

Flow topology classification of limit cycles

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(Received 3 January 2026; accepted 1 July 2026; published 20 August 2026)

Recent advancements in topological analysis offer a powerful way to classify how phases of nonlinear bosonic resonators organize. These tools are rooted in the Morse-Smale flow topology framework, where a graph-based construction describes how steady-state solutions are arranged in phase space. This construction remains incomplete as self-sustained oscillations in the form of limit cycles can impact the global connectivity of phase-space flows and the corresponding Morse-Smale topological classification. In this work, we extend the bosonic classification scheme to include limit cycles as fundamental topological elements. We illustrate the approach with a minimal nonlinear Van der Pol resonator model, where limit cycles coexist with steady-state solutions. Our results provide a unified topological description of stationary and time-periodic phases in nonlinear bosonic systems, with direct relevance to photonic, superconducting, and optomechanical platforms. Our findings raise fundamental questions regarding synchronization and the extension of flow topology to the quantum regime.

DOI: [10.1103/fwpm-41bj](https://doi.org/10.1103/fwpm-41bj)

I. INTRODUCTION

Topological phases of matter reveal physical behavior that is governed by global, quantized invariants rather than local order parameters [1–7]. These indices fix quantized response functions that remain robust against disorder and other small perturbations, and often manifest through anomalies that reflect violated conservation laws [8]. Classic examples include the quantum Hall effect [1,9], topological insulators protected by time-reversal symmetry [4,5], higher-order phases connected to topology in auxiliary dimensions [2,10–15], and topological effects in quasiperiodic structures [16–18]. While these ideas originated in closed and energy-conserving settings, they now extend to open systems where gain and loss play an essential role. Effective *non-Hermitian Hamiltonians* illustrate this trend and highlight the need to generalize notions of topology when spectra become complex and boundary sensitivity increases [19–23]. Concepts such as the spectral winding number [24–27] and non-Bloch band theory [28–32] have been developed to address these challenges and show that topological reasoning remains powerful away from equilibrium.

Moving to many-body bosonic systems, the introduction of nonlinearities forces a departure from the standard approach of classifying phases through the spectrum of Hermitian or non-Hermitian quadratic operators. Nonlinear dynamics reshapes the interplay between drive and loss and

generates distinct nonequilibrium phases, each with its own linear (quadratic) environment, yet different in their global organization in phase space. Dissipative phase transitions provide a natural arena for this behavior [33–38], since nonlinearities and competing forces conspire to produce multiple nonequilibrium steady states (NESS) and metastability across numerous platforms ranging from Kerr oscillators [36,39–41] and Bose-Hubbard lattices [42,43] to polariton fluids [44,45] and superconducting arrays [46–51]. To capture the global topological structure of such nonlinear dynamics, a graph-based classification framework was recently introduced based on the Morse-Smale flow topology theory [52,53]. Originally developed to classify watershed networks, this graph index records how attractors, repellers, and saddles connect through the vector field and therefore characterizes the variety and arrangement of dynamical phases in phase space. Transitions between regions with different graph indexes define flow topology transitions, which correspond to qualitative rearrangements of the NESS landscape, which have been observed in experiments [53]. Crucially, while bifurcations lead to different graphs, this construction is sensitive to all structural transitions, even those that do not entail bifurcations. This framework also extends to the quantum regime, where its structure persists beyond semiclassical theory [54].

The flow topology framework characterizes how steady-state solutions organize in the phase space of nonlinear resonators, but it does not account for periodic attractors. Such so-called limit cycles (LCs) are isolated, self-sustained periodic solutions of nonlinear equations, with an oscillation period fixed by the internal balance of forces [55]. Asymptotic stability and the absence of a preferred time origin make them robust organizing centers of nonlinear dynamics. They arise broadly in out-of-equilibrium platforms, including light-matter ensembles [56–66], parametric and electromechanical oscillators [39,67–69], and superconducting or

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optomechanical devices [70–75]. Because LCs reshape how regions of phase space connect, any topological classification of nonlinear resonator flows remains incomplete without their inclusion. We note that the original graph construction for watershed networks naturally includes elements that represent LCs [52,76]; thus, a consistent topological description of resonator flows should be extendable to include them.

In this work, we extend the flow-topology-based classification of nonlinear bosonic resonators to include limit cycles as topological elements of the phase-space organization. Building on the Morse-Smale graph framework, we first show how periodic attractors modify the global connectivity of flows and give rise to classes of flow topology beyond fixed-point-based classifications. We demonstrate this extension by constructing a minimal nonlinear resonator scenario that realizes an extended van der Pol (vdP) dynamics, in which stable LCs coexist and compete with steady-state solutions. Then, we enhance the graph invariant for bosonic systems, by directly incorporating the LCs in the construction. Our results establish a unified topological description of stationary and time-periodic phases in nonlinear bosonic systems, with direct relevance across photonic, superconducting, and optomechanical platforms. They further raise open questions about how such flow topological structures persist in the quantum regime and how they constrain synchronization and collective dynamics in many-body bosonic systems.

