG. 3. Effects of fluctuations on average density. (a) The average density across a generic MI-SF phase transition with $\mu/U = (\mu/U)_c/2$ in 1D. The different curves for $\langle n \rangle$ are derived from Bogoliubov (red) and field-theoretical saddle-point (blue) approximations, in the presence (solid) and absence (dashed) of fluctuations ([1,23,31] and Appendix I). In the presence of fluctuations, both approximations lead to a similar values for $\langle n \rangle$ over a larger region of the SF phase. The contribution of fluctuations around the field-theoretical saddle point vanishes in the MI phase, demonstrating a jump discontinuity at the phase transition, and then decreasing into the SF phase. (b) The jump in average density in the presence of fluctuations at the quantum phase transition curves across the first two Mott lobes (shown in the inset). Within the Mott lobes, the density is constant independently of dimension (blue), while in the SF phase near the quantum phase transition it jumps to a finite, dimension-dependent value shown in light red (1D chain), red (2D square lattice), and dark red [three-dimensional (3D) cubic lattice]. The jump is mostly positive (negative) for $\mu < \mu_c$ ($\mu > \mu_c$), while across the multicritical points ($\mu = \mu_c$, vertical black dot-dashed lines) the density is continuous. The reduction in the contributions of fluctuations in larger dimensions is consistent with the exactness of the mean-field approximation in infinite dimension.

find that the poles of the Green's matrix play the role of an inherent regulator of the fluctuations, revealing a jump discontinuity of the density across generic points of the quantum phase transition, decreasing with spatial dimension.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: FIELD THEORY—PATH INTEGRAL APPROACH

We study the quantum partition function using the coherent state path integral approach [32,33],

$$Z = \int D[\phi, \phi^*] \exp(-S_E[\phi, \phi^*]), \quad \text{with}$$

$$S_E[\phi, \phi^*] = \sum_a \phi_a^* (\partial_\tau - \mu) \phi_a + \frac{U}{2} \sum_a |\phi_a|^4 - \sum_{a,b} \phi_a^* \mathcal{J}_{ab} \phi_b, \quad (\text{A1})$$

with ϕ the bosonic field representing the collection $\{\phi_i(\tau)\}$, for all sites i and at imaginary time $\tau \in [0, \beta)$, each being the complex eigenvalue of the instantaneous coherent state of the operator $\hat{a}_i(\tau)$. The sums over indices $a, b \equiv \{i_a, \tau_a\}, \{i_b, \tau_b\}$ include a sum over site indices and imaginary time

integrations. The HST is then used to decouple the sites at the expense of introducing an auxiliary field, ψ , which via Gaussian integration imposes the mean-field constraint $\psi_i = \sum_j \mathcal{J}_{ij} \phi_j$. Following integration over ϕ and ϕ^* , the partition function is $Z = \int D[\psi, \psi^*] \exp(-S_E[\psi, \psi^*])$, with

$$S_E[\psi, \psi^*] = \sum_{a,b} \psi_a^* \mathcal{J}_{ab}^{-1} \psi_b - \ln \left\langle \exp \left(\sum_a \psi_a^* \phi_a + \phi_a^* \psi_a \right) \right\rangle_{\text{loc}}. \quad (\text{A2})$$

Here, $\langle \cdots \rangle_{\text{loc}}$ is the thermal average taken with respect to $\hat{H}_{\text{loc}}$ [Eq. (1)].

Due to interactions, the average cannot be evaluated exactly, but instead is approximated in the standard approach [17] using an expansion of the logarithm in ψ and ψ^* . The resulting Landau-like action is a functional of these fields, which is number conserving due to the U(1) symmetry of $\hat{H}_{\text{loc}}$. The quantum phase transition from MI to SF is marked by a change in sign of the coefficient of $|\psi|^2$ [18]. The different phases are characterized by the OP $\langle \psi \rangle$, where the average is performed with respect to $S_E[\psi, \psi^*]$ expanded up to at least fourth order, $|\psi|^4$, predicting a vanishing OP in the MI phase and a finite OP in the SF phase, as anticipated. The cumulant expansion thus predicts the same phase diagram [Fig. 2(a)] as the standard mean-field or GMF approximations [1].

APPENDIX B: FAILURE OF THE CUMULANT EXPANSION AROUND THE MOTT INSULATOR

The order parameter that describes the ground state is the solution of the saddle-point approximation for the path integral formulation of the partition function, i.e., $\partial_\psi S_E[\psi, \psi^*]|_{\psi=\psi^{(0)}} = 0$. Assuming a constant OP, ψ_0 , and

using the cumulant expansion, the action contains only products of $|\psi^{(0)}|^2$ due to the U(1) symmetry. The expansion coefficients are proportional to the static and homogeneous Fourier components of the tunneling matrix and connected n -particle correlators of the bosonic field ϕ ,

$$S[\psi^{(0)}, \psi^{*(0)}] = a_1 |\psi^{(0)}|^2 (\mathcal{J}_{\text{hom}}^{-1} + G_{c,\text{hom}}^{(1)}) + \sum_{n=2}^{\infty} a_n |\psi^{(0)}|^{2n} G_{c,\text{hom}}^{(n)}, \quad (\text{B1})$$

where the coefficients a_n are strictly combinatorial and the connected correlators are related to the normal correlators,

$$G^{(n)} = - \left\langle \mathcal{T} \left(\prod_{\ell=1}^n \hat{a}_{i_\ell}(\tau_\ell) \right) \left(\prod_{\ell=n+1}^{2n} \hat{a}_{i'_\ell}^\dagger(\tau'_\ell) \right) \right\rangle_{\text{loc}}, \quad (\text{B2})$$

where the subscript loc indicates averaging with respect to the local Hamiltonian, $\hat{H}_{\text{loc}}$, of Eq. (1).

