

Distinct reentrant transitions in a quasiperiodic Raman lattice

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We investigate a one-dimensional lattice with spin-orbit coupling (SOC) and a Zeeman potential containing uniform and quasiperiodic components. By tuning SOC, anomalous mobility edges emerge that separate critical from noncritical states, yielding a reentrant transition between two mixed phases, $M_1 \rightarrow M_2 \rightarrow M_1$, where M_1 (M_2) lacks (hosts) anomalous mobility edges. A *reentrant criticality transition*, defined as multiple entries into the critical phase, is identified. As a counterpart to reentrant delocalization/localization transitions, it completes the basic framework of reentrant phenomena across extended, localized, and critical states. The uniform Zeeman potential drives a reentrant delocalization transition, arising from the splitting of the localized region induced by the shift of mobility edges. This reveals a distinct pathway for reentrant phenomena beyond hybridization mechanisms.

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Introduction. In the field of Anderson localization [1], the competition between different types of quantum states has been extensively studied [2–8], particularly in quasiperiodic systems [9–13]. Many interesting phenomena are discovered, e.g., mobility edges [14], well-defined critical energies that separate extended states from localized ones. Beyond the conventional picture of mobility edges, recent studies reveal the proliferation of critical states in the spectrum, inducing the emergence of anomalous mobility edges [10,11], also known as new mobility edges [12,13]. This phenomenon goes beyond the conventional mobility-edge paradigm and allows for rich coexistence manners among extended, localized, and critical states. These rich coexistence phenomena have been both theoretically predicted and experimentally confirmed across a variety of physical systems [15–19].

Recently, an intriguing phenomenon has been discovered in modified quasiperiodic systems, known as the reentrant delocalization/localization transition [20–22]. In contrast to the general quasiperiodic systems, where the adjustment of a single parameter induces only a delocalization/localization transition [3–8], the modified quasiperiodic systems exhibit two or more distinct delocalization/localization transitions. This reentrant behavior generally originates from hybridization mechanisms, which can be driven by an enhanced density of states [20–25]. The relevant experimental verifi-

cation and advances in photonic platforms further emphasize the practical relevance of reentrant transitions [20,25,26], which has broad significance for exploring complex delocalization/localization transitions in superconducting simulators [27–29], cold atoms [15–18,30–32], and other platforms. These reentrant behaviors highlight a complex landscape of localization physics, which is crucial for understanding the interplay between disorder-induced localization and hybridization-driven delocalization.

Quasiperiodic systems host extended, critical, and localized states, yet reentrant phenomena involving critical states remain unexplored. The inclusion of critical states is expected to generate richer competition, leading to a more complex scenario for reentrant behavior. Furthermore, beyond the hybridization mechanism, it remains an open question whether other mechanisms can drive reentrant transitions. In this Letter, we study the reentrant transition in a quasiperiodic Raman lattice. In the presence of a Zeeman potential, spin-orbit coupling (SOC) induces a reentrant transition between mixed phases, distinguished by the absence or presence of anomalous mobility edges. Remarkably, the system repeatedly enters the purely critical phase, giving rise to a *reentrant criticality transition*. By tuning the Zeeman potential, we demonstrate a reentrant delocalization transition via direct control of mobility edges, revealing a distinct origin of reentrant behaviors. Since the model we adopted is feasible in optical Raman lattice experiments [30–32], all phenomena can be realized experimentally.

Model and quantities. We investigate a one-dimensional lattice model with SOC and a Zeeman potential, described by the Hamiltonian [34]

$$\hat{H} = -t \sum_j (\hat{c}_{j,\uparrow}^\dagger \hat{c}_{j+1,\uparrow} - \hat{c}_{j,\downarrow}^\dagger \hat{c}_{j+1,\downarrow} + \text{H.c.})$$

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$$\begin{aligned}
& + \Delta \sum_j (\hat{c}_{j,\uparrow}^\dagger \hat{c}_{j+1,\downarrow} - \hat{c}_{j,\uparrow}^\dagger \hat{c}_{j-1,\downarrow} + \text{H.c.}) \\
& + \sum_j V_j (\hat{n}_{j,\uparrow} - \hat{n}_{j,\downarrow}),
\end{aligned} \quad (1)$$

where $\hat{c}_{j,\sigma}^\dagger$ ($\hat{c}_{j,\sigma}$) is the creation (annihilation) operator with spin $\sigma = \uparrow, \downarrow$ at the site j and $\hat{n}_{j,\sigma} = \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma}$ is the number operator. t and Δ denote the strengths of nearest-neighbor spin-conserved hopping and SOC induced spin-flip hopping, respectively. The Zeeman potential is given by

$$V_j = V \cos(2\pi\alpha j + \phi) + W, \quad (2)$$

where V (W) represents the amplitude of the quasiperiodic (uniform) component. α is taken as the ratio of adjacent Fibonacci numbers, $\alpha = F_{m-1}/F_m$ [30,35,36]. For $m \rightarrow \infty$, α approaches the inverse of the golden ratio, $\alpha = (\sqrt{5} - 1)/2$. The random phase factor $\phi \in [0, 2\pi)$ generates different examples of the quasiperiodic Zeeman potential. Unless otherwise specified, we set $t = 1$ as the energy unit and adopt periodic boundary conditions. The system size is selected as a Fibonacci number $L = F_m$. Specifically, we take $L = 2584$ with $\alpha = 1597/2584$.