This article is structured as follows. In Sec. II, we summarize the principles of two-dimensional (2D) vector-flow topology in the literature, with and without LCs, starting from the example of a modified van der Pol oscillator. We describe how to build the topological graph invariant following Ref. [53], which combines the scheme from Ref. [52] and the concept of chirality [36,77]. Then, we summarize the topological classification of systems sustaining LCs in Ref. [52]. In Sec. III, we present our main results, where we include the

topological classification of LCs into the bosonic resonator framework, encoding chirality and nesting relations. Finally, in Sec. IV, we summarize our findings and propose extensions of our work.

II. 2D FLOW TOPOLOGY AND LIMIT CYCLES

A. Modified van der Pol oscillator

We start our discussion by considering a vdP oscillator, modified by including additional polynomial potential terms in the equation [78]

$$\ddot{x} + \omega_0^2 x + \gamma \dot{x} - \mu(1 - x^2)\dot{x} + \alpha_3 x^3 + \alpha_5 x^5 = 0, \quad (1)$$

where $x(t)$ is the oscillator's displacement at natural angular frequency ω_0 , γ controls the linear damping, $\mu > 0$ is the standard vdP parameter that controls the linear gain channel alongside nonlinear damping, and $\alpha_{3,5}$ are cubic and quintic nonlinearities, respectively. Equation (1) can be recast into two coupled first-order differential equations,

$$\begin{aligned} \dot{x} &= p, \\ \dot{p} &= -\omega_0^2 x + (\mu - \gamma)p - \mu x^2 p - \alpha_3 x^3 - \alpha_5 x^5, \end{aligned} \quad (2)$$

that define a two-dimensional vector flow. Depending on the parameter values, such vector flows can exhibit distinct patterns and features, giving rise to a plethora of possible dynamical phases (see Fig. 1). In the following, we explain how each phase can be classified within our framework. The classification scheme proposed in Ref. [53] is solely based on fixed points, which are the steady-state solutions satisfying $\dot{x} = \dot{p} = 0$ and correspond to attractors, saddles, or repellers. Notice that the system can host multiple coexisting steady-state solutions [see Figs. 1(d)–1(f)].

In the following, we first consider the parameter regime where $\gamma > \mu$. There, the system lacks LCs. Depending on

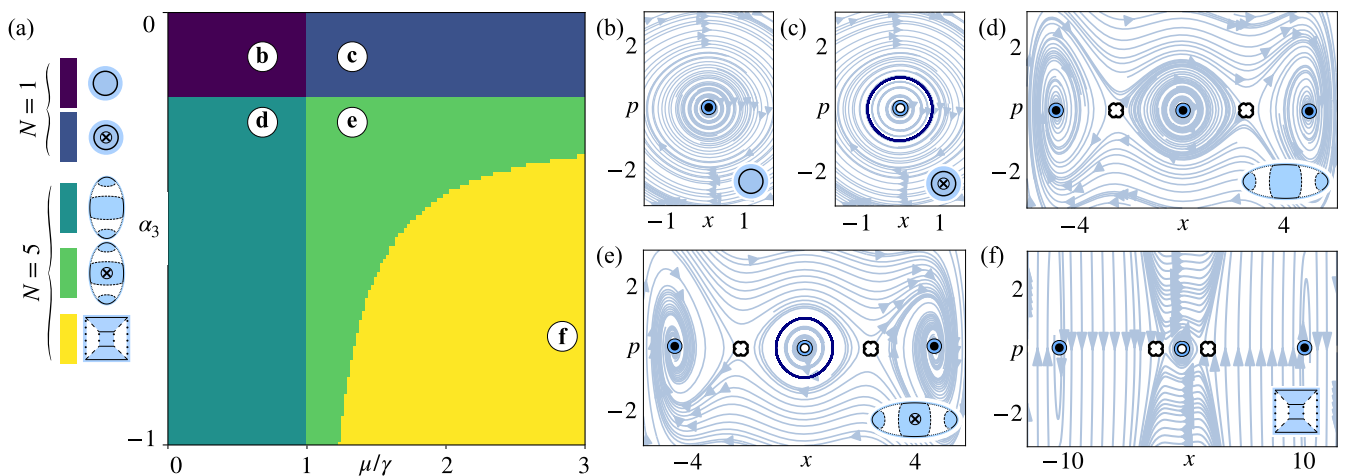


FIG. 1. Vector-flow topology phase diagram. (a) Phase diagram of the modified vdP (1) as a function of α_3 and the ratio μ/γ , with colors indicating distinct vector-flow topologies. Each region corresponds to a different global organization of phase-space trajectories; see legend where N denotes the number of steady-state solutions. (b)–(f) Representative phase-space streamline plots [cf. Eq. (2)] at the parameter points marked by bullets in panel (a); filled dots, empty circles, and cross markers correspond to attractors, repellers, and saddles, respectively. The color around the dots/circles marks the chirality around the steady-state solution with blue, red, and white corresponding to clockwise (CW), counterclockwise (CCW), and no winding, respectively. Dark blue lines trace CW-rotating limit cycles. Insets show the associated graph indices characterizing the flow topology in each regime. In all panels, $\omega_0 = 1$, $\gamma = 0.03$, and $\alpha_5 = 0.01$.

