Finite-cutoff holographic thermodynamics

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We construct the first self-consistent framework for holographic thermodynamics in finite-cutoff holography, thereby capturing gravitational thermodynamics beyond the anti-de Sitter (AdS) boundary. We formulate two distinct first laws of thermodynamics: for the gravitational bulk, a Schwarzschild-AdS black hole with a finite Dirichlet cutoff, and for its dual T^2 deformed conformal field theory (CFT) on the boundary. The central innovation is promoting the deformation parameter to a thermodynamic variable on the boundary, which maps precisely to the cutoff radius in the bulk. We demonstrate exact duality between these two thermodynamic laws, establishing a complete holographic dictionary. This framework reveals that the T^2 deformation never lowers the confinement-deconfinement phase transition temperature below that of the original undeformed CFT, which acts as a global minimum. These results offer insights into gravitational thermodynamics with boundary constraints and quantum gravity in finite spacetime regions.

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I. INTRODUCTION

The equivalence principle and the relativistic paradigm introduce a fundamentally new geometric framework for understanding gravity at macroscopic and cosmological scales. Contradistinctively, the development of the standard model has demonstrated that the electromagnetic, weak, and strong nuclear forces are effectively described by gauge theories grounded in the principle of gauge symmetry [1,2]. Another foundational concept—the holographic principle [2,3]—states that gravitational physics is dual to quantum physics. This principle is embodied by the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [4–8], which proposes that a gravitational theory in a $(d+1)$-dimensional AdS spacetime (e.g., an AdS black hole) is equivalent to a strongly coupled gauge theory (at finite temperature in the case of a black hole) defined on its d-dimensional conformal boundary.

The thermodynamics of AdS black holes has been extensively studied following the discovery of the Hawking-Page phase transition in Schwarzschild-AdS black holes [9]. Phase structures analogous to the van der Waals liquid-gas system have been identified for charged Reissner-Nordström AdS black holes, exhibiting first-order phase transitions and critical phenomena [10–12]. Significant advances have been achieved by treating the cosmological constant as a thermodynamic variable, revealing phenomena such as fluid/superfluid phase transitions [13], black hole microstructures [14], and more [15–20]. The subject is now known as black hole chemistry [21,22].

The holographic interpretation of black hole chemistry (i.e., holographic thermodynamics) has recently garnered significantly more attention. Holographically, the Hawking-Page phase transition of the Schwarzschild-AdS (SAdS) black hole corresponds to the confinement-deconfinement phase transition of the quark-gluon plasma in the boundary CFT [23], and was subsequently understood as a solid-liquid transition in the context of black hole chemistry [24]. Exploratory studies [25–29] indicate that the thermodynamic pressure P and its conjugate volume V in the bulk correspond to the central charge c and its conjugate chemical potential μ on the boundary. Given that the background metric for the CFT is obtained through conformal completion of the AdS bulk, there is flexibility in rescaling the spatial radius of the boundary CFT. Specifically, the boundary radius R may either match the AdS radius ℓ directly [30] or differ by a conformal factor $\omega = R/\ell$ [5,6]. Rather than varying Newton's constant [31] or fixing the spatial volume of the boundary CFT [32], a

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recent study has treated the conformal rescaling factor as an independent thermodynamic parameter [33]. This approach established an exact duality between the first law of extended black hole thermodynamics in the bulk and the first law in the dual CFT, ensuring independent variations of the central charge and spatial volume in the dual theory.

Up to now, holographic thermodynamics has fundamentally built upon the AdS/CFT correspondence, where the CFT is ultraviolet (UV) complete, located at spatial infinity. In this setup, an external observer measures properties of gravity contained within an infinite AdS enclosure. A key question is: how can we capture the perspective of observers localized in a region of AdS space, defining bulk observables at some finite distance while making sense of gravity beyond the region's boundary yet well below the AdS scale? Specifically, can holographic thermodynamics within a finite-cutoff boundary be defined?

It is a longstanding challenge to generalize AdS/CFT duality to finite-cutoff holography, where an effective quantum field theory with a UV cutoff corresponds to a dual gravity with a finite radial cutoff [34,35] (i.e., a framework where a quantum field theory with a sharp energy cutoff is posited to be dual to a gravitational theory defined only within a finite region of spacetime, bounded by a Dirichlet wall). To deal with specified spatial boundary conditions, the framework of the holographic Wilsonian renormalization group (RG) was developed [36,37], where radial flow in the bulk geometry corresponds to RG flow on the boundary theory, and a radial cutoff in the bulk is interpreted as a Wilsonian cutoff in the dual field theory [38–43]. On the other hand, the $T\bar{T}$ deformed two-dimensional CFT, which possesses a large central charge and represents the simplest nontrivial integrable deformation of two-dimensional CFTs [44–48], admits an exact holographic RG flow equation that fully describes the effective UV cutoff in terms of an irrelevant deformation parameter. Applying the $T\bar{T}$ deformation for a higher dimensional CFT has been explored in Refs. [48–50]; specifically, a holographic formulation of a T^2 deformation has been proposed [51,52]. Here, we take the first step in extending holographic black hole chemistry in the traditional AdS/CFT setting to finite-cutoff holographic thermodynamics, within the framework of T^2 deformation. An extension to the $T\bar{T}$ case is similar.

