

Solid-state analog of gravitational redshift: Transport signatures of massless Dirac fermions in tilted Dirac cone heterostructures

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(Received 22 March 2025; accepted 23 June 2025; published 11 August 2025)

A Dirac electron passing through a heterojunction with spatially variable tilt experiences an effective curved spacetime. In this work, we show how this experience is manifested in its conductance. We investigate the propagation of electron waves in a two-dimensional tilted Dirac cone heterostructure where tilt depends on the coordinate z along the junction. The resulting Dirac equation in an emergent curved spacetime for the spinor $\psi(z)$ can be efficiently solved using a fourth-order Runge-Kutta numerical method by a transformation to a suitable spinor φ where the resulting Dirac cone looks locally upright. The spatial texture of the tilt induces oscillatory behaviors in key physical quantities such as the norm $|\varphi(z)|^2$ of the wave function, polar and azimuthal angles $\Theta(z)$ and $\Phi(z)$ of the pseudospin, and the integrated transmission τ where oscillation wavelengths get shorter (longer) in stronger (weaker) tilt regions. Such an oscillatory behavior that is reminiscent of *gigantic gravitational redshift* is an indicator of an underlying spacetime metric that can be probed in tunneling experiments. We derive analytical approximations for the position-dependent wave numbers $\Delta k_z(z)$ that explain the redshift patterns and corroborate it with numerical simulations. For a tilt bump spread over length scale ℓ , upon increasing ℓ , the *amplitude* of redshifted oscillations reduces whereas the number of peaks increases. The scale invariance of the Dirac equation allows us to probe these aspects of ℓ dependence by a voltage sweep in transmission experiments. Smooth variations of the tilt reduces impedance mismatch of the electron waves, thereby giving rise to very high transmission rates. This concept can be used in combination with a sigmoid-shaped tilt texture for a gigantic redshift or blueshift engineering of the transmitted waves, depending on whether the sigmoid is downswing or upswing.

DOI: [10.1103/PhysRevResearch.7.033140](https://doi.org/10.1103/PhysRevResearch.7.033140)

I. INTRODUCTION

Tilted Dirac fermions [1] can arise when two bands arbitrarily cross. They also appear in many two-dimensional systems [2] from organic conductors [3] to the (110) surface of elemental tungsten [4] to 8Pmmn borophene [5].

Tilting as a band structure aspect can drive many interesting properties, such as tunneling valley Hall effect [6], anisotropic optical conductivity [7], plasmonic gains [8], tilt-induced kink in plasmon modes [9,10], hyperbolic plasmon modes [11,12], and anomalous heat flow [13]. While the Dirac point in noninteracting systems is protected by appropriately generalized chiral symmetry [14], upon including many-body interactions, interesting phenomena such as chiral excitonic instability [15] and chiral symmetry breaking [16] may be possible. Also adding disorder and interactions sets interesting competition between the tilt parameter and Coulomb interactions [17]. The tilt effects show up in many measurements

such as thermal difference reflectivity [18], RKKY interaction mediated by tilted Dirac fermions [19], spin transport [20], and in-plane conductivity [21].

While upright Dirac cones represent solid-state realization of an emergent Lorentz invariance which is associated with the corresponding emergent Minkowski spacetime with coordinates $x^\mu = (v_F t, x^1, \dots, x^d)^T$ and the metric $ds^2 = -v_F^2 dt^2 + d\vec{x}^2$, the tilted Dirac fermions are associated with the metric $ds^2 = -v_F^2 dt^2 + (d\vec{x} - \vec{\zeta} v_F dt)^2$ [22] known as Painlevé-Gulstrand spacetime, where in the solid-state case, $\vec{\zeta}$ is determined by the tilting of the Dirac cone. In this way, interesting aspects of curved spacetime are brought into solids possessing a tilting texture $\vec{\zeta}(\vec{x})$ [23–30]. The structure of the above Painlevé-Gulstrand (PG) spacetime is defined by shape of the function ζ in space or time. $|\zeta| < 1$ defines the exterior of the black hole that upon crossing the event horizon at $\zeta = 1$ enters the interior of black hole with $\zeta > 1$ [25–28].

The tilt of the Dirac cone can be manipulated, which will then correspond to the solid-state physics way of controlling the above PG metric. For example, in 8Pmmn borophene, it has been shown that replacing boron atoms arranged on chainlike threads, e.g., by carbon atoms, significantly modifies

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¹Here d is the spatial dimension.

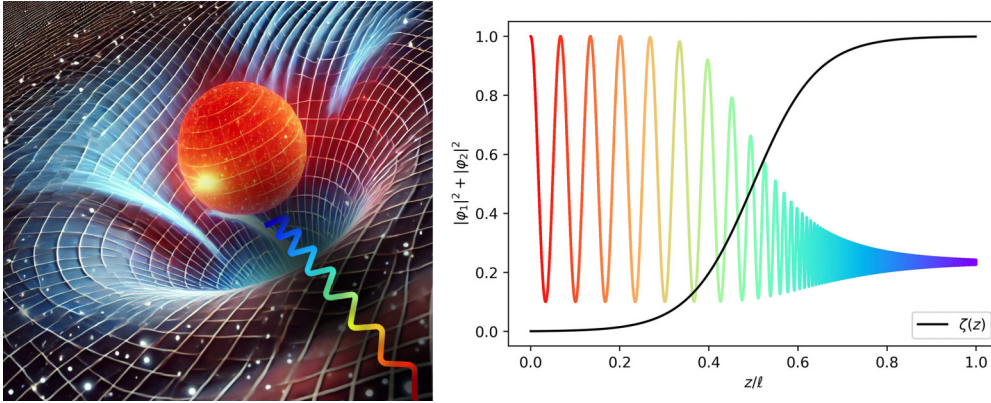


FIG. 1. Left: Exaggerated schematic representation of the gravitational redshift: Although photons do not have mass, as a result of curved spacetime, an emitted photon near a gravitational source loses its kinetic energy $h\nu$ as it overcomes the gravity to move away from stronger gravity regions to weaker gravity regions, thereby looking red in the less gravitating region. The true gravitational redshift is on the scale of $\Delta\nu = 10^{-15}\nu_0$. Right: Probability distribution for an electron crossing a region with a tilting profile varying from $\zeta = 0$ to $\zeta = 0.999$. In the smaller ζ region, the oscillations have longer wavelengths (i.e., redshifted), which can be detected in conductance. The relative redshift in its present solid-state incarnation instead of 10^{-15} can be ~ 1 or even larger.

the tilting [31]. As such, a texture of dopants (random or deterministic) will generate a corresponding curved spacetime that can be random [32] or deterministic. Another class of Dirac cones that can be manipulated by external magnetic influence is spin-orbit coupled Dirac cones, e.g., on the surface of topological insulators [33]. In such systems, a textured magnetic influence such as a domain wall (DW) deposited on the surface of a topological insulator or the (110) surface of tungsten [4] is expected to create spatial variations by tens of percents of tilt [34].

Given the above possibility of synthesizing PG metrics in solids, how do the corresponding gravity-like aspects manifest in spectroscopies? One of the salient predictions of Einstein's general relativity was that the photons (despite being massless), when climbing up a gravitational potential, lose their energy $h\nu$, the value of which is set by the source producing the radiation on the earth surface, thereby getting red-shifted [35] as in Fig. 1. Within Newtonian mechanics, this would make sense only if the photon had mass. For a massless particle, the gravitational redshift can be explained in terms of an underlying curved spacetime. This tiny shift [36] involving fractional frequency changes of the order 10^{-15} was measured in 1959 by Pound and Rebka [37] using high-accuracy measurement enabled by Mössbauer effect.

The purpose of this paper is to theoretically demonstrate a similar effect in a solid-state setup for massless Dirac electrons *without using general relativity's computational tools*. The remarkable aspect of the present solid-state gravitational redshift is that the fractional change in this case can be on the scale of ~ 1 or larger. Such a redshift-like behavior of Dirac electrons in solids can be similarly considered to be one possible indicator of an underlying emergent spacetime geometry. A change of variable from the original Dirac spinor $\psi(z)$ to a convenient spinor $\varphi(z)$ acts like a change of frame of reference from the substrate frame where the transport experiment is being conducted to a locally nontilted frame of reference.

Recent advances in the theoretical interpretation of zero-bias photocurrents in spin-orbit-coupled surface Dirac cones

of topological insulators—attributed to frame transformations [34]—promises exciting prospects for inducing strong position-dependent tilts via magnetic domain walls. The smooth variation of tilt not only minimizes impedance mismatch, leading to nearly perfect transmission, but also unlocks the potential for tilted Dirac cone junctions to act as tunable devices capable of red- or blueshifting electron waves, depending on whether the tilt profile follows an upswing or downswing along the junction. This introduces another concept for wavelength up- or down-conversion in *conducting* media, harnessing abstract gravitational concepts to enable useful applications—distinct from traditional semiconductor-based approaches.

The paper is organized as follows: In Sec. II, we discuss the boundary conditions and the transformation that simplifies the Dirac equation in a variable tilt background. In Sec. III, we formulate the transport coefficients in the transformed spinor basis. In Sec. IV, we present numerical results and our WKB-like approximate analytic treatment. In Sec. V, we compute the transmission coefficient that is accessible in transport measurements. We end the paper in Sec. VI by a summary and outlook.

II. ALTERNATIVE REPRESENTATION OF THE HAMILTONIAN

In this section, we derive an alternative representation of the Hamiltonian that can comfortably give the continuity equation necessary to describe the behavior of massless Dirac fermions in two-dimensional (2D) materials with position-dependent tilting. Let us start with the conventional expression of the Hamiltonian for massless Dirac fermions in a 2D system with a constant tilt in the z direction [34],

$$\begin{aligned}\mathcal{H} &= v_F \sigma_x k_x + (v_F \sigma_z + v_t \sigma_0) k_z \\ &= -i v_F \sigma_x \frac{\partial}{\partial x} - i (v_F \sigma_z + v_t \sigma_0) \frac{\partial}{\partial z},\end{aligned}\quad (1)$$

where v_F and v_t denote the Fermi and tilt velocities, respectively, while σ_x and σ_z represent the well-known Pauli

matrices, and σ_0 is the 2×2 identity matrix. For later convenience, we have rotated the standard representation of the Dirac equation in the xy plan around the x axis to obtain a representation in the xz plane. This form of the Hamiltonian has the advantage that it can be rewritten as

$$\mathcal{H} = -iv_F \left(\sigma_x \frac{\partial}{\partial x} + \mathcal{M} \sigma_z \frac{\partial}{\partial z} \mathcal{M} \right), \quad (2)$$

where

$$\mathcal{M} = \begin{bmatrix} \sqrt{1+\zeta} & 0 \\ 0 & \sqrt{1-\zeta} \end{bmatrix}, \quad (3)$$

and $\zeta = v_t/v_F$ is the tilting parameter that is driven by the coefficient v_t of the unit matrix σ_0 that has been now absorbed into the above matrix $\mathcal{M}$. This transformation is an essential step of the present work that facilitates derivation of analytical expressions, along with plausible numerical solutions of the ensuing wave equations. The above form can now be nicely generalized to a situation with spatially variable ζ . As particles propagate through the medium, the Hamiltonian must be constructed to ensure the corresponding continuity equation is satisfied. With this crucial requirement in mind, and based on the Hamiltonian provided in Eq. (2), it is reasonable to introduce the desired Hamiltonian as

$$\mathcal{H} = -iv_F \left[\sigma_x \frac{\partial}{\partial x} + \mathcal{M}(z) \sigma_z \frac{\partial}{\partial z} \mathcal{M}(z) \right]. \quad (4)$$

It is important to note that, since the tilting parameter ζ is a function of z , the matrix $\mathcal{M}$ also depends on z , and the derivative with respect to z acts on this matrix as well. This equation is similar to the way position-dependent mass in semiconducting heterostructure is handled [38–40], where *ad hoc* generalizations can be possible by allowing the arbitrary powers $(1 \pm \zeta)^\alpha$ with $\alpha = a, b$ for the two $\mathcal{M}$ such that $a + b = 1$ in the spirit of corresponding position-dependent mass problems [40]. Such ambiguity disappears by introducing the $\mathcal{M}$ given in Eq. (3). This can be considered an advantage of the approach based on gravitational analogy. Similar forms of the Hamiltonian as in Eq. (4) based on curved space field theory concepts have been used in Refs. [41,42] for the Dirac equation with a z -dependent Fermi velocity. In fact, Eq. (4) for $\alpha = 1/2$, at the expense of introducing a z -dependent matrix $\mathcal{M}$, appears like a Dirac equation for a locally upright Dirac cone. In that respect, it can be considered a solid-state physicist's way of introducing Einstein's vielbeins (or frame fields) [43]. As we will see in Eq. (26), this transformation allows us to write current densities that in the new (local) basis looks like corresponding expressions for upright Dirac cones. Therefore, this important equation secretly has a frame field content.

Using our representation of the Hamiltonian in Eq. (4), the continuity equation is

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \quad (5)$$

where the probability density, ρ , and the probability current density $\mathbf{J}$ are

$$\begin{aligned} \rho &= \Psi^\dagger \Psi, \\ \mathbf{J} &= v_F [\Psi^\dagger \sigma_x \Psi] \mathbf{e}_x + v_F [\Psi^\dagger (\sigma_z + \zeta(z) \sigma_0) \Psi] \mathbf{e}_z, \end{aligned} \quad (6)$$

where Ψ is the two-component (spinor) wave function, and $\mathbf{e}_x$ and $\mathbf{e}_z$ are unit vectors in the x and z directions, respectively. A detailed derivation of the current operator $\mathbf{J}$ is presented in Appendix A. In the general case, Ψ , ρ , and $\mathbf{J}$ are functions of both position and time. However, for the sake of brevity, this dependence is not explicitly shown in the equations above. The Hamiltonian in Eq. (4), the continuity equation in Eq. (5), and the expressions for ρ and $\mathbf{J}$ in Eqs. (6) are all mutually consistent. Furthermore, the Hamiltonian (4) is Hermitian in the sense that $\langle \Psi | H | \Phi \rangle = \langle \Phi | H | \Psi \rangle$ [44].

The specific case where ζ is constant has been thoroughly studied in our previous work [45]. That study examined the behavior of massless Dirac particles as they cross the boundary between two materials with different tilt parameters, the occurrence of Klein tunneling, and the spin rotation angle of these particles after crossing the interface. In the present work, we extend this investigation to the case where ζ varies as a function of z .

III. TRANSMISSION PROBABILITY AND KLEIN TUNNELING

Using the Hamiltonian and the associated continuity equation for tilted Dirac cones with spatially varying tilt parameters, we can explore the transport in a heterostructure composed schematically, depicted in Fig. 2. In this figure, three regions labeled 1, 3 are separated by a region labeled 2 within which the tilt varies in space. The corresponding tilts, in general, can be ζ_1 , $\zeta_2(z)$, and ζ_3 . The middle layer has a thickness of ℓ , while the outer layers extend infinitely. In the setup we have studied, region 1 is assumed to have no tilt, i.e., $\zeta_1 = 0$ where the Dirac fermions are represented by upright Dirac cones. In region 2, the tilt continuously varies starting from $\zeta_1 = 0$ on its left end and continuously reaches ζ_3 on its right end. The shape of variations can be any continuous function $\zeta_2(z)$ schematically displaced in Fig. 2.

The Schrödinger eigenvalue equation, which governs the behavior of massless Dirac fermions in the various regions of the setup depicted in Fig. 2, is expressed in its general form as

$$\mathcal{H}u(x, z) = Eu(x, z), \quad (7)$$

where $u(x, z)$ represents the eigenfunction of $\mathcal{H}$ corresponding to the energy eigenvalue E . The z axis is considered to be normal to the interface of the media. In this configuration, due to the translation invariance in the x direction, k_x is conserved, and $u(x, z)$ will be of the following form:

$$u(x, z) = \psi(z) e^{ik_x x}. \quad (8)$$

For this class of wave functions, the Schrödinger eigenvalue equation for the middle region simplifies to

$$\mathcal{M}(z) \sigma_z \frac{d}{dz} \mathcal{M}(z) \psi(z) = i \left(\frac{E}{v_F} - k_x \sigma_x \right) \psi(z). \quad (9)$$

Defining the new spinor $\varphi(z)$ by

$$\varphi(z) = \mathcal{M}(z) \psi(z), \quad (10)$$

Eq. (9) becomes

$$\mathcal{M}^2(z) \sigma_z \frac{d}{dz} \varphi(z) = i \left[\frac{E}{v_F} - k_x \mathcal{M}(z) \sigma_x \mathcal{M}^{-1}(z) \right] \varphi(z). \quad (11)$$

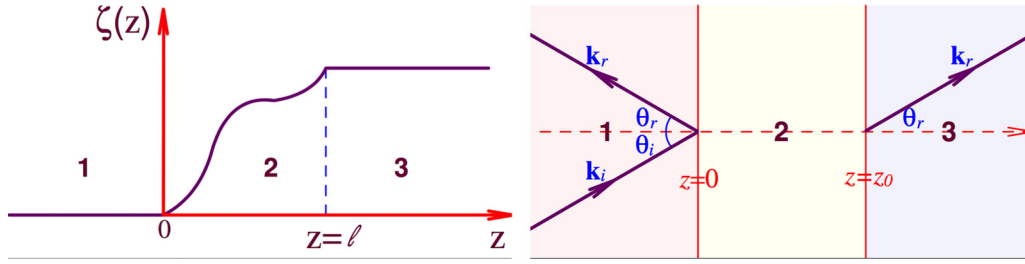


FIG. 2. Left: Schematic illustration of a heterojunction composed of three materials with different tilt parameters. In the outer regions, the tilting parameter ζ is constant, while in the middle layer it varies as an arbitrary function of z . The tilting parameter of the middle layer, $\zeta(z)$, smoothly varies and takes the constant values $\zeta_1 = 0$ and $\zeta_3 \neq 0$ at the boundaries $z = 0$ and $z = \ell$, respectively. Right: Incident, reflected, and transmitted electron waves. Note that the indicated angles belong to wave vectors that in the presence of tilt does not necessarily coincide with the direction of group velocity.

To develop some intuition into the structure of solutions, first consider the normal incidence of massless Dirac fermions from region 1 onto the interface. In this case, the particles travel along the z axis, so $k_x = 0$. Furthermore, a complete scattering state can be considered by setting the spinor part of the wave function $\varphi(z)$ and, consequently, $\psi(z)$ to have only one nonzero component, i.e., their spinor part is proportional to either $[1 \ 0]^T$ or $[0 \ 1]^T$. For example, consider the first case where $\varphi^T(z) = (1 \ 0)\varphi_+(z)$, which corresponds to the boundary conditions

$$\lim_{z \rightarrow -\infty} \zeta(z) = 0 \quad \text{and} \quad \psi_{\text{in}} = \psi(z \rightarrow -\infty) = e^{ik_z z}. \quad (12)$$

Equation (11) reduces to

$$\mathcal{M}_{11}^2(z) \frac{d}{dz} \varphi_+(z) = i \frac{E}{v_F} \varphi_+(z), \quad (13)$$

where $\mathcal{M}_{ij}(z)$ represents the element in row i and column j of the matrix $\mathcal{M}$. The solution to this equation, under the boundary conditions of Eqs. (12), is

$$\varphi_+(z) = e^{i\alpha_+(z)}, \quad (14)$$

where

$$\alpha_+(z) = -\frac{E}{v_F} \int_{-\infty}^z \frac{dz}{\mathcal{M}_{11}^2(z)}.$$

Finally, using Eq. (10), $\psi_+(z)$ is determined as

$$\psi_+(z) = \frac{1}{\mathcal{M}_{11}(z)} e^{i\alpha_+(z)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (15)$$

Equation (15) shows that the z component of the probability current density, J_z , remains constant, signifying the occurrence of Klein tunneling. This is because when the incident wave is such that only one of the components as in Eq. (15) is nonzero, dynamics under the wave equation cannot generate another spinor component. In this scenario, which corresponds to normal incidence at the interface, there is a perfect transmission probability, allowing all particles to traverse the central slab and enter the third medium.

If instead the boundary conditions are

$$\lim_{z \rightarrow +\infty} \zeta(z) = 0 \quad \text{and} \quad \psi_{\text{in}} = \psi(z \rightarrow +\infty) = e^{-ik_z z}, \quad (16)$$

then in the above equations, $\mathcal{M}_{11}(z)$ should be replaced by $\mathcal{M}_{22}(z)$ and the corresponding σ_z eigenspinor must be replaced by $[0 \ 1]^T$.

Next, let us consider a more general case. As illustrated in the right panel of Fig. 2, we assume that massless Dirac fermions travel through medium 1 and strike the interface with the middle slab at an arbitrary angle. Our goal is to calculate the reflection and transmission probabilities for these states as they traverse the intermediate region.

Referring to the right panel of Fig. 2, the wave function of the system in the outer regions is

$$\psi(z) = \begin{cases} \frac{1}{t} \psi_i(z) + \frac{r}{t} \psi_r(z), & z < 0 \\ \psi_t(z), & z > z_0, \end{cases} \quad (17)$$

where

$$\begin{aligned} \psi_i(z) &= \begin{bmatrix} \cos \frac{\theta_i}{2} \\ \sin \frac{\theta_i}{2} \end{bmatrix} e^{ik_{iz}z}, \\ \psi_r(z) &= \begin{bmatrix} -\sin \frac{\theta_r}{2} \\ \cos \frac{\theta_r}{2} \end{bmatrix} e^{-ik_{rz}z}, \\ \psi_t(z) &= \begin{bmatrix} \cos \frac{\theta_t}{2} \\ \sin \frac{\theta_t}{2} \end{bmatrix} e^{ik_{tz}z}, \end{aligned} \quad (18)$$

where subscripts i , r , and t refer to incident, reflected, and transmitted waves, respectively. The coefficients r and t represent the reflection and transmission amplitudes. The angles θ_i , θ_r , and θ_t are termed the angles of incidence, reflection, and transmission, respectively,² as shown in Fig. 2. Correspondingly, the wave vectors associated with the incident, reflected, and transmitted waves are denoted as $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$.

The components of these wave vectors in the z direction (normal to the interface) are k_{iz} , k_{rz} , and k_{tz} , while their components in the x direction are k_{ix} , k_{rx} , and k_{tx} . Due to translational invariance along the x direction, all the x components of the wave vectors are equal and therefore they are all specified with a single value k_x :

$$k_x = k_{ix} = k_{rx} = k_{tx}. \quad (19)$$

It is crucial to note that the value of the z components of the wave vectors adjust themselves to satisfy the boundary

²Note that for the upright Dirac cones, the direction of the wave vectors are the same as the group velocities, whereas in the tilted Dirac cone case they are not the same. We choose to parametrize the angular variables referring to the wave vectors [45].

conditions that ensure continuity equation. The value of k_{tz} depends on the tilting parameter in region 3. Moreover, the energy E is also converted in all regions and in region 1 where there is no tilting, we have

$$E = \sqrt{k_{tz}^2 + k_x^2} = \sqrt{k_{rz}^2 + k_x^2}, \quad (20)$$

allowing us to easily determine k_{tz} and k_{rz} .

To compute k_{tz} , we note that in region 3, the Schrödinger eigenvalue equation from the Hamiltonian in Eq. (1) is

$$(k_{tz}\sigma_z + k_x\sigma_x)\psi_t(z) = (E - \zeta_3 k_{tz})\psi_t(z). \quad (21)$$

The eigenvalues of the operator on the left-hand side are $\pm\sqrt{k_x^2 + k_{tz}^2}$. Since we are considering waves propagating to the right ($k_{tz} > 0$), we choose the positive eigenvalue. Setting this eigenvalue equal to $E - \zeta_3 k_{tz}$, we obtain

$$E = \zeta_3 k_{tz} + \sqrt{k_x^2 + k_{tz}^2}, \quad (22)$$

which is an implicit equation for k_{tz} that can be solved for k_{tz} . With the z components of the incident, reflected, and transmitted wave vectors, along with the known angles of incidence, reflection, and transmission, the wave functions $\psi_i(z)$, $\psi_r(z)$, and $\psi_t(z)$ are uniquely determined.