the nonlinear parameter condition $\alpha_3 \leq -\sqrt{4\alpha_5}$, we observe either an elementary flow with a single steady state [see Fig. 1(b)] or a nonelementary flow containing three clockwise attractors and two saddles [see Fig. 1(d)]. In these cases, the only topological features are steady-state solutions. Therefore, such flows can be readily topologically classified using the approach of Ref. [53]. Moving on, we focus on the limit where $\gamma < \mu$. This regime recovers the standard van der Pol resonator when $\alpha_{3,5} = 0$, where the gain channel $-\mu\dot{x}$ drives the oscillator to higher amplitudes and the nonlinear damping $\mu x^2\dot{x}$ saturates the motion. There, the system exhibits a CW attracting LC around the origin. This structure operates as a single isolated feature [see Fig. 1(c)] or coexists directly with steady-state solutions [see Fig. 1(e)]. In order for the flow lines of the inner region to converge to the LC, the CW attractor at the origin dynamically changes stability to become a source. Finally, we examine the limit where the ratio between μ and γ increases further and interplays with the nonlinear potentials. There, the LC is no longer sustained and vanishes entirely [see Fig. 1(f)]. The source at the origin retains its stability. In the most general physical case, LCs coexist with steady-state solutions. The central result of this work is to extend the flow topology classification to such systems where both types of critical elements occur. As an example, we obtain a complete phase diagram of the model (1), which differs from that presented in Ref. [78], as we base our study solely on exact vector-flow topology rather than approximate solution-finding approaches. Combined, we can classify the model's topological phase diagram [see Fig. 1(a)].

B. Bosonic vector flows without limit cycles

The vector flows in Figs. 1(b), 1(d), and 1(f) can be topologically classified following Ref. [53]. There, a graph invariant extends the Morse-Smale framework [52] to out-of-equilibrium nonlinear resonator dynamics. We succinctly review here this skeleton construction (see Fig. 2). Because all generic flow lines are incoming from infinity, we first perform a compactification to map infinity to a virtual source. Consequently, this verifies that the entire vector flow maps onto a sphere, with all trajectories descending from the apical source. The procedure then requires defining the separatrices of the original flow. These special trajectories connect critical points and partition the phase space into isolated asymptotic basins. Starting directly from the saddle points, we utilize forward and backward time evolution to numerically generate these boundaries [see Fig. 2(a)]. Dashed curves evolve forward to the attractors, while dotted curves trace backward to the virtual source. In a subsequent triangulation step, we add representative trajectories (solid lines) directly connecting the virtual source to the attractors. This divides the phase space into distinct triangular faces. Crucially, this ensemble of separatrices, steady states, and faces constitutes the topologically robust Morse-Smale complex (MSC) [52].

Due to dense separatrix winding in Fig. 2(a), the raw spatial subdivision remains visually opaque. We therefore deploy a smoothly unraveled schematic to reveal the precise domain boundaries [see Figs. 2(b) and 2(c)]. We, then, construct the topological planar graph invariant directly as the dual of the Morse-Smale complex. The nodes represent the