II. FINITE-CUTOFF HOLOGRAPHY

A CFT defined at the boundary $\partial\mathcal{B}$ at infinity is holographically dual to quantum gravity in the asymptotically AdS bulk region $\mathcal{B}$. $T\bar{T}$ deformation of two-dimensional CFTs is an irrelevant and integrable deformation [44,46]. It is constructed from the stress-energy tensor $T_{\mu\nu}$ of the deformed quantum theory. Observables like the energy spectrum of the deformed theory can be explicitly expressed in terms of the undeformed theory. The generalization of $T\bar{T}$ deformations to higher-dimensional effective field theories with large degrees of freedom and finite UV cutoffs, dual to gravitational theories coupled with matter fields in the bulk, has been recently explored [51,52]. Unlike the two-dimensional scenario, higher-dimensional T^2 deformations explicitly involve background fields. Physical observables such as the energy spectrum [34,46,47,52], stress-energy tensor correlation functions [35,52–57], and entanglement entropy [58,59] are computable within this deformation framework. Generically, the action S_λ for a perturbed d-dimensional CFT is governed by an exactly solvable one-parameter flow equation

$$\frac{\mathrm{d}S_\lambda}{\mathrm{d}\lambda} = \int \mathrm{d}^d x\sqrt{\gamma}\mathcal{F}(T_{ij}, G_{ij}), \tag{1}$$

where $S_{\lambda=0}$ is for the undeformed theory, and $\mathcal{F}$ is a scalar function of the stress-energy tensor T_{ij} and the background metric G_{ij}. The extent of the deformation is characterized by the double trace deformation parameter λ. Negative values of λ yield theories with a super-Hagedorn asymptotic density of states, resulting in ill-defined canonical ensembles [45], whereas $\lambda > 0$ corresponds holographically to an AdS bulk geometry truncated by a finite radial cutoff at the Dirichlet boundary where the dual deformed CFT resides [34,41,51,52,60–63]. While the Dirichlet problem for general relativity is known to be ill-posed for generic boundary data [64], we emphasize that our analysis is restricted to the highly symmetric case of a spherical boundary in a Schwarzschild-AdS background. This special class of boundary conditions is known to lead to a well-posed problem, with recent work establishing local-in-time well-posedness under specific conditions on the boundary's extrinsic geometry that are satisfied in our setup [65–67].

Physical observables such as the energy spectrum [34,46,47,52], stress-energy tensor correlation functions [35,52–57], and entanglement entropy [58,59] are computable within this deformation framework. From the holographic viewpoint, a positively deformed CFT corresponds precisely to a bulk geometry truncated by a finite cutoff [34,41,51,52,60–63], described by the Fefferman-Graham expansion [68,69], the AdS bulk geometry is described by

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}(\mathrm{d}r^2 + g_{ij}(x, r)\mathrm{d}x^i\mathrm{d}x^j), \quad r > r_c, \tag{2}$$

where r_c is the cutoff radius, ℓ is the AdS length scale, x^i are a set of the boundary coordinates, and the metric $g_{ij}(x, r) = g_{(0)ij} + r^2 g_{(2)ij} + r^3 g_{(3)ij} + \cdots$ admits an expansion in powers of the radial coordinate r, which in turn yields the boundary stress-energy tensor [70]. The holographic dictionary relating gravity and effective field theory is succinctly expressed as

$$Z_{\mathrm{EFT}}[\lambda, \gamma_{ij}] = Z_{\mathrm{grav}}\big[g_{(0)} = r_c^2\gamma_{ij}\big] \tag{3}$$

with γ_{ij} denoting the metric of the deformed CFT. This equivalence indicates that the generating functional of the deformed CFT defined by Eq. (1) matches the gravitational path integral for a finite cutoff geometry (2), applicable both to pure gravity [34] and to gravity theories coupled with matter [52]. To be specific, we focus on the case where $d = 3$ in what follows.

