

## Entanglement halos

Nadir Samos Sáenz de Buruaga <sup>1,2,\*</sup>, Silvia N. Santalla <sup>3</sup>, Germán Sierra <sup>4,5</sup> and Javier Rodríguez-Laguna <sup>6</sup>

<sup>1</sup>CeFEMA, LaPMET, *Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal*

<sup>2</sup>CIAF, Departamento de Física Teórica, *Universidad Autónoma de Madrid, Madrid, Spain*

<sup>3</sup>Dto. Física, and GISC, *Universidad Carlos III de Madrid, Leganés, Madrid, Spain*

<sup>4</sup>*Instituto de Física Teórica, UAM-CSIC, Universidad Autónoma de Madrid, Cantoblanco, Madrid, Spain*

<sup>5</sup>*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*

<sup>6</sup>Dto. Física Fundamental, *Universidad Nacional de Educación a Distancia (UNED), Madrid, Spain*



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We introduce the concept of *entanglement halos*, a set of strongly entangled distant sites within the ground state of a quantum many-body system. Such halos emerge in starlike systems with exponentially decaying couplings, as we show using both free fermions and the spin-1/2 antiferromagnetic Heisenberg model. Depending on the central connectivity, entanglement halos may exhibit trivial and nontrivial symmetry-protected topological features. Our findings highlight how geometry and connectivity can generate complex entanglement structures with rich physical content, which can be experimentally accessible via state-of-the-art technologies.

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### I. INTRODUCTION

The integration of quantum information [1] concepts into condensed matter has opened a fertile, interdisciplinary, and rapidly evolving field, bridging diverse areas such as quantum technologies, quantum matter and optics [2], and even more fundamental ones such as quantum gravity [3]. At the heart of many of these developments lies quantum entanglement—the most striking manifestation of nonclassical correlations [4].

Indeed, entanglement underpins the parellism in operations that may allow quantum speedup [5], it guides in the distinction of phases of matter beyond the Landau paradigm—such as topological [6] and symmetry-protected topological (SPT) [7,8] orders—and through the holographic principle—nongravitational systems with sufficient entanglement may exhibit the characteristic signatures of quantum gravity, suggesting that entanglement can play a role in emergent spacetime geometries [9,10].

A central principle in the study of quantum many-body systems is the area law for entanglement entropy (EE) [11,12], which, in one dimension, implies that the EE remains independent of the system size. This property underlies the efficient description of ground states using matrix product states (MPS) [13]. A notable exception to the one-dimensional area law arises in systems whose low-energy behavior is governed by conformal field theory, where the EE exhibits a logarithmic violation of the area law [14,15]. Even stronger violations can occur in certain inhomogeneous spin chains, where the entanglement entropy grows more rapidly with system size.

A paradigmatic example is the *rainbow chain* [16–18] of  $2L$  fermionic sites or spins whose couplings decay exponentially from the center toward the extremes. For strong inhomogeneity, the strong disorder renormalization group [19,20] shows that symmetrically placed sites around the center become strongly entangled, thus yielding a set of concentric valence bonds.

The emergence of these nonlocal Bell states constitutes the simplest example of what we call the *entanglement halos*. It should be noted that the concept of a *halo* is well established in physics, denoting a ring- or shell-like distribution. In astrophysics, it refers to the real-space glow resulting from incoherent small-angle scattering off interstellar dust [21]. In ultracold atom physics, a halo designates the momentum-space shell formed when two dilute Bose-Einstein condensates collide and scatter coherently, producing correlated atom pairs [22–24]. Here, we designate the entanglement halo as a well-defined entanglement structure within the ground state among certain constituents that do not need to be spatially close.

In this work, we explore the emergence of multiparticle entanglement halos in star-graph geometries due to the inhomogeneity of the local interactions, delving in the influence of the underlying geometry and connectivity of the system, which originates trivial and nontrivial SPT phases. We discuss that such engineered quantum systems can be realized with current state-of-the-art technologies, opening avenues to probe entanglement structures in synthetic geometries.

Long-range entanglement between distant degrees of freedom has also been discussed in other contexts, such as quantum spin liquids, where it emerges from frustration and gauge constraints rather than from geometry or coupling inhomogeneity [25,26]. Taken together, these results highlight that spatial distance alone is not a reliable predictor of entanglement structure in many-body ground states.

The paper is organized as follows. In Sec. II, we introduce the geometries and the inhomogeneous spin and free-fermion

\*Contact author: nadir.samos@uam.es

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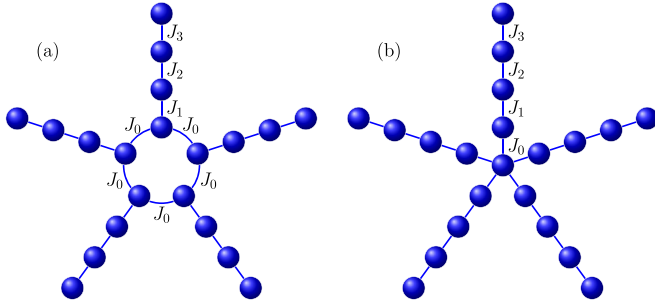


FIG. 1. Star geometries of  $n_B = 5$  branches and  $\ell = 4$ . (a) Ring star with  $N = n_B \ell = 20$  and (b) site star comprising  $N = n_B \ell + 1 = 21$  sites.

models considered in this work. In Sec. III, we present the different types of entanglement halos, illustrating how the same organizing concept manifests itself differently depending on geometry and symmetry. We also discuss possible experimental realizations. Finally, we summarize our conclusions and outline prospects for future work in Sec. IV.

## II. SETTING THE SCENE

### A. Ring and site stars

We consider a star-graph geometry consisting of  $n_B$  chains, referred to as *branches*, each comprising  $\ell$  nodes that host local degrees of freedom such as spins or fermions. We shall show that their physical properties depend critically on how the branches are connected, rather than on the nature of the degrees of freedom.

We distinguish between two types of graph geometries. In the *ring-star* geometry, the  $n_B$  branches are connected through an internal ring that links their innermost sites, as illustrated in Fig. 1(a). In contrast, the *site-star* geometry connects the  $n_B$  branches through an additional central site, as shown in Fig. 1(b). Consequently, the ring-star (site-star) configuration contains  $N = n_B \ell$  ( $N = n_B \ell + 1$ ) sites.

Thus, each star consists of concentric *rings*, which are indexed outward as  $r \in \{0, \dots, \ell\}$ , with  $r = 0$  reserved for the central site (if it exists). Branches, on the other hand, are labeled as  $p \in \{1, \dots, n_B\}$ . The  $r$ th site along  $p$ th branch is labeled as  $i = (r - 1)n_B + p$ , and the index  $i = 0$  is reserved for the central site (if it exists).

Such stars have been considered previously in the literature, both in the homogeneous [27,28] and the disordered case [29]. Also, the possibility of a single detached ring was considered within a particular case in Ref. [30]. Starlike lattices provide a natural framework in order to extend the rainbow chain to models with SU(3) symmetry, in which color singlets play the role of Bell pairs [31].

### B. Models

We analyze two representative models, interacting and noninteracting. Given a graph with  $N$  sites, we consider a particle-conserving free-fermionic model given by the Hamiltonian

$$H_{\text{FF}} = - \sum_{\langle i, j \rangle} J_{ij} c_i^\dagger c_j + \text{H.c.}, \quad (1)$$

where  $\langle i, j \rangle$  denotes the neighborhood relation in the graph,  $J_{ij} \in \mathbb{R}^+$  and  $c_i^\dagger$  ( $c_j$ ) denote the creation (annihilation) operators on sites  $i$  ( $j$ ), which fulfill the anticommutation relations  $\{c_i, c_j^\dagger\} = \delta_{ij}$ . Through this work, we analyze the many-body ground state (GS) at half filling [32].

In the interacting case, we consider the antiferromagnetic (AF) spin-1/2 Heisenberg model, defined by the Hamiltonian

$$H_{\text{Heis}} = \sum_{\langle i, j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \quad \mathbf{S}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z). \quad (2)$$

### C. Inhomogeneity

We consider strongly inhomogeneous couplings with radial symmetry  $J_{ij} = J_r$  whenever  $i$  and  $j$  belong to rings  $r$  and  $r + 1$ , as illustrated in Fig. 1. In addition, we enforce a strong hierarchy on the coupling strengths  $J_0 \gg J_1 \gg J_2 \gg \dots \gg J_{\ell-1}$ . Specifically, inspired by previous work [18,33], we shall consider an exponential decay of the couplings strength,

$$J_r = \begin{cases} J_0 = e^{-h/2}, \\ J_{r>0} = e^{-hr}, \end{cases} \quad r \in \{1, \dots, \ell - 1\}, \quad (3)$$

where  $h \in \mathbb{R}^+$  is an inhomogeneity parameter that reduces to the homogeneous case if  $h = 0$ .

In the case of the ring-star geometry with  $n_B = 2$  legs; i.e., a chain, the ground state of Hamiltonians (1) and (2) with couplings (3) can be determined in the  $h \gg 1$  limit using the strong-disorder renormalization group (SDRG) approach. Originally developed for disordered chains, this RG remains highly accurate as long as the system exhibits a strong energy hierarchy. The method proceeds by iteratively decimating low-energy degrees of freedom, assumed to be spatially localized. These degrees of freedom are approximately disentangled from the rest of the system, resulting in a GS that is essentially a product state. In this case, with this method one obtains the rainbow state [16–18,34].

## III. ENTANGLEMENT HALOS

### A. Single-ring entanglement halos

Let us consider a ring star, such as the one shown in Fig. 1(a), and assume a strong inhomogeneity regime  $h \gg 1$ . For both the interacting and noninteracting models, the central ring  $r = 1$  forms a closed chain, which is described by the Hamiltonian

$$\begin{aligned} H_{\text{FF}}^{(1)} &= -J_0 \sum_{p=1}^{n_B} c_p^\dagger c_{p+1} + \text{H.c.}, \quad c_{n_B+1} \equiv c_1, \\ H_{\text{Heis}}^{(1)} &= J_0 \sum_{p=1}^{n_B} \mathbf{S}_p \cdot \mathbf{S}_{p+1}, \quad \mathbf{S}_{n_B+1} \equiv \mathbf{S}_1. \end{aligned} \quad (4)$$

We first consider the case in which the GS  $|\psi_0^{(1)}\rangle$  of  $H^{(1)}$  is unique. This is the case when  $n_B$  is even for the Heisenberg model or  $n_B \not\equiv 0 \pmod{4}$  for the free-fermion model. The coupling hierarchy,  $J_0 \gg J_1$ , ensures that the innermost ring described by  $|\psi_0^{(1)}\rangle$  detaches from the rest of the system, weakly affecting neighboring sites in the ring  $r = 2$ .

Following the RG spirit, we show in Appendix A that, for our current choice of inhomogeneities (3), at each RG

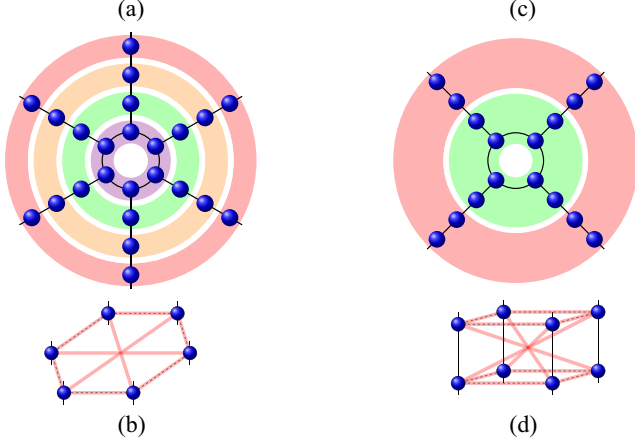


FIG. 2. Entanglement structure of the free-fermionic ring stars. (a) Entanglement halos in a  $n_B = 2 \bmod 4$  free-fermionic ring star. Sites along each ring get entangled, while different rings factorize. (b) Folding the ring star yields a tower of rings, whose internal entanglement pattern shows connections toward odd neighbors. (c) Entanglement double halos in a  $n_B = 0 \bmod 4$  free-fermionic ring star. Sites along each ring pair will get entangled. (d) Two rings along the folded tower, showing the internal entanglement pattern: odd neighbors in the same floor, even neighbors across floors.

step  $r$ , one can detach the inner ring  $r$ , described by the GS  $|\psi_0^{(r)}\rangle$  of an effective Hamiltonian  $H^{(r-1)}$  obtained via second-order perturbation theory. As a result, the many-body GS is approximately described by a product state:

$$|\Psi_{\text{GS}}\rangle \approx \bigotimes_{r=1}^{\ell} |\psi_0^{(r)}\rangle, \quad (5)$$

where we identify each  $|\psi_0^{(r)}\rangle$  as forming an *entanglement halo*, as illustrated in Fig. 2(a).

Numerical evidence for the validity of these results is shown in Fig. 3, where we plot the EE of the  $r$ th ring  $A_r$  for both (1) free-fermionic and (2) Heisenberg ring stars using  $n_B = 6$  and different values of  $h$  and  $\ell$ . Notice that  $S(A_r)$  decreases as  $h$  increases approaching zero in the limit  $h \gg 1$ , which supports the factorization in Eq. (5).

### B. Double-ring entanglement halos

Whenever the GS of  $H^{(1)}$  is degenerate, e.g., for Heisenberg ring stars with odd  $n_B$  (due to geometric frustration) or free-fermionic stars with  $n_B \equiv 0 \bmod 4$ , the entanglement structure becomes more involved. As shown in Appendix A, the degree(s) of freedom associated with the degeneracy are transferred to the  $r = 2$  ring, thus yielding a more complex Hamiltonian  $H^{(2)}$ . Yet, the GS of  $H^{(2)}$  is unique, yielding a well-defined global state  $|\psi^{(1,2)}\rangle$  for the pair of rings, which detaches again from the rest of the system. The effective Hamiltonian  $H^{(3)}$  for the third ring is proportional to the first one,  $H^{(1)}$ , and the double-halo structure reappears, giving rise to a product state many-body GS:

$$|\Psi_{\text{GS}}\rangle \approx \bigotimes_{m=1}^{\ell/2} |\psi_0^{(2m-1, 2m)}\rangle. \quad (6)$$

Evidence for this structure is shown in Fig. 3, where we observe the EE of central blocks  $S(B_r)$ , which contain all sites inside ring  $r$ . We observe an alternating pattern, for both (3) free-fermionic ring stars ( $\ell = 7$ ,  $n_B = 8$ ) and (4) Heisenberg ring stars ( $n_B = 3$ ,  $\ell = 6$ ,  $h = 6$ ). We should emphasize that double-ring halos for  $n_B = 0 \bmod 4$  free-fermionic ring stars are associated with an accidental degeneracy, which can be lifted with an internal magnetic flux through the central ring.

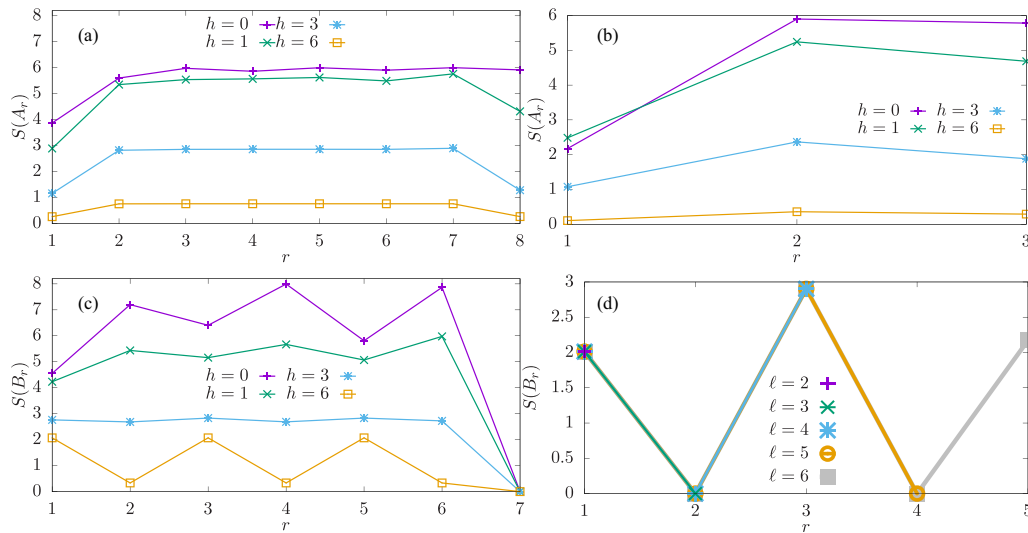


FIG. 3. Top: Ring stars presenting single-ring entanglement halos, with  $n_B = 6$ . EE of the  $r$ th ring,  $S(A_r)$ , for (a) free-fermionic with  $\ell = 8$ , and (b) Heisenberg ring stars with  $\ell = 3$ , using different values of  $h$ . Bottom: Ring stars presenting double-ring entanglement halos. EE of central blocks, containing up to the  $r$ th ring,  $S(B_r)$ , for (c) the free-fermionic case with  $n_B = 8$ ,  $\ell = 7$ , and different values of  $h$ , and (d) the Heisenberg case with  $n_B = 3$  and  $\ell = 6$  and  $h = 6$ . The double-ring structure is made manifest in the alternation in  $S(B_r)$ , which vanishes for even  $r$ .

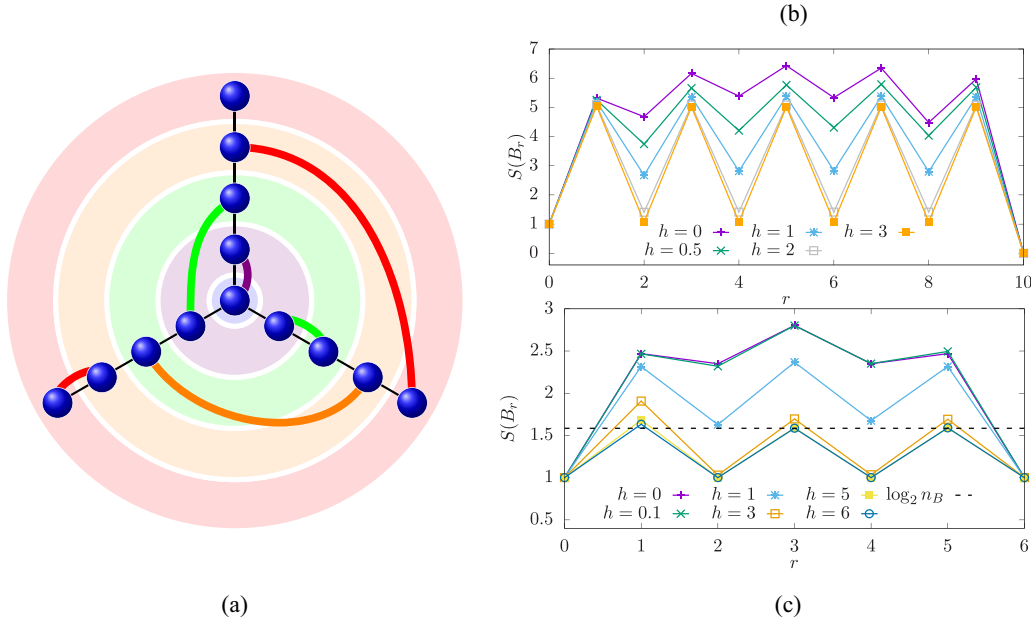


FIG. 4. Twisted halos in site stars. (a) Diagram showing a typical bond structure. Notice that sites never establish a bond within the same ring. (b) EE of central blocks,  $S(B_r)$ , for different values of  $h$  for a free-fermion system with  $n_B = 6$  and  $\ell = 10$  rings. (c) Same observable for a Heisenberg system with  $n_B = 3$  and  $\ell = 7$ .

In summary, the GS of ring stars in strong inhomogeneity regime is always a product state of either single-ring or double-ring halos, showing a topologically trivial structure.

### C. Twisted entanglement halos

We follow by considering the site-star geometry shown in Fig. 1(b) and evaluate the GS of both models with couplings given by Eq. (3) in the regime of  $h \gg 1$ . The inner—most energetic—subsystem is given now by the Hamiltonians

$$H_{\text{FF}}^{(1)} = -J_0 \sum_{i=1}^{n_B} c_0^\dagger c_i + \text{H.c.}, \quad (7)$$

$$H_{\text{Heis}}^{(1)} = J_0 \sum_{i=1}^{n_B} \mathbf{S}_0 \cdot \mathbf{S}_i,$$

for fermions and spins, respectively. In similarity with the double-ring halo, the GS of Eq. (7) presents degeneracy. Indeed, we can write an effective Hamiltonian for the second ring,  $H^{(2)}$ , which includes the degrees of freedom associated with this degeneracy. Yet, in opposition to the double-ring halo case, now  $H^{(2)}$  also presents degeneracy, which extends along the RG procedure. Thus, the GS of the site star is not factorizable.

Let us first consider the free-fermionic case. The single-body energies of Hamiltonian  $H^{(1)}$  are  $\pm J_0 \sqrt{n_B}$ , plus  $n_B - 1$  zero modes. Consequently, the many-body GS manifold presents a large degeneracy,  $2^{n_B-1}$ . A single-body real-space RG (see Ref. [35]) carries the  $n_B - 1$  zero modes to the  $r = 2$  ring and aggregates them to the effective Hamiltonian  $H^{(2)}$ , which only has a single zero mode, and then the pattern repeats itself. Therefore, the EE of odd central blocks,  $S(B_{2m-1}) = n_B - 1$ , and the EE of even central blocks is  $S(B_{2m}) = 1$ , as shown numerically in Fig. 4(a) for a site star

with  $\ell = 10$  and  $n_B = 6$ . Since the Hamiltonian is invariant under time-reversal, particle-hole and it enjoys sublattice symmetry, it belongs to the Altland-Zirnbauer BDI symmetry class, characterized by both timereversal and particle-hole symmetries squaring to +1 [36], same as the paradigmatic Su-Schrieffer-Hager model [37].

This entanglement structure can be physically understood as follows: The central site is entangled with one site in the first ring, while the remaining  $n_B - 1$  sites in ring  $r = 1$  are maximally entangled with sites in the second ring. Only one site on the ring  $r = 2$  then connects to ring  $r = 3$ , etc. In stark contrast to the ring-star case, sites within each ring remain completely uncorrelated, while displaying strong entanglement across adjacent rings. This defines a reversed entanglement pattern, which we refer to as *twisted halos*: structures internally detached but maximally entangled with neighboring layers, as illustrated in Fig. 4(b).

Let us now consider the Heisenberg site star. Exploiting the SU(2) symmetry of the model, we write  $H^{(1)}$  in terms of Casimir operators,

$$H_{\text{Heis}}^{(1)} = \frac{J_0}{2} (\mathfrak{S}_1^2 - \mathbf{S}_0^2 - \mathcal{S}_1^2), \quad (8)$$

where  $\mathcal{S}_1 \equiv \sum_{i=1}^{n_B} \mathbf{S}_i$  denotes the composition of all spins in the first ring,  $r = 1$ , and  $\mathfrak{S}_1 \equiv \mathbf{S}_0 + \mathcal{S}_1$  is obtained by further addition of the innermost spin. In the above form, it is clear that the minimal energy of Eq. (8) is obtained when  $\mathcal{S}_1 = n_B/2$  is maximal and  $\mathfrak{S}_1 = (n_B - 1)/2$  is minimal, leading to a GS manifold of dimension  $n_B$ . As detailed in Appendix B, we devise an RG procedure in which each step  $r$  yields an effective Hamiltonian describing the spins in ring  $r + 1$  coupled with an effective spin  $\mathfrak{S}_r$  built from the inner spins,

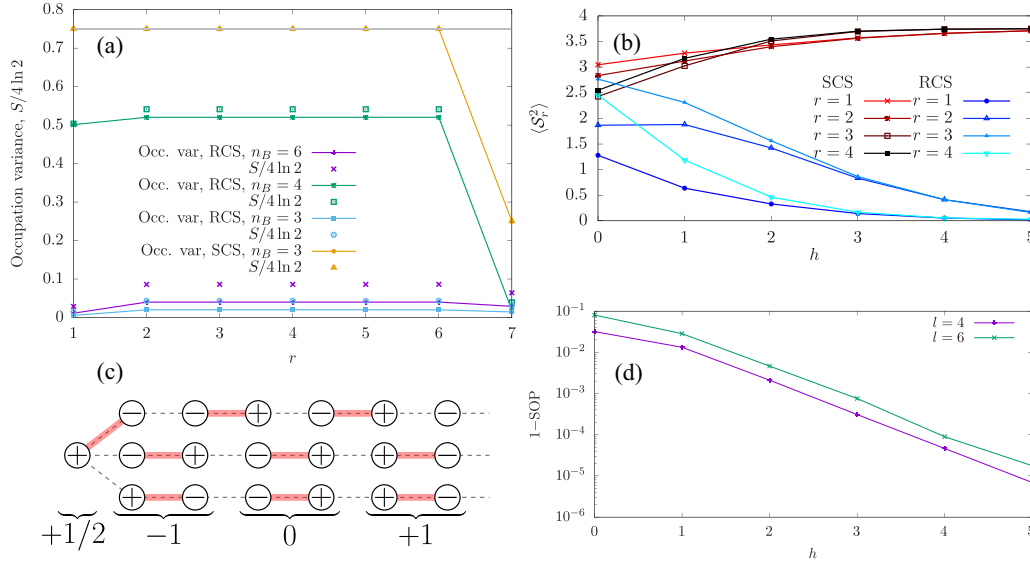


FIG. 5. Observables which highlight the different phases. (a) Variance of the number of fermions on each ring, along with the EE divided by  $4 \ln 2$ . The horizontal dashed line at  $3/4$  marks the theoretical prediction for site-centered rings with  $n_B = 3$ . (b) Expected value of  $\langle S_r^2 \rangle$  for each ring of a site star (ring star) with  $n_B = 3$  ( $n_B = 4$ ) as a function of  $h$ . In the strong inhomogeneity, it converges to  $S_r^2 = 15/4$  ( $S_r^2 = 0$ ). (c) Illustration of the SOP and the Haldane phase within a  $n_B = 3$  site star. (d)  $1-\text{SOP}$  for ring pairs of a site star with  $n_B = 3$  and  $\ell = 4$  and  $6$ , also as a function of  $h$ , showing an exponential decay toward zero.

$H^{(r+1)} = \tilde{J}_r \mathfrak{S}_r \cdot \mathcal{S}_{r+1}$ , where

$$\mathcal{S}_r \equiv \sum_{i=1}^{n_B} \mathbf{S}_{(r-1)n_B+i}, \quad \mathfrak{S}_r \equiv \mathfrak{S}_{r-1} + \mathcal{S}_r. \quad (9)$$

Hence, each RG step requires that all spins in ring  $r + 1$  combine into a maximal total spin  $\mathcal{S}_{r+1}$ . Thus, the Heisenberg twisted halos are not internally disentangled, because the spins-1/2 within each ring fulfill a condition: They add up to a total spin of  $n_B/2$ . However, analogously to the free fermion case, each halo is coupled to a spin  $\mathfrak{S}_r$  composed of all inner halos, resembling a Matryoshka doll. Energy minimization imposes this spin to be minimal, resulting in an alternation  $\mathfrak{S}_r = (n_B - 1)/2$  (respectively,  $\mathfrak{S}_r = 1/2$ ) for odd (even) RG steps  $r$ . Furthermore, the effective coupling  $\tilde{J}_m = \xi^{(m)} J_m$  is fully determined by Wigner-Eckart's theorem, yielding  $\xi^{(m)} = (n_B + 2)/(n_B + 1)$  [respectively,  $(n_B + 2)/(3n_B)$ ] for odd (even) steps.

The nested structure of the renormalized spins  $\mathfrak{S}_r$  becomes more transparent if it is written in terms of a matrix product state (MPS) of  $\ell$  halos with an additional left boundary spin  $S_0 = 1/2$  (see details in Appendix B)

$$|\mathfrak{S}_\ell\rangle = \sum_{s_0 \dots s_\ell} \Gamma^{s_0} \Gamma^{s_1} \tilde{\Gamma}^{s_1} \dots \tilde{\Gamma}^{s_{\ell-1}} \Gamma^{s_\ell} |s_0 s_1 s_2 \dots s_\ell\rangle, \quad (10)$$

where the bond dimension is given by the virtual spins  $\mathfrak{S}_r$ , alternating therefore between 2 and  $n_B$ . Hence, the MPS is defined by two types of tensors,  $\Gamma^{S_m}$  and  $\tilde{\Gamma}^{S_m}$ , with respective dimensions  $2 \times n_B$  and  $n_B \times 2$ , obtained from the corresponding Clebsch-Gordan coefficients. For all values of  $n_B$ , we observe that pairs of consecutive rings have spin-1, and their effective MPS corresponds to the Haldane phase. The  $n_B = 2$  case is special in that  $\Gamma^{S_m} = \tilde{\Gamma}^{S_m}$  and single rings present the Affleck-Kennedy-Lieb-Tasaki (AKLT) structure [38]. Therefore, the EE between the neighboring rings alternates between

$S(B_{2m}) = 1$  and  $S(B_{2m-1}) = \ln n_B$ , since each halo has total spin  $n_B/2$ . This prediction is numerically checked in Fig. 4(c), where  $S(B_r)$  is shown as a function of  $r$  for different values of  $h$  for a Heisenberg site star with  $n_B = 3$  and  $\ell = 7$ .

#### D. Observables

EE measures are still challenging, even though they are slowly becoming commonplace [39,40]. Thus, it is convenient to provide other experimentally accessible observables in order to check the validity of our theoretical predictions regarding the entanglement halos.

For free-fermionic systems, the variance in the number of particles inside a region is a good proxy measure to its EE [41,42], based on the bound  $S_A \geq 4 \ln 2 \sigma_{N_A}^2$ . This variance is shown in Fig. 5(a) for different ring and site stars, along with the entropy of the corresponding blocks divided by  $4 \ln 2$ , highlighting how tight the bound is in practice.

In the Heisenberg case, the total spin of certain blocks of spins highlights the different phases. As shown in Fig. 5(b), the entanglement halos of ring stars with even  $n_B$  tend to form a singlet with  $S_r^2 = 0$ , whereas in site-star geometry, each halo exhibits maximal spin  $S_r^2 = (n_B/2)$ .

Yet, the hallmark of the Haldane phase is given by the string order parameter (SOP) [43]. The total spin of a consecutive pair of twisted halos,  $\mathbf{S}_{P,k} = \mathbf{S}_{2k-1} + \mathbf{S}_{2k}$ , will be one, independently of  $n_B$ . By measuring the Z components of the central spins and for each pair,  $\{S_{0,0}^z, S_{P,1}^z, S_{P,2}^z, \dots, S_{P,\ell/2}^z\}$ , we observe that it follows a diluted Néel pattern, i.e., a perfect sign alternation when the zeros are removed. An illustration is provided in Fig. 5(c), where we can see a sign assignment for the different spins-1/2 in a  $n_B = 3$  site star within a possible bond configuration and the corresponding  $S_{P,k}^z$  values. We may define the SOP in this system as the probability of only getting diluted-Néel configurations. We may define the SOP

in this system as the probability of measuring a diluted-Néel configuration. Let  $P(S_0^z, S_{P,1}^z, \dots, S_{P,\ell/2}^z)$  be the probability of making a joint measurement which yields all those values for the central spin and the total spin of each ring pair, and let  $\mathcal{F}(S_0^z, S_{P,1}^z, \dots, S_{P,\ell/2}^z) = 1$  if nonzero values alternate sign, and 0 otherwise. Then, we write

$$\text{SOP} = \sum_{\substack{S_0^z \in \pm 1/2, \\ S_{P,k}^z \in \{-1, 0, +1\}}} P(S_0^z, S_{P,1}^z, \dots, S_{P,\ell/2}^z). \quad (11)$$

Figure 5(d) shows  $1 - \text{SOP}$  for site stars with  $n_B = 3$  and  $\ell = 4$  and 6, showing an exponential decay toward zero with  $h$ .

### E. Experimental implementation

Engineering the ring star and site-stars topologies along with the required decay in the couplings poses an interesting experimental challenge. Among the available technologies for quantum simulators, we propose the use of *synthetic dimensions* in ultracold atomic systems [44–46], employing an internal degree of freedom such as, e.g., the hyperfine levels of a suitable atomic species in order to represent the different nodes along a branch, with suitably tailored Raman intensities inducing the required couplings. Thus, each of the  $n_B$  branches can be implemented inside a single atom, which can be strategically placed in an optical lattice in order to build the final ring star or site star. Another promising direction is the coupling of several coplanar waveguide resonators to produce artificial photonic materials in an effective curved space [47], including hyperbolic lattices, as required by our systems, where the inhomogeneity may be seen as a lattice proxy of spacetime curvature [48,49].

### IV. CONCLUSIONS

In this work, we have presented the concept of entanglement halo as the emergence of an entanglement structure among constituents that are distant according to the notion of locality given by their Hamiltonian.

We exemplified the notion using inhomogeneous free fermion and antiferromagnetic Heisenberg spin-1/2 arranged in two different star-graph geometries, highlighting the impact of geometry and its interplay with inhomogeneity.

The most straightforward example of entanglement halos is provided by the single and double rings appearing in ring stars with exponentially decaying hoppings, both for free-fermionic and spins, depending on the number of branches,  $n_B$  as illustrated in Fig. 2. These findings may have implications for quantum communication protocols based on spatially structured entanglement. However, the expected fragility of these halos under decoherence points to the need for further study regarding their robustness and controllability.

For site-star systems, the many-body GS is far from being a product state, and no subsystem is disentangled from the rest. Yet, its structure is transparent in terms of *twisted halos*, in which sites within a ring are (relatively or completely) disentangled among themselves, and strongly entangled with neighboring adjacent rings, as depicted in Fig. 4(b). Crucially,

twisted halos constitute a realization of nonlocal SPT phases. In particular, we observe the emergence of the Haldane phase independently of the number of branches. While this phase has been previously implemented in spin-1/2 ladders [50,51], our work proposes a nonlocal realization.

While genuine multipartite entanglement has been proposed as a diagnostic in topologically ordered quantum spin liquids [26], the entanglement halos discussed here are instead characterized by their geometric organization and effective decoupling, and a detailed multipartite analysis is left for future work.

We suggest an experimental implementation based on the use of synthetic dimensions, within the reach of state-of-the-art ultracold atomic technologies, or artificial photonic lattices, along with relevant proxy observables.

Finally, although the exponential decay of the couplings may seem experimentally unrealistic, it may be possible to find optimal polynomial decay yielding similar phenomena. In any case, research in these many-body systems may lead to insights into the interface of strongly interacting systems, quantum mechanics, and gravity.

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### DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

### APPENDIX A: RING-STAR GEOMETRY

We will first discuss the many-body approach for free-fermionic systems, which can be extended to interacting systems, such as the Heisenberg model, with the necessary changes. We expect the spin-1/2 case to have the same qualitative behavior, but much more involved to treat analytically, since it requires applying perturbation theory to the ground

state of the AF Heisenberg chain. We, however, support our claims with the numerical evidence presented in the main text.

### 1. Nondegenerate ground state

Let us consider the free-fermionic Hamiltonian (1) with couplings given by Eq. (3) on a ring star with  $n_B$  branches. We will require the ground state of the system to be nondegenerate, in order to obtain single halos. This implies, in practice, that  $n_B = 2 \bmod 4$ , in the absence of a central flux. If the inhomogeneity is big enough,  $h \gg 1$ , it is a good approximation to focus on the subsystem composed of central ring, which contains the strongest couplings,

$$H^{(1)} = - \sum_{j=1}^{n_B} t (c_{1,j}^\dagger c_{1,j+1} + \text{H.c.}), \quad (\text{A1})$$

and treat as a perturbation their first neighbors

$$H_2 = - \sum_{j=1}^{n_B} J_1 (c_{1,j}^\dagger c_{2,j} + \text{H.c.}). \quad (\text{A2})$$

The GS of  $H^{(1)}$  is easily obtained making use of the translational invariance of the model and switching to momentum space

$$|\psi_0^1\rangle = \prod_{k \in \Omega_1} d_{1,k}^\dagger |0\rangle, \quad (\text{A3})$$

with

$$d_{1,k} = \sum_{j=1}^{n_B} U_{k,j} c_{1,j} = \frac{1}{\sqrt{n_B}} \sum_{j=1}^{n_B} e^{ijk} c_{1,j}, \quad (\text{A4})$$

and  $\Omega_1 = \{m \in [-n_B/2 + 1, n_B/2] \text{ s.t. } \epsilon_0 = -2t \cos(2m\pi/n_B) < 0\}$ . We introduce the superindex 1 to emphasize that this state belongs to the Hilbert space associated with the level-1 operators  $\{c_{1,j}\}$ . Prior to studying the effect of  $H_2$ , we extend the eigenstates of  $H^{(1)}$ :  $|\psi_k^1\rangle \rightarrow |\psi_k^1, I\rangle$ , where  $|I\rangle$  corresponds to a generic state belonging to the Hilbert space spanned by the level-2 operators  $c_{2,j}$   $j = 1, \dots, n_B$ .

It is easy to check that  $\langle \psi_0^1, I | H_2 | \psi_0^1, I \rangle = 0$ . The matrix element of the degenerate second-order contribution is

$$B_{I,J} = \sum_{i \neq 0} \sum_{I'} \frac{1}{E_0 - E_i} \langle \psi_0^1, I | H_2 | \psi_i^1, I' \rangle \langle \psi_i^1, I' | H_2 | \psi_0^1, J \rangle. \quad (\text{A5})$$

Plugging Eq. (A2) above and expanding yields

$$\begin{aligned} B_{I,J} = & \sum_{i \neq 0} \frac{J_1^2}{E_0 - E_i} \sum_{j,j'=1}^{n_B} \langle I | c_{2,j} c_{2,j'} | J \rangle \langle \psi_0^1 | c_{1,j}^\dagger | \psi_i^1 \rangle \langle \psi_i^1 | c_{1,j'}^\dagger | \psi_0^1 \rangle + \langle I | c_{2,j}^\dagger c_{2,j'} | J \rangle \langle \psi_0^1 | c_{1,j} | \psi_i^1 \rangle \langle \psi_i^1 | c_{1,j'} | \psi_0^1 \rangle \\ & \times \langle I | c_{2,j} c_{2,j'}^\dagger | J \rangle \langle \psi_0^1 | c_{1,j}^\dagger | \psi_i^1 \rangle \langle \psi_i^1 | c_{1,j'} | \psi_0^1 \rangle + \langle I | c_{2,j}^\dagger c_{2,j'}^\dagger | J \rangle \langle \psi_0^1 | c_{1,j} | \psi_i^1 \rangle \langle \psi_i^1 | c_{1,j'}^\dagger | \psi_0^1 \rangle. \end{aligned} \quad (\text{A6})$$

Inverting the relation Eq. (A4) and substituting above, we obtain

$$\begin{aligned} B_{I,J} = & J_1^2 \sum_{i \neq 0} \frac{1}{E_0 - E_i} \sum_{j,j'=1}^{n_B} \sum_{k,k'} \langle I | c_{2,j} c_{2,j'} | J \rangle \langle \psi_0^1 | d_{0,k}^\dagger | \psi_i^1 \rangle \langle \psi_i^1 | d_{0,k'}^\dagger | \psi_0^1 \rangle U_{k,j} U_{k',j'} \\ & + \langle I | c_{2,j}^\dagger c_{2,j'} | J \rangle \langle \psi_0^1 | d_{0,k} | \psi_i^1 \rangle \langle \psi_i^1 | d_{0,k'}^\dagger | \psi_0^1 \rangle U_{k,j} U_{k',j'}^* + \langle I | c_{2,j} c_{2,j'}^\dagger | J \rangle \langle \psi_0^1 | d_{0,k}^\dagger | \psi_i^1 \rangle \langle \psi_i^1 | d_{0,k'} | \psi_0^1 \rangle U_{k,j}^* U_{k',j'} \\ & + \langle I | c_{2,j}^\dagger c_{2,j'}^\dagger | J \rangle \langle \psi_0^1 | d_{0,k} | \psi_i^1 \rangle \langle \psi_i^1 | d_{0,k'} | \psi_0^1 \rangle U_{k,j}^* U_{k',j'}^*. \end{aligned} \quad (\text{A7})$$

Note that the above expression can be nonzero for those excited states that differ in just one fermion with energy  $E_i = E_0 \pm \epsilon_k$ . With this consideration, we obtain the following:

$$B_{I,J} = -J_1^2 \sum_{j,j'=1}^{n_B} \sum_{k,k'} \frac{1}{\epsilon_k} (\langle I | c_{2,j}^\dagger c_{2,j'} | J \rangle U_{k,j} U_{k',j'}^* - \langle I | c_{2,j} c_{2,j'}^\dagger | J \rangle \psi_0^1 U_{k,j}^* U_{k',j'}) \delta(k - k'). \quad (\text{A8})$$

Finally, we particularize this result for the closed chain  $U_{k,j} = e^{ijk}/\sqrt{n_B}$  and  $\epsilon_k = -2t \cos(k)$ :

$$B_{I,J} = \frac{J^2}{2tn_B} \sum_{j,j'=1}^{n_B} \sum_k \left( \frac{e^{ik(j-j')}}{\cos(k)} \langle I | c_{2,j}^\dagger c_{2,j'} | J \rangle + \text{H.c.} \right) + \text{const.}, \quad (\text{A9})$$

where the additional  $1/2$  compensates the double counting in the sum. It can be shown that

$$\sum_{m=-2n}^{2n+1} \frac{e^{i \frac{m\pi}{2n+1} \Delta}}{\cos(\frac{m\pi}{2n+1})} = \begin{cases} 0 & \text{if } \Delta = 2p, \\ (-1)^p n_B, & \text{if } \Delta = 2p + 1. \end{cases} \quad (\text{A10})$$

Thus, we have obtained an effective Hamiltonian that describes a system of free fermions lying exclusively at level-2 with alternating long-range couplings between odd neighbors:

$$H^{(2)} = J^{(2)} \left( \sum_{j=1}^{n_B} \sum_{l=1}^n (-1)^{l+1} c_{2,j}^\dagger c_{2,j+2l-1} + \frac{(-1)^n}{2} c_{2,j}^\dagger c_{2,j+2n+1} + \text{H.c.} \right), \quad J^{(2)} = \frac{J_1^2}{2t}, \quad n_B + j = n_B. \quad (\text{A11})$$

Thus, the level-1 is described by the state  $|\psi_0^1\rangle$  Eq. (A3), and it is disentangled from the rest, yielding a star with  $N - 1$  levels of  $n_B$  fermions each [52].

Observe that given the strength of the couplings, Eq. (3), the effective Hamiltonian obtained above presents the strongest hopping interactions  $|J^{(2)}| = e^{-\frac{3h}{2}}/2$ . Hence, in the next RG step, we focus on this system and its neighbor level-3,

$$H_3 = - \sum_{j=1}^{n_B} J_1 (c_{2,j}^\dagger c_{3,j} + \text{H.c.}). \quad (\text{A12})$$

First, we diagonalize  $H^{(2)}$  by switching to momentum space, Eq. (A4), yielding

$$H^{(2)} = \sum_k \epsilon(k) d_{2,k}^\dagger d_{2,k}, \quad \epsilon(k) = J^{(2)} \sum_{l=1}^n 2 \cos(k(2l-1)) + (-1)^n \cos(k(2n+1)). \quad (\text{A13})$$

Imposing the boundary conditions  $e^{ik(m+N)} = e^{ikm}$ , it can be shown that the single-body energies are

$$\epsilon(k) = J^{(2)} (-1)^n \cos(2kn + k) + 2 \sec(k) \sin^2 \left( kn - \frac{\pi n}{2} \right), \quad k = \frac{m\pi}{2n+1}, \quad m = -2n+1, \dots, 2n+2. \quad (\text{A14})$$

As mentioned previously, the GS is obtained by filling up the negative energies,

$$|\psi_0^2\rangle = \prod_{k \in \Omega_2} d_{2,k}^\dagger |0\rangle, \quad \text{with } \Omega_2 = \{k \text{ s.t. } \epsilon(k) < 0\}. \quad (\text{A15})$$

Notice that  $H^{(2)}$  presents sublattice symmetry, meaning that  $|\psi_0^2\rangle$  describes a state of  $n_B/2 = 2n+1$  fermions. Like before, we work on  $H_3$ , Eq. (A12), using second-order perturbation theory, Eq. (A8). Taking into account that

$$\begin{aligned} & \sum_{m=-2n}^{2n+1} \frac{e^{i \frac{m\pi}{2n+1} \Delta}}{(-1)^n \cos(2kn + k) + 2 \sec(k) \sin^2 \left( kn - \frac{\pi n}{2} \right)} \\ &= \frac{n_B}{2} (\delta(\Delta \pm 1) + \delta(\Delta \pm (n_B - 1))), \end{aligned} \quad (\text{A16})$$

we obtain the effective Hamiltonian

$$H^{(3)} = J^{(3)} \sum_{j=1}^{n_B} t (c_{3,j}^\dagger c_{3,j+1} + \text{H.c.}), \quad J^{(3)} = -\frac{J_3^2}{J^{(2)}}, \quad (\text{A17})$$

that describes a closed chain of first-neighbor couplings lying exclusively at level-3. The level-2 is given by the state  $|\psi_0^2\rangle$ , Eq. (A15), which becomes detached from the star. Observe that because of the strength couplings Eq. (3) of the system, the whole process of the RG is an iteration of the two steps detailed above.

## 2. Degenerate ground state

The degenerate case is more involved. In this section, we will cover the  $n_B = 4$  case for free-fermionic ring stars, although the procedure can be generalized to odd number of branches and other cases.

Let us generalize the SDRG procedure in the following way. Consider a hopping single-body system composed of two types of sites:  $L_0$  (with  $n_0$  sites) and  $L_1$  (with  $n_1$  sites). Sites in

$L_0$  are strongly linked among themselves, with links of order  $J_0$ , while sites in  $L_1$  only possess links to sites in  $L_0$ , of order  $J_1 \ll J_0$ . Let  $P_0$  be the projector on sites of  $L_0$ , and  $P_1$  the projector on sites of  $L_1$ . Our single-body Hamiltonian will be  $H = H_0 + V$ , where  $H_0$  contains the internal links of  $L_0$  and  $V = H_{0,1}$  contains the links joining  $L_0$  and  $L_1$ .

We may obtain the eigenstates of the hopping system as a perturbative series in  $V$ . To zero order, we just consider the eigenstates of  $H_0$ . Due to sublattice symmetry, the negative eigenvalues will be occupied. Let us assume, for the time being, that the only zero modes of  $H_0$  are given by the states located on the sites in  $L_1$ . In that case, which we will call *nondegenerate*, we can write the effective Hamiltonian for  $L_1$  as a perturbation series. To first order, we have  $\tilde{H}_1^{(1)} \approx P_1 V P_1$ , which is zero because  $V$  does not contain links joining sites in  $L_1$ . Thus, we must proceed to the second order,

$$\tilde{H}_1^{(2)} \approx P_1 V P_0 H_0^{-1} P_0 V P_1. \quad (\text{A18})$$

Yet, in this section we will focus on the degenerate case, in which  $H_0$  has more zero modes, i.e.,  $\dim(\text{Ker}(H_0)) > n_1$ . These modes must be considered along with the sites in  $L_1$  in order to write an effective Hamiltonian. Let  $P_Z$  be the projector on all the zero modes of  $H_0$ . Thus, we consider the first-order effective Hamiltonian

$$\tilde{H}_1^{(1)} \approx P_Z V P_Z, \quad (\text{A19})$$

which must be diagonalized. The negative-energy modes of  $\tilde{H}_1^{(1)}$  must be occupied in the GS, and the positive-energy modes must be empty. Then, what happens with the zero modes? Let  $Q_Z$  denote the projector on these remaining zero modes. We must build a second-order perturbation theory effective Hamiltonian

$$\tilde{H}_1^{(2)} \approx Q_Z V (1 - Q_Z) H_0^{-1} (1 - Q_Z) V Q_Z. \quad (\text{A20})$$

Once we have obtained an effective Hamiltonian for  $L_1$ , we may iterate the procedure. Yet, there is an important *caveat*: in some cases, zero modes may travel along the renormalization procedure, and cross the full system.

In order to make the calculations concrete, let  $|u_k\rangle = \sum_i U_{ki}|i\rangle$  be the  $k$ th eigenvector of  $H_0$ , with energy  $\varepsilon_k$ . In the nondegenerate case, we have

$$\langle i|\tilde{H}_1|j\rangle = \sum_k \frac{\langle i|V|u_k\rangle\langle u_k|V|j\rangle}{\varepsilon_k} = \sum_{k,l,m} \frac{U_{kl}\tilde{U}_{km}V_{il}V_{jm}}{\varepsilon_k}, \quad (\text{A21})$$

where  $l$  and  $m$  range over sites of  $L_0$ . The GS on  $L_0$  is built from the negative-energy modes, and the process is then iterated. The final GS is approximately a tensor product of GS on the different levels,

$$|\Psi\rangle \approx |\psi_0\rangle \otimes |\psi_1\rangle \otimes \cdots |\psi_n\rangle. \quad (\text{A22})$$

In the degenerate case, the previous expression need not be correct, since zero modes may hybridize between different levels. In this case, we proceed as follows:

(1) We combine the zero modes of  $L_0$  with the sites of  $L_1$ , and write the first-order perturbation matrix for them,  $\tilde{H}_1^{(1)}$ .

(2) We diagonalize  $\tilde{H}_1^{(1)}$ , and separate the remaining zero modes.

The nonzero modes acquire an energy and are detached from the subspace. They are occupied if their energy is negative.

(3) We obtain the effective second-order Hamiltonian for the zero modes remaining,  $\tilde{H}_1^{(2)}$ , which may combine sites from  $L_0$  and  $L_1$ .

(4) We fill up the negative energy modes.

(5) If any zero mode survives, it must be upgraded to the next level.

Rigorously, the surviving zero modes might disappear at higher orders in perturbation theory. We should upgrade them only if we are sure that they are exact, and that is only true if they are protected by a symmetry.

The aforementioned procedure suggests its use on *hierarchical graphs* with the following conditions. The lattice sites must be decomposed into *levels*,  $\mathcal{G} = L_0 \cup L_1 \cup L_2 \cdots \cup L_n$ , with  $L_k \cap L_l = \emptyset$ , and  $|L_k| = n_k$ . Sites within level  $L_k$  with  $k > 0$  are not linked among themselves, i.e.,  $J_{i,j} = 0$  if  $i, j \in L_k$  and  $k > 0$ . Furthermore, the energies associated with the Hamiltonian  $H_{k,k+1}$  are ordered  $O(J_k)$ , and  $J_0 \gg J_1 \gg J_2 \gg \cdots$ . The ring-centered and site-centered rainbow stars defined above fulfill these conditions.

For example, let us consider the free-fermionic case with  $n_B = 4$ . The spectrum of the single-body Hamiltonian is degenerate,  $H_0$  has two zero modes, given as follows:

$$\begin{aligned} \langle i|u_1\rangle &= \{1/2, 1/2, 1/2, 1/2\}, & \varepsilon_1 &= -J_0, \\ \langle i|u_2\rangle &= \{0, 1/\sqrt{2}, 0, 1/\sqrt{2}\}, & \varepsilon_2 &= 0, \\ \langle i|u_3\rangle &= \{1/\sqrt{2}, 0, 1/\sqrt{2}, 0\}, & \varepsilon_3 &= 0, \\ \langle i|u_4\rangle &= \{1/2, -1/2, 1/2, -1/2\}, & \varepsilon_4 &= +J_0. \end{aligned} \quad (\text{A23})$$

Thus, modes  $|u_2\rangle$  and  $|u_3\rangle$  are *upgraded* to level 1, and we write down the perturbation matrix  $V$  for the six states. Therefore, it contains nonzero elements. We diagonalize it

by realizing that it splits into two  $3 \times 3$  blocks. Four states acquire a first-order correction, hybridizing sites from levels 0 and 1:

$$\begin{aligned} |V_1\rangle &= \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}|1_1\rangle + \frac{1}{2}|3_1\rangle, & E_1 &= -J_1, \\ |V_2\rangle &= \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}|1_1\rangle + \frac{1}{2}|3_1\rangle, & E_2 &= +J_1, \\ |V_3\rangle &= \frac{1}{\sqrt{2}}|u_3\rangle + \frac{1}{2}|2_1\rangle + \frac{1}{2}|4_1\rangle, & E_3 &= -J_1, \\ |V_4\rangle &= \frac{1}{\sqrt{2}}|u_3\rangle + \frac{1}{2}|2_1\rangle + \frac{1}{2}|4_1\rangle, & E_4 &= +J_1. \end{aligned} \quad (\text{A24})$$

Furthermore, there are two surviving zero modes,  $|Z_1\rangle$  and  $|Z_2\rangle$ ,

$$\begin{aligned} |Z_1\rangle &= \frac{1}{\sqrt{2}}(|1_1\rangle - |3_1\rangle), \\ |Z_2\rangle &= \frac{1}{\sqrt{2}}(|2_1\rangle - |4_1\rangle). \end{aligned} \quad (\text{A25})$$

We may then obtain the effective second-order effective Hamiltonian for these two zero modes, summing over all other states, using Eq. (A20) and obtain a single nonzero matrix element,

$$\langle Z_1|\tilde{H}_1^{(2)}|Z_2\rangle = \frac{J_1^2}{2J_0}. \quad (\text{A26})$$

Thus, we prove that  $\tilde{H}_1^{(2)}$  is nondegenerate, and thus the second ring *closes up*. The first two rings,  $L_0$  and  $L_1$ , can be characterized by a single wavefunction, disentangled from the rest of the chain.

## APPENDIX B: SITE-STAR GEOMETRY

Since in the previous section, we considered the free fermion model, we shall analyze here the spin-1/2 system. It is worth noting that a similar single-body RG to the one described in the previous section can be found for the free fermion site star.

Let us consider now a fully SU(2)-symmetric Heisenberg model on site stars with  $n_B$  branches, using the couplings Eq. (3). The most energetic subsystem is described by the Hamiltonian

$$H^{(1)} = J_0 \sum_{i=1}^{n_B} \mathbf{S}_0 \cdot \mathbf{S}_i, \quad (\text{B1})$$

We denote with calligraphic  $\mathcal{S}_1$  the spin obtained by coupling all spins belonging to level 1

$$\mathcal{S}_1 \equiv \sum_{i=1}^{n_B} \mathbf{S}_i, \quad (\text{B2})$$

and the spin  $\mathfrak{S}_1$  obtained by coupling the above with the innermost spin  $\mathbf{S}_0$

$$\mathfrak{S}_1 \equiv \mathbf{S}_0 + \mathcal{S}_1. \quad (\text{B3})$$

Equipped with them, we can rewrite Eq. (B1). Indeed, we can rewrite Eq. (B1) in terms of Casimir operators

$$H^{(1)} = J_0 \mathbf{S}_0 \cdot \left( \sum_{i=1}^{n_B} \mathbf{S}_i \right) = J_0 \mathbf{S}_0 \cdot \mathbf{S}_1 = \frac{J_0}{2} (\mathbf{S}_1^2 - \mathbf{S}_0^2 - \mathbf{S}_1^2), \quad (\text{B4})$$

from which it follows that the ground-state energy is obtained when  $\mathcal{S}_1 = n_B/2$  is maximal and  $\mathfrak{S}_1 = (n_B - 1)/2$  is minimal. The leading perturbation of Eq. (B4) is

$$W_1 = J_1 \sum_{i=1}^{n_B} \mathbf{S}_i \cdot \mathbf{S}_{i+n_B}, \quad (\text{B5})$$

which must be written in terms of the effective spin  $\mathfrak{S}_1$ . Working in the  $z$ -component basis, we can write

$$\langle \mathfrak{S}_1 | \mathbf{S}_i^a | \tilde{\mathfrak{S}}_1 \rangle = \xi_{1,i}^a \langle \mathfrak{S}_1 | \mathbf{S}_1^a | \tilde{\mathfrak{S}}_1 \rangle, \quad (\text{B6})$$

with  $a = x, y, z$ . Since we consider no anisotropy, the coefficient cannot depend on the spin component  $\xi_{1,i}^a = \xi_i$ . Moreover, the site-star geometry is invariant under operations of the symmetric group of  $n_B$  elements. Hence, the coefficient cannot depend on the branch index  $\xi_{1,i} \equiv \xi_1$ . Using the above, we can compute  $\xi_{1,0}^z$  and then use the fact that

$$\xi_{1,0} + \sum_{i=1}^{n_B} \xi_{1,i} = 1 \Rightarrow \xi_1 = \frac{1 - \xi_{1,0}}{n_B}. \quad (\text{B7})$$

Hence, we need to compute

$$\xi_{1,0}^z = \frac{\langle \mathfrak{S}_1 | \mathbf{S}_0^z | \tilde{\mathfrak{S}}_1 \rangle}{\langle \mathfrak{S}_1 | \mathbf{S}_1^z | \tilde{\mathfrak{S}}_1 \rangle} = \frac{1}{\mathfrak{S}_1} \langle \mathfrak{S}_1 | \mathbf{S}_0^z | \mathfrak{S}_1 \rangle. \quad (\text{B8})$$

We can infer from the Wigner-Eckhart theorem that, since  $\mathbf{S}$  is a vector operator, and

$$\begin{aligned} \langle \mathfrak{S}_1, \mathfrak{S}_1 | \mathbf{S}_0^z | \mathfrak{S}_1, \mathfrak{S}_1 \rangle \\ = \langle \mathfrak{S}_1, \mathfrak{S}_1; 1, 0 | \mathfrak{S}_1, \mathfrak{S}_1 \rangle \langle \mathfrak{S}_1 \| \mathbf{S}_0 \| \mathfrak{S}_1 \rangle, \end{aligned} \quad (\text{B9})$$

that  $\langle \mathbf{S}_0^x \rangle = \langle \mathbf{S}_0^y \rangle = 0$ , meaning that  $\mathbf{S}_0 = \alpha \mathfrak{S}_0$ . Then, it follows that

$$\begin{aligned} \alpha &= \frac{\langle \mathfrak{S}_1 \cdot \mathbf{S}_0 \rangle}{\langle \mathfrak{S}_1^2 \rangle} = \frac{\langle \mathbf{S}_0^2 \rangle + \langle \mathbf{S}_1 \cdot \mathbf{S}_0 \rangle}{\langle \mathfrak{S}_1^2 \rangle} \\ &= \frac{\langle \mathfrak{S}_1^2 \rangle + \langle \mathbf{S}_0^2 \rangle - \langle \mathbf{S}_1^2 \rangle}{2\langle \mathfrak{S}_1^2 \rangle} = -\frac{1}{n_B + 1}. \end{aligned} \quad (\text{B10})$$

Now, we can use the fact that  $\langle \mathbf{S}_0^z \rangle = \alpha \langle \mathfrak{S}_1^z \rangle = \mathfrak{S}_1 \alpha$  and write Eq. (B8)  $\xi_0 = -1/(n_B + 1)$  and therefore

$$\xi_{1,i} \equiv \xi_1 = \frac{1}{n_B} \frac{n_B + 2}{n_B + 1}. \quad (\text{B11})$$

Hence, we can write the perturbation Eq. (B5) as

$$\begin{aligned} W_1 &= J_1 \sum_{i=1}^{n_B} \mathbf{S}_i \cdot \mathbf{S}_{i+n_B} = J_1 \xi_1 \mathfrak{S}_1 \cdot \left( \sum_{i=1}^{n_B} \mathbf{S}_{i+n_B} \right) \\ &= \tilde{J}_1 \mathfrak{S}_1 \cdot \mathbf{S}_2 \equiv H^{(2)}, \end{aligned} \quad (\text{B12})$$

recovering the functional form of the unperturbed Hamiltonian Eq. (B1) with a renormalized coupling  $\tilde{J}_1 = J_1 \xi_1$ . When  $\tilde{J}_1 \gg J_2$ , the same reasoning that we presented above applies, and the ground state of Eq. (B12) is obtained by minimizing the total spin  $\mathfrak{S}_2 = \mathfrak{S}_1 + \mathbf{S}_2$ , which yields  $\mathfrak{S}_2 = 1/2$ . The leading perturbation is then

$$H^{(3)} = \tilde{J}_2 \mathfrak{S}_2 \cdot \mathbf{S}_3. \quad (\text{B13})$$

Using the same reasoning above

$$\langle \mathfrak{S}_2 | \mathfrak{S}_1^z | \tilde{\mathfrak{S}}_2 \rangle = \xi_{2,i}^z \langle \mathfrak{S}_2 | \mathfrak{S}_2^z | \tilde{\mathfrak{S}}_2 \rangle, \quad (\text{B14})$$

and using an analogous reasoning  $\mathfrak{S}_1 = \alpha \mathfrak{S}_2$  with

$$\begin{aligned} \alpha &= \frac{\langle \mathfrak{S}_2^2 \rangle + \langle \mathfrak{S}_1^2 \rangle - \langle \mathbf{S}_2^2 \rangle}{2\langle \mathfrak{S}_2^2 \rangle} \\ &= \frac{1}{2} \frac{3 + n_B^2 - 1 - n_B^2 - 2n_B}{3} = -\frac{n_B - 1}{3}, \end{aligned} \quad (\text{B15})$$

with a renormalized coupling  $\tilde{J}_2 = \xi_2 J_2$ . Using Eq. (B6),

$$\xi_{2,i} \equiv \xi_2 = \frac{2 + n_B}{3n_B}. \quad (\text{B16})$$

To gain some intuition, consider the case of  $n_B = 3$  and  $\ell = 1$ . Following the prescription, we impose that the central spins  $\mathbf{S}_1 = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$  combine, producing the maximal spin  $\mathcal{S} = \frac{3}{2}$ . Then, we combine  $\mathbf{S}_0 + \mathbf{S}_1$  to yield the minimal value,  $\mathfrak{S}_1 = 1$ :

$$|1, 1\rangle = \sqrt{\frac{3}{4}} |+++\rangle - \sqrt{\frac{1}{12}} (|+++-\rangle + |++-+\rangle + |+-++\rangle), \quad (\text{B17})$$

$$|1, 0\rangle = \sqrt{\frac{1}{6}} (|-++-\rangle + |-+-+\rangle + |--++\rangle - |++--\rangle - |+-+--\rangle - |-+--+ \rangle), \quad (\text{B18})$$

$$|1, -1\rangle = \sqrt{\frac{1}{12}} (|-++-\rangle + |--++\rangle + |--+-\rangle) - \sqrt{\frac{3}{4}} |--+-\rangle, \quad (\text{B19})$$

where we take the convention  $|\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\rangle$ ,  $\mathbf{S}_i = \pm \frac{1}{2} \equiv \pm$ . Using the above, we can compute  $\xi_1$

$$\xi_{1,0} = \frac{\langle \tilde{m} | \mathbf{S}_0^a | \tilde{n} \rangle}{\langle \tilde{m} | \mathfrak{S}_1^a | \tilde{n} \rangle} = -\frac{1}{4}, \quad a = x, y, z, \quad \tilde{m}, \tilde{n} = 1, 0, -1, \quad (\text{B20})$$

$$\xi_i = \frac{\langle \tilde{m} | \mathbf{S}_i^a | \tilde{n} \rangle}{\langle \tilde{m} | \mathfrak{S}_1^a | \tilde{n} \rangle} = \frac{5}{12}, \quad i = 1, 2, 3. \quad (\text{B21})$$

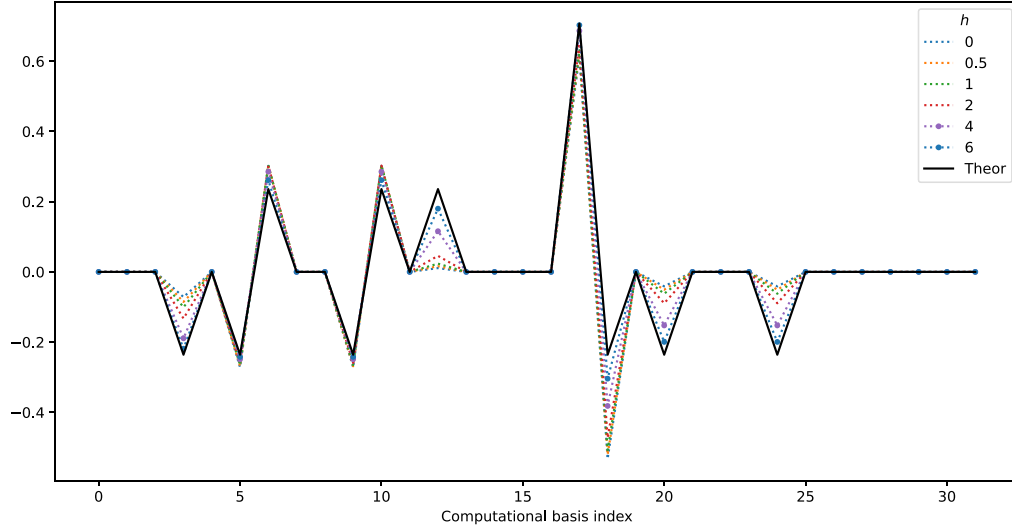


FIG. 6. The black line corresponds to the state Eq. (B22). As the inhomogeneity increases, the GS approaches this state.

Observe that  $\sum_{i=0}^3 \xi_{i,1}^a = 1$   $a = x, y, z$  as a consequence of preserving SU(2) and it is the value given by Eq. (B11). Also, let us add an additional site yielding a total  $\mathfrak{S}_1 + \mathbf{S}_4$  spin-1/2. In Fig. 6, we plot the ground state in the computational basis for different values of the inhomogeneity parameter  $h$  and the RG prediction:

$$|1/2\rangle = \sqrt{\frac{1}{2}}| - + + + - \rangle + \sqrt{\frac{1}{18}}(-| + + + - - \rangle - | + + - + - \rangle - | + - + + - \rangle - | - + + - + \rangle - | - + - + + \rangle - | - - + + + \rangle + | + + - - + \rangle + | + - + - + \rangle + | + - - + + \rangle). \quad (\text{B22})$$

### Tensor network representation

In this section, we will write the GS of the site-star Heisenberg model as an MPS. Given the Matrioshka structure of the described RG procedure, let us start writing the final state in the Z component basis  $|\mathfrak{S}_\ell, \mathfrak{S}_\ell\rangle \equiv |\mathfrak{S}_\ell\rangle$  using closure relations:

$$\begin{aligned} |\mathfrak{S}_\ell\rangle &= \sum_{\mathfrak{S}_{\ell-1}, S_\ell} \langle \mathfrak{S}_{\ell-1}; S_\ell | \mathfrak{S}_\ell \rangle |\mathfrak{S}_{\ell-1}; S_\ell\rangle \\ &\equiv \sum_{\mathfrak{S}_{\ell-1}, S_\ell} C_{\mathfrak{S}_{\ell-1} S_\ell}^{\mathfrak{S}_\ell} |\mathfrak{S}_{\ell-1}; S_\ell\rangle. \end{aligned} \quad (\text{B23})$$

We can iterate this procedure  $\ell - 1$  times:

$$|\mathfrak{S}_\ell\rangle = \sum_{\substack{\mathfrak{S}_1 \dots \mathfrak{S}_{\ell-1} \\ S_1 \dots S_\ell}} C_{\mathfrak{S}_0 S_1}^{\mathfrak{S}_1} C_{\mathfrak{S}_1 S_2}^{\mathfrak{S}_2} \dots C_{\mathfrak{S}_{\ell-1} S_\ell}^{\mathfrak{S}_\ell} |s_0; S_1; \dots; S_\ell\rangle, \quad (\text{B24})$$

where we have rewritten  $\mathfrak{S}_0 \equiv s_0$  as the central physical spin. Observe that the interchange of the indices

$$C_{M_{\mathfrak{S}_{m-1}} M_{S_m}}^{M_{\mathfrak{S}_m}} \longrightarrow \Gamma_{M_{\mathfrak{S}_{m-1}} M_{\mathfrak{S}_m}},$$

allows to see Eq. (B24) as a product of matrices:

$$\begin{aligned} |\mathfrak{S}_\ell\rangle &= \sum_{s_0, S_1, \dots, S_\ell} \sum_{\mathfrak{S}_1, \dots, \mathfrak{S}_{\ell-1}} \Gamma_{s_0 \mathfrak{S}_1}^{S_1} \Gamma_{\mathfrak{S}_1 S_2}^{S_2} \dots \Gamma_{\mathfrak{S}_{\ell-1} S_\ell}^{S_\ell} \\ &\quad \times |s_0, S_1, S_2, \dots, S_\ell\rangle \end{aligned} \quad (\text{B25})$$

$$= \sum_{s_0, S_1, \dots, S_\ell} \Gamma^{s_0} \Gamma^{S_1} \Gamma^{S_2} \dots \Gamma^{S_\ell} |s_0, S_1, S_2, \dots, S_\ell\rangle, \quad (\text{B26})$$

where we have added the boundary vector  $\Gamma_{s_0 \mathfrak{S}_1}^{S_1}$ . Expression (B26) MPS with open boundary conditions consisting of  $\ell + 1$  spins, with the initial spin representing the physical spin at site 0 and the subsequent spins being obtained as maximal composition of the spins lying at the same level  $m = 1, \dots, \ell$ . The bond dimension is given by the renormalized spins and therefore alternates between  $\mathfrak{S} = 1/2$  and  $\mathfrak{S} = (n_B - 1)/2$ . As a consequence, the matrices  $\Gamma$  consist of appropriate Clebsch-Gordan coefficients and are rectangular  $2 \times n_B$ .

To gain insight, let us consider the system of  $n_B = 2$  branches. In this case, the physical spins are  $S = 1$  and the bond dimension is always 2, since  $\mathfrak{S} = 1/2$ . As can be seen in Table I, we obtain the MPS tensors by reading the suitable

TABLE I. Clebsch-Gordan coefficients  $1 \otimes 1/2$ .

|                |        |                                                         |                                                         |
|----------------|--------|---------------------------------------------------------|---------------------------------------------------------|
| $1 \times 1/2$ |        | $\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$              | $\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$  |
| 1              | $1/2$  | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$                  | $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}$    |
|                | $-1/2$ | $\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$ | $\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ |
|                | 0      | $\begin{bmatrix} 2/3 & -1/3 \\ 0 & -1/2 \end{bmatrix}$  | $\begin{bmatrix} 1/3 & 1/3 \\ -1 & 1/2 \end{bmatrix}$   |
|                | $-1/2$ | $\begin{bmatrix} 2/3 & 1/3 \\ -1 & -1/2 \end{bmatrix}$  | $\begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}$             |
|                | $1/2$  | $\begin{bmatrix} 1/3 & -2/3 \\ -1 & -1/2 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$                  |

Clebsch-Gordan coefficients.

$$\mathbf{\Gamma}^1 = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}, \quad \mathbf{\Gamma}^0 = \begin{pmatrix} -\sqrt{\frac{1}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} \end{pmatrix}, \quad \mathbf{\Gamma}^{-1} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}. \quad (\text{B27})$$

That corresponds to the well-known tensors of the AKLT state. Observe, however, the crucial difference: Whereas in this case, the physical and virtual indices of the MPS description align with the physical degrees of freedom of the AKLT Hamiltonian, which describes the dynamics of spin 1 particles, our case is the opposite. The physical indices of the MPS do not appear in the spin-1/2 inhomogeneous Hamiltonian described in the main text. They are indeed virtual spins formed to minimise the energy.

Now, we may consider the star of  $n_B = 3$  branches. In this case, the MPS description involves spins  $S = 3/2$ , and the matrices are rectangular, since we have the alternation  $\mathfrak{S} = 1/2$  and  $\mathfrak{S} = 1$ . Hence, we extract the corresponding tensors of the odd spins from the Clebsch-Gordan coefficients  $1/2 \times 3/2$  (see Table II):

$$\mathbf{\Gamma}^{-3/2} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}, \quad \mathbf{\Gamma}^{-1/2} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \mathbf{\Gamma}^{1/2} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad \mathbf{\Gamma}^{3/2} = \begin{pmatrix} 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{B28})$$

On the other hand, the tensors describing spins at even positions are obtained from the Clebsch-Gordan coefficients  $1 \times 3/2$  (see Table II), taking the form

$$\tilde{\mathbf{\Gamma}}^{-3/2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \tilde{\mathbf{\Gamma}}^{-1/2} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \tilde{\mathbf{\Gamma}}^{1/2} = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{\Gamma}}^{3/2} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{B29})$$

Crucially, notice that the product of two consecutive tensors  $\mathbf{\Gamma}^{S_1} \tilde{\mathbf{\Gamma}}^{S_2}$  yields nonzero  $2 \times 2$  matrices provided  $S_1 + S_2 = \pm 1, 0$ . This suggests that a pair of neighboring spins behaves as a spin 1. To further progress, we compute the transfer matrix of the product

$$T \equiv \sum_{S_1 S_2} (\mathbf{\Gamma}^{S_1} \otimes (\mathbf{\Gamma}^{S_1})^*) (\tilde{\mathbf{\Gamma}}^{S_2} \otimes (\tilde{\mathbf{\Gamma}}^{S_2})^*) = \begin{pmatrix} \frac{7}{12} & 0 & 0 & \frac{5}{12} \\ 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ \frac{5}{12} & 0 & 0 & \frac{7}{12} \end{pmatrix}. \quad (\text{B30})$$

Proceeding like before, we read the coefficients from the Clebsch-Gordan Table II.

TABLE II. Clebsch-Gordan coefficients. Left:  $3/2 \times 1/2$  used for obtaining  $\mathbf{\Gamma}$  tensors. Right:  $3/2 \times 1$  used for obtaining  $\tilde{\mathbf{\Gamma}}$  tensors.

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |
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| $3/2 \times 1/2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |  |  |  |  |  |  |  |  |  | $3/2 \times 1$                                                                                                                |  |  |  |  |  |  |  |  |  | $5/2$                                                                                                                         |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |
| $\begin{array}{ c c c c c c c c c c }\hline & & 2 & & & & & & & \\\hline 3/2 & 1/2 & 1 & & & & & & & \\\hline\end{array}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |  |  |  |  |  |  |  |  |  | $\begin{array}{ c c c c c c c c c c }\hline & & 2 & & 1 & & & & & \\\hline 3/2 & 1/2 & 1 & & 1 & & & & & \\\hline\end{array}$ |  |  |  |  |  |  |  |  |  | $\begin{array}{ c c c c c c c c c c }\hline & & 2 & & 1 & & & & & \\\hline 3/2 & 1/2 & 1 & & 1 & & & & & \\\hline\end{array}$ |  |  |  |  |  |  |  |  |  | $\begin{array}{ c c c c c c c c c c }\hline & & 5/2 & & & & & & & \\\hline 3/2 & 1/2 & 1 & & 1 & & & & & \\\hline\end{array}$ |  |  |  |  |  |  |  |  |  | $\begin{array}{ c c c c c c c c c c }\hline & & 5/2 & & 3/2 & & & & & \\\hline 3/2 & 1/2 & 1 & & 1 & & & & & \\\hline\end{array}$ |  |  |  |  |  |  |  |  |  | $\begin{array}{ c c c c c c c c c c }\hline & & 5/2 & & 3/2 & & & & & \\\hline 3/2 & 1/2 & 1 & & 1 & & & & & \\\hline\end{array}$ |  |  |  |  |  |  |  |  |  | $\begin{array}{ c c c c c c c c c c }\hline & & 5/2 & & 3/2 & & & & & \\\hline 3/2 & 1/2 & 1 & & 1 & & & & & \\\hline\end{array}$ |  |  |  |  |  |  |  |  |  |
| $\begin{array}{ c c c c c c c c c c }\hline & & 3/2 & & -1/2 & & 1/4 & & 3/4 & & 2 & & 1 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & &amp$ |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                               |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |                                                                                                                                   |  |  |  |  |  |  |  |  |  |

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