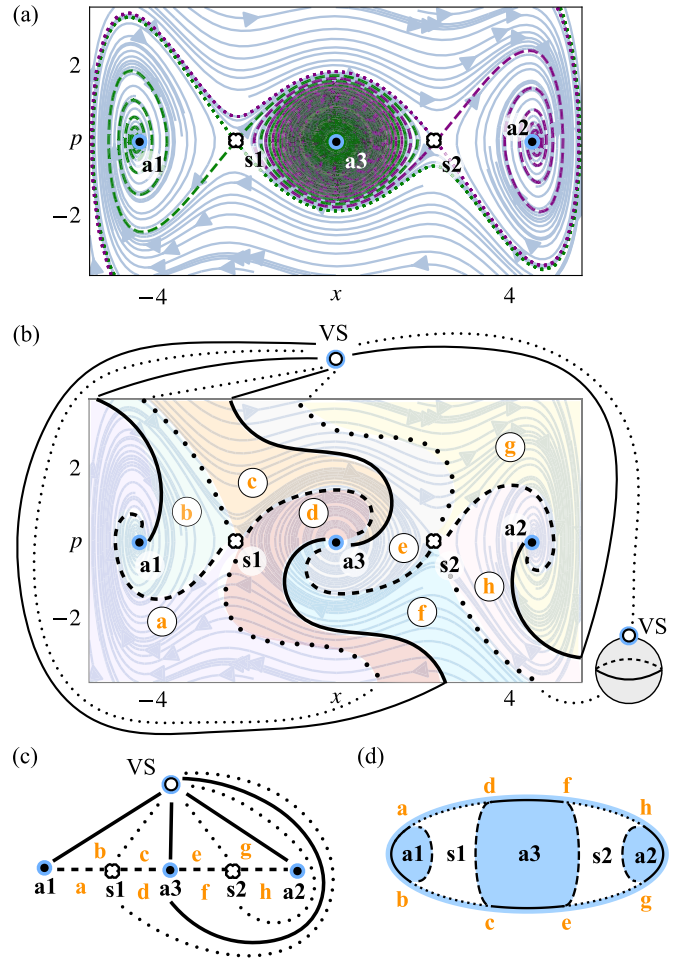


FIG. 2. Topological graph invariant without LCs. Construction of the vector-flow graph invariant for the flow corresponding to Fig. 1(d) [52,53]. (a) Phase-space flow portrait partitioned by separatrices into distinct asymptotic regions. Dashed (dotted) lines denote separatrices leading to attractors (originating from the virtual source). (b) Triangulated Morse-Smale complex assuming a compactification onto a hemisphere (inset) by mapping infinity to a virtual source (VS). Colored patches with orange labels designate distinct triangular faces formed by adding representative connecting trajectories (solid lines). (c) Schematic representation of the MSC. (d) Topological planar graph invariant dual to the MSC. Nodes correspond to MSC faces, connected by corresponding shared edges. Blue shading indicates CW chirality for the attractors and the bounding VS halo.

triangular faces, whereas the connecting lines encode their shared topological edges. For example, adjacent faces sharing one solid and one dashed separatrix are correspondingly connected by a solid and a dashed line in the dual graph [see Fig. 2(d)]. Two flows are topologically equivalent if and only if their corresponding graphs are isomorphic. Such isomorphism guarantees that one flow can be smoothly deformed into the other without macroscopically rearranging the separatrix skeleton.

In the standard Morse-Smale framework, this classification ignores local rotational components. However, in nonlinear bosonic systems, trajectories generically wind clockwise or

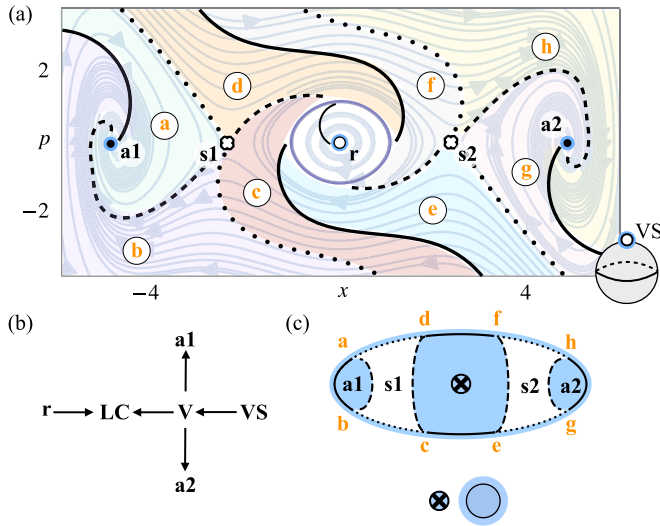


FIG. 3. Extension of the graph invariant formalism to LCs. Topological graph invariant construction for the flow corresponding to Fig. 1(e) featuring a CW attracting limit cycle around the origin. (a) Phase space subdivided by schematized separatrices (three dashed styles) into distinct asymptotic regions, omitting the virtual source (see the Supplemental Material [80]). (b) Standard invariant representation for a single LC [52]. The LC separates the phase space into an inner (r) and an outer (V) region. Arrows denote flow directionality toward the attracting LC. Steady-state solutions (a_1 , a_2 , VS) define the elementary regions of V . (c) Proposed augmented graph invariant [53]. A crossed (double-lined) circular marker denotes an inner region repelling (attracting) toward the enclosing LC. Colored faces and the bounding halo encode the LC orientation.

counterclockwise around steady-state solutions. This winding reflects a phase reference set either by the external drive or by inherent causality (see also Appendix) [77,79]. We therefore evaluate this rotational chirality via the linearized system at each steady state. To incorporate this constraint, each region of the planar graph receives a specific color encoding its local flow chirality. We utilize blue (red) shading to denote (counter)clockwise attractors, while transitions via exceptional points into nonchiral overdamped regions remain uncolored [53]. A surrounding halo encodes the chirality of the virtual source at infinity, dictated by the sign of the highest nonlinear potential power. With this addition, the colored dual graph provides a complete topological classification. Distinct flow phases result in nonisomorphic graphs, dictating that any local chirality inversion necessarily marks a macroscopic topological transition.