For $\Delta = 0$, Eq. (1) reduces to two separated Aubry-André (AA) models for each spin [37], which exhibit a transition from a purely extended phase to a purely Anderson localized phase. The transition occurs at $V/t = 2$ [37]. In this regime, the properties of the system are independent of W . For $\Delta \neq 0$ and $W = 0$, the inclusion of SOC can induce a purely critical phase [30] and the phase diagram is consistent with that of the non-Abelian AA model with p -wave superfluidity [38]. Upon the introduction of W , band splitting arises, weakening the correlation between different spin components and competing with SOC [33].

To characterize different states, we employ multiple diagnostic quantities. The core one is the inverse participation ratio (IPR), defined as $\mathcal{I}_n = \sum_{j=0}^{L-1} (u_{j,n}^4 + v_{j,n}^4)$ [38–40], where n is the energy level index and $u_{j,n}$ ($v_{j,n}$) denotes the amplitude of the spin- $\uparrow$ (spin- $\downarrow$) component of the eigenstate $|\Psi^{(n)}\rangle = \sum_{j=0}^{L-1} (u_{j,n} \hat{c}_{j,\uparrow}^\dagger + v_{j,n} \hat{c}_{j,\downarrow}^\dagger) |\text{vac}\rangle$. IPR is associated with the normalized participation ratio, $\mathcal{N}_n = (2L\mathcal{I}_n)^{-1}$ [21,41], and the fractal dimension, $D_n = -\ln(\mathcal{I}_n)/\ln(2L)$ [42–46], where $2L$ corresponds to the Hilbert space dimension. To distinguish pure phases from mixed phases, we calculate the composite participation ratio $\eta = \log_{10}[\langle \mathcal{I}_n \rangle \langle \mathcal{N}_n \rangle]$ [47–49], where $\langle \cdot \rangle$ denotes averaging over the entire spectrum.

Phase diagrams. Figure 1(a) presents a global phase diagram influenced by both W and Δ for $V/t = 2$. It contains purely critical phases (C) and purely extended phases (E), while no purely localized phases (L) are observed, which only appear for large V [33]. In addition, two mixed phases, denoted as M_1 and M_2 , are identified. The M_1 (M_2) phase corresponds to the regions without (with) anomalous mobility edges. Specifically, M_1 consists of a mixture of localized and extended states, whereas M_2 mixes critical states with localized states, extended states, or both. Along the abscissa ($W/t = 0$), the system undergoes a transition from C to E as Δ increases, with the transition point located at $\Delta/t = 2$ [30,38]. In contrast, along the ordinate ($\Delta/t = 0$), Eq. (1)

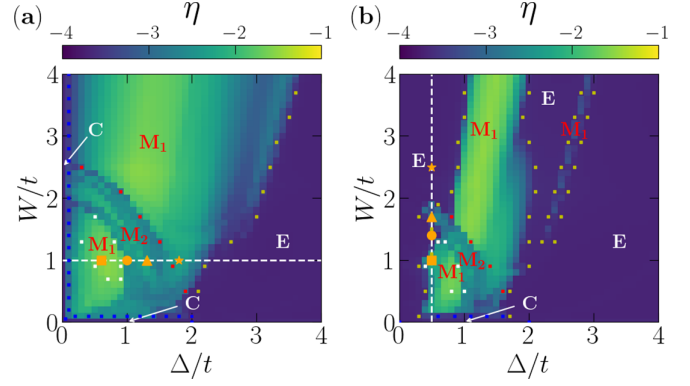


FIG. 1. Phase diagrams in the W - Δ plane for (a) $V/t = 2.0$ and (b) $V/t = 1.4$, respectively. Color indicates the value of η . White letter C (E) marks the purely critical phase (purely extended phase). M_1 and M_2 denote the mixed phases without and with anomalous mobility edges. White dotted lines mark $W/t = 1$ in panel (a) and $\Delta/t = 0.5$ in panel (b). Orange markers in panels (a) and (b) indicate points further analyzed in Figs. 2 and 3, and Fig. 5, respectively. η is used only to distinguish pure and mixed phases. The distinction between pure phases relies on fractal dimensions, and colored dot markers indicate transition points between different phases extracted from finite-size scaling analysis (see the Supplemental Material [33]).

reduces to the AA model [37], where W does not change the properties of the system. Since $V/t = 2$ is the critical point of the AA model, the entire ordinate is in C. In the limiting case of $\Delta/t \rightarrow \infty$, the system resides in E for finite W [33]. Therefore, tuning Δ can induce a C-E transition. In this process, conventional and anomalous mobility edges emerge. Interestingly, the M_1 phase is not contiguous, which is divided into two separate regions by the M_2 phase. This causes the system to experience a reentrant transition of M_1 as Δ increases. The fundamental reason for this phenomenon is analyzed below. Moreover, a *reentrant criticality transition* is observed in the regime of perturbative W . In Fig. 1(b), we show another phase diagram for $V/t = 1.4$. In this scenario, the abscissa undergoes an E-C-E transition as Δ increases [30,38], whereas the ordinate is in E. Compared to Fig. 1(a), E encroaches upon the survival region of mixed phases, resulting in a significant suppression of mixed phases. Note that the stability of mixed phases varies with W and Δ . It results in the M_1 phase being divided into three regions [33]. More importantly, it causes the system to undergo a reentrant delocalization transition as W increases for weak Δ , e.g., $\Delta/t = 0.5$, marked by the white dotted line in Fig. 1(b). When perturbative random disorder and interactions are introduced, the phase diagram is not substantially modified, indicating the robustness of the reentrant behavior.

Conventional and anomalous mobility edges. In Fig. 2(a), we show the spectrum as a function of Δ for $V/t = 2$ and $W/t = 1$. This demonstrates that localized states appear during the C-E transition, leading to conventional and anomalous mobility edges. To distinguish two classes of anomalous mobility edges, we label them aME_1 and aME_2 , defined as the critical energies separating critical states from localized and extended states in the spectrum, respectively. In Figs. 2(b)–2(d), we use 1000 samples to calculate the mean fractal

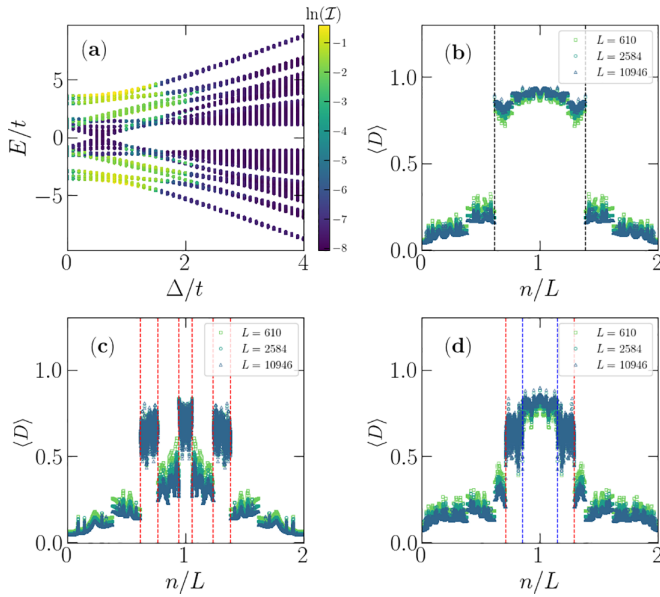


FIG. 2. (a) Spectrum as a function of Δ , along the white dotted line in Fig. 1(a). (b)–(d) Mean fractal dimension $\langle D \rangle$ at various energy levels for $\Delta/t = 0.6, 1.0$, and 1.3 , as indicated by orange markers $\blacksquare, \bullet$, and $\blacktriangle$ in Fig. 1(a), respectively. Other cases for $\Delta/t = 1.4, 1.8$, and 3.0 are provided in the Supplemental Material [33]. Black dotted lines mark conventional mobility edges, whereas red and blue dotted lines denote two distinct classes of anomalous mobility edges.

dimension $\langle D \rangle$ for various energy levels to show more details of mobility edges. In Fig. 2(b), $\langle D \rangle$ approaches unity as the system size increases in the region $0.618 < n/L < 1.382$ [33], indicating extended states [42–46]. Outside this region, $\langle D \rangle$ decreases with the system size and is expected to approach zero in the thermodynamic limit [33], signaling localized states [42–46]. The separation points, marked by black dotted lines, correspond to conventional mobility edges. Figure 2(c) illustrates anomalous mobility edges aME_1 , where in the critical regions $0.618 < n/L < 0.766$, $0.944 < n/L < 1.056$, and $1.234 < n/L < 1.382$, $\langle D \rangle$ approaches a finite value within $0 < \langle D \rangle < 1$ as the system size increases [33], revealing the presence of critical states. The behavior of localized states outside the critical regions remains consistent with that in Fig. 2(b). The anomalous mobility edges aME_2 are shown in Fig. 2(d), marked by blue dotted lines. Here, extended states emerge within $0.854 < n/L < 1.146$, owing to the enhanced density of states near the middle of the spectrum at $\Delta/t = 1.3$, as illustrated in Fig. 2(a). This is consistent with the reemergence mechanism of extended states during reentrant localization transitions [21,24], where the increased density promotes overlap among nonextended states, facilitating the formation of extended states. When we increase Δ to reduce the density of states, these extended states are destroyed and reduced to critical states again [33]. As Δ further increases, the system reenters an M_1 phase before entering E. The coexistence behavior in this regime is similar to that in Fig. 2(b) [33]. It is worth noting that the rich coexistence phenomena remain robust against weak random disorder and interaction perturbations [33].