The tunneling matrix coefficient is given by

$$\mathcal{J}_{\text{hom}}^{-1} = \int_0^\beta d\tau_1 d\tau_2 \sum_{i_1, i_2} (\mathcal{J}^{-1})_{i_1 - i_2} \delta(\tau_1 - \tau_2) = N\beta \left[\sum_{\Delta i} (\mathcal{J}^{-1})_{\Delta i} \right] = N\beta (zJ)^{-1}, \quad (\text{B3})$$

while dimensional analysis shows that $G_{c,\text{hom}}^{(n)}$ scales as $N\beta \varepsilon^{1-2n}$, where ε is some energy scale of the local Hamiltonian, i.e., U or μ . Considering the action for contours of constant $\mu/U = \text{const}$, we only have U as an energy scale, and we can claim that $G_{c,\text{hom}}^{(n)} \propto U^{1-2n}$. Substituting this scaling in the action, we have

$$S[\psi^{(0)}, \psi^{*(0)}] = |\psi^{(0)}|^2 (b_0 + b_1 U^{-1}) + \sum_{n=2}^{\infty} b_n |\psi^{(0)}|^{2n} U^{1-2n} = U \left[\frac{|\psi^{(0)}|^2}{U^2} (U b_0 + b_1) + \sum_{n=2}^{\infty} b_n \frac{|\psi^{(0)}|^{2n}}{U^{2n}} \right]. \quad (\text{B4})$$

The coefficients b_n are proportional to $N\beta$ and are U independent, with b_0 depending on the tunneling amplitude J . The order parameter that minimizes the action is then a function of coefficients b_n and U . For any finite cumulant expansion, one has

$$|\psi_0|^2 = U^2 F(U b_0 + b_1, b_{n \geq 2}) \xrightarrow{U \rightarrow 0} 0. \quad (\text{B5})$$

The cumulant expansion is therefore not an accurate approximation for the BHM in the SF phase. While higher-order expansions may lead to better results near the phase transition, and may increase the range of validity with zJ/U at the cost of computational complexity, attaining the $U \rightarrow 0$ limit requires the retention of all orders of the expansion of $\ln \langle \exp(\sum_a \phi_a \psi_a^* + \psi_a \phi_a^*) \rangle_{\text{loc}}$.

APPENDIX C: CONSISTENT FIELD THEORY DERIVATION

Per the previous Appendix, deep in the SF phase, the cumulant expansion predicts a vanishing OP as $J/U \rightarrow \infty$ to any order in $|\psi|^2$, which contradicts the Bogoliubov approximation of a weakly interacting Bose gas [23], and leads to a nonphysical excitation spectrum [34].

This failure of the cumulant expansion is due to expansion around a local maximum at $\psi = 0$. In the SF phase, the OP acquires a finite value, and the auxiliary field must instead be expanded around this nonzero background. Thus, we take $\psi = \psi^{(0)} + \delta\psi$ in the action following the HST, whose associated Hamiltonian becomes Eq. (2); $\psi^{(0)}$ is the constant, homogeneous saddle-point solution $\langle \phi \rangle_{\text{MF}} = (zJ)^{-1} \psi^{(0)}$, with z the coordination number, evaluated self-consistently with respect to the action following the HST. Equivalently, $\langle \cdots \rangle_{\text{MF}}$ is the thermal average with respect to Eq. (2), as stated in the paragraph preceding Eq. (2).

Next, we study the deviations from the GMF approximation, $\delta\psi$, which correspond to the excitations, with partition function $Z = \int D[\delta\psi, \delta\psi^*] \exp(-S_E[\delta\psi, \delta\psi^*])$ and

$$S_E[\delta\psi, \delta\psi^*] = \sum_{a,b} (\delta\psi_a^* \mathcal{J}_{ab}^{-1} \psi_b^{(0)} + \psi_a^{(0)} \mathcal{J}_{ab}^{-1} \delta\psi_b) + \sum_{a,b} \delta\psi_a^* \mathcal{J}_{ab}^{-1} \delta\psi_b - \ln \left\langle \exp \left(\sum_a \delta\psi_a^* \phi_a + \phi_a^* \delta\psi_a \right) \right\rangle_{\text{MF}}. \quad (\text{C1})$$