C. Limit cycles in standard Morse-Smale graphs

The Morse-Smale framework extends beyond gradientlike vector fields to incorporate rotational elements. Specifically, the planar graph invariant admits a natural extension to flows supporting LCs [52]. From a flow topological perspective, an isolated LC plays a role analogous to a steady-state solution, in that it acts as a terminal object for trajectories within its basin [cf. Fig. 3(a)]. The fundamental difference from a fixed point is that an LC separates the phase space into an inner and an outer region. Consequently, the LC

functions as an interface linking the topological invariants of the two bounded subdomains. This macroscopic structure enriches the topological landscape by introducing an additional organizing element that mediates how separatrices connect across regions.

To incorporate this structure into the standard classification [52], the vector flow is decomposed into elementary regions bounded by invariant transverse curves. Regions containing only steady-state solutions are treated conventionally. Conversely, regions containing a single attracting or repelling LC define distinct elementary components. The global flow is then reconstructed by specifying connections across these boundaries. There, the orientation of trajectories fixes the directionality of the graph edges [see Fig. 3(b)]. With respect to the LC, we identify explicit inner (r) and outer (V) regions. Because the current inner region contains a single repelling source, we identify it as elementary. Since the LC is attracting, both inner and outer flows converge toward it. This dynamics is marked by directed arrows, $r \rightarrow LC$ and $LC \leftarrow V$. To accommodate the structural complexity of V , its internal elementary regions are also directionally highlighted. Specifically, $V \rightarrow a_1$ and $V \rightarrow a_2$ dictate that V feeds into the pointlike attractors, while $V \leftarrow VS$ marks the origin at the virtual source.

Standard formulations require specifying how the LC winds relative to adjacent saddle regions. Within the vector-flow formalism [52], this is indicated by a $\pm$ label on the corresponding graph arrows. On an oriented surface, this reduces to the difference in winding relative to the overall flow chirality. Correspondingly, for phase-space flows bounded on a disk, this calculation simplifies to evaluating the winding directly relative to the virtual source chirality. There, this explicit detail proves useful when classifying multiple or nested LCs. We therefore omit it here within our single-cycle case study, as it provides no additional information. In the following, we propose an augmented graph invariant that directly captures both regional boundaries and local chiralities. This is rendered possible through the chirality-encoding framework of Ref. [53], which already embeds the rotational character within each region.

III. LIMIT CYCLES IN NONLINEAR BOSONIC SYSTEMS

In nonlinear bosonic systems, the rotational flow components carry a deep physical meaning related to causality of excitations (see Appendix). Therefore, in Ref. [53], Morse-Smale graphs were augmented to encode the chirality information of steady states. Here, we extend these graphs to include LCs and their chiralities [see Fig. 3(c)]. Specifically, we augment the colored-graph formalism to retain information on (1) the number of subregions into which the manifold is divided, (2) their internal topological structure, and (3) the chirality around each critical point or limit cycle.

Consider the flow in Fig. 3(a), which contains a single LC. We first construct the graph invariants associated with the inner and outer regions, treating the LC as a fixed point. In this spirit, the upper panel of Fig. 3(c) corresponds to the node V of the invariant in Fig. 3(b), while the lower panel is referred to the inner region (r). As we show later, the inner region of the LC can host nonelementary flows just like

the outside does in this case. Nevertheless, the entire interior region always acts globally as an attractor/repeller, opposite to the stability of the LC itself. We specify this behavior in our graph with a small black marker, namely, an empty double-lined circle when attractive (not relevant in this case) and a crossed circle when repelling (relevant in this case). To identify where the LC is located in the graph invariant and what is the behavior of its inner region, we repeat this symbol in the corresponding region of the upper panel. Notice that the stability of the LC itself is also visible from the edges of the corresponding graph invariant region, as they are built via separatrices. The chirality is encoded by introducing the same coloring scheme as in Fig. 2. The procedure we described above yields a complete topological description of the vector flow that remains invariant under smooth deformations.