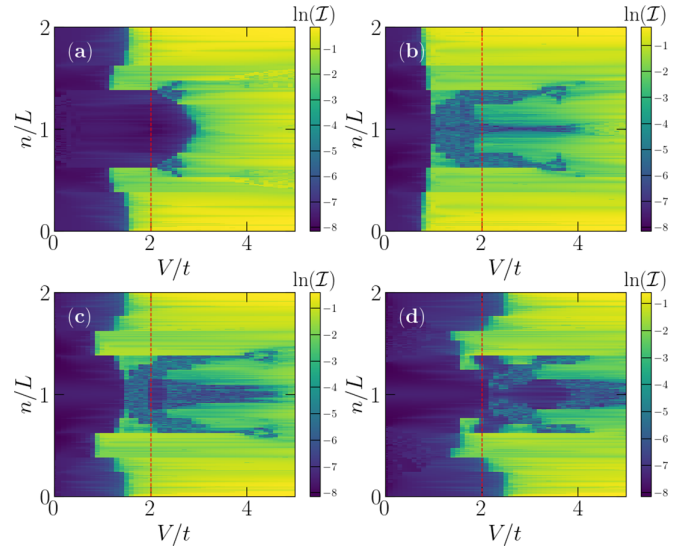


FIG. 3. (a)–(d) The IPR associated with the energy level index as a function of V/t for $W/t = 1$ and $\Delta/t = 0.6, 1.0, 1.3$, and 1.8 , respectively. Red dotted lines mark $V/t = 2$, corresponding to orange markers $\blacksquare, \bullet, \blacktriangle$, and $\star$ in Fig. 1(a).

To clarify the origin of the reentrant transition in Fig. 2(a), we fix $W/t = 1$ to investigate the localization transition induced by the quasiperiodic Zeeman potential for $\Delta/t = 0.6, 1.0, 1.3$, and 1.8 , as shown in Figs. 3(a)–3(d). For $\Delta/t = 0.6$, the system undergoes an E–L transition as V/t increases. During this process, both conventional and anomalous mobility edges emerge in the spectrum. In comparison to the case of $W/t = 0$ discussed in the Supplemental Material [33], the addition of the uniform Zeeman potential W in Fig. 3(a) causes extended and localized states to encroach upon regions originally occupied by critical states. This change results in the emergence of localized states at the specific amplitude of the quasiperiodic Zeeman potential, e.g., $V/t = 2$, indicated by the red dotted line in Fig. 3(a). When $\Delta/t = 1.0$ in Fig. 3(b), E is suppressed relative to the case of $\Delta/t = 0.6$, and the system forms localized states under a weaker quasiperiodic Zeeman potential. Moreover, the increase of Δ is more conducive to the existence of critical states [30,33,38], promoting the formation of anomalous mobility edges. As Δ continues to increase, it produces the opposite effect in Fig. 3(c), where a larger Δ facilitates the survival of E, requiring stronger V to drive the localization transition. This trend is consistent with the case of $W/t = 0$ [33]. Remarkably, as V increases, the extended states vanish completely at $V/t = 1.4$ and then reappear near the middle of the spectrum within $1.9 < V/t < 2.4$, following the same mechanism discussed in Fig. 2(d). For even larger Δ , the localization transition with both conventional and anomalous mobility edges shifts to higher amplitudes of the quasiperiodic Zeeman potential in Fig. 3(d). When we fix $V/t = 2$ (red dotted lines) in Figs. 3(a)–3(d), the system experiences mixed phases $M_1 \rightarrow M_2 \rightarrow M_2 \rightarrow M_1$ as Δ increases. Fundamentally, the above phenomena, whether localized states emerge during the critical-extended transition or the reentrant transition of M_1 , all originate from the non-monotonic role of SOC. The enhancement of SOC initially suppresses extended states, thereby promoting critical and