Expanding to quadratic order in $\delta\psi$, the overall linear term vanishes due to the cancellation of explicit linear terms in the action and linear terms in the cumulant expansion, as expected from the expansion of the action around its minimum value. As Eq. (2) does not conserve the number of bosons, the resulting action for the excitations contains anomalous terms,

$$S_E[\delta\psi, \delta\psi^*] \approx \sum_{a,b} \delta\psi_a^* (\mathcal{J}_{ab}^{-1} - \langle \delta\phi_a \delta\phi_b^* \rangle_{\text{MF}}) \delta\psi_b - \frac{1}{2} \sum_{a,b} \delta\psi_a^* \langle \delta\phi_a \delta\phi_b \rangle_{\text{MF}} \delta\psi_b^* - \frac{1}{2} \sum_{a,b} \delta\psi_a \langle \delta\phi_a^* \delta\phi_b^* \rangle_{\text{MF}} \delta\psi_b. \quad (\text{C2})$$

Here, $\delta\phi = \phi - \langle \phi \rangle_{\text{MF}}$ is the fluctuation of the original bosonic field around the mean-field average. Working in momentum-frequency domain leads to Eqs. (3) and (4).

APPENDIX D: EXCITATIONS—SPECTRUM AND CHARACTERIZATION

In the MI phase, G'_{MI} vanishes due to the U(1) symmetry of the MF Hamiltonian, while

$$G_{\text{MI}}(\omega) = -\frac{n_0}{\omega + \mu - U(n_0 - 1)} + \frac{n_0 + 1}{\omega + \mu - U n_0}, \quad (\text{D1})$$

where $n_0 = \lceil \max(0, \mu/U) \rceil$ is the occupation number in the corresponding Mott lobe in the phase diagram. The spectrum is composed of the two lowest gapped bands found in the GMF, i.e.,

$$\omega_{\pm, \mathbf{k}} = \frac{1}{2} \sqrt{U^2 - 4\mathcal{J}_{\mathbf{k}} U \left(n_0 + \frac{1}{2} \right) + \mathcal{J}_{\mathbf{k}}^2} \pm \left[U \left(n_0 - \frac{1}{2} \right) - \mu - \frac{\mathcal{J}_{\mathbf{k}}}{2} \right], \quad (\text{D2})$$

where $n_0 = \lceil \mu/U \rceil$ is the commensurate density in the Mott lobe. In the corresponding kernel, one of the bands λ, λ' is of pure particle excitations ($0 < \mathcal{U}_{\lambda, \mathbf{k}} = \delta\psi_{\mathbf{k}}(\omega_{\lambda, \mathbf{k}}) < \infty, \mathcal{V}_{\lambda, \mathbf{k}} = \delta\psi_{-\mathbf{k}}(-\omega_{\lambda, \mathbf{k}}) = 0$), while the other is of pure hole ($\mathcal{U}_{\lambda', \mathbf{k}} = \delta\psi_{\mathbf{k}}(\omega_{\lambda', \mathbf{k}}) = 0, 0 < \mathcal{V}_{\lambda', \mathbf{k}} = \delta\psi_{-\mathbf{k}}(-\omega_{\lambda', \mathbf{k}}) < \infty$), depending on the chemical potential. The GMF approximation, in addition to the two gapped bands, finds higher multiparticle flat bands, which are absent in the field theory approach

as they are not described by the one-particle correlator $\langle \delta\psi_{\mathbf{k}}(\omega) \delta\psi_{\mathbf{k}}^*(\omega) \rangle_{\text{MI}}$.

In the SF phase, the spectrum is evaluated numerically and both G and G' are sums of $2n_{\text{max}}$ rational functions, leading to a polynomial equation of degree n_{max} for the variable ω^2 . The resulting spectrum is n_{max} bands identical to the spectrum found in the GMF approximation [Fig. 1(a)] [1]. Each band, due to the anomalous Green's function, includes excitations that are a superposition of a particle and a hole ($0 \leq |\mathcal{U}_{\lambda, \mathbf{k}}|, |\mathcal{V}_{\lambda, \mathbf{k}}| < \infty$) [11].

Alternatively, we diagonalize the Green's matrix directly, $\tilde{\mathcal{G}} = \text{diag}(G_+(\mathbf{k}, i\omega), G_-(-\mathbf{k}, -i\omega))$, in the basis of the new fields $(\varphi_{+, \mathbf{k}}(i\omega), \varphi_{-, \mathbf{k}}(i\omega))^T$. Each of the equations for the excitation spectrum, $G_{\pm}(\mathbf{k}, \omega) = 0$, gives $n_{\text{max}}/2$ bands with the kernel of $(1, 0)^T$ or $(0, 1)^T$. The particle/hole amplitudes form the diagonalizing matrix.

We note that while the ratio $\mathcal{U}_{\lambda, \mathbf{k}}/\mathcal{V}_{\lambda, \mathbf{k}}$ is enough to characterize the excitation, hence the kernel need not be normalized, the definite values satisfy an interbranch normalization condition [11, 15].