Having established the admissible topological phases of flows with LCs, we now construct explicit examples with increasing complexity. This can be achieved by extending the original model (1) with additional terms. Indeed, the vdP oscillator provides a versatile framework in which a wide range of flow topologies can be realized. First, by generalizing the nonlinear gain-loss term to an arbitrary even polynomial in x , one can generate any desired number of LCs. Second, by adjusting the potential, an arbitrary number of steady states can be introduced and distributed among the regions separated by the LCs in accordance with the topological constraints. By appropriately combining polynomial gain-loss terms with tailored potentials, one can systematically construct flows realizing a wide range of admissible topological phases, two of which are portrayed in Figs. 4(a) and 4(b). First, we consider Fig. 1(c) and modify it in Fig. 4(a) to show a flow with three nested LCs generated by four alternating gain-loss regions. Concretely, we set $\gamma = 0$ and replace the term $\mu(1 - x^2)\dot{x}$ in Eq. (1) by

$$\mu(a_0 + a_2x^2 + a_4x^4 + a_6x^6)\dot{x}, \quad (3)$$

where $a_0, a_4 > 0$ and $a_2, a_6 < 0$. The roots of this polynomial provide good estimates for the LC radii; by choosing them sufficiently separated, one can introduce an arbitrary number of LCs. Their stability can be inverted by changing the overall sign. The resulting graph invariant is shown in Fig. 4(c). Due to the deep nesting of LCs, in this case, we use an equivalent representation of the graph invariant. Here, the LCs separate phase space into elementary regions only. Thus, we directly nest the corresponding elementary subgraphs.

The second example we propose is a modification of the more complex region [Fig. 1(e)], introducing additional structure within the LC itself [Fig. 4(b)]. For the practical construction, we start from a triple-attractor picture. As the LC grows and approaches the saddles, it vanishes in the transition from Fig. 1(e) to Fig. 1(f). In this symmetric configuration, the LC cannot manifest beyond the saddles, as it would contain a subflow that is not a valid elementary region within the framework of topological classification. However, the combinatorial nature of the graph invariant predicts the existence of a very similar flow, where the LC can host several steady-state solutions and still manifest as a stable orbit: We can build a case where we asymmetrically shift two steady states toward the interior of the LC. This allows the LC to persist and yields a flow with distinct structures inside and outside

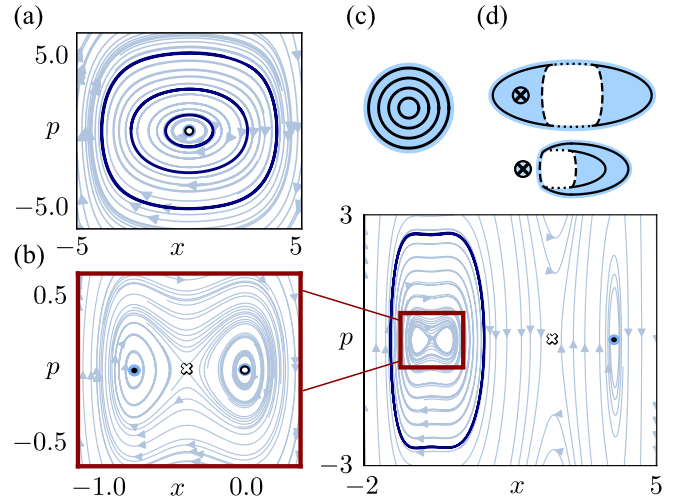


FIG. 4. Designing exemplary flows involving nested LCs and nonelemental structure on the inside and outside of a LC. (a) Phase portrait featuring two attracting LCs bounding an intermediate repelling LC and a central source, generated by alternating nonlinear drives and losses $\mu \sum_{n \in 2\mathbb{N}} a_n x^n$. Parameters are $a_0 = 15$, $a_2 = -45$, $a_4 = 15$, $a_6 = -1$, and $\mu = 0.0004$, under a single-well potential $0.5x^2 + 0.06x^6$. (b) Topological vector flow exhibiting a nonatomic structure with saddles located both inside and outside an attracting LC. The dynamics combine linear gain and nonlinear damping $\mu(1 - x^2)\dot{x}$ within a triple-well potential $((x - 4)^2(x + 0.75)^2x^2)/6$, with the $x < 0$ steady-state solutions shifted toward the origin. Drive and loss parameters are $\mu = 0.054$ and $\gamma = 0.03$. (c) Nested graph invariant representation for the flow in panel (a). Isolated nested subgraphs denote intermediate LCs of corresponding stability rather than standard pointlike sinks or sources. (d) Graph invariant representation for the flow in panel (b).

the cycle. The resulting graph invariant is shown Fig. 4(d). Notice that the flows in Figs. 1(e) and 4(b) have the same critical elements. Nevertheless, the systems belong to different topological phases governed by distinct graph connectivities.

IV. CONCLUSION AND OUTLOOK

We extended the flow topological classification of driven-dissipative nonlinear resonators to include limit cycles as fundamental topological features. By generalizing the Morse-Smale graph construction beyond steady-state solutions, we show how self-sustained oscillations reorganize phase-space connectivity and generate flow topological phases that cannot be captured by fixed-point-based indices alone. Using a minimal extension of the van der Pol resonator, we demonstrate how the emergence and bifurcation of LCs modify global flow topology and induce flow topology transitions associated with qualitative rearrangements of dynamical basins. To encode these structures, we modify the graph invariant of Ref. [53]. This generalized construction offers specific advantages: (1) It captures both global connectivity and local flow details, preventing the concealment of complex inner saddle regions; (2) it uniquely identifies the position and local directionality of each invariant element; and (3) it specifies the limit cycle orientation. There, this orientation metric becomes necessary when classifying multiple coexisting cycles. Unlike the

standard graphical representation of Ref. [52], where distinct critical elements remain visually degenerate, our formulation isolates each attractor, source, and limit cycle. Correspondingly, we uniquely label each critical feature in Fig. 3(b) (see the Supplemental Material [80]).