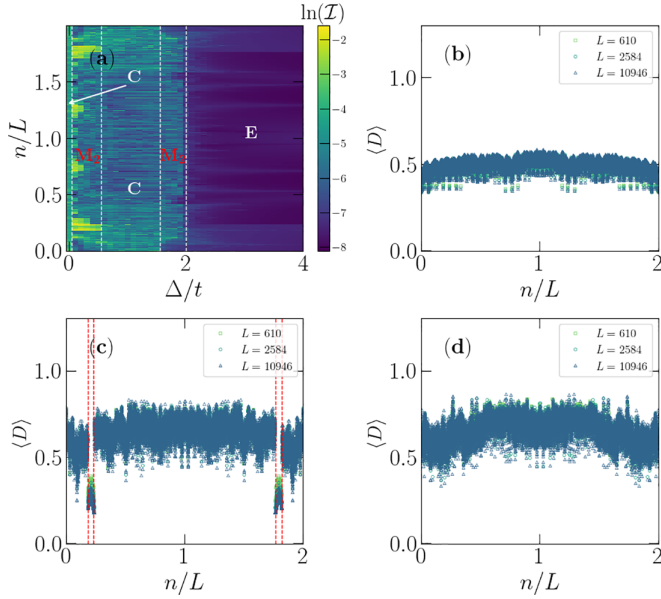


FIG. 4. (a) IPR as a function of Δ for $V/t = 2$ and $W/t = 0.01$. (b)–(d) $\langle D \rangle$ at various energy levels for $\Delta/t = 0.0, 0.4$, and 0.8 , respectively. Other parameters are the same as those in Fig. 1(a). The white dotted lines in panel (a) mark boundaries between different phases, whereas the red dotted lines in panel (c) indicate anomalous mobility edges aME_1 .

localized phases. In contrast, further increasing SOC favors the reemergence of extended states while suppressing critical and localized ones. Moreover, SOC regulates the density of states and induces hybridization-driven delocalization, which further contributes to the observed reentrant behavior.

Reentrant transitions. Figure 4(a) displays the IPR associated with the energy level index n as a function of Δ for $V/t = 2$ and $W/t = 0.01$, where W is chosen sufficiently small to preserve the purely critical phase [33]. As Δ decreases from 4 to 0, the system undergoes an $E-M_2-C-M_2-C$ transition, entering the purely critical phase C twice. Notably, the two C originate from distinct mechanisms. For $\Delta/t = 0$, the variation of W does not affect the nature of the system, and the choice of $V/t = 2$ ensures that the system remains in C regardless of W . In contrast, for $\Delta > 0$, W competes with Δ , leading to the breakdown of part of C into the M_2 phase. The surviving C in the region $0.5 < \Delta/t < 1.6$, together with C at $\Delta/t = 0$, gives rise to a *reentrant criticality transition* as Δ decreases. Moreover, Fig. 4(a) also shows that noncritical states penetrate C from multiple locations, resulting in two distinct M_2 phases. In Figs. 4(b) and 4(d), we present $\langle D \rangle$ as a function of the energy level for $\Delta/t = 0$ and $\Delta/t = 0.8$, respectively. As the system size increases, the fractal dimensions of all states approach values within $0 < \langle D \rangle < 1$, indicating that the system resides in C . In contrast, for $\Delta/t = 0.4$ in Fig. 4(c), both localized and critical states coexist in the spectrum, producing anomalous mobility edges aME_1 (red dotted lines).

In addition to the *reentrant criticality transition*, we also observe a *reentrant delocalization transition*, as shown in Fig. 1(b). To explore the mechanism behind this phenomenon, we vary Δ/t from 0.8 to 0.5 and plot the spectra as a function of W in Fig. 5. When $\Delta/t = 0.8$, extended states appear near

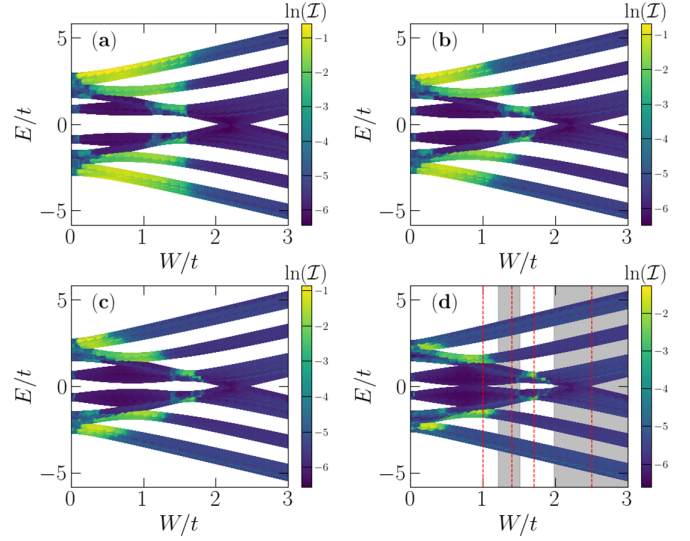


FIG. 5. (a)–(d) Spectra as a function of W/t for $\Delta/t = 0.8, 0.7, 0.6$ and 0.5 , sliced from Fig. 1(b). In panel (d), the gray regions mark purely extended phases E and red dotted lines correspond to orange markers $\blacksquare, \bullet, \blacktriangle, \star$ in Fig. 1(b).