First, let us introduce the holographic black hole chemistry. The standard holographic dictionary in four-dimensional spacetime is attained via Eq. (3) in the limit $r_c \to \infty$. The background metric of the seed CFT in static coordinates is given by

$$\mathrm{d}s^2 = \omega^2\big(-\mathrm{d}t^2 + \ell^2\mathrm{d}\Omega_2^2\big), \tag{4}$$

which can be obtained by a Weyl rescaling of the asymptotic AdS metric [71,72]. Here, ω is a variable conformal factor, and $\mathrm{d}\Omega_2^2$ is the line element of a unit sphere. The thermodynamic first law

$$\mathrm{d}M = T\mathrm{d}S + V\mathrm{d}P + \cdots \tag{5}$$

in the gravitational bulk is holographically dual to

$$\mathrm{d}\bar{E} = \bar{T}\mathrm{d}\bar{S} - \bar{P}\mathrm{d}\bar{V} + \mu\mathrm{d}c + \cdots \tag{6}$$

on the conformal boundary at spatial infinity [33,73,74]. Here, M, T, S, V, P, respectively, represent the bulk mass, temperature, entropy, thermodynamic volume, and thermodynamic pressure, while $\bar{E}, \bar{T}, \bar{S}, \bar{V}, \bar{P}$ denote their boundary counterparts. Note that there is an extra pair (c, μ) of conjugate variables, with c being the central charge and μ its conjugate chemical potential, relative to the bulk. As pointed out in Ref. [75], the internal energy of the large-N SU(N) gauge theories with conformal symmetry depends not only on the extensive quantities $\bar{S}$ and $\bar{V}$, but also on the intensive quantity $c \sim N^2$. This is consistent with the scenario of working in the extended phase space, where $P = 3/(8\pi G\ell^2)$ relates the AdS radius to the thermodynamic pressure, with Newton's constant G. Variation of the pressure is equivalent to varying central charge and volume of the CFT. The standard extrapolated holographic dictionary reads [23,76–78]

$$M = \omega\bar{E}, \quad S = \bar{S}, \quad c = \frac{\ell^2}{4G}. \tag{7}$$

The holographic temperature relation is

$$T = \omega\bar{T} \tag{8}$$

via the first laws for the bulk and boundary.

Turning to the finite-cutoff holographic thermodynamics, we now consider a deformed CFT residing at a Dirichlet wall at $r = r_c$. To consistently implement finite-cutoff holographic thermodynamics, the standard thermodynamic quantities must be suitably adapted. On the gravitational side, the bulk mass M in Eq. (5) must be replaced by a cutoff-dependent energy $\mathcal{E}(r_c)$, which approaches M in Eq. (5) in the asymptotic limit as

$$\lim_{r_c\to\infty} \mathcal{E}(r_c) = M. \tag{9}$$

Correspondingly, the energy $\bar{E}$ of the undeformed boundary seed CFT in Eq. (6) is generalized to a deformed energy $\bar{\mathcal{E}}(\lambda)$, which satisfies the UV consistency condition

$$\lim_{\lambda\to 0} \bar{\mathcal{E}}(\lambda) = \bar{E}. \tag{10}$$

In light of Eq. (7), the dictionary between the bulk mass and boundary energies becomes

$$\mathcal{E}(r_c) = \omega\bar{\mathcal{E}}(\lambda). \tag{11}$$

Note the conformal factor ω is independent of both r_c and λ. Due to Eq. (3), the holographic relation for the Bekenstein-Hawking entropy S and the entropy $\bar{S}$ for the deformed CFT remains unchanged [34]. Moreover, the Brown-Henneaux-like relation [79] in Eq. (7) still holds.

Based on the above basic requirements of Eqs. (9)–(11), we are now ready to construct finite-cutoff holographic thermodynamics. To be specific, in the bulk, we consider a four-dimensional SAdS black hole subject to a finite Dirichlet cutoff; on the cutoff surface, the dual theory is a three-dimensional dual T^2 deformed CFT.

III. FINITE-CUTOFF HOLOGRAPHIC THERMODYNAMICS: BOUNDARY SIDE

The background geometry of the T^2 deformed CFT is specified by [52]

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + R^2\mathrm{d}\Omega_2^2, \tag{12}$$

where $R^2\mathrm{d}\Omega_2^2$ denotes the standard metric on a two-sphere of radius R. The spatial volume of the deformed CFT is thus

$$\bar{V} = 4\pi R^2. \tag{13}$$

This expression matches the volume inferred from Eq. (4) provided the conformal factor satisfies $\omega = R/\ell$. Note that in so doing, we have defined $\tau = \omega t$. The finite-size energy spectrum of a large-N T^2 deformed three-dimensional CFT is given by

$$\bar{\mathcal{E}} = \frac{4\pi R^2}{3\lambda}\left(1 + \frac{3\alpha\lambda^{2/3}}{R^2} - \sqrt{1 - \frac{3\lambda\bar{E}}{2\pi R^2} + \frac{6\alpha\lambda^{2/3}}{R^2}}\right), \tag{14}$$

