

Modeling instances of partially coherent fields with an arbitrary Wigner distribution function

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We introduce a general-purpose numerical algorithm for simulating single-shot realizations of partially coherent electromagnetic radiation, based solely on its known statistical correlations. The algorithm operates by shaping complex Gaussian noise using an arbitrary Wigner distribution function, without assuming factorability in the corresponding Fourier domains. This enables the generation of field realizations for a broad class of sources, including thermal light, synchrotron radiation, and, notably, free-electron laser (FEL) pulses in the linear regime. Compared to Monte Carlo particle-based methods, our approach circumvents the need for detailed modeling of emission mechanisms and offers a substantially more efficient means of generating large ensembles of stochastic fields. As a case study, we simulate FEL pulses with spectrally nontrivial coherence properties (e.g., frequency chirps), thereby demonstrating the versatility of the algorithm. We analytically confirm that ensemble-averaged intensity and correlation properties are correctly reproduced. The algorithm provides a computationally lightweight framework for modeling partially coherent radiation with an arbitrary Wigner distribution function and can serve both as a theoretical tool and as a practical simulation resource for users' experiments at FELs.

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I. INTRODUCTION

Research on synchrotron radiation sources and free-electron lasers (FELs) has led to significant advances in radiation generation, producing beams with unprecedented properties such as high brilliance, ultrashort pulse durations [1–9], exceptional radiation coherence, and broad tunability, far exceeding the capabilities of conventional laboratory x-ray sources. Advances in accelerator physics have pushed the design of storage rings to their performance limits [10,11], enabling a qualitative leap in transverse coherence at higher photon energies compared to previous-generation facilities, as illustrated by comparing the design parameters of PETRA III [12] and PETRA IV [13]. Over the past few decades, FELs [14–16] have been extensively studied, with numerous successful commissioning milestones [17–24]. Often (though not always [25–35]), FELs operating in the self-amplified spontaneous emission (SASE) regime emit radiation with high transverse coherence while exhibiting more complex temporal characteristics [36–39]. Recent efforts have concentrated on pushing the limits of pulse duration into the subfemtosecond regime [2,3,7,40,41].

SASE FELs and synchrotron radiation sources represent canonical examples of partially coherent sources, exhibiting partial coherence in both transverse and

longitudinal dimensions. The coherence properties of synchrotron radiation have been extensively investigated, both theoretically [42–45] (and in numerous additional works cited therein) and experimentally. For instance, transverse coherence was characterized using classical intensity interferometry methods to extract the electron beam size [46] and the pulse duration [47]. Studies of the coherence properties of SASE FELs began in parallel with experimental efforts toward achieving lasing. These properties were first described theoretically [48,49], and subsequently explored in a wide range of experimental studies, including Refs. [50–55], among others. The list of references cited here is not exhaustive; a comprehensive review of coherence studies in SASE FELs and synchrotron radiation would warrant a dedicated survey. In this work, we do not consider seeded and self-seeded FELs, which have different statistical properties.

With this rapid progress in undulator-based physics, simulation tools are adapting to accurately reflect the properties of the emitted radiation [56–61]. A straightforward approach to simulating partially coherent synchrotron radiation fields is to use Monte Carlo-like methods that model the entire emission process on a particle-by-particle basis. Such simulations, especially when performed with full three-dimensional FEL codes [56–60], are time consuming and may be impractical for generating large ensembles of statistically independent realizations. Fortunately, not all applications require the same level of detail. In fact, although these tools are essential for optimizing the performance of facilities by the accelerator physics experts, they may not be optimal for the users of these facilities due to the complexity and specificity of the underlying accelerator physics. This is particularly true for cutting-edge lasing schemes, where the simulations themselves represent the state of the art in computational modeling with Genesis 1.3 [56], such as simulations of attosecond

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experiments [40] or cases involving radiation with frequency chirps [62]. For modeling such experiments in the context of simulating x-ray-light-matter interactions, only a robust statistical characterization of the radiation pulses is required, as noted in Ref. [63]. This characterization must be sufficiently accurate to represent real radiation properties, while details related to simulating the emission process itself should be avoided whenever possible when simulating users' experiments.

As a matter of fact, both synchrotron and FEL radiation can be described using statistical optics [44,64,65]. These stochastic sources emit electromagnetic fields that fluctuate both spatially (across the beam profile) and temporally (along the pulse), making it impossible to predict the exact electric field distribution of a single realization. Instead, such radiation is characterized by its statistical properties, such as average intensity profiles and correlation functions. These can also be expressed using Wigner function.

A rigorous phase-space representation for describing stochastic electromagnetic fields has been established in recent developments of Wigner-function-based formalisms. Originally formulated in quantum mechanics as a quasiprobability distribution in position-momentum space [66], with more recent applications reviewed in Ref. [67], the Wigner function was later adopted in optics and electromagnetism as a tool to represent partially coherent fields [68–70] both spatially and temporally. The extensions of this formalism have been applied to model stochastic field propagation and measurement [71–75], including the introduction of vector and tensor Wigner functions that account for polarization and energy transport [73]. Many of these findings directly translate to the field of synchrotron radiation [45,76] and free-electron lasers [62,77–79], providing both a more informative representation for field analysis and a practical framework for measurements.

In this work, we present an algorithm for modeling single-shot realizations of partially coherent radiation based on an arbitrary, *a priori* known Wigner distribution function. This corresponds to having complete information about the correlation function of a Gaussian random field, without requiring factorizability in spatial or frequency domains. The proposed algorithm generates a realization of a stochastic field by shaping complex Gaussian noise using the Wigner function as a mathematical support. This allows for the simulation of fields with arbitrary coherence structure, including spatially or temporally inhomogeneous sources.

Similar noise-based approaches have previously been developed for specific cases of predominantly incoherent radiation. These include simulations of cosmic microwave background fields [80], radiation from bending magnets [81], and, beyond the field of optics, photographic film grain emulation [82] and ocean surfaces in computer graphics [83]. In the context of partially coherent optical fields, such methods are used in modeling stochastic radiation from SASE FELs [63,84]. In our previous work [85], we proposed an algorithm for undulator radiation, based on shaping the field from a complex Gaussian noise using intensity supports in real-space and inverse-space domains to model quasihomogeneous sources.

However, these algorithms were limited to factorizable Wigner functions and therefore are only applicable to quasi-

homogeneous and quasistationary fields. In contrast, the algorithm presented here is not restricted to any particular coherence structure of a field and can be applied to simulate radiation with an arbitrary Wigner function shape for a variety of radiation sources. This is particularly the case in experimental scenarios encountered at FEL facilities, where the longitudinal phase space of the electron beam is imprinted one to one onto the radiation structure [62]. This longitudinal phase space typically exhibits nontrivial shapes due to nonlinear effects occurring during acceleration, e.g., Refs. [86,87].

With appropriate adaptation, the same formalism can be extended beyond optics, offering a general framework for simulating stochastic processes, which are described by Gaussian random fields. In summary, noise-based simulation methods are widely used across disciplines, yet they remain underutilized in optics for modeling the stochastic properties of light.

II. QUASISTATIONARITY AND QUASIHOMOGENEITY OF STOCHASTIC RADIATION SOURCES

Synchrotron radiation sources and FELs produce intrinsically nonstationary pulses that are partially coherent. Normally, statistical optics is derived for *stationary processes* [64,65], whereas finding common laws for *nonstationary* sources is challenging and typically only done for specific cases. For example, in certain instances, both synchrotron radiation and SASE FEL can be considered as *quasistationary* sources [39,88]. This means that the process appears stationary within a specific time interval, which is much shorter than the overall pulse duration and longer than coherence length, though it remains finite in time. This observation is useful as it allows us to derive expressions for quasistationary processes that are very similar to those for stationary processes [44,89]. In this section, we primarily follow the notation of Refs. [64,65], but we extend it to account for the intrinsically nonstationary nature of synchrotron radiation and SASE FEL within this quasistationary approximation [89]. This happens when the longitudinal phase space of the electron beam is chirp less. We then turn to cases beyond the quasistationary approximation. This regime is commonly observed in SASE FELs, particularly when the longitudinal phase space of the electron beam exhibits chirp with nontrivial shapes. The Wigner function formalism has been used to describe such radiation properties, for example, in Ref. [62].

The concept of *quasistationarity* closely parallels the concept of *quasihomogeneity*, as reflected in the mathematical similarity between two fundamental theorems in statistical optics: the van Cittert-Zernike (VCZ) and Wiener-Khinchin (WK) theorems, along with their generalized versions for “quasistationary” and “quasihomogeneous” sources. These theorems are mathematically analogous, as we discuss in Appendixes A and B, and are, in fact, special cases of a more general result from harmonic analysis—Bochner’s theorