

Fast and exact approach for stabilizer Rényi entropy via the XOR-FWHT algorithm

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Quantum advantage is widely understood to rely on key quantum resources beyond entanglement, among which nonstabilizerness (quantum “magic”) plays a central role in enabling universal quantum computation. However, the deterministic evaluation of the second-order stabilizer Rényi entropy for generic many-body quantum states remains computationally challenging, with brute-force methods scaling as $\mathcal{O}(8^N) = \mathcal{O}(d^3)$ for an N -qubit state with $d = 2^N$. Here, we develop an exact algebraic reformulation and a deterministic numerical algorithm, numerically exact up to floating-point roundoff, that reduces this cost to $\mathcal{O}(N4^N) = \mathcal{O}(d^2 \log_2 d)$ while retaining natural parallelism. This is an exponential speedup, although the scaling remains exponential in the number of qubits. This advance enables high-precision calculations for generic state vectors at medium system sizes and provides a practical tool for investigating the scaling and nonequilibrium dynamics of quantum magic in many-body systems.

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Introduction. Quantum advantage is commonly understood within the framework of resource theories, in which the key resources include quantum entanglement [1,2] and *nonstabilizerness* [3–6], the latter also known as quantum *magic*. Entanglement alone, however, is not sufficient to guarantee quantum advantage, since even highly entangled stabilizer states remain efficiently simulable on a classical computer under the Gottesman-Knill theorem [7,8]. Magic is therefore regarded as an independent resource required for universal quantum computation beyond the Clifford setting. Its operational cost is architecture dependent: Non-Clifford states or gates are costly to generate or implement in experiments and in some fault-tolerant architectures, while other architectures may place the dominant overhead on different logical operations [6,9–12].

In the recent years, nonstabilizerness in many-body quantum systems has attracted growing interest. At the interface of quantum information and nonequilibrium many-body physics, issues such as scrambling, operator growth, and complex quantum dynamics have been extensively explored [13–21]. In this context, magic and related quantifiers of nonstabilizerness have been used to characterize many-body quantum states and their dynamical behavior [6,22–32]. A broad range of magic measures has been proposed, including quantifiers based on quasiprobability representations, robustness-type

measures, and measures related to operational or simulation overhead [4,33–38].

However, the direct evaluation of many of these measures requires costly optimization over stabilizer or quasiprobability decompositions, which generally restricts their numerical application to small-scale systems. Stabilizer Rényi entropies (SREs), on the other hand, provide a complementary, optimization-free framework for quantifying nonstabilizer resources [5,39,40]. This distinction does not imply polynomial-time computability for generic state-vector inputs: Direct evaluation of the second-order SRE (2-SRE) still costs $\mathcal{O}(8^N) = \mathcal{O}(d^3)$, with $d = 2^N$. Among them, the second-order SRE is the simplest nontrivial case, possessing an explicit quartic structure in its representation in terms of Pauli expectation values. This quantity has been widely studied in a variety of settings, including quantum phase transitions [23,24,41–45] and nonequilibrium dynamics [46–58].

Direct numerical evaluation of the 2-SRE remains highly expensive. The summation runs over the full set of Pauli strings, whose cardinality grows exponentially with system size as $|\mathcal{P}_N| \sim 4^N$; for a generic state-vector input, a straightforward brute-force implementation has overall complexity $\mathcal{O}(8^N)$ and is in practice usually restricted to relatively small system sizes, typically $N \lesssim 14$ [59–62]. Specialized exact calculations up to $N = 22$ qubits have nevertheless been reported for random-circuit states [63]. To overcome this bottleneck, several computational strategies have been developed for the 2-SRE and related quantities. Tensor-network approaches are effective when the quantum state admits a low-entanglement representation or a controlled bond dimension, although their accuracy is limited by the quality of the state representation and truncation errors [41,45,64–73]. Sampling and Monte Carlo methods avoid explicit enumeration of all 4^N terms, but at the price of statistical uncertainty [23,74–77]. For states with special algebraic structure, such as Gaussian states, substantially faster evaluations can be achieved,

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but the applicability is then restricted to these structured families [42,50,78–82]. Overall, state-vector, tensor-network, structured-state, and sampling methods are complementary approaches with distinct applicability regimes. It is therefore highly desirable to develop a numerical method for computing the 2-SRE that does not rely on *a priori* structural assumptions about the quantum state, while retaining deterministic evaluation and predictable computational scaling.