where, using the Brown-Henneaux-like relation (7) for c, $\alpha = \ell^{4/3}/(4\sqrt[3]{6}\pi^{2/3}G^{2/3}) = c^{2/3}/(2\sqrt[3]{3}\pi^{2/3})$. Note that according to Eq. (14), a shock singularity [46] emerges for $\lambda > \pi\sqrt{c}\bar{V}^{3/2}/3\bar{S}^{3/2}$. The quantity $\bar{\mathcal{E}}$ has some differences compared to the energy spectrum given in Ref. [52]. The consistency condition (10) is satisfied by replacing M/R in Ref. [52] with $\bar{E}$, which also renders the energy $\bar{\mathcal{E}}$ as a homogeneous function. Most importantly, we now have $\bar{E} = \bar{E}(R)$, and by using the holographic dictionary (7) for the seed CFT, it can be explicitly specified as [73,80,81]

$$\bar{E} = \frac{4\pi c\bar{S} + \bar{S}^2}{2\pi\sqrt{c\bar{S}\mathcal{V}_0}}, \tag{15}$$

where $\mathcal{V}_0$ is the volume of the undeformed CFT. Since the conformal factor ω can be chosen freely, we identify $\mathcal{V}_0 = \bar{V}$. These reformulations of the original energy spectrum obtained in Ref. [52] still make Eq. (14) a solution of the T^2 flow equation, and it also makes it possible to construct a first law for the deformed CFT.

Substituting this explicit form of $\bar{E}$ into Eq. (14) and considering Eq. (13), we obtain the first law

$$\mathrm{d}\bar{\mathcal{E}} = \bar{T}\mathrm{d}\bar{S} - \bar{P}\mathrm{d}\bar{V} + \mu\mathrm{d}c + \nu\mathrm{d}\lambda \tag{16}$$

for deformed CFT dual to the cutoff AdS bulk, where we have chemical potentials μ, ν conjugate to the central charge c and the deformation parameter λ. All conjugate quantities in Eq. (16) can be directly attained via differentiation using Eq. (14) (see Appendix A for details). The corresponding holographic Euler relation for the deformed CFT reads

$$\bar{\mathcal{E}} = \bar{T}\bar{S} + \mu c - \nu\lambda. \tag{17}$$

This relation reduces to the one for the seed CFT as $\lambda \to 0$ and it is a result of the scaling law

$$\bar{\mathcal{E}}(\zeta\bar{S},\ \bar{V},\ \zeta c,\ \zeta^{-1}\lambda) = \zeta\bar{\mathcal{E}}(\bar{S},\ \bar{V},\ c,\ \lambda), \tag{18}$$

where ζ is a dimensionless parameter. The equation of state for the seed CFT is $\bar{P} = \bar{E}/(2\bar{V})$ [33]. This must be modified for the deformed CFT. We find another scaling law

$$\bar{\mathcal{E}}(\bar{S},\ \zeta^{2/3}\bar{V},\ c,\ \zeta\lambda) = \zeta^{-1/3}\bar{\mathcal{E}}(\bar{S},\ \bar{V},\ c,\ \lambda) \tag{19}$$

that yields the equation of state of the deformed CFT as

$$\bar{P} = \frac{\bar{\mathcal{E}}}{2\bar{V}} + \frac{3\lambda\nu}{2\bar{V}}. \tag{20}$$

According to Eq. (10), the standard equation of state of the seed CFT can be restored for $\lambda \to 0$.

IV. FINITE-CUTOFF HOLOGRAPHIC THERMODYNAMICS: BULK SIDE

In the framework of finite-cutoff holography, the T^2 deformation of the three-dimensional CFT is holographically dual to the SAdS black hole with a finite Dirichlet radial cutoff, and there is an agreement between the dressed energy spectrum (14) of the deformed field theory and the quasilocal energy of the SAdS black hole evaluated within the region $r < r_c$, given by

$$\mathcal{M} = \frac{r_c}{G}\left(\frac{\ell}{2r_c} + \frac{r_c}{\ell} - \sqrt{1 - \frac{2GM}{r_c} + \frac{r_c^2}{\ell^2}}\right), \tag{21}$$

where M denotes the mass of the SAdS black hole without a radial cutoff. We note that there is a degree of freedom of choice for the zero-point configuration ε_0 for the quasilocal energy [82]. In Eq. (21), we have chosen $\varepsilon_0 = -(\ell/(2r_c) + r_c/\ell)/(4\pi r_c)$ by implementing constraints of erecting a self-consistent holographic dictionary so that the energy vanishes at $r_c \to \infty$. At large r_c, $\mathcal{M} \to M\ell/r_c + \mathcal{O}(1/r_c^2)$, we thus rescale the quasilocal energy (21) by a dimensionless factor r_c/ℓ, which results in a rescaled bulk mass

$$\mathcal{E} \equiv \frac{r_c\mathcal{M}}{\ell} = \frac{\sqrt{A_c}(-2\sqrt{2}\sqrt{\mathbb{F}GP} + 4GPA_c + 3)}{12\sqrt{\pi}G}, \tag{22}$$

where $\mathbb{F} = A_c(2GPA_c + 3) - \sqrt{A_c}(2\sqrt{GS}(3 + 8G^2PS))$ with $A_c \equiv 4\pi r_c^2$ representing the size of the transverse space, i.e., the area of the cutoff surface.

The advantage of this redefined observable, which is invariant under all gauge transformations of the theory [83], is that for $A_c \to \infty$, it reduces to the ADM mass M of the SAdS black hole as $\lim_{A_c\to\infty}\mathcal{E}(A_c) \to M$. For the vacuum AdS case, it also reduces to zero energy. A similar rescaling was also used to calculate the holographic mass of accelerating black holes [84]. Note that the boundary stress tensor T_{ij} is related to the bulk Brown-York tensor $\widetilde{T}_{ij}$ via $r_c\widetilde{T}_{ij} = T_{ij}$. The stripping off of the asymptotically decaying factor ℓ/r_c in Eq. (22) for $\mathcal{M}$ can be viewed as an approach to extracting the holographic mass of cutoff SAdS black hole, similar to methods for extracting the full CFT stress-energy tensor from the corresponding supergravity solutions [85].

With the well-defined rescaled mass $\mathcal{E}$, it is possible to form a quasilocal first law for the finite-cutoff spacetime. Given that the mass of the SAdS black hole without a cutoff is [86]

$$M = \frac{\sqrt{S}(8G^2PS + 3)}{6\sqrt{\pi G}}, \tag{23}$$

the first law of thermodynamics for the cutoff SAdS black hole takes the form

$$\mathrm{d}\mathcal{E} = T\mathrm{d}S + V\mathrm{d}P + \tau\mathrm{d}A_c, \tag{24}$$

and the associated Smarr relation is

$$\mathcal{E} = 2TS - 2VP + 2\tau A_c. \tag{25}$$

The conjugate quantities, temperature T, thermodynamic volume V, and tension τ, can be obtained via differentiation after substituting the bulk mass (22) into (24) (see Appendix A for their explicit forms). Note that the temperature is independent of the zero-point configuration ε_0.

To establish the relation in Eq. (11), which connects the redefined energy $\bar{\mathcal{E}}$ of the deformed CFT in Eq. (14) to the modified bulk mass $\mathcal{E}$ in Eq. (22) via a conformal scaling parameter ω, we relate the cutoff radius r_c to the deformation parameter λ through the holographic dictionary

$$\frac{\lambda}{R^3} = \frac{4\pi G\ell}{3r_c^3}. \tag{26}$$

This dictionary makes $\ell\mathcal{E} = R\bar{\mathcal{E}}$; in the limit $r_c \to \infty$, $\lambda \to 0$, we have $\ell M = R\bar{E}$, consistent with Eq. (7). The thermodynamic first law (16) for the deformed CFT and its bulk counterpart (24) can be derived from each other. To illustrate this connection, we outline how the former can be obtained from the latter. For the left-hand side of Eq. (24), we have $\mathrm{d}\mathcal{E} = \mathrm{d}(R\bar{\mathcal{E}}/\ell)$. For the right-hand side of Eq. (24), expressing all the bulk quantities S, P, A_c, ℓ, r_c in terms of the boundary counterparts, we just solve $\mathrm{d}\bar{\mathcal{E}}$ to find that the $T\mathrm{d}S$ term becomes $\bar{T}\mathrm{d}\bar{S}$, the $V\mathrm{d}P$ term along with part of $\tau\mathrm{d}A_c$ term become $-\bar{P}\mathrm{d}\bar{V} + \mu\mathrm{d}c$, and $\nu\mathrm{d}\lambda$ term arises from the remaining part of the $\tau\mathrm{d}A_c$ term (see Appendix B for details).

V. PHASE TRANSITIONS

Hawking-Page transitions, originally discovered between pure AdS and SAdS for $\lambda = 0$, also persist in cutoff scenarios with nonvanishing λ. The (rescaled) coexistence curve $(T(\lambda), S(\lambda))$ (phase boundary) follows from the condition $G = 0$ for the cutoff Gibbs free energy $G(S, P, A_c) = \mathcal{E} - TS$, as the vacuum AdS background has vanishing free energy. Through holographic duality, this transition corresponds to a confinement/deconfinement transition in the dual cutoff CFT, captured by the same coexistence curve $(\bar{T}(\lambda), \bar{S}(\lambda))$ obtained from the intersection of $\bar{G} = 0$ and $\bar{G}(\bar{S}, \bar{V}, c, \lambda) = \bar{\mathcal{E}} - \bar{T}\bar{S}$. Figure 1 shows the coexistence curve traced out by the boundary confinement/deconfinement and bulk Hawking-Page phase transition points. Due to holographic duality, the $\bar{T}$ vs $\bar{S}$ and T/ω vs S curves coincide and are plotted together. Starting from the undeformed CFT at X_0 ($\lambda = 0$), increasing the deformation parameter λ drives the system along a trajectory $X_0 \to X_1 \to X_2 \to X_0'$ before continuing along the branch toward X_3, with X_0' coincides with X_0, and $\mathrm{d}\bar{T}/\mathrm{d}\bar{S} = 0, \infty$ at X_1, X_2, respectively. The region above the curve is broadly the deconfinement (black hole) phase and below is the confinement (AdS vacuum) phase. This curve reveals several

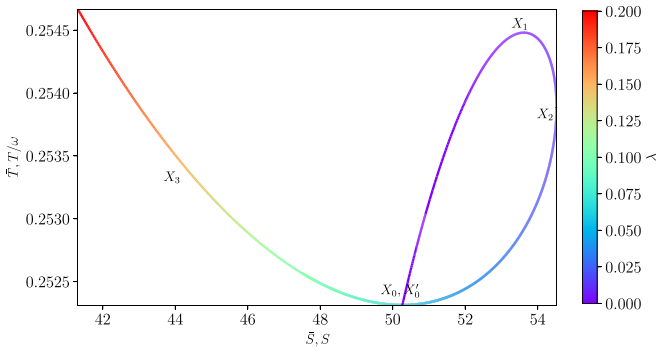


FIG. 1. Coexistence curve of the boundary confinement/deconfinement and bulk Hawking-Page phase transitions with $c = 4, \bar{V} = 20$.

features. A single phase transition temperature $\bar{T}$ may correspond to as many as three distinct phase transition entropies $\bar{S}$. At the temperature of local maximum X_1, two distinct $\bar{S}$ are possible. Crucially, the system possesses a global minimum phase transition temperature at the point X_0. This minimum corresponds to a single entropy value $\bar{S}$ but two distinct deformation states: the undeformed seed CFT ($\lambda = 0$) and a specific deformed CFT ($\lambda \neq 0$). This demonstrates that the T^2 deformation never lowers the phase transition temperature below that of the undeformed case.

VI. DISCUSSION

We have constructed a self-consistent framework for holographic thermodynamics, extending the AdS/CFT correspondence to a finite cutoff. Our approach hinges on formulating two distinct, parallel first laws of thermodynamics: one for the gravitational bulk, a Schwarzschild-AdS black hole with a Dirichlet cutoff, and one for the dual boundary, a T^2 deformed CFT. The central innovation is the promotion of the deformation parameter to a thermodynamic variable, which we demonstrated maps precisely to the bulk cutoff radius. We have shown the exact correspondence between these two thermodynamic laws, establishing a complete holographic dictionary. This framework, rooted in the bulk-boundary duality, reveals a phase structure. The key result in this framework is that the confinement-deconfinement phase transition temperature of the undeformed CFT acts as a global minimum, a benchmark that the deformation can only increase. As such, we have provided the first complete description of holographic thermodynamics for a deformed CFT, opening a pathway for studying quantum gravity in finite spacetime settings.

Based on our results, several directions for future research can be pursued. Our analysis can be readily extended to holographic thermodynamics for a CFT in general dimensions, including $d = 2$, where $T\bar{T}$ holographic thermodynamics can be constructed. The framework can also be applied to other types of black holes (e.g., charged black holes). Additionally, it is natural to explore the Penrose inequality [87] and the reverse isoperimetric inequality [88] within cutoff holographic thermodynamics. The current study in AdS backgrounds can be extended to other contexts, such as cosmological holography [89–95], Cauchy holography [96–99], and black holes with Lifshitz and hyperscaling violating asymptotics [100]. Furthermore, extensions to holographic thermodynamics with a negative deformation parameter λ [101] can be considered; the Hagedorn singularity that can be calculated via the torus partition function [102] may arise and be related to the phase transition in the bulk. Investigating subleading orders in $1/N$ beyond the classical level presents an additional challenge for future work. Finally, extend the current work to other boundary conditions [101,103–108] and other relevant interpretations of black hole thermodynamics from holographic perfect fluid [109], particularly for the finite-cutoff scenario [110], warrant further study.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EXPLICIT EXPRESSIONS FOR THERMODYNAMIC QUANTITIES