the middle of the spectrum for small W/t , in contrast to the case of $\Delta/t = 1.0$ [33]. This behavior can be attributed to two primary factors. First, the reduction of Δ/t facilitates the transition from critical states to extended states [38,50]. In the limit $\Delta/t \rightarrow 0$, Eq. (1) reduces to the AA model, where all states are extended for $V/t = 1.4$ [37]. Second, critical states in the middle of the spectrum possess larger fractal dimensions, indicating greater spatial extension, and are thus more likely to evolve into extended states as Δ/t decreases. This trend is also clearly illustrated in Figs. 5(a)–5(d), where the region of extended states around the spectrum center expands systematically with decreasing Δ/t . Crucially, Figs. 5(a)–5(d) demonstrate that the region where localized states exist shrinks as Δ/t decreases. When $\Delta/t = 0.5$ in Fig. 5(d), localized states vanish entirely within the region $W/t \in [1.21, 1.51]$. Although a finite W makes the system violate the pure phase criterion [2], the absence of overlap between mobility edges and the spectrum restores a pure phase [33]. As a result, the localized region is split into two parts, causing two distinct delocalization transitions. Specifically, the first transition into E occurs at $W/t = 1.21$, which is subsequently destroyed at $W/t = 1.51$. Upon further increasing W/t , the system reenters E at $W/t = 1.98$, thereby exhibiting the reentrant phenomenon. Experimentally, this reentrant delocalization can be probed via wave-packet dynamics and the retention of initial-state memory, which provide direct signatures of mobility-edge shifts, and the specific experimental steps and details are shown in the Supplemental Material [33].

Conclusions. In this Letter, we investigate reentrant transitions in an optical Raman lattice. In the presence of the Zeeman potentials, SOC generates anomalous mobility edges that drive reentrant transitions between mixed phases. We also observe a *reentrant criticality transition*, supplementing the basic framework of reentrant phenomena in quasiperiodic systems. Moreover, we identify a reentrant delocalization transition induced by the uniform Zeeman potential. This

reentrance occurs because the region supporting localized states enclosed by mobility edges fails to overlap with energy bands within a certain region, leading to the disappearance of localized states. This mechanism presents a way to induce reentrant transitions.

The underlying model studied here is supported by well-established experimental schemes [30], and its basic phase diagram has already been realized in ultracold atomic systems [19] and programmable superconducting simulators [29]. The inclusion of a uniform Zeeman splitting introduces richer reentrant phenomena and provides a promising platform for further theoretical and experimental research. It is also

worthwhile to introduce this work into interacting systems and non-Hermitian systems to explore complex reentrant transitions.

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Data availability. The data that support the findings of this article are openly available [51], embargo periods may apply.