Fast Walsh-Hadamard transforms (FWHTs) have previously been used to accelerate stabilizer-overlap subroutines in robustness-of-magic calculations [83]. That application and the present one share the exploitation of binary Hadamard structure but act on different computational objects: Here, transforms of XOR-shifted wavefunction correlations return the complete Pauli expectation spectrum entering the 2-SRE. Closely related Walsh-Hadamard-based formulations for state-vector magic calculations have also been developed in Refs. [84,85]. These works additionally consider sampling estimators and variance reduction, qutrit-mana calculations, mixed-state extensions, and high-performance implementations.

In this Letter, we fill this gap by presenting an exact algebraic reformulation and a deterministic numerical algorithm for evaluating the 2-SRE of a generic N -qubit pure state represented as a full state vector. We systematically rewrite the Pauli-string summation in the 2-SRE expression into a bitstring form and reveal convolution and correlation structures over the group $\mathbb{Z}_2^N$, namely, XOR convolutions [86] and correlations over bitstrings, which enables us to exploit the orthogonality identities and the convolution theorem associated with the Walsh-Hadamard transform. This yields an exact formula amenable to deterministic numerical implementation, in which the computation reduces to repeated fast Walsh-Hadamard transforms [87], pointwise operations, and index reshuffling. Accordingly, we refer to the algorithm proposed in this work as the XOR-FWHT algorithm. The algorithm requires repeated FWHTs of length d and reduces the runtime from $\mathcal{O}(8^N)$ to $\mathcal{O}(N4^N)$ with modest additional memory overhead and natural parallelism over the shift index. This is an exponential speedup, but the cost remains exponential in N . On the stated 72-core CPU implementation, we demonstrate deterministic calculations, numerically exact up to floating-point roundoff, up to $N = 20$ for generic state vectors. This is an implementation benchmark rather than a record system-size claim.

Algebraic reformulation of the 2-SRE. Throughout this work, logarithms in the SRE are taken to base 2. The SRE of pure states, originally introduced in Ref. [5], is defined from the full set of Pauli expectation values as

$$\mathcal{M}_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \log_2 \left(\frac{1}{d} \sum_{P \in \mathcal{P}_N} |\langle \psi | P | \psi \rangle|^{2\alpha} \right), \quad (1)$$

where N is the number of qubits, $d = 2^N$, and α is the Rényi index. Here, $\mathcal{P}_N$ denotes the N -qubit Pauli group generated by tensor products of the single-qubit operators $\{\mathbf{1}, X, Y, Z\}$. Since $|\mathcal{P}_N| = 4^N$, a direct evaluation of Eq. (1) is generically exponentially costly. For $\alpha \geq 2$, the SRE is a magic monotone [5,40,64]. In particular, it satisfies several key properties: (1)

faithfulness, namely, $\mathcal{M}_\alpha(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ is a stabilizer state; (2) *invariance under Clifford transformations*, i.e., for all $C \in \mathcal{C}_N$, $\mathcal{M}_\alpha(C|\psi\rangle) = \mathcal{M}_\alpha(|\psi\rangle)$; and (3) *additivity* under tensor products, $\mathcal{M}_\alpha(|\phi\rangle \otimes |\psi\rangle) = \mathcal{M}_\alpha(|\phi\rangle) + \mathcal{M}_\alpha(|\psi\rangle)$. Here, $\mathcal{C}_N$ denotes the Clifford group.