Our notation preserves the topological invariant while satisfying the requirement to associate each critical point with its exact graphical position, an essential capability for characterizing macroscopic nonlinear bosonic systems. This extension opens several natural directions. Classically, it provides a systematic framework to analyze synchronization, mode locking, and multicycle dynamics in coupled resonator networks. Quantum mechanically, it raises questions about how periodic attractors appear in quantum steady states and trajectories, and whether their flow topological signatures persist under strong fluctuations. More broadly, incorporating LCs advances flow topology toward a fully dynamical classification of nonequilibrium phases, connecting topology, nonlinear dynamics, and open quantum systems.

ACKNOWLEDGMENTS

We acknowledge funding from the Deutsche Forschungsgemeinschaft (DFG) through Projects No. 449653034, No. 521530974, and No. 545605411, as well as via SFB 1432 (Project No. 425217212). We also acknowledge support from the Swiss National Science Foundation (SNSF) through the Sinergia Grant No. CRSII5_206008/1.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: SYMPLECTIC NORM AND CHIRALITY

Morse-Smale vector fields can possess rotational elements. Consequently, trajectories may approach attractors or limit cycles with either a CW or CCW chirality. The rotational character of a vector field $\vec{F}(x, y) = (\dot{x}, \dot{y})$ at its critical points is typically reflected in the Jacobian eigenvalues λ_i ($i \in \{1, 2\}$), each identifying a principal direction [81]. The real part of each eigenvalue determines the stability of the direction, while the imaginary part corresponds to the frequency of oscillation along the corresponding principal axis:

(1) If $\text{Re}(\lambda_i) > 0$, the corresponding direction is unstable. If it is $\text{Re}(\lambda_i) < 0$, then the direction is stable (indeed, $\text{Re}(\lambda_i) > 0 \forall i$ at maxima and $\text{Re}(\lambda_i) < 0 \forall i$ at minima).

(2) If $\text{Im}(\lambda_i) = 0$, the flow is locally nonrotational. If $\text{Im}(\lambda_i) \neq 0$, then the flow winds CW or CCW around the critical point.

By definition, saddles in two-dimensional flows have purely real eigenvalues, of which one is positive and one is negative. On the other hand, a minimum (maximum) can have nonzero imaginary part, thus becoming an attracting (repelling) focus.

Additional information on the rotational sense of the flow can be inferred from its curl:

$$\text{curl}(F) = \partial_x \dot{y} - \partial_y \dot{x}. \quad (\text{A1})$$

In the systems considered here, the sign of the curl is associated with the rotation of the local flow: A negative curl corresponds to a CW rotation, while a positive curl reflects a CCW rotation. It is important, however, that our flow is globally rotating and not only around the fixed points. This induces a bias in the curl, which is nonzero even at nonoscillatory fixed points (e.g., saddles). This implies that the curl cannot be used as a signature of whether the flow is locally spiraling or not, but only of whether it is co- or counterrotating with respect to the background. On the other hand, the linearized dynamics captured by the Jacobian distinguish between spiraling or nonspiraling trajectories.

Thus, in order to fully understand whether and how the field is locally rotating, we can combine the Jacobian analysis and the curl evaluating as follows:

(1) When $\lambda_i \in \mathbb{R}$, the flow is locally nonrotational.

(2) When $\lambda_i \in \mathbb{C}$, the flow locally possesses vorticity. We can, thus, use to curl to determine the winding direction.

1. Symplectic norm

In nonlinear bosonic systems, the rotational character of the flow is determined by the potential. For instance, in a nonlinear parametric resonator, the sense of the background rotation is attributed to the sign of the dominating term in the rotating Hamiltonian, which is proportional to the nonlinearity. The local rotational sense (chirality) around a critical point can be interpreted in terms of the nature of the corresponding excitation. This is done via the symplectic norm ds^2 [36,77,79], specifically its real sign:

$$\text{sgn}(ds^2) \in \{-1, 0, 1\}. \quad (\text{A2})$$

When an excitation is created on top of a stationary state with a negative (positive) symplectic norm, it reduces (increases) the energy of the system relative to the rotating frame, that is, relative to an excitation at the driving frequency ω (or at the frequency at which the rotating frame is moving). This establishes a one-to-one correspondence between chirality and nature of the excitation. An excitation behaves particlelike in the rotating frame if it moves *faster* than the reference and, hence, forward in time. In this case, we talk about CW chirality, corresponding to a positive ds^2 . If, instead, the excitation behaves holelike, it moves *slower* than the reference, and therefore backward in time. As a consequence, its chirality will be counterclockwise, and ds^2 will be negative. If the real sign of the symplectic norm is zero (namely, the Jacobian eigenvalues are purely real), the excitation is overdamped and the flow is, therefore, locally nonrotating.