APPENDIX E: LOW-ENERGY EXPANSION

Expanding the Green's functions to low orders in momenta and energy yields the low-energy effective theory for excitations, primarily the gapless mode. The Green's matrix is expanded to give the following:

$$\mathcal{G}(\mathbf{k}, i\omega) \approx \begin{pmatrix} (zJ)^{-1} + G + \frac{k^2}{2} \partial_{\mathbf{k}}^2 \mathcal{J}_{\mathbf{k}}^{-1} + i\omega \partial_{i\omega} G - \frac{\omega^2}{2} \partial_{i\omega}^2 G & G' + i\omega \partial_{i\omega} G' - \frac{\omega^2}{2} \partial_{i\omega}^2 G' \\ G' - i\omega \partial_{i\omega} G' - \frac{\omega^2}{2} \partial_{i\omega}^2 G' & (zJ)^{-1} + G + \frac{k^2}{2} \partial_{\mathbf{k}}^2 \mathcal{J}_{\mathbf{k}}^{-1} - i\omega \partial_{i\omega} G - \frac{\omega^2}{2} \partial_{i\omega}^2 G \end{pmatrix}. \quad (\text{E1})$$

Here, $G, G', \mathcal{J}_{\mathbf{k}}$, and their derivatives with respect to $\mathbf{k}$ and $i\omega$ are evaluated at $\mathbf{k} = 0, i\omega = 0$ and represent real numbers. We can take $\partial_{i\omega} G' = 0$, since it is a coefficient of a full derivative term in the time-domain action and thus should vanish (this is also demonstrated numerically throughout the phase diagram). Transforming to the real space-time domain gives Eq. (5).

APPENDIX F: PURE AMPLITUDE AND PHASE HAMILTONIAN

We now turn to derive Eq. (6). Starting from the Euclidean Lagrangian density [Eq. (5)], writing $\delta\psi(\mathbf{r}, \tau) = h(\mathbf{r}, \tau) + ig(\mathbf{r}, \tau)$, we get

$$\mathcal{L}[h(\mathbf{r}, \tau), g(\mathbf{r}, \tau)] = (M + \Delta)h^2 + (M - \Delta)g^2 + K_s((\nabla h)^2 + (\nabla g)^2) + iZ(g\partial_{\tau}h - h\partial_{\tau}g) + (\partial_{\tau}h)^2(K_{\tau} + \tilde{K}_{\tau}) + (\partial_{\tau}g)^2(K_{\tau} - \tilde{K}_{\tau}). \quad (\text{F1})$$

We write the real-time Lagrangian density using a Wick rotation $\mathcal{L}_E(\tau) \rightarrow \mathcal{L}(t) = -\mathcal{L}_E(it)$:

$$\mathcal{L}[h(\mathbf{r}, t), g(\mathbf{r}, t)] = -(M + \Delta)h^2 - (M - \Delta)g^2 - K_s((\nabla h)^2 + (\nabla g)^2) - Z(g\partial_t h - h\partial_t g) + (\partial_t h)^2(K_{\tau} + \tilde{K}_{\tau}) + (\partial_t g)^2(K_{\tau} - \tilde{K}_{\tau}). \quad (\text{F2})$$

The canonical momenta are

$$p_h = \partial_{\partial_t h} \mathcal{L} = -Zg + 2(K_{\tau} + \tilde{K}_{\tau})\partial_t h, \quad (\text{F3})$$

$$p_g = \partial_{\partial_t g} \mathcal{L} = +Zh + 2(K_{\tau} - \tilde{K}_{\tau})\partial_t g. \quad (\text{F4})$$

The Legendre transform leads to the Hamiltonian density

$$\mathcal{H} = \frac{(p_h + Zg)^2}{(K_{\tau} + \tilde{K}_{\tau})} + (M + \Delta)h^2 + K_s(\nabla h)^2 + \frac{(p_g - Zh)^2}{(K_{\tau} - \tilde{K}_{\tau})} + (M - \Delta)g^2 + K_s(\nabla g)^2. \quad (\text{F5})$$

As a final step, we take the Fourier transform to find the Hamiltonian density in momentum space

$$\mathcal{H}_{\mathbf{k}}^+ = \frac{|p_{h, \mathbf{k}} + Zg_{\mathbf{k}}|^2}{4(K_{\tau} + \tilde{K}_{\tau})} + \frac{|p_{g, \mathbf{k}} - Zh_{\mathbf{k}}|^2}{4(K_{\tau} - \tilde{K}_{\tau})} + (M + \Delta + k^2 K_s)|h_{\mathbf{k}}|^2 + (M - \Delta + k^2 K_s)|g_{\mathbf{k}}|^2. \quad (\text{F6})$$

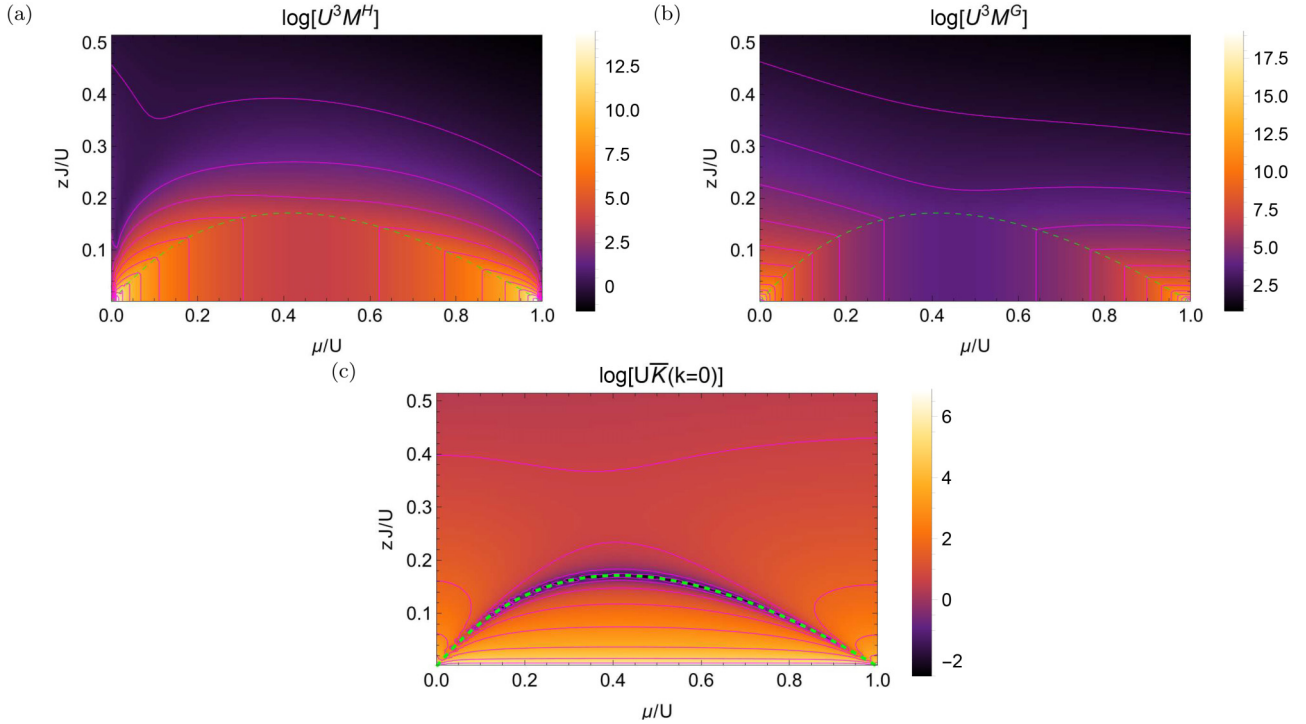


FIG. 4. Dimensionless parameters for the low-energy effective Hamiltonian: (a) mass of the amplitude mode, (b) mass of phase mode, and (c) mean spring constant $2\bar{K} = \bar{K}_h + \bar{K}_g$. Masses are equal in the MI phase. The MI-SF phase transition is depicted by a green dashed line, and contour lines are in magenta.

This Hamiltonian is in the form of an anisotropic two-dimensional harmonic oscillator with a uniform magnetic field, Eq. (6), with the different parameters defined following Eq. (6) plotted in Fig. 4 and the magnetic field in Fig. 1(a). The Hamilton equation of motion is

$$\frac{d}{dt} \begin{pmatrix} h_{\mathbf{k}} \\ g_{\mathbf{k}} \\ p_{h,\mathbf{k}} \\ p_{g,\mathbf{k}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mathcal{B}}{2\mathcal{M}_h} & \frac{1}{\mathcal{M}_h} & 0 \\ -\frac{\mathcal{B}}{2\mathcal{M}_g} & 0 & 0 & \frac{1}{\mathcal{M}_g} \\ -\frac{\mathcal{B}^2}{4\mathcal{M}_g} - \mathcal{K}_h(\mathbf{k}) & 0 & 0 & \frac{\mathcal{B}}{2\mathcal{M}_g} \\ 0 & -\frac{\mathcal{B}^2}{4\mathcal{M}_h} - \mathcal{K}_g(\mathbf{k}) & -\frac{\mathcal{B}}{2\mathcal{M}_h} & 0 \end{pmatrix} \begin{pmatrix} h_{-\mathbf{k}} \\ g_{-\mathbf{k}} \\ p_{h,-\mathbf{k}} \\ p_{g,-\mathbf{k}} \end{pmatrix}. \quad (\text{F7})$$

The eigenvectors of this equation define new variables a_+ , a_- , and their complex conjugate, such that their time evolution is $a_{\pm}(t) = a_{\pm}(0) \exp(-i\omega_{\pm}t)$ with the frequencies $\omega_{\pm}$. Defining the harmonic and cyclotron frequencies, $\omega_{0,i}(\mathbf{k}) = \sqrt{\mathcal{K}_i(\mathbf{k})/\mathcal{M}_i}$ and $\omega_{c,i} = \mathcal{B}/\mathcal{M}_i$, respectively, these frequencies are

$$2\omega_{\pm}^2 = \omega_{c,h}\omega_{c,g} + \omega_{0,h}^2 + \omega_{0,g}^2 \pm \sqrt{(\omega_{c,h}\omega_{c,g} + \omega_{0,h}^2 + \omega_{0,g}^2)^2 - 4\omega_{0,h}^2\omega_{0,g}^2}. \quad (\text{F8})$$

They agree with the spectrum of the low-energy effective theory to order k^2 .

APPENDIX G: EXCITATIONS OF THE LOW-ENERGY EFFECTIVE THEORY

We study the low-energy excitations by diagonalizing the effective action [Eq. (E1)]. In the MI phase, the Green's function assumes a diagonal form, and the effective action is simply

$$S^{(\text{MI})}[\varphi_+, \varphi_-] = \frac{1}{2} \sum_{\mathbf{k}, \omega} (M + k^2 K_s + i\omega Z + \omega^2 K_{\tau}) |\varphi_{+,\mathbf{k}}(i\omega)|^2 + \frac{1}{2} \sum_{\mathbf{k}, \omega} (M + k^2 K_s - i\omega Z + \omega^2 K_{\tau}) |\varphi_{-,\mathbf{k}}(i\omega)|^2. \quad (\text{G1})$$

The spectrum to second order in $\mathbf{k}$ is

$$\omega_{\pm} \approx \frac{\pm Z + \sqrt{Z^2 + 4K_{\tau}M}}{2K_{\tau}} + \frac{k^2 K_s}{\sqrt{Z^2 + 4K_{\tau}M}}, \quad (\text{G2})$$

which is of the form $\omega_{\pm} \approx \epsilon_{\pm}^{(\text{MI})} + k^2/2m^{(\text{MI})}$, where $\epsilon_{\pm}^{(\text{MI})} > 0$ and $m^{(\text{MI})} > 0$.

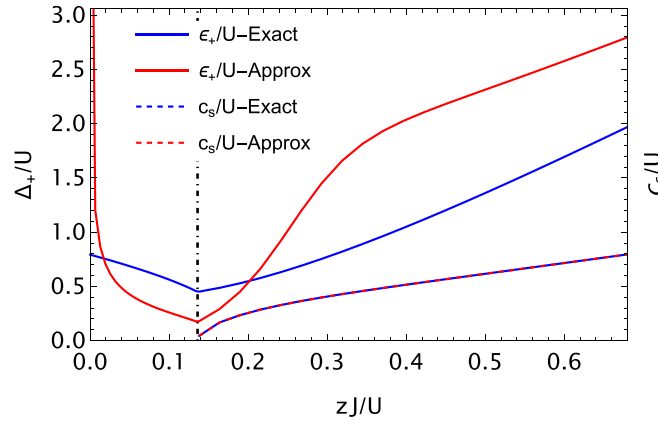


FIG. 5. Comparison of full theory vs effective low-energy theory. Speed of sound of the gapless mode c_s and the gap of the upper band $\epsilon_+ = \omega_{+, \mathbf{k}=0}$ calculated using the full Green's matrix and using the low-energy effective theory [Eqs. (G5) and (G7)]. The discrepancy in the gap but not in the speed of sound is due to the gap being outside of the low-energy EFT radius of convergence [Fig. 1(a)].

In the SF phase, the action following diagonalization and up to second order in ω and $\mathbf{k}$ is

$$S^{(\text{SF})}[\varphi_+, \varphi_-] = \frac{1}{2} \sum_{\mathbf{k}, \omega} \left[M + \Delta + k^2 K_s - \frac{\omega^2 Z^2 - 2\Delta(K_\tau + \tilde{K}_\tau)}{\Delta} \right] |\varphi_{+, \mathbf{k}}(i\omega)|^2 + \frac{1}{2} \sum_{\mathbf{k}, \omega} \left[M - \Delta + k^2 K_s + \frac{\omega^2 Z^2 + 2\Delta(K_\tau - \tilde{K}_\tau)}{\Delta} \right] |\varphi_{-, \mathbf{k}}(i\omega)|^2. \quad (\text{G3})$$

Furthermore, in the SF phase, we observe numerically that $G + (zJ)^{-1} = G'$, i.e., $M = \Delta$, and the action is cast in the form of a Klein-Gordon action with one gapless mode and one massive mode,

$$S^{(\text{SF})}[\varphi_+, \varphi_-] = \frac{\rho_s^{(1)}}{2} \sum_{\mathbf{k}, \omega} [(m_+^{(\text{SF})} c_+)^2 + k^2 + c_+^{-2} \omega^2] |\varphi_{+, \mathbf{k}}(i\omega)|^2 + \frac{\rho_s^{(1)}}{2} \sum_{\mathbf{k}, \omega} [k^2 + c_-^{-2} \omega^2] |\varphi_{-, \mathbf{k}}(i\omega)|^2. \quad (\text{G4})$$

Here, the overall coefficient is $\rho_s^{(1)} = K_s = G^{(2,0)}/2$, the sound velocities are

$$c_{\pm}^{-2} = \mp \frac{Z^2 \mp 2\Delta(K_\tau \pm \tilde{K}_\tau)}{2K_s \Delta}, \quad (\text{G5})$$

and the upper band mass is

$$m_+^{(\text{SF})} = \sqrt{-\frac{Z^2 - 2\Delta(K_\tau + \tilde{K}_\tau)}{K_s^2}} > 0. \quad (\text{G6})$$

Thus, the gap of the massive Klein-Gordon field is the rest mass

$$\epsilon_+^{(\text{SF})} = m_+^{(\text{SF})} c_+^2 = \sqrt{\frac{-4\Delta^2}{Z^2 - 2\Delta(K_\tau + \tilde{K}_\tau)}} > 0. \quad (\text{G7})$$

We compare in Fig. 5 the significant characteristics of the modes, i.e., the speed of sound of the gapless mode and the energy gap of the massive mode, derived from the full Green's matrix and the low-energy effective theory.

APPENDIX H: RADIUS OF CONVERGENCE OF THE EXPANSION IN FREQUENCY

Here, we provide details regarding the radius of convergence of the frequency expansion. We aim to demonstrate that the radius of convergence is set by the distance to the energies $\omega_{\mathbf{k}_r}$ with $\mathcal{J}_{\mathbf{k}_r} = 0$, denoted by a dashed line in Fig. 1(a). Starting from the Green's matrix

$$\mathcal{G}(\mathbf{k}, i\omega) = \begin{pmatrix} \mathcal{J}_{\mathbf{k}}^{-1} + G(i\omega) & G'(i\omega) \\ G'(-i\omega) & \mathcal{J}_{-\mathbf{k}}^{-1} + G(-i\omega) \end{pmatrix}, \quad (\text{H1})$$

we write the normal and anomalous Green's functions as rational functions, $G(i\omega) = G_n(i\omega)/G_d(i\omega)$ and $G'(i\omega) = G'_n(i\omega)/G'_d(i\omega)$, to get

$$\mathcal{G}(\mathbf{k}, i\omega) = \begin{pmatrix} \mathcal{J}_{\mathbf{k}}^{-1} + \frac{G_n(i\omega)}{G_d(i\omega)} & \frac{G'_n(i\omega)}{G'_d(i\omega)} \\ \frac{G'_n(-i\omega)}{G'_d(-i\omega)} & \mathcal{J}_{-\mathbf{k}}^{-1} + \frac{G_n(-i\omega)}{G_d(-i\omega)} \end{pmatrix}. \quad (\text{H2})$$

The spectrum and radius of convergence are given by the roots and poles of the determinant, respectively (after the analytic continuation $i\omega \rightarrow \omega + 0^+$), with the determinant given by

$$\det[\mathcal{G}(\mathbf{k}, \omega)] = \frac{1}{\mathcal{J}_{\mathbf{k}}^2} \frac{G'_d(-\omega)G'_d(\omega)[G_d(-\omega) - \mathcal{J}_{\mathbf{k}}G_n(-\omega)][G_d(\omega) - \mathcal{J}_{\mathbf{k}}G_n(\omega)] - \mathcal{J}_{\mathbf{k}}^2 G'_n(-\omega)G'_n(\omega)G_d(-\omega)G_d(\omega)}{G'_d(-\omega)G'_d(\omega)G_d(-\omega)G_d(\omega)}. \quad (\text{H3})$$

The determinant's poles are dispersionless and are given by a vanishing denominator G_d and G'_d of the normal and anomalous Green's functions. Furthermore, at the limit $\mathcal{J}_{\mathbf{k}} \rightarrow 0$, the determinant's numerator is identical to the denominator; i.e., the poles and the roots satisfy the same equation and thus coincide. We have that the poles setting the radius of convergence coincide with the roots giving the spectrum, i.e., $\omega_{\text{poles}} = \omega_{\mathbf{k}_r}$ for $\mathcal{J}_{\mathbf{k}_r} = 0$.

APPENDIX I: BEYOND MEAN-FIELD AVERAGE DENSITY

1. Bogoliubov approximation

The discrete Gross-Pitaevskii equation, valid in the weakly interacting limit, is the saddle-point approximation for the field ϕ under the partition function [Eq. (A1)],

$$[(\partial_\tau - \mu + U|\phi_{i,\tau}|^2)\delta_{ij} - \mathcal{J}_{ij}]\phi_{j,\tau} = 0. \quad (\text{I1})$$

Under the assumption of a constant uniform field $\phi_{j,\tau} = \phi^{(0)}$, one retrieves the solution $|\phi_0|^2 = (\mu + zJ)/U$ [23].

The Bogoliubov spectrum can be derived similarly to Eqs. (3) and (4) by expanding the field around its MF value, $\phi = \phi^{(0)} + \delta\phi$, to arrive at the partition function

$$Z \approx \exp^{-1} \left(-\beta N \frac{(\mu + zJ)^2}{2U} \right) \int D[\delta\phi, \delta\phi^*] \exp(-S_E[\delta\phi, \delta\phi^*]), \quad \text{with}$$

$$S_E[\delta\phi, \delta\phi^*] = \frac{1}{2} \sum_{\mathbf{k}, \omega_n} \delta\Phi_{\mathbf{k}}^\dagger(i\omega) \begin{pmatrix} [(i\omega_n + \mu + 2zJ) - \mathcal{J}_{\mathbf{k}}]e^{-i\omega_n\eta} & (\mu + zJ) \\ (\mu + zJ) & [(-i\omega_n + \mu + 2zJ) - \mathcal{J}_{-\mathbf{k}}]e^{i\omega_n\eta} \end{pmatrix} \delta\Phi_{\mathbf{k}}(i\omega), \quad (\text{I2})$$

the excitation action and with $\delta\Phi_{\mathbf{k}}^\dagger(i\omega) = (\delta\phi_{\mathbf{k},\omega}^*, \delta\phi_{-\mathbf{k},-\omega})$. The convergence factors $\exp(\pm i\omega_n\eta)$ are necessary to obtain the correct behavior when taking the limit $\eta \rightarrow 0^+$ [30].