In this work, we focus on the case $\alpha = 2$ [88], which already corresponds to the simplest nontrivial and most widely studied member of the SRE family. Starting from the definition of the SRE in Eq. (1), we specialize to the case $\alpha = 2$ and briefly derive the working expression used below, leaving the more detailed steps to the Supplemental Material [89]. To expose the hidden convolution structure underlying the 2-SRE, we first rewrite the Pauli-string summation in a binary bitstring representation. Noting that $Y = iXZ$, and that the overall phase of a Pauli string does not affect $|\langle \psi | P | \psi \rangle|^4$, we may, without loss of generality, ignore the global phase and retain only the $X^s Z^z$ structure. Any N -qubit Pauli string can then be written uniquely as $P = X_1^{s_1} \cdots X_N^{s_N} Z_1^{z_1} \cdots Z_N^{z_N} \equiv X^s Z^z$, where $s := (s_1, \dots, s_N)$, $z := (z_1, \dots, z_N) \in \mathbb{Z}_2^N$. Therefore, summing over all Pauli strings $P \in \mathcal{P}_N$ is equivalent to summing over all $s, z \in \mathbb{Z}_2^N$. We write the quantum state in the computational bitstring basis as $|\psi\rangle = \sum_{t \in \mathbb{Z}_2^N} \phi_t |t\rangle$, where $|t\rangle$ denotes the computational basis state labeled by the length- N transform-input bitstring $t = (t_1, \dots, t_N)$. The actions of the Pauli operators X^s and Z^z on the computational basis are, respectively, given by $X^s |t\rangle = |t \oplus s\rangle$, $Z^z |t\rangle = (-1)^{z \cdot t} |t\rangle$, where $\oplus$ denotes bitwise addition modulo 2 and $z \cdot t = \sum_{j=1}^N z_j t_j \pmod{2}$ is the binary inner product. Hence,

$$P|\psi\rangle = \sum_t \phi_t (-1)^{z \cdot t} |t \oplus s\rangle. \quad (2)$$

Taking the overlap with $\langle \psi |$, we obtain

$$\langle \psi | X^s Z^z | \psi \rangle = \sum_t (-1)^{z \cdot t} \phi_{t \oplus s}^* \phi_t. \quad (3)$$

This expression already makes the key structure visible: For fixed XOR shift s , the dependence on the transform-output index z takes the form of a Walsh-Hadamard kernel, while the amplitudes appear in the XOR-shifted correlation $\phi_{t \oplus s}^* \phi_t$. The binary representation of Pauli strings and their action on computational-basis states are standard; the present formulation exploits these ingredients in the quartic 2-SRE through XOR correlation and Walsh-Hadamard structure over $\mathbb{Z}_2^N$.

To see the first nontrivial consequence of this reformulation, let $g_s(t) := \phi_{t \oplus s}^* \phi_t$, $A_s(z) := \sum_t (-1)^{z \cdot t} g_s(t)$. Then, the 2-SRE involves sums of $|A_s(z)|^4$ over all $z \in \mathbb{Z}_2^N$. Expanding the fourth power and using the orthogonality of the Walsh-Hadamard characters, $\sum_{z \in \mathbb{Z}_2^N} (-1)^{z \cdot q} = 2^N \delta_{q,0}$, one finds

$$\sum_z |A_s(z)|^4 = 2^N \sum_{\substack{t_1, t_2, t_3, t_4 \\ t_1 \oplus t_2 \oplus t_3 \oplus t_4 = 0}} g_s(t_1) g_s^*(t_2) g_s(t_3) g_s^*(t_4), \quad (4)$$

where $g_s(t) = \phi_{t \oplus s}^* \phi_t$. Thus, the summation over z projects the originally unconstrained quartic sum onto an XOR-constrained subspace. This XOR constraint is the crucial structural step: It is what ultimately allows the quartic Pauli-string summation to be recast into a form amenable to Walsh-Hadamard acceleration.

Proceeding from Eq. (4), one can further reorganize the constrained sum into XOR-correlation functions and then apply the convolution and Parseval properties of the Walsh-Hadamard transform. For a function $f : \mathbb{Z}_2^N \rightarrow \mathbb{C}$, we use the unnormalized Walsh-Hadamard transform

$$[\mathcal{H}f](z) = \sum_{t \in \mathbb{Z}_2^N} (-1)^{z \cdot t} f(t). \quad (5)$$

The FWHT [87] evaluates this same map through N stages of butterfly updates $(a, b) \mapsto (a + b, a - b)$. For a vector of length $d = 2^N$, its arithmetic cost is $\mathcal{O}(d \log_2 d) = \mathcal{O}(N2^N)$, rather than the $\mathcal{O}(d^2)$ cost of direct multiplication by the Hadamard matrix. The numerical implementation uses the unnormalized convention in Eq. (5); the normalized WHT used in analytical identities in the Supplemental Material [89] is related by

$$\text{FWHT}(f) = \mathcal{H}f = \sqrt{d} \text{WHT}(f). \quad (6)$$

Since $A_s(z) = [\mathcal{H}g_s](z)$, the exact working formula is

$$\mathcal{M}_2(|\psi\rangle) = -\log_2 \left[\frac{1}{d} \sum_{s, z \in \mathbb{Z}_2^N} |[\mathcal{H}g_s](z)|^4 \right]. \quad (7)$$

The XOR-shift index s , transform-input index t , and transform-output index z therefore have distinct roles throughout Eqs. (3)–(7). A more detailed derivation is provided in Sec. I of the Supplemental Material [89]. Equation (7) is the key outcome of the derivation: It shows that deterministic evaluation of the 2-SRE for a generic state vector can be reduced to repeated unnormalized FWHTs acting on XOR-shifted amplitude correlators.

XOR-FWHT algorithm for the 2-SREs. By using Eq. (7), we design Algorithm 1 for $\mathcal{M}_2$ of an arbitrarily chosen state. It is worth emphasizing that the bitstrings s , t , and z are represented by their integer encodings in the numerical implementation, which is straightforward for a computer to handle. The implementation requires performing $d = 2^N$ FWHTs, each of length d , leading to an overall time complexity of $\mathcal{O}(2^N \times N2^N) = \mathcal{O}(N4^N) = \mathcal{O}(d^2 \log_2 d)$, compared with the brute-force cost $\mathcal{O}(8^N) = \mathcal{O}(d^3)$. Thus, the speedup is exponential, while the computation remains exponential in N . The additional memory overhead originates from storing the reordered conjugate vector $\overline{\phi[t \oplus s]}$ ($t = 0, \dots, d-1$); this overhead does not accumulate with s and is therefore acceptable. Moreover, the computations corresponding to different values of s are mutually independent, making the algorithm naturally well suited to parallel implementation. Implementation details are provided in Sec. II E of the Supplemental Material [89].

We benchmarked the performance of the algorithm using PYTHON/NUMPY with Numba just-in-time compilation, double-precision complex arithmetic, and parallelization over the independent shift index s . The workstation was equipped with two Intel Xeon Platinum 8352V processors (36 cores per processor, 72 cores in total). The reported values are wall-clock times for a single evaluation of the 2-SRE on one Haar-random input state. For different system sizes N , the run times are as follows: approximately 0.88 s for $N = 14$; about 12 s for $N = 16$; about 300 s for $N = 18$; and about

ALGORITHM 1. XOR-FWHT algorithm for $\mathcal{M}_2$.

Input: normalized state vector $\phi[0], \dots, \phi[d-1]$ in the computational basis, where $d = 2^N$.

Output: $\mathcal{M}_2$.

```

1:  $r \leftarrow 0$ 
2:  $t \leftarrow [0, 1, \dots, d-1]$ 
3: for  $s = 0$  to  $d-1$ 
4:   construct the length- $d$  complex array  $G$  by
        $G[t] \leftarrow \overline{\phi[t \oplus s]} \phi[t]$  ( $t = 0, \dots, d-1$ )
5:    $\widehat{G} \leftarrow \text{FWHT}(G)$  ▷ unnormalized transform
6:    $r \leftarrow r + \sum_{z=0}^{d-1} |\widehat{G}[z]|^4$ 
7: end for
8: Return  $\mathcal{M}_2 \leftarrow -\log_2(r/d)$ 

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2 h for $N = 20$. These numerical results are consistent with the theoretical time complexity $\mathcal{O}(N4^N)$ of the algorithm. In Fig. 1(a), we compare the wall-clock time of Algorithm 1 with that of a direct brute-force approach. The $N = 20$ result is a benchmark of this specified 72-core CPU implementation rather than a record system-size claim, since exact calculations up to $N = 22$ qubits were reported previously [63]. The numerical procedure is deterministic and introduces neither sampling nor truncation error; numerical “exactness” here means exactness up to floating-point roundoff. The corresponding implementation is also provided in the Supplemental Material [89].

As a validation of the numerical implementation, we evaluate the 2-SRE of Haar-random states. A Haar-random pure state is a pure quantum state drawn uniformly from Hilbert space according to the Haar measure. In many-body quantum physics, Haar-random pure states serve as benchmark models for characterizing the typical properties of generic pure states in quantum many-body systems [90]. Haar-random states exhibit near-Page-law volume-law entanglement, which

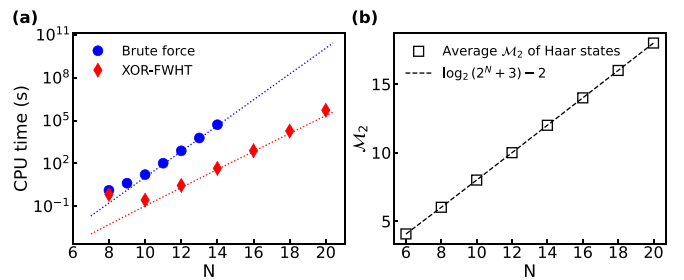


FIG. 1. (a) Exponential speedup of our XOR-FWHT implementation (red) in the wall-clock time required for evaluating $\mathcal{M}_2$ of random pure states, in comparison with the brute-force implementation (blue). The dashed lines indicate the corresponding theoretical scaling behaviors, $\mathcal{O}(8^N)$ and $\mathcal{O}(N4^N)$, respectively. (b) The sample-averaged 2-SRE $\mathcal{M}_2$ of Haar-random pure states as a function of the system size N . For each N , we independently sample 20–100 Haar-random states and compute the average. The dashed line denotes the annealed Haar benchmark $\mathcal{M}_{2,\text{ann}}^{\text{Haar}} = -\log_2 \mathbb{E}_{\text{Haar}}[\zeta_2] = \log_2(2^N + 3) - 2$, where $\zeta_2 = d^{-1} \sum_{P \in \mathcal{P}_N} |\langle \psi | P | \psi \rangle|^4$ [52]. At finite d , this benchmark need not be exactly identical to $\mathbb{E}_{\text{Haar}}[\mathcal{M}_2]$, although the two become close as the distribution concentrates.

causes tensor-network-based methods for evaluating $\mathcal{M}_2$ to encounter an exponentially growing bond-dimension bottleneck as the system size increases. A volume-law 2-SRE need not be Haar-typical: For example, a single layer of local T gates acting on $|+\rangle^{\otimes N}$ already produces the extensive value $N \log_2(4/3)$, whose coefficient remains below the Haar-typical coefficient. The Haar result is therefore a benchmark for typical generic pure states rather than a criterion for extensivity or maximality. In contrast, the XOR-FWHT algorithm proposed in this work does not rely on the bond dimension of a tensor-network representation, but instead acts directly on the state vector itself, thereby enabling deterministic evaluation of $\mathcal{M}_2$ for Haar-random states. As shown in Fig. 1(b), the numerical sample means are in good agreement with the annealed Haar benchmark [52], validating the applicability of the algorithm to highly entangled states. At finite d , the annealed value $-\log_2 \mathbb{E}_{\text{Haar}}[\zeta_2]$ need not be exactly equal to $\mathbb{E}_{\text{Haar}}[-\log_2 \zeta_2]$, although concentration makes the two close at large dimension.

Physical application examples. In this section, we numerically present several representative physical examples by using Algorithm 1 for evaluating 2-SRE.

Example 1. Hamiltonian quench dynamics. We consider the transverse-field Ising model with an additional longitudinal field [54,91–96] (TFIM with longitudinal field), whose Hamiltonian is given by

$$H_{\text{TFIM+LF}} = -J \sum_{i=0}^{N-2} Z_i Z_{i+1} - h_x \sum_{i=0}^{N-1} X_i - h_z \sum_{i=0}^{N-1} Z_i, \quad (8)$$

where we impose open boundary conditions, set $J = 1$, $h_x = 1.5$, and $h_z = 1.5$, and consider system sizes $N = 14$ and 16.

The initial state is chosen as a tensor product of random single-qubit states,

$$|\psi_0\rangle = \bigotimes_{i=1}^N \left(\cos \frac{\theta_i}{2} |\uparrow\rangle + e^{i\phi_i} \sin \frac{\theta_i}{2} |\downarrow\rangle \right), \quad (9)$$

where the angles are sampled independently as $\theta_i \sim \text{Uniform}[0, \pi]$ and $\phi_i \sim \text{Uniform}[0, 2\pi)$. This specifies the angular ensemble used here; it is not the single-qubit Haar measure, for which $\cos \theta_i$ would be uniform on $[-1, 1]$. Although these initial states are unentangled, they are generally nonstabilizer states and therefore carry local nonstabilizer resources. Starting from 100 different random initial states, we perform the real-time evolution using a matrix-exponential-action method inspired by the Krylov-subspace approach. In practice, we implement $e^{-iHt}|\psi_0\rangle$ via `scipy.sparse.linalg.expm_multiply`. We then compute $\mathcal{M}_2$ during the evolution and average the results over all initial states.

The resulting dynamical behavior is shown in Fig. 2. The ensemble-averaged $\mathcal{M}_2(t)$ grows rapidly and approaches the full-Hilbert-space Haar reference from below over the simulated interval. Panel (b) resolves the small difference from this reference, which continues to decrease at the latest times shown. Because the Hamiltonian evolution conserves energy, the full Haar ensemble serves as an unconstrained benchmark. Conservation laws, symmetries, and other dynamical constraints can restrict anticoncentration and produce stabilizer

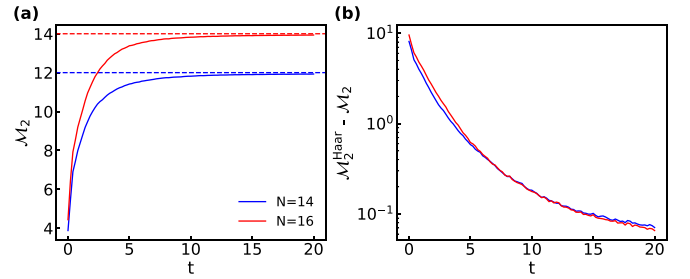


FIG. 2. Time evolution of the 2-SRE $\mathcal{M}_2$ for $N = 14$ and 16 spins evolved under the transverse-field Ising model with a longitudinal field, computed using our XOR-FWHT algorithm with time step $\Delta t = 0.2$. (a) Starting from 100 random product states, $\mathcal{M}_2(t)$ grows rapidly and approaches the full-Hilbert-space Haar reference (dashed line) from below. (b) The finite-time difference from the unconstrained Haar reference is resolved over the displayed interval.

entropies below full-Hilbert-space Haar-typical values [53]. This example demonstrates that the XOR-FWHT algorithm can resolve small differences in the 2-SRE of time-evolved many-body states.

Example 2. Random unitary circuit quench dynamics. We extend our analysis to dynamical quantum states generated by local random unitary circuits. Unlike Haar-random states sampled directly from Hilbert space, these states are produced progressively from a simple product state through nonequilibrium unitary evolution. As such, they exhibit generic features of many-body dynamics. Random unitary circuits themselves constitute a standard framework for the study of nonequilibrium many-body physics, and have been widely employed to characterize operator spreading and scrambling [14,19,97], entanglement growth, equilibration, and typicality [90,98,99], as well as the emergence of approximate unitary designs and Haar-like behavior [100–102]. More recently, they have also become an important platform for investigating the dynamics of magic [46,57,63]. Therefore, studying the time evolution of the 2-SRE in this setting enables us to track how the 2-SRE gradually approaches its Haar-typical value during the dynamical process. Specifically, we consider dynamical quantum states generated by a one-dimensional nearest-neighbor brickwork random unitary circuit on an N -qubit chain with open boundary conditions, initialized in the product state $|\psi_0\rangle = |\uparrow\rangle^{\otimes N}$. At each discrete time step, the evolution consists of two successive layers of two-qubit unitary gates acting on the even and odd bonds, respectively; all two-qubit gates are sampled independently from the Haar measure on $U(4)$. The quantum state then evolves according to $|\psi_t\rangle = U_t |\psi_{t-1}\rangle$, where U_t denotes the brickwork random circuit for the t th time step. At each time step, we record the full quantum state and evaluate the corresponding 2-SRE.

As illustrated in Fig. 3(b), we first simulate the random-circuit dynamics for system sizes $N = 12, 14, 16$ using `TensorCircuit-NG` [103,104], and then feed the resulting state vectors at each discrete time step into our XOR-FWHT algorithm to evaluate $\mathcal{M}_2$ deterministically, up to floating-point roundoff. As the circuit evolves, the state rapidly develops strong entanglement and gradually approaches Haar-like behavior, making the evaluation of $\mathcal{M}_2$ increasingly challenging for methods that rely on low-entanglement

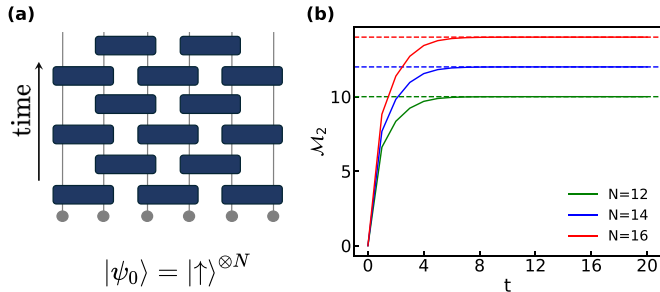


FIG. 3. (a) Schematic of the one-dimensional brickwork random unitary circuit. The system is initialized in the product state $|\psi_0\rangle = |\uparrow\rangle^{\otimes N}$ and evolves under layers of nearest-neighbor Haar-random two-qubit gates. (b) Dynamics of the 2-SRE $\mathcal{M}_2$ for system sizes $N = 12, 14, 16$, obtained by first simulating the circuit evolution and then feeding the resulting state vectors into our XOR-FWHT algorithm, with averaging over 100 circuit realizations. The dashed horizontal lines indicate the corresponding full-Hilbert-space annealed Haar benchmarks.

tensor-network representations. For all system sizes considered, $\mathcal{M}_2$ grows rapidly from the initial product state and approaches the corresponding Haar-random-state benchmark over the displayed circuit depths. This example extends the Haar-state benchmark to a genuinely dynamical setting and shows that our method can efficiently and accurately evaluate the 2-SRE for generic random-circuit states while directly resolving their approach to Haar-typical magic.

Conclusion and outlook. In this work, we have derived an exact algebraic reformulation of the 2-SRE $\mathcal{M}_2$ and implemented it as a deterministic state-vector algorithm that is numerically exact up to floating-point roundoff. Its computational complexity is $\mathcal{O}(N4^N) = \mathcal{O}(d^2 \log_2 d)$, thereby achieving an exponential speedup over the brute-force cost $\mathcal{O}(8^N) = \mathcal{O}(d^3)$ while remaining exponential in N . Starting from the definition of the SRE, we reorganized the Pauli spectrum into XOR-shifted correlations evaluated by the FWHT, without stochastic sampling or state-structure

assumptions. The benchmarks and applications demonstrate deterministic evaluation for generic, highly entangled state vectors.

The present algorithm is designed for states supplied as full state vectors. Tensor-network contractions, structured-state formulas, and sampling methods remain preferable in their respective regimes; no generic complexity advantage is claimed for combining the present state-vector FWHT procedure with a tensor-network representation.

Looking ahead, several promising directions remain for further exploration. One direct application of the present algorithm is the systematic study of the dynamical behavior of $\mathcal{M}_2$ in a variety of interacting quantum many-body models [53,59,96,105–107], as well as its interplay with other physical quantities such as entanglement [57,108,109] and operator spreading [16,97,110]. Beyond these immediate applications, the complete Pauli expectation spectrum returned by the transforms can also enter SRE moments at other Rényi indices. The derivation, complexity benchmarks, and physical applications presented here focus on $\alpha = 2$; a systematic study of other indices is left for future work. It would also be interesting to apply this kind of algorithmic idea on real quantum hardware [111].

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Data availability. The numerical data that support the findings of this study are publicly available [112].

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