- [1] P. W. Anderson, Absence of diffusion in certain random lattices, *Phys. Rev.* **109**, 1492 (1958).
- [2] X.-C. Zhou, B.-C. Yao, Y. Wang, Y. Wang, Y. Wei, Q. Zhou, and X.-J. Liu, The fundamental localization phases in quasiperiodic systems: A unified framework and exact results, *Sci. Bull.* **71**, 1654 (2026).
- [3] G. Semeghini, M. Landini, P. Castilho, S. Roy, G. Spagnolli, A. Trenkwalder, M. Fattori, M. Inguscio, and G. Modugno, Measurement of the mobility edge for 3D Anderson localization, *Nat. Phys.* **11**, 554 (2015).
- [4] M. Pasek, G. Orso, and D. Delande, Anderson localization of ultracold atoms: Where is the mobility edge? *Phys. Rev. Lett.* **118**, 170403 (2017).
- [5] S. Longhi, Dephasing-induced mobility edges in quasicrystals, *Phys. Rev. Lett.* **132**, 236301 (2024).
- [6] S. Ganeshan, J. H. Pixley, and S. Das Sarma, Nearest neighbor tight binding models with an exact mobility edge in one dimension, *Phys. Rev. Lett.* **114**, 146601 (2015).
- [7] Y. Wang, X. Xia, L. Zhang, H. Yao, S. Chen, J. You, Q. Zhou, and X.-J. Liu, One-dimensional quasiperiodic mosaic lattice with exact mobility edges, *Phys. Rev. Lett.* **125**, 196604 (2020).
- [8] J. Biddle and S. Das Sarma, Predicted mobility edges in one-dimensional incommensurate optical lattices: An exactly solvable model of Anderson localization, *Phys. Rev. Lett.* **104**, 070601 (2010).
- [9] H. Yao, A. Khoudli, L. Bresque, and L. Sanchez-Palencia, Critical behavior and fractality in shallow one-dimensional quasiperiodic potentials, *Phys. Rev. Lett.* **123**, 070405 (2019).
- [10] T. Liu, X. Xia, S. Longhi, and L. Sanchez-Palencia, Anomalous mobility edges in one-dimensional quasiperiodic models, *SciPost Phys.* **12**, 027 (2022).
- [11] Z. Wang, Y. Zhang, L. Wang, and S. Chen, Engineering mobility in quasiperiodic lattices with exact mobility edges, *Phys. Rev. B* **108**, 174202 (2023).
- [12] X.-C. Zhou, Y. Wang, T.-F. J. Poon, Q. Zhou, and X.-J. Liu, Exact new mobility edges between critical and localized states, *Phys. Rev. Lett.* **131**, 176401 (2023).
- [13] Y. Wang and Q. Zhou, Exact new mobility edges, *arXiv:2501.17523*.
- [14] N. Mott, Electrons in disordered structures, *Adv. Phys.* **16**, 49 (1967).
- [15] H. P. Lüschen, S. Scherg, T. Kohlert, M. Schreiber, P. Bordia, X. Li, S. Das Sarma, and I. Bloch, Single-particle mobility edge in a one-dimensional quasiperiodic optical lattice, *Phys. Rev. Lett.* **120**, 160404 (2018).
- [16] F. A. An, K. Padavić, E. J. Meier, S. Hegde, S. Ganeshan, J. H. Pixley, S. Vishveshwara, and B. Gadway, Interactions and mobility edges: Observing the generalized Aubry-André model, *Phys. Rev. Lett.* **126**, 040603 (2021).
- [17] Y. Wang, J.-H. Zhang, Y. Li, J. Wu, W. Liu, F. Mei, Y. Hu, L. Xiao, J. Ma, C. Chin, and S. Jia, Observation of interaction-induced mobility edge in an atomic Aubry-André wire, *Phys. Rev. Lett.* **129**, 103401 (2022).
- [18] Y.-J. Chang, J.-H. Zhang, Y.-H. Lu, Y.-Y. Yang, F. Mei, J. Ma, S. Jia, and X.-M. Jin, Observation of photonic mobility edge phases, *Phys. Rev. Lett.* **134**, 053601 (2025).
- [19] T. Xiao, D. Xie, Z. Dong, T. Chen, W. Yi, and B. Yan, Observation of topological phase with critical localization in a quasi-periodic lattice, *Sci. Bull.* **66**, 2175 (2021).
- [20] V. Goblots, A. Štrkalj, N. Pernet, J. L. Lado, C. Dorow, A. Lemaître, L. Le Gratiet, A. Harouri, I. Sagnes, S. Ravets, *et al.*, Emergence of criticality through a cascade of delocalization transitions in quasiperiodic chains, *Nat. Phys.* **16**, 832 (2020).
- [21] S. Roy, T. Mishra, B. Tanatar, and S. Basu, Reentrant localization transition in a quasiperiodic chain, *Phys. Rev. Lett.* **126**, 106803 (2021).
- [22] A. Padhan, M. K. Giri, S. Mondal, and T. Mishra, Emergence of multiple localization transitions in a one-dimensional quasiperiodic lattice, *Phys. Rev. B* **105**, L220201 (2022).
- [23] A. Štrkalj, Elmer V. H. Doggen, I. V. Gornyi, and O. Zeitler, Many-body localization in the interpolating Aubry-André-Fibonacci model, *Phys. Rev. Res.* **3**, 033257 (2021).
- [24] X. Wei, Reentrant localization transition induced by a composite potential, *Phys. Rev. B* **111**, 134204 (2025).
- [25] Z.-S. Xu, J. Gao, A. Iovan, I. M. Khaymovich, V. Zwiller, and A. W. Elshaari, Observation of reentrant metal-insulator transition in a random-dimer disordered SSH lattice, *npj Nanophotonics* **1**, 8 (2024).
- [26] S. Vaidya, C. Jörg, K. Linn, M. Goh, and M. C. Rechtsman, Reentrant delocalization transition in one-dimensional photonic quasicrystals, *Phys. Rev. Res.* **5**, 033170 (2023).
- [27] Q. Guo, C. Cheng, Z.-H. Sun, Z. Song, H. Li, Z. Wang, W. Ren, H. Dong, D. Zheng, Y.-R. Zhang, *et al.*, Observation of energy-resolved many-body localization, *Nat. Phys.* **17**, 234 (2021).
- [28] Y.-H. Shi, Z.-H. Sun, Y.-Y. Wang, Z.-A. Wang, Y.-R. Zhang, W.-G. Ma, H.-T. Liu, K. Zhao, J.-C. Song, G.-H. Liang, *et al.*, Probing spin hydrodynamics on a superconducting quantum simulator, *Nat. Commun.* **15**, 7573 (2024).

- [29] X. Li, H. Xu, J. Wang, L.-Z. Tang, D.-W. Zhang, C. Yang, T. Su, C. Wang, Z. Mi, W. Sun, *et al.*, Mapping the topology-localization phase diagram with quasiperiodic disorder using a programmable superconducting simulator, *Phys. Rev. Res.* **6**, L042038 (2024).
- [30] Y. Wang, L. Zhang, S. Niu, D. Yu, and X.-J. Liu, Realization and detection of nonergodic critical phases in an optical Raman lattice, *Phys. Rev. Lett.* **125**, 073204 (2020).
- [31] Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Ji, Y. Deng, S. Chen, X.-J. Liu, and J.-W. Pan, Realization of two-dimensional spin-orbit coupling for Bose-Einstein condensates, *Science* **354**, 83 (2016).
- [32] B. Song, C. He, S. Niu, L. Zhang, Z. Ren, X.-J. Liu, and G.-B. Jo, Observation of nodal-line semimetal with ultracold fermions in an optical lattice, *Nat. Phys.* **15**, 911 (2019).
- [33] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/g15k-q96k> for details on (1) matrix form of the Hamiltonian, (2) multifractal analysis of different states, (3) transition points in the phase diagram and (4) in the limiting cases, (5) reentrant phenomenon and pure phase criterion, (6) the effect of W on the purely critical phase, (7) the influence of disorder and interactions on phase diagrams, (8) experimental detection, (9) destruction of the reemerged extended states, (10) the transition in the case of $W/t = 0$, (11) spectrum for $\Delta/t = 1$ and typical states, (12) Δ -induced transition for large W , (13) the phase diagram for $V/t = 2.5$, and (14) reentrant delocalization transitions, which includes Refs. [2,15,20,30,38,52–54].
- [34] X.-J. Liu, Z.-X. Liu, and M. Cheng, Manipulating topological edge spins in a one-dimensional optical lattice, *Phys. Rev. Lett.* **110**, 076401 (2013).
- [35] M. Kohmoto, Metal-insulator transition and scaling for incommensurate systems, *Phys. Rev. Lett.* **51**, 1198 (1983).
- [36] J. Zhao, Y. Zhao, J.-G. Wang, Y. Li, and X.-D. Bai, Phase transition of a non-Abelian quasiperiodic mosaic lattice model with p -wave superfluidity, *Phys. Rev. B* **108**, 054204 (2023).
- [37] S. Aubry and G. André, Analyticity breaking and Anderson localization in incommensurate lattices, *Ann. Isr. Phys. Soc.* **3**, 133 (1980).
- [38] J. Wang, X.-J. Liu, G. Xianlong, and H. Hu, Phase diagram of a non-Abelian Aubry-André-Harper model with p -wave superfluidity, *Phys. Rev. B* **93**, 104504 (2016).
- [39] M. Yahyavi, B. Hetényi, and B. Tanatar, Generalized Aubry-André-Harper model with modulated hopping and p -wave pairing, *Phys. Rev. B* **100**, 064202 (2019).
- [40] Q.-B. Zeng, S. Chen, and R. Lü, Generalized Aubry-André-Harper model with p -wave superconducting pairing, *Phys. Rev. B* **94**, 125408 (2016).
- [41] X. Li, X. Li, and S. Das Sarma, Mobility edges in one-dimensional bichromatic incommensurate potentials, *Phys. Rev. B* **96**, 085119 (2017).
- [42] R. Ghosh, M. Sarkar, and I. M. Khaymovich, Reentrant localization induced by short-range hopping in the fractal Rosenzweig-Porter model, *Phys. Rev. B* **111**, L220102 (2025).
- [43] D. J. Luitz, F. Alet, and N. Laflorencie, Universal behavior beyond multifractality in quantum many-body systems, *Phys. Rev. Lett.* **112**, 057203 (2014).
- [44] G. De Tomasi and I. M. Khaymovich, Multifractality meets entanglement: Relation for nonergodic extended states, *Phys. Rev. Lett.* **124**, 200602 (2020).
- [45] R. Ji and Z. Xu, Fibonacci-modulation-induced multiple topological Anderson insulators, *Commun. Phys.* **8**, 336 (2025).
- [46] G.-Q. Zhang, L.-Z. Tang, L. Quezada, S.-H. Dong, and D.-W. Zhang, Reentrant topological phases and spin density wave induced by 1D moiré potentials, *Commun. Phys.* **8**, 275 (2025).
- [47] X. Li and S. Das Sarma, Mobility edge and intermediate phase in one-dimensional incommensurate lattice potentials, *Phys. Rev. B* **101**, 064203 (2020).
- [48] R. Qi, J. Cao, and X.-P. Jiang, Multiple localization transitions and novel quantum phases induced by a staggered on-site potential, *Phys. Rev. B* **107**, 224201 (2023).
- [49] E. Guan, G. Wang, X.-W. Guan, and X. Cai, Reentrant localization and mobility edges in a spinful Aubry-André-Harper model with a non-Abelian potential, *Phys. Rev. A* **108**, 033305 (2023).
- [50] T. Lv, T.-C. Yi, L. Li, G. Sun, and W.-L. You, Quantum criticality and universality in the p -wave-paired Aubry-André-Harper model, *Phys. Rev. A* **105**, 013315 (2022).
- [51] T. L. G. X. Xingbo Wei, Z. Xie and Y. Zhang, Distinct reentrant transitions in a quasi-periodic Raman lattice, Zenodo, 2025, 10.5281/zenodo.17082777.
- [52] F. Pietracaprina, N. Macé, D. J. Luitz, and F. Alet, Shift-invert diagonalization of large many-body localizing spin chains, *SciPost Phys.* **5**, 045 (2018).
- [53] X. Wei, C. Cheng, G. Xianlong, and R. Mondaini, Investigating many-body mobility edges in isolated quantum systems, *Phys. Rev. B* **99**, 165137 (2019).
- [54] H. Tabanelli, C. Castelnovo, and A. Štrkalj, Reentrant localization transitions and anomalous spectral properties in off-diagonal quasiperiodic systems, *Phys. Rev. B* **110**, 184208 (2024).