The Bogoliubov spectrum is given by the roots of the determinant of the Green's matrix, while the kernel vectors comprise the particle and hole strengths. Integrating the excitation action and taking the logarithm of the partition function, we obtain the Landau free energy

$$\frac{1}{N} \Omega_{\text{Bog}} = \frac{-\ln Z}{\beta N} = -\frac{(\mu + zJ)^2}{2U} + \frac{1}{2N\beta} \text{tr}_{\mathbf{k}, \omega_n} \ln \begin{pmatrix} [(i\omega_n + \mu + 2zJ) - \mathcal{J}_{\mathbf{k}}]e^{-i\omega_n\eta} & (\mu + zJ) \\ (\mu + zJ) & [(-i\omega_n + \mu + 2zJ) - \mathcal{J}_{-\mathbf{k}}]e^{i\omega_n\eta} \end{pmatrix}. \quad (\text{I3})$$

The first term is the mean-field energy and the second term is the contribution of the excitations. Summing the Matsubara frequencies, we have a zero-point energy term and a thermal term that vanishes at zero temperature [30], which we omit for $T \rightarrow 0$:

$$\Omega_{\text{Bog}} = \frac{-\ln Z}{\beta N} = -\frac{(\mu + zJ)^2}{2U} + \frac{1}{2N} \text{tr}_{\mathbf{k}} (\omega_{\mathbf{k}} - \mu - 2zJ + \mathcal{J}_{\mathbf{k}}). \quad (\text{I4})$$

Here, $\omega_{\mathbf{k}} = \sqrt{(\mu + 2zJ - \mathcal{J}_{\mathbf{k}})^2 - (\mu + zJ)^2}$ is the Bogoliubov excitation spectrum. The zero-point energy sum is regularized due to the convergence factors, leading to a finite result.

Summing over the momentum, we can find an explicit solution to the 1D case:

$$\begin{aligned} \frac{1}{N} \Omega_{\text{Bog,1D}} &= -\frac{(\mu + 2J)^2}{2U} + \frac{1}{4\pi} \int_{-\pi}^{\pi} (\omega_k - \mu - 4J + \mathcal{J}_k) dk \\ &= -\frac{(\mu + 2J)^2}{2U} + \frac{1}{\pi} \sqrt{2J(2J + \mu)} + \frac{(4J + \mu)}{2\pi} \arctan \left(2\sqrt{\frac{2J}{\mu} \left(\frac{2J}{\mu} + 1 \right)} \right), \end{aligned} \quad (\text{I5})$$

which gives rise to the average density

$$\langle n \rangle_{\text{Bog,1D}} = -\frac{1}{N} \frac{\partial \Omega_{1D}}{\partial \mu} = \frac{\mu + 2J}{U} + \frac{1}{2} - \frac{1}{2\pi} \arctan \left(2\sqrt{\frac{2J}{\mu} \left(\frac{2J}{\mu} + 1 \right)} \right). \quad (\text{I6})$$

2. Field-theoretical approach

For the field-theoretical approach, we present in the main text, the partition function to integrate does not require any convergence factors; rather the regulator appears naturally in the Green's matrix when its determinant [Eq. (H3)] is cast in a form that takes into account both the zeros and the roots

$$\det [\mathcal{G}(\mathbf{k}, \omega)] = \frac{1}{\mathcal{J}_{\mathbf{k}}^2} \prod_{\lambda} \frac{i\omega_n - \omega_{\lambda, \mathbf{k}}}{i\omega_n - \omega_{\lambda, \mathbf{k}_r}}. \quad (17)$$

When taking the logarithm of the partition function, we have for the Landau free energy

$$\frac{1}{N} \Omega = \frac{1}{N} \Omega_{\text{mf}} + \frac{1}{2N} \sum_{\lambda > 0, \mathbf{k}} (\omega_{\lambda, \mathbf{k}} - \omega_{\lambda, \mathbf{k}_r}). \quad (18)$$

In the MI, the two branches [Eq. (D2)] lead to the following form for this Landau free energy:

$$\frac{1}{N} \Omega = \frac{1}{N} \Omega_{\text{mf}}^{\text{MI}} + \frac{1}{2N} \sum_{\mathbf{k}} \left(\sqrt{U^2 - 4\mathcal{J}_{\mathbf{k}} U \left(n_0 + \frac{1}{2} \right) + \mathcal{J}_{\mathbf{k}}^2 - U} \right). \quad (19)$$

The fluctuation contribution is independent of μ , so it does not contribute to the average density in the MI. Interestingly, this vanishing is not the result of the cancellation of μ between particle and hole energies. Instead, the cancellation of μ occurs for each branch separately due to the presence of the regulator.

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