To fully characterize the wave function of the system outside the central slab, it is crucial to determine the amplitudes r and t defined in Eq. (17). This requires a detailed analysis of the wave function behavior within the central region. By examining this wave function and applying the continuity conditions, the values of r and t can be precisely determined. To achieve this, we utilize Schrödinger's equation for a fixed energy E , as given by Eq. (11) for the central slab region. In this region, the wave function $\varphi(z)$ is represented as a two-component spinor:

$$\varphi(z) = \begin{bmatrix} \varphi_1(z) \\ \varphi_2(z) \end{bmatrix}. \quad (23)$$

In the middle slab ($0 \leq z \leq \ell$), Eq. (11) can be separated into two coupled first-order differential equations governing $\varphi_1(z)$ and $\varphi_2(z)$:

$$\begin{aligned} A(z)\frac{d}{dz}\varphi_1(z) - iB(z)\varphi_1(z) + ik_x\varphi_2(z) &= 0, \\ A(z)\frac{d}{dz}\varphi_2(z) + iC(z)\varphi_2(z) - ik_x\varphi_1(z) &= 0, \end{aligned} \quad (24)$$

where the coefficients $A(z)$, $B(z)$, and $C(z)$ are defined as

$$\begin{aligned} A(z) &= \mathcal{M}_{11}(z)\mathcal{M}_{22}(z), \\ B(z) &= \left(\frac{E}{v_F}\right)\frac{\mathcal{M}_{22}(z)}{\mathcal{M}_{11}(z)}, \\ C(z) &= \left(\frac{E}{v_F}\right)\frac{\mathcal{M}_{11}(z)}{\mathcal{M}_{22}(z)}. \end{aligned} \quad (25)$$

For arbitrary variable functions of $\zeta(z)$ satisfying the appropriate boundary conditions (as illustrated in Fig. 2), these coupled first-order differential equations can be numerically solved to desired accuracy. The boundary conditions are derived from the continuity of the z component of the probability current density at the interfaces. Using the expression for $\mathbf{J}$

from Eqs. (6), we can express the z component of the current density as

$$\begin{aligned} J_z &= v_F \psi^\dagger(z)(\sigma_z + \zeta(z)\sigma_0)\psi(z) \\ &= v_F \psi^\dagger(z)\mathcal{M}(z)\sigma_z\mathcal{M}(z)\psi(z) \\ &= v_F \varphi^\dagger(z)\sigma_z\varphi(z). \end{aligned} \quad (26)$$

As pointed out below Eq. (4), the last line looks like the current density for a locally upright Dirac cone. Therefore the new spinor $\varphi(z)$ defined by Eq. (10) can be interpreted as a spinor in a locally Minkowski spacetime. As such, the matrix $\mathcal{M}(z)$ defined in Eq. (4) encapsulates a transformation to a locally flat Minkowski spacetime.

To proceed with the question of boundary conditions, since $J_z = v_F \varphi^\dagger(z)\sigma_z\varphi(z)$, it follows that $\sqrt{v_F}\varphi(z)$ is continuous across all regions, including at the interfaces. In particular, at the interface between regions 2 and 3 ($z = \ell$), this continuity implies

$$\sqrt{v_{2F}} \begin{bmatrix} \varphi_1(\ell) \\ \varphi_2(\ell) \end{bmatrix} = \sqrt{v_{3F}} \begin{bmatrix} \cos(\theta_t/2) \\ \sin(\theta_t/2) \end{bmatrix}, \quad (27)$$

where v_{2F} and v_{3F} are the Fermi velocities in regions 2 and 3, respectively. This boundary condition is essential to obtain unique numerical solutions $\varphi_1(z)$ and $\varphi_2(z)$ of the coupled differential equations (24). In this work, we will assume that all regions have the same Fermi velocity v_F .

By applying the continuity condition at the boundary $z = 0$, the following expression is derived:

$$\frac{r}{t} \begin{bmatrix} \sin(\frac{\theta_i}{2}) \\ \cos(\frac{\theta_i}{2}) \end{bmatrix} + \frac{1}{t} \begin{bmatrix} \cos(\frac{\theta_i}{2}) \\ \sin(\frac{\theta_i}{2}) \end{bmatrix} = \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \end{bmatrix}. \quad (28)$$

This equation consists of two coupled algebraic equations, the solution of which readily gives the amplitudes of reflection (r) and transmission (t):

$$\begin{aligned} \frac{1}{t} &= \frac{\varphi_1(0)\cos(\frac{\theta_i}{2}) - \varphi_2(0)\sin(\frac{\theta_i}{2})}{\cos(\theta_i)}, \\ \frac{r}{t} &= \frac{\varphi_1(0)\sin(\frac{\theta_i}{2}) - \varphi_2(0)\cos(\frac{\theta_i}{2})}{\cos(\theta_i)}. \end{aligned} \quad (29)$$

With these amplitudes, the reflection and transmission probabilities R and T can be readily computed:

$$\begin{aligned} R &= \frac{J_r}{J_{\text{in}}} = |r|^2, \\ T &= \frac{J_t}{J_{\text{in}}} = |t|^2 \left(\frac{\varphi_1^2(0) - \varphi_2^2(0)}{\cos^2(\theta_i)} \right), \end{aligned} \quad (30)$$

Needless to say, they satisfy $R + T = 1$.

IV. RESULTS AND DISCUSSION

In this section, we perform a detailed analysis of the solution to the set of differential equations in Eqs. (24). We begin by examining a heterojunction composed of three regions where regions 1 and 3 have two uniform tilts as shown in Fig. 3. The middle layer 2 has a position-dependent tilt. In our work, we have considered two distinct tilting textures ζ_b

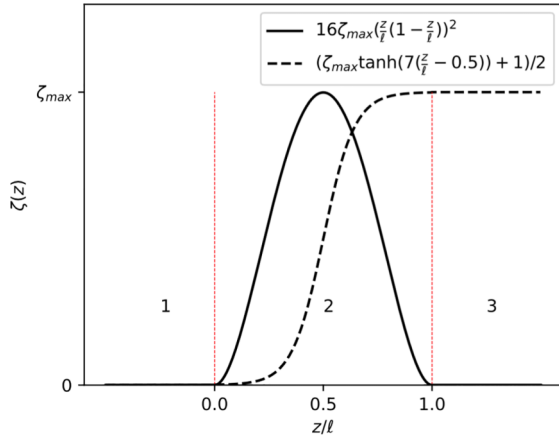


FIG. 3. Two distinct tilting functions, $\zeta_b(z)$ (solid lines) and $\zeta_s(z)$ (dashed lines), describing a bump- and sigmoid-shaped profile of tilting in the middle layer of the heterojunction used in this study. The functional forms are indicated in the figure and Eqs. (31) and (32) represent situations where stronger tilt is concentrated in the middle and right sides of the junction, respectively.

and ζ_s for bump- and sigmoid-shaped functions given by

$$\zeta_b(z) = 16\zeta_{\max} \left(\frac{z}{\ell} \left(1 - \frac{z}{\ell} \right) \right)^2, \quad (31)$$

where $\zeta_{\max}$ is a parameter that we take at most to be 1^- , and ℓ represents the length of the middle layer, measured in units of $\lambda_E = \hbar v_F / E$. For a typical $v_F \sim 10^6$ m/s in Dirac materials, at energy of $E = 1$ eV, one obtains $\lambda = 6.56$ Å. So, at 1 eV energy, a typical value of $\ell = 50$ in this unit corresponds to 32.8 nm. Note that due to linear energy-momentum scaling, fixing the energy E fixes this natural unit λ_E for the length. Therefore, the length ℓ and energy E are not independent variables. Studying the problem at higher (lower) E or smaller (larger) λ_E means that a fixed value, e.g., $\ell = 50$ in units of λ_E will correspond to longer (shorter) physical lengths for the variable-tilt region. For the sigmoid-shaped tilting profile, we choose

$$\zeta_s(z) = \frac{\zeta_{\max} \tanh(7(\frac{z}{\ell} - 0.5)) + 1}{2}, \quad (32)$$

which introduces an alternative tilting profile. Both functions are displayed in Fig. 3 where $\zeta_b(z)$ is shown by solid lines and $\zeta_s(z)$ by dashed line.

Next, we solve the set of Eqs. (24) by considering an incoming electron with a given wave vector $\mathbf{k}_i = (k_{ix}, k_{iz}) = (\cos(\theta_i), \sin(\theta_i))$. It is clear that $\theta_i = \theta_t$, and therefore $[\varphi_1^{(\ell)}, \varphi_2^{(\ell)}]$ will be determined according to Eq. (27). This provides the boundary conditions for the equations at $z = \ell$ and allows us to numerically find the solutions of these equations.

Due to the complex structure of the spinors $\varphi_1(z)$ and $\varphi_2(z)$, the final solutions of the system are parameterized as

$$\begin{bmatrix} \varphi_1(z) \\ \varphi_2(z) \end{bmatrix} = |\varphi(z)| \begin{bmatrix} \cos(\Theta(z)/2) \\ \sin(\Theta(z)/2)e^{i\Phi(z)} \end{bmatrix}, \quad (33)$$

where $|\varphi(z)|^2 = |\varphi_1(z)|^2 + |\varphi_2(z)|^2$, where $\Theta(z) \in [0, \pi)$ and $\Phi(z) \in [0, 2\pi)$ are the usual spherical angles which for a

spinor are given by

$$\Theta(z) = \tan^{-1} \left(\frac{|\varphi_2(z)|}{|\varphi_1(z)|} \right),$$

$$\Phi(z) = \tan^{-1} \left(\frac{\text{Im}(\varphi_2)}{\text{Re}(\varphi_2)} \right) - \tan^{-1} \left(\frac{\text{Im}(\varphi_1)}{\text{Re}(\varphi_1)} \right).$$

The angles Θ and Φ can be regarded as the polar and azimuthal angles of the spinors in locally flat Minkowski spacetime. In the following, we present numerical results for $|\varphi(z)|$, $\Theta(z)$, and $\Phi(z)$ obtained using the fourth-order Runge-Kutta method.

We choose the parameter of the incident wave $\theta_i = \frac{\pi}{2} - 0.1$ and take the length of the middle layer to be $\ell = 50$ (in units of λ_E). The corresponding numerical solutions are presented in Fig. 4. The first row of panels, Figs. 4(a)–4(c), displays $|\varphi(z)|$, $\Theta_\varphi(z)$, and $\Phi_\varphi(z)$ as functions of z/ℓ for four different values of the tilt heights: $\zeta_{\max} = 0.1, 0.5, 0.9, 0.999$, which specifies the height of the bump. As can be seen upon increasing the magnitude of the bump, more oscillations are induced to all these quantities as a function of the dimensionless length z/ℓ . Another salient feature of all the figures of the top row that represent the spinor components and the spherical angles as a function of position is that for large enough $\zeta_{\max}$ where enough oscillations are formed, the wavelengths of the oscillations are smaller in the center of the bump whereas the wavelengths are larger (i.e., redshifted) away from the bump. Note that the normalization of φ is such that the continuity equation holds. The physically observable transmission coefficients are calculated based on the ratios of the amplitudes at the beginning and end of the middle region, and this normalization does not pose any issues.

As pointed out, the spinor φ represents the wave function in a local frame. It is useful to transform it back to the original frame (that can be dubbed a substrate frame to emphasize its solid state context) by the defining relation $\psi = \mathcal{M}^{-1}\varphi$. The results are shown in the second row as Figs. 4(d)–4(e). For clarity, we use the notations Θ_φ and Φ_φ for the spherical angles of φ , and Θ_ψ and Φ_ψ for those of ψ .

Notably, in both frames the oscillations are faster in the regions where the local value of ζ is larger, which is reminiscent of a gravitational redshift pattern. As can be seen, this behavior manifests itself in both the local frame (φ) and substrate frame (ψ). In particular, the oscillation patterns of the azimuthal angles Φ_ψ and Φ_φ are quite similar. Note that the azimuthal angles are with respect to the z axis that also denotes the direction of transport. Therefore, the azimuthal angles both represent transverse precession in the local frame and the substrate frames.

The behavior of the polar angle that represents the orientation of the spinor with respect to the motion direction is, however, quite different, which can be seen from the comparison of panels (c) and (f). Let us begin by noting the similarities in the wavelength of the oscillations: both panels clearly share the redshift behavior. In the substrate frame, as can be seen in Fig. 4(f), by increasing the bump strength $\zeta_{\max}$ in the center of the middle region where ζ can reach its $\zeta_{\max}$ value, the average value around which the oscillations is taking place is lifted upward. For $\zeta_{\max} = 0.999$ (blue curve) that is very close to the horizon value of $\zeta = 1$, this enhancement of the Θ

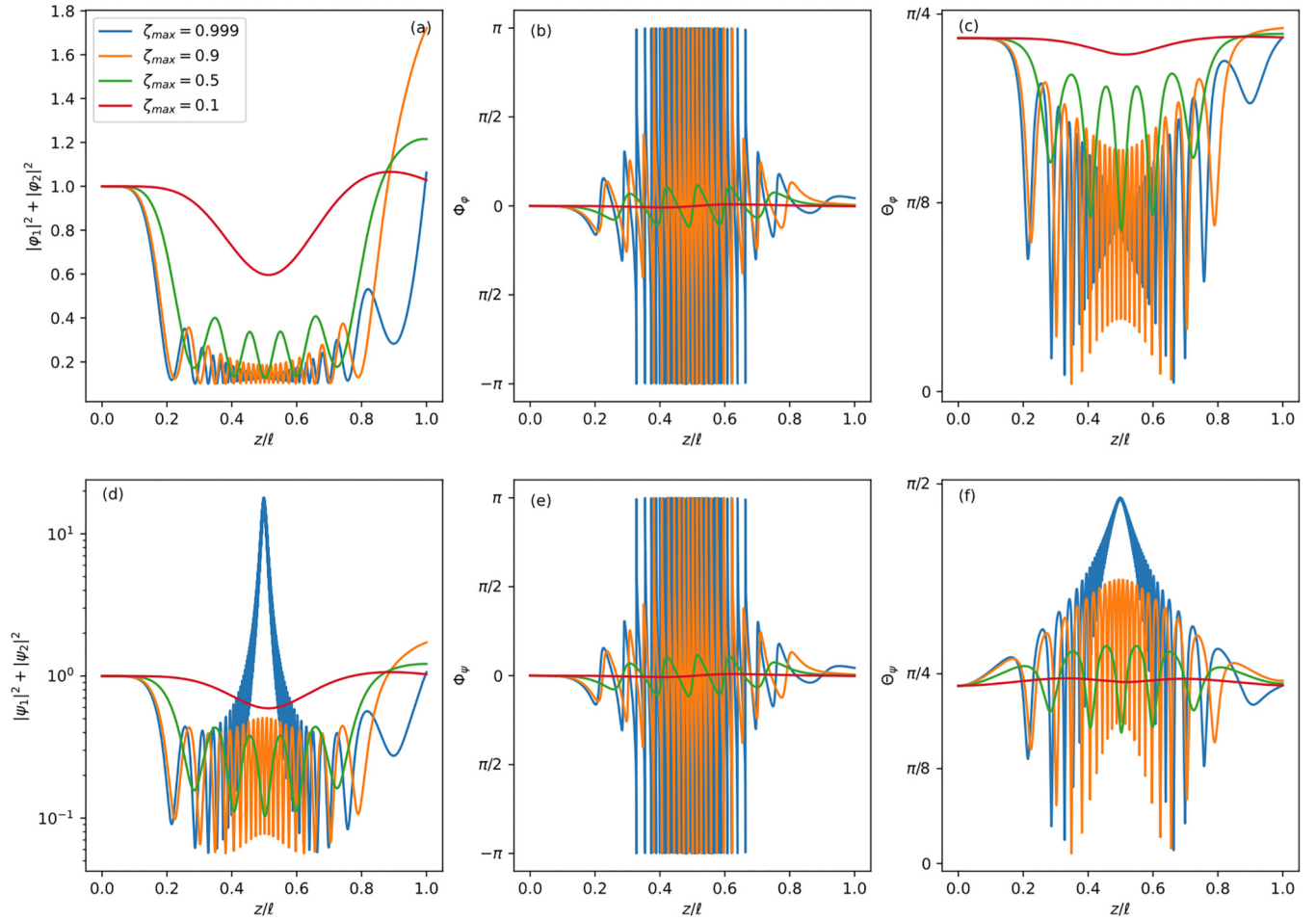


FIG. 4. Numerical solutions of spinor amplitude, polar angle, and azimuthal angle in local frame (first row) and substrate frame (second row) as functions of z/ℓ for $\theta_i = \frac{\pi}{2} - 0.1$, $\ell = 50$, $E = 1 \times \hbar v_F/\ell$, and various tilt parameters $\zeta_{\max} = 0.1, 0.5, 0.9, 0.999$. $|\varphi(z)|$, $\Theta_\varphi(z)$, and $\Phi_\varphi(z)$ are given in (a)–(c). $|\psi(z)|$, $\Theta_\psi(z)$, and $\Phi_\psi(z)$ are given in (d)–(f). Increasing $\zeta_{\max}$ leads to an enhanced oscillation frequency across all panels, reminiscent of gravitational redshift. The energy $E = 1$ sets the natural unit of energy and also the unit of length $\lambda_E = \hbar v_F/E$.

and its tendency to the equator value of $\pi/2$ is more manifest in Fig. 4(d). This means that in the substrate frame, first the horizon is indicated as locking the polar angle to the equatorial value, and, second, the nodding amplitude is suppressed more and more. The suppression of the nodding amplitude is also seen in Fig. 4(c), which represents the polar angle in the local frame. However, in this frame the saturation value of the polar angle is $\approx \pi/8$, which is much smaller than the equatorial value.

A comparison can also be made between Figs. 4(a) and 4(d), which represents the total probability constructed from both components of the spinors in local and substrate frames, respectively. Again, as can be seen, both frames clearly display the redshift pattern in their wavelengths. In the local frame, as can be seen in Fig. 4(a), the average value around which the oscillations are taking place approaches a fixed value at the center where ζ has its maximum value. The approach to horizon is remarkably reflected in the behavior of the probability $|\psi|^2$ in Fig. 4(d) as a strong peak [note the logarithmic scale in Fig. 4(d)]. Comparing Fig. 4(d) against Fig. 4(a), it can be seen that the approach to the critical value $\zeta = 1$ can be diagnosed in the substrate coordinate in which the solid-state measurements are done. The local frame that

moves along with the electron wave is only a mathematically convenient frame for the computation.

Combining the information from Θ and the probability if one has a local probe of pseudospin, the enhancement of the local ζ can be inferred from both the nodding amplitude as well as the strength of the signal that will be directly controlled by the square of the wave function.

Analytical result for the position-dependent oscillations

Can one understand the above numerical results analytically? It turns out that the answer is yes, and a very illuminating physical picture emerges. To address the physical origin of position-dependent or, equivalently, $\zeta(z)$ -dependent oscillations, we focus on a small enough interval Δz of z where the function $\zeta(z)$ can be approximated as constant. Within this interval, the Schrödinger equation (9) for φ , derived from Eq. (23), takes the form

$$i\mathcal{M}^2\sigma_z\frac{d}{dz}\varphi(z) = (-E\sigma_0 + k_x\mathcal{M}\sigma_x\mathcal{M}^{-1})\varphi(z), \quad (34)$$

where σ_0 is the 2×2 identity matrix. Now, in the spirit of WKB approximation, we seek solutions of the form

$\varphi(z) \propto e^{-ik_z z}$, where $k_z(z)$ depends on z and hence the local value of ζ . Substituting this ansatz into Eq. (34), we obtain two eigenvalues for k_z , denoted by k_z^+ and k_z^- , which are given by

$$k_z^\pm = \frac{-\zeta E \pm \sqrt{E^2 - (1 - \zeta^2)k_x^2}}{1 - \zeta^2}. \quad (35)$$

By transforming the Pauli matrix to the local frame, namely, $\sigma'_{x,z} = \mathcal{M}\sigma_{x,z}\mathcal{M}^{-1}$, we have $\sigma'_z = \sigma_z$, $\{\sigma'_x, \sigma'_z\} = 0$, and $(\sigma'_x)^2 = 1$. As a result, the eigenvalues k_z^+ and k_z^- in Eq. (34) are identical to those of

$$i\mathcal{M}\sigma_z \frac{d}{dz}\psi(z) = (-E\sigma_0 + k_x\sigma_x)\psi(z), \quad (36)$$

where the spinors in the two frames are related by $\varphi = \mathcal{M}\psi$.

Let the eigenfunctions of Eq. (36) be ψ^+ and ψ^- and those of Eq. (34) be φ^+ and φ^- . The relationship between these eigenfunctions is given by

$$\langle \varphi^+ | \varphi^- \rangle = 2\zeta(\psi_1^+)^* \psi_2^-, \quad (37)$$

where the subscripts 1, 2 on the right-hand side indicate the first and second spinor *components*. This relation shows that the eigenfunctions φ^+ and φ^- in the local frame are not orthogonal and the amount of nonorthogonality is determined controlled by the amount of tilt parameter ζ vanishing for $\zeta \rightarrow 0$.

If the initial state of the system is chosen as a linear combination of the eigenbasis, i.e., $\varphi(z=0) = \alpha_+\varphi^+ + \alpha_-\varphi^-$, the norm of $\varphi(z)$ varies with z as

$$|\varphi(z)|^2 = 2\zeta\alpha_+\alpha_- \sin(\Delta k_z z), \quad (38)$$

where $\Delta k_z = k_z^+ - k_z^-$. This result demonstrates that $|\varphi(z)|^2$ oscillates periodically as a function of z , and for a sufficiently large length, it may even return to its initial value. This analysis even allows us to extrapolate to neighboring point $z + \Delta z$. The solution in an adjacent interval can be expressed as

$$\varphi(z + \Delta z) = e^{i\Lambda(z)\Delta z}\varphi(z), \quad (39)$$

where $\Lambda(z) = k^+|\varphi^+\rangle\langle\varphi^+| + k^-|\varphi^-\rangle\langle\varphi^-|$. It is evident that even small changes in z alter the norm of φ . Furthermore, it can be shown that the expectation value of σ_z is conserved, i.e., $\langle \varphi(z) | \sigma'_z | \varphi(z) \rangle = \text{const}$, since $\frac{\partial}{\partial z} \langle \varphi(z) | \sigma'_z | \varphi(z) \rangle = 0$. This result is consistent with the earlier discussion on current density conservation. Algebraically it derives from the fact that since both σ_z and $\mathcal{M}$ are diagonal matrices, the Pauli matrix σ_z in both substrate frame and local frame is the same. While the norm of φ oscillates with z , it can also acquire a phase factor due to precession around the z -axis.

To corroborate the above analytical result with numerical computations, in Fig. 5 for a middle layer with $\ell = 200$ we examine the variation in $|\varphi_1|^2 + |\varphi_2|^2$ for the two tilting functions ζ_b [top panel, Fig. 5(a)] and ζ_s [bottom panel, Fig. 5(b)] defined in Eqs. (31) and (32) and for the incident wave with $\theta_i = \pi/2 - 0.1$. The oscillation peaks in Fig. 5 are marked by orange crosses. By locating the positions of the adjacent peaks, we determined the wavelength $\lambda(z)$ from which the wave number $\Delta k_z(z)$ is obtained as $2\pi/\lambda(z)$. In the insets, the wave numbers for both ζ_b and ζ_s are compared with the

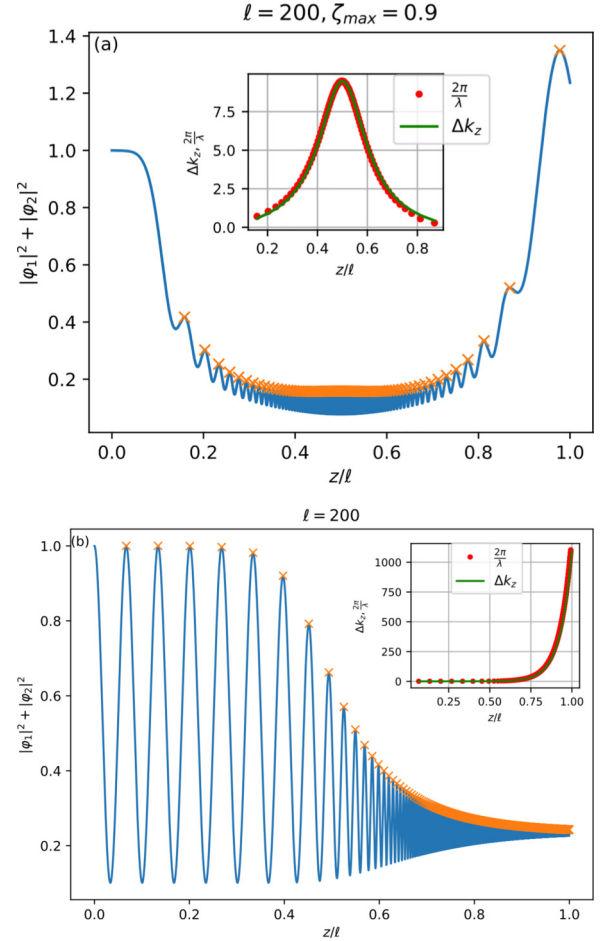


FIG. 5. Variation of oscillation wavelengths for tilting functions $\zeta_b(z)$ and $\zeta_s(z)$. The top panel (a) shows the oscillations of $|\varphi_1|^2 + |\varphi_2|^2$ for bump profile ζ_b as a function of z/ℓ with $\ell = 200$, where the peaks of the oscillations are marked by crosses ($\times$). The bottom panel (b) presents the same analysis for the sigmoid profile ζ_s . Insets in both panels show the calculated wave number $\Delta k_z(z)$ (red dots), derived from the peak separations, alongside the analytic expression for $\Delta k_z(z)$ (green curve), demonstrating excellent agreement between the numerical and analytical results. The value of k_x is such that it corresponds to $\theta_i = \pi/2 - 0.1$. Energy is $E = 1$ and $\zeta_{\max} = 0.9$.

following analytic expression $\Delta k_z(z)$:

$$\Delta k_z(z) = \frac{2\sqrt{1 - (1 - \zeta^2(z))\sin^2(\theta_i)}}{1 - \zeta^2(z)}. \quad (40)$$

As can be seen, very good agreement between the analytic wave numbers and numerical results for both tilting functions can be achieved. Figure 5(b) further indicates that the redshift behavior is not limited to the bumped tilt texture. Such a redshift behavior can be considered a probe of the local value of ζ . Our analysis confirms that similar wave numbers can be obtained for other quantities, such as Φ and Θ .

Before concluding this section, it is worth addressing why for bump profile ζ_b , which is symmetric around $z/\ell = 0.5$, the solutions are not symmetric; specifically, why are the solutions at the boundaries $z/\ell = 0$ and $z/\ell = 1$ not identical, even though layers 1 and 3 are equivalent? The

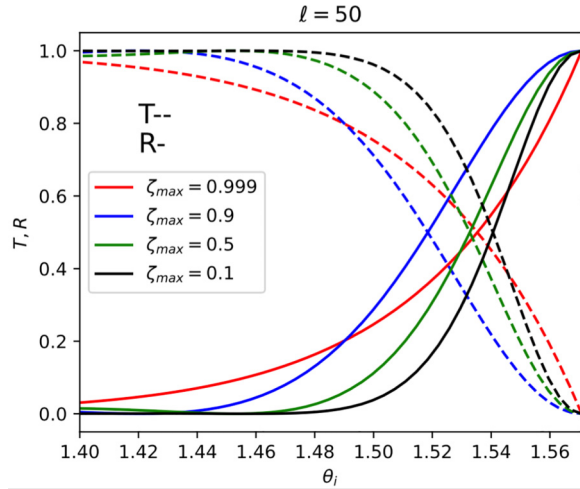


FIG. 6. Transmission (T) and reflection (R) coefficients as functions of the angle θ_i for a fixed middle layer length $\ell = 50$ for the tilting bump ζ_b profile for values of the tilting bump height parameter $\zeta_{\max} = 0.1, 0.5, 0.9, 0.999$ as indicated. Energy is $E = 1 \times \hbar v_F / \ell$.

asymmetry arises from the reflected waves: while the solutions of the equations in the set (24) are periodic, there is always a reflected wave, and not all of the incident waves are transmitted. This partial reflection leads to an asymmetrical profile in the solution. Figure 6 illustrates the transmission (T) and reflection (R) coefficients, calculated using Eqs. (30), for a fixed $\ell = 50$ and different values of $\zeta_{\max}$, as functions of θ_i in an interval close to $\pi/2$. Furthermore, as pointed out, the spinor φ corresponds to a local frame. This local frame at every point can be considered an upright Dirac cone moving with velocity ζv_F [34]. As such, one moving direction is preferred and hence the two directions along and opposite to moving velocity $v_F \hat{z}$ are not equivalent, although the profile of *magnitude* ζ looks symmetric.

V. TRANSPORT MEASUREMENT OF RED-SHIFT PATTERN

To suggest a tangible experimental trace of the above redshift features, in Fig. 7 we have computed the integrated transmission τ that quantifies the fraction of the incident wave transmitted through the tilted region by

$$\tau = \frac{2}{\pi} \int_0^{\pi/2} T(\theta_i) d\theta_i,$$

where $T(\theta_i)$ is the angle-dependent transmission coefficient. For this analysis, we evaluate τ at various positions z/ℓ within the tilted region, representing the fraction of the length that the wave has traversed.

Figure 7 shows τ as a function of z/ℓ for the tilting function $\zeta(z)$. As expected, the redshift behavior obtained for a fixed value of the incident wave parameter θ_i persists after integrating over θ_i . Again, the left-right asymmetry continues to hold in τ , illustrating the complex interplay between wave reflection and transmission within the tilted region. These oscillations arise from the interference of waves as they propagate and reflect, and they persist even though the external layers are equivalent. The inset of Fig. 7 highlights

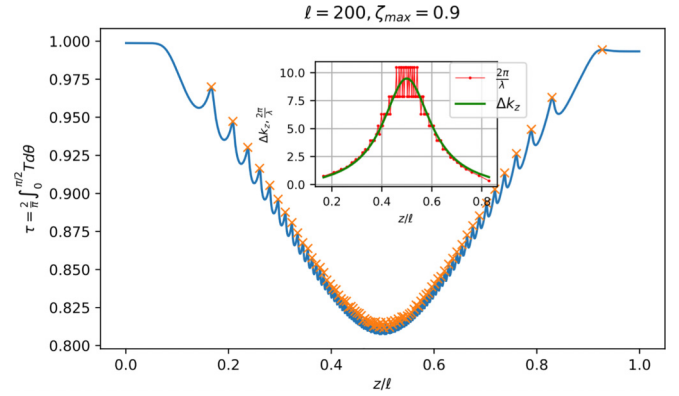


FIG. 7. Integrated transmission τ as a function of z/ℓ representing the fraction of the tilted region traversed by the incident wave. The oscillations in τ reflect interference effects caused by wave propagation through the tilted region. The inset shows the wave number $2\pi/\lambda$ extracted from τ , which coincides with Δk_z , demonstrating consistent agreement with the wavenumber derived analytically (see Fig. 5). Energy is $E = 1$ and $\ell = 200$ in units of $\hbar v_F / E$.

the corresponding wave number, extracted from the oscillatory behavior of τ . Interestingly, the wave number matches perfectly with Δk_z , as previously discussed and shown in the inset of Fig. 5. But here we are dealing with the θ_i -integrated version of the numerical and analytical results. This agreement confirms the consistency of the observed oscillatory behavior across different physical quantities, further validating our analytical predictions regarding Δk_z .

The above results are obtained for a fixed value of the energy $\ell = 200$. For the fixed $E = 1$ that also fixes the unit of length $\hbar v_F / E$, it means that the physical length of the

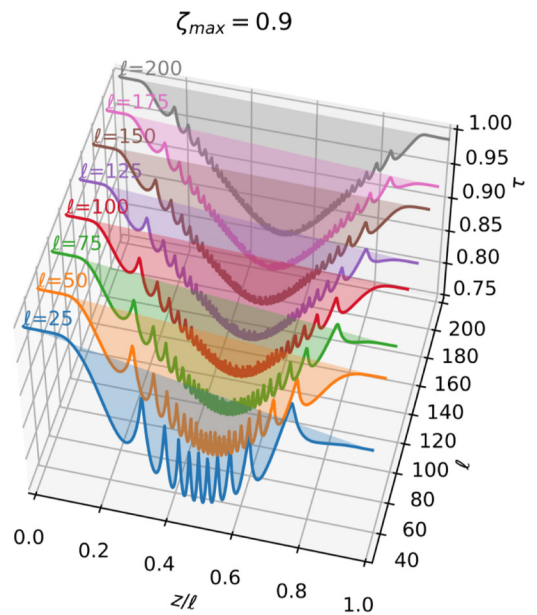


FIG. 8. Intensity plot of the conductance τ as a function of z/ℓ and energy E (in units of $\hbar v_F / \ell$). The plot reveals the oscillatory behavior of τ , arising from interference effects during wave propagation through the tilted region. This plot corresponds to the bump profile with $\zeta_{\max} = 0.9$.

TABLE I. Number of conduction peaks n_p versus the length ℓ of the junction in Fig. 8.

n_p	10	20	30	40	50	60	70	79
ℓ	25	50	75	100	125	150	175	200

middle layer hosting the variable tilt medium is $200\hbar v_F/E$. For example, when $E = 1$ eV and $v_F \sim 10^6$ m/s, $\ell = 200$ would correspond to 131 nm. One could ask what happens when the physical junction length becomes thinner. This computation has been plotted in Fig. 8 where the integrated transmission has been plotted for smaller length scales down to nearly an order of magnitude smaller length scale of $\ell = 25$ that at $E = 1$ eV corresponds to 16.3 nm. The following points can be observed in the figures:

(1) For all the above length scales, the redshift pattern is manifest, i.e., by moving away from the bump, the wave length of the oscillations increases.

(2) For all junction lengths, the transmission at the end of junction is around ~ 0.98 . This should be contrasted with a situation where ζ abruptly changes across the junction [45] where, as can be seen in Fig. 7 of Ref. [45], for a constant tilt of $\zeta = 0.9$ the transmission drops below 0.7.

(3) The number of peaks scales linearly with ℓ as given in Table I and fits remarkably well to the formula $n_p = 0.4\ell$. This means that although the wavelength undergoes a redshift pattern away from the bump center, the total number of oscillations for the bump-shaped tilting profile manages to scale with the length of the junction.

(4) As can be clearly seen, by spreading the bump over a longer junction, the *amplitude* of the oscillations steadily decreases for larger ℓ . One can extrapolate from Fig. 8 that for very large junction lengths ℓ , the transmission curve becomes a smooth curve.

Property (2) above rests on the fact that gradual change in ζ gives rise to gradual change in velocity, producing a small impedance mismatch [46], thereby diminishing the reflection and enhancing the transmission. This property can be employed as an engineering concept in designing media to convert electron wave numbers while they nearly perfectly transmit. The wave length of the incident and transmitted waves are red/blueshifted depending on the step direction of the sigmoid profile. This concept is similar to the graded-index optical fiber [47,48].

For larger ℓ , the bump in Fig. 3 spreads over a longer distance, and therefore the spatial variations of ζ , namely, $\partial_z \zeta$ on average becomes smaller. In such a situation, property (4) above means that the amplitude of the oscillations are controlled by the derivative $\partial_z \zeta$. Such derivatives appear in the Christoffel connections that govern the motion of waves in the ray-optics limit.

In the experimental setup measuring the conductance, one usually synthesizes one sample with a fixed ℓ and there is not much room to change the ℓ . How can the above results be useful? As pointed out, the linear scaling of energy and momentum implies that for a physical junction with a given length, the dimensionless number ℓ that counts the number of $\lambda_E = \hbar v_F/E$ fitting into the physical junction can be decreased by increasing λ_E . Therefore, moving to lower E

corresponds to moving from the gray plot to blue plot in Fig. 8. This can be done in the laboratory by sweeping the voltage in a scanning tunneling spectroscopy: Scanning along the junction can probe various z/ℓ positions, whereas scanning the energy from low to high voltages amounts to scanning from blue to gray curves of Fig. 8. Measuring the tunneling current at a fixed energy as a function of z/ℓ will give a typical redshift pattern of the form displayed in Fig. 7. Increasing the voltage and scanning again, the junction width would lead to (i) an increase in the number of peaks and (ii) a decrease in the amplitude of oscillations as in Fig. 8. For example, doubling the voltage and scanning the junction width $z/\ell = 0$ to $z/\ell = 1$ is expected to double the number of peaks within the junction as suggested by Table I.

VI. SUMMARY AND OUTLOOK

Tilt of Dirac cones is a proxy for an emergent space-time metric where tilting parameter directly appears in the off-diagonal components of the spacetime metric that mix space and time coordinates. When an underlying metric is present, it must show up in various quantities. In this work, we have computed the transmission of electron waves through a region where tilting has spatial profiles of the bump and sigmoid shapes in Fig. 3. We find features in the amplitude of the wave function, spherical angles of the pseudospin, and the transmission coefficients that resemble the gravitational redshift. It means that the wavelength of oscillatory behavior is smaller in regions with larger tilt ζ and get (red) shifted to larger wavelengths in regions with smaller ζ .

The ability to red- or blueshift electron waves, in addition to demonstrating a general-relativistic effect in the laboratory, can find potential application: Since due to very low impedance mismatch that arises from the gradual change of tilting, the reflection is diminished and the medium nearly perfectly transmits the waves—a sigmoid tilting profile can be used to perfectly transmit electron waves and deliver them with red/blueshifted wave lengths.

Recent experimental evidence for the tilting induced by magnetic influence in the spin-orbit coupled Dirac cones at the surface of topological insulators [33,34] is a promising platform to imprint spatially variable textures of tilting ζ by a corresponding texture of the magnetization. Table I giving $n_p = 0.4\ell$ immediately implies that the number of oscillations in the central region $[0.4\ell, 0.6\ell]$ is almost $0.5n_p$, whereas in the absence of redshift it is expected to be $0.2n_p$ since the length of this interval is 0.2ℓ . Therefore, the wavelength of conductance oscillations in the middle is ≈ 2.5 times smaller than the edges or the edges are 250% redshifted with respect to the middle region for the bump profile. Within the above scenario, it should be possible to imprint a tilt texture by magnetic DWs where the typical DW width 100 – 150 nm will correspond to $\ell \sim 200$ at $E = 1$ eV. In the above scenario, the tilting is first order in external magnetic field, whereas shift in the chemical potential is second order in the magnetic field and therefore can be neglected [34]. The present proof of principle study indicates that aspects of the underlying spacetime geometry can be detected by standard solid-state spectroscopic tools. In the spin-orbit coupled Dirac cones, interesting oscillatory behavior of pseudospin orientation and

the corresponding redshift effects is expected to find applications in spintronics devices.

ACKNOWLEDGMENT

S.A.J. acknowledges the support of Hendrik J. Bluhm.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: DERIVATION OF THE CURRENT OPERATOR FROM THE CONTINUITY EQUATION

We begin with the definition of the probability density:

$$\rho = \Psi^\dagger \Psi.$$

Taking the time derivative, we obtain

$$\frac{\partial \rho}{\partial t} = \frac{\partial \Psi^\dagger}{\partial t} \Psi + \Psi^\dagger \frac{\partial \Psi}{\partial t}.$$

Assuming that the wave function Ψ satisfies the time-dependent Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi,$$

we also have the Hermitian conjugate:

$$-i\hbar \frac{\partial \Psi^\dagger}{\partial t} = \Psi^\dagger H.$$

Substituting the Hamiltonian from Eq. (2), the equation of motion becomes

$$\frac{\partial \Psi}{\partial t} = \frac{v_F}{i\hbar} \left(\sigma_x \frac{\partial}{\partial x} + \mathcal{M} \sigma_z \frac{\partial}{\partial z} \mathcal{M}^{-1} \right) \Psi.$$

Using the identity

$$\mathcal{M} \sigma_z \mathcal{M}^{-1} = \sigma_z + \zeta(z) \sigma_0,$$

we derive the continuity equation in the familiar form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

from which the current density operator can be directly identified as

$$\mathbf{J} = v_F [\Psi^\dagger \sigma_x \Psi] \mathbf{e}_x + v_F [\Psi^\dagger (\sigma_z + \zeta(z) \sigma_0) \Psi] \mathbf{e}_z.$$

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