In this spirit, the symplectic norm recovers the same information as the curl (with an inverted sign). We can analyze the Jacobian spectrum of an attractor and

(1) if $\text{Im}(\lambda_i) = 0$, we conclude that it is overdamped (indeed, the real sign of its symplectic norm would be zero);

(2) if $\text{Im}(\lambda_i) \neq 0$, we can use the sign of the symplectic norm (or the curl) to determine the rotation sense and, analogously, the nature of the corresponding excitation.

2. Relation to the exceptional points

Consider an attractor in a rotating flow, whose local chirality is CW and consistent with the background rotation sense. We can perform a smooth parameter deformation and observe that, while no other critical point appears and the background chirality remains CW, the local chirality around the point changes. Specifically,

(1) the flow initially rotates CW. There, $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) < 0$ and $\text{Im}(\lambda_i) \neq 0$;

(2) $\text{Im}(\lambda_i) = 0$ and the flow locally is gradientlike. There, $\text{Re}(\lambda_1) \neq \text{Re}(\lambda_2)$, while both remain negative (granting that the point is always attracting);

(3) $\text{Im}(\lambda_i) \neq 0$ and the flow resumes rotation, this time in the CCW sense. Once again, $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) < 0$ and $\text{Im}(\lambda_i) \neq 0$.

The points at which the real and imaginary parts of the Jacobian eigenvalues coalesce or diverge are known as *exceptional points*. This detail, combined with the excitation nature interpretation, is of the utmost importance: CW, CCW, and overdamped local flow behavior cannot belong to the same topological phase, when nonlinear bosonic resonators are studied. This motivates the introduction of the coloring scheme in Ref. [53].

3. Practical calculation of the symplectic norm

We now briefly describe how to practically compute the symplectic norm. We notice that this calculation can be performed from the eigenspectrum of the dynamical matrix or the Jacobian, due to the relation between them. Here, for a better analogy with the vector-flow topology literature, we follow the approach via the Jacobian. For further details, we recommend Refs. [36,77,79].

We start by linearizing the slow-flow equations. At a given fixed point $\mathbf{x}_0$, the Jacobian matrix describes the linearized

forces acting in the rotating frame:

$$\mathbb{J} = \begin{pmatrix} \partial_x \dot{x} & \partial_y \dot{x} \\ \partial_x \dot{y} & \partial_y \dot{y} \end{pmatrix} \bigg|_{\mathbf{x}_0}. \quad (\text{A3})$$

For a given set of parameters, the elements of the eigenvectors and eigenvalues are functions of the coordinates of the steady state $\mathbf{x}_0$. Namely,

$$\mathbf{w}_j \equiv \begin{pmatrix} w_{ju} \\ w_{jl} \end{pmatrix} = \begin{pmatrix} w_{ju} \\ 1 \end{pmatrix}, \quad (\text{A4})$$

where $j \in \{1, 2\}$, and w_{ju} and w_{jl} are all functions of $\mathbf{x}_0$ and the system's parameters. We use the indices u and l to denote the upper and the lower entries of the eigenvector. Analogously, the corresponding eigenvalues are indicated as μ_j . If the fixed points are rotating, the associated Jacobian eigenvectors and eigenvalues form complex conjugate pairs. Specifically,

$$\mathbf{w}_{\pm} = \begin{pmatrix} a \pm ib \\ 1 \end{pmatrix} \quad \text{and} \quad \mu_{\pm} = -\frac{\gamma}{2} \pm ic, \quad (\text{A5})$$

where a, b , and $c \in \mathbb{R}$. In particular, c is known as the complex frequency. In terms of the Jacobian eigenvectors, the symplectic norm ds^2 is defined as follows:

$$ds^2 \equiv i(w_{1u}w_{2l} - w_{2u}w_{1l}), \quad (\text{A6})$$

where w_{1u} denotes the upper entry of the first Jacobian eigenvalue (and analogously for the other terms). For the rotating fixed points, such definition can be recast in the following expression:

$$ds^2 \equiv i(w_{+u}w_{-l} - w_{-u}w_{+l}). \quad (\text{A7})$$

As anticipated, we are interested in the real sign of the symplectic norm: If $ds^2 < 0$, the excitation is holelike (CCW); if $ds^2 > 0$, the excitation is particlelike (CW). Otherwise, it is in an overdamped regime [36,39,82,83].

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