We show explicit expressions for the thermodynamic quantities in the first laws for the T^2 deformed boundary field theory and the finite-cutoff Schwarzschild-AdS black hole in what follows. For the T^2 deformed CFT, the first law reads

$$\mathrm{d}\bar{\mathcal{E}} = \bar{T}\mathrm{d}\bar{S} - \bar{P}\mathrm{d}\bar{V} + \mu\mathrm{d}c + \nu\mathrm{d}\lambda, \tag{A1}$$

where the energy $\bar{\mathcal{E}}$, temperature $\bar{T}$, pressure $\bar{P}$, chemical potentials μ, ν conjugate to entropy $\bar{S}$, volume $\bar{V}$, central charge c, and deformation parameter λ, respectively, are

$$\bar{\mathcal{E}} = \frac{\bar{V}}{3\lambda}\left(-\sqrt{\frac{2c^{2/3}\lambda^{2/3}K}{\bar{V}} - \frac{6\bar{E}\lambda}{\bar{V}} + 1} + \frac{c^{2/3}\lambda^{2/3}K}{\bar{V}} + 1\right), \tag{A2}$$

$$\bar{T} = \left(\frac{\partial\bar{\mathcal{E}}}{\partial\bar{S}}\right)_{\bar{V},c,\lambda} = \frac{\bar{V}\bar{E}'(\bar{S})}{\sqrt{\bar{V}(-6\lambda\bar{E}+\bar{V}+2K(c\lambda)^{2/3})}}, \tag{A3}$$

$$\bar{P} = -\left(\frac{\partial\bar{\mathcal{E}}}{\partial\bar{V}}\right)_{\bar{S},c,\lambda} = -\frac{\bar{V}(3\lambda\bar{E}'(\bar{V})-1)+3\lambda\bar{E}+\nu_0\sqrt{\bar{V}}-K(c\lambda)^{2/3}}{3\lambda\nu_0\sqrt{\bar{V}}}, \tag{A4}$$

$$\mu = \left(\frac{\partial\bar{\mathcal{E}}}{\partial c}\right)_{\bar{S},\bar{V},\lambda} = \frac{\sqrt{\bar{V}}(9\sqrt[3]{c\lambda}\bar{E}'(c)-2K)+2K\nu_0}{9\nu_0\sqrt[3]{c\lambda}}, \tag{A5}$$

$$\nu = \left(\frac{\partial\bar{\mathcal{E}}}{\partial\lambda}\right)_{\bar{S},\bar{V},c} = \frac{\sqrt{\bar{V}}(4K(c\lambda)^{2/3}-9E\lambda)-3\nu_0\bar{V}+3\bar{V}^{3/2}-K\nu_0(c\lambda)^{2/3}}{9\lambda^2\nu_0}, \tag{A6}$$

where

$$K = \frac{6\pi}{\sqrt[3]{3}\pi^{2/3}}, \quad \nu_0 = \sqrt{\bar{V}+2K(c\lambda)^{2/3}-6E\lambda}, \tag{A7}$$

$$E = \frac{4\pi c\bar{S}+\bar{S}^2}{2\pi\sqrt{c\bar{S}\bar{V}}}, \quad \bar{E}'(\bar{S}) = \frac{3\bar{S}+4\pi c}{4\pi\sqrt{c\bar{S}\bar{V}}}, \quad \bar{E}'(\bar{V}) = -\frac{c\bar{S}^2(\bar{S}+4\pi c)}{4\pi(c\bar{S}\bar{V})^{3/2}}, \quad \bar{E}'(c) = \frac{\bar{S}(4\pi c-\bar{S})}{4\pi c\sqrt{c\bar{S}\bar{V}}}. \tag{A8}$$

The first law in the finite-cutoff bulk is

$$\mathrm{d}\mathcal{E} = T\,\mathrm{d}S + V\,\mathrm{d}P + \tau\,\mathrm{d}A_c, \tag{A9}$$

where the quasilocal mass $\mathcal{E}$, temperature T, volume V, tension τ conjugate to entropy S, pressure P, and cutoff surface area A_c, respectively, are

$$\mathcal{E} = \frac{\sqrt{A_c}(4PA_c-2\nu_1+3)}{12\sqrt{\pi}}, \qquad T = \left(\frac{\partial\mathcal{E}}{\partial S}\right)_{P,A_c} = \frac{2PA_cM'(S)}{\nu_1}, \tag{A10}$$

$$V = \left(\frac{\partial\mathcal{E}}{\partial P}\right)_{S,A_c} = \frac{A_c(\sqrt{A_c}(-4PA_c+2\nu_1-3)+12\sqrt{\pi}(PM'(P)+M))}{6\sqrt{\pi}\nu_1}, \tag{A11}$$

$$\tau = \left(\frac{\partial\mathcal{E}}{\partial A_c}\right)_{S,P} = \frac{24\sqrt{\pi}MP\sqrt{A_c}-16P^2A_c^2+12(\nu_1-1)PA_c+\nu_1(3-2\nu_1)}{24\sqrt{\pi}\nu_1\sqrt{A_c}}, \tag{A12}$$

with

$$\nu_1 = \sqrt{2PA_c(2PA_c+3)-24\sqrt{\pi}MP\sqrt{A_c}}, \quad M = \frac{\sqrt{S}(8PS+3)}{6\sqrt{\pi}}, \quad M'(S) = \frac{8PS+1}{4\sqrt{\pi}\sqrt{S}}, \quad M'(P) = \frac{4S^{3/2}}{3\sqrt{\pi}}. \tag{A13}$$

We have set Newton's constant $G = 1$ for simplicity.

APPENDIX B: DETAILED DERIVATION OF THE FIRST LAW FROM THE BULK TO THE BOUNDARY

Here, we show in detail how the first law (A1) for the T^2 deformed CFT can be derived from first law (A9) for the Schwarzschild-AdS spacetime with finite-cutoff. The relations we will use include

$$\frac{\lambda}{R^3} = \frac{4\pi G\ell}{3r_c^3}, \quad c = \frac{\ell^2}{4G}, \quad \alpha = \frac{c^{2/3}}{2\sqrt[3]{3}\pi^{2/3}}, \quad P = \frac{3}{8\pi G\ell^2}, \quad \bar{V} = 4\pi R^2. \tag{B1}$$

On the left-hand side of the bulk first law (A9), we have

$$\mathrm{d}\mathcal{E} = \mathrm{d}\left(\frac{R\bar{\mathcal{E}}}{\ell}\right) = \frac{\bar{\mathcal{E}}}{\ell}\mathrm{d}R + \frac{R}{\ell}\mathrm{d}\bar{\mathcal{E}} - \frac{R\bar{\mathcal{E}}}{\ell^2}\mathrm{d}\ell. \tag{B2}$$

Applying Eq. (B1), each term on the right-hand side of Eq. (A9) turns to be

$$T\,\mathrm{d}S = T\,\mathrm{d}\bar{S}, \quad V\,\mathrm{d}P = -\frac{3V}{4\pi G\ell^3}\mathrm{d}\ell, \quad \tau\,\mathrm{d}A_c = 8\pi r_c\tau\,\mathrm{d}r_c. \tag{B3}$$

Combing Eqs. (A9), (B2), and (B3), we get

$$\mathrm{d}\bar{\mathcal{E}} = -\frac{\bar{\mathcal{E}}}{R}\mathrm{d}R + \frac{\bar{\mathcal{E}}}{\ell}\mathrm{d}\ell + \frac{T\ell}{R}\mathrm{d}\bar{S} + \frac{8\pi\tau\ell r_c}{R}\mathrm{d}r_c - \frac{3V}{4\pi GR\ell^2}\mathrm{d}\ell. \tag{B4}$$

Furthermore, $\mathrm{d}r_c$ term can be split into

$$\frac{8\pi\tau\ell r_c}{R}\mathrm{d}r_c = \frac{16\sqrt[3]{2}\pi^{5/3}G^{2/3}\ell^{2/3}\tau(-\ell R\mathrm{d}\lambda + \lambda R\mathrm{d}\ell + 3\lambda\ell\mathrm{d}R)}{3^{5/3}\lambda^{5/3}}. \tag{B5}$$

Substituting Eq. (B5) back into Eq. (B4) and then collect like terms, we have relations

$$\frac{T\ell}{R}\mathrm{d}\bar{S} = \bar{T}\mathrm{d}\bar{S}, \qquad \left(\frac{16\sqrt[3]{2}\pi^{5/3}G^{2/3}\tau\ell^{5/3}}{3^{2/3}\lambda^{2/3}} - \frac{\bar{\mathcal{E}}}{R}\right)\mathrm{d}R = -\bar{P}\mathrm{d}\bar{V}, \tag{B6}$$

$$\left(\frac{\bar{\mathcal{E}}}{\ell} + \frac{16\sqrt[3]{2}\pi^{5/3}G^{2/3}R\tau\ell^{2/3}}{3^{5/3}\lambda^{2/3}} - \frac{3V}{4\pi GR\ell^2}\right)\mathrm{d}\ell = \mu\mathrm{d}c, -\frac{16\sqrt[3]{2}\pi^{5/3}G^{2/3}R\tau\ell^{5/3}}{3^{5/3}\lambda^{5/3}}\mathrm{d}\lambda = \nu\mathrm{d}\lambda. \tag{B7}$$

Then, Eq. (B4) will be reduced to the boundary first law (A1). Note that the holographic dictionary for the temperature $\bar{T} = T\ell/R = T/\omega$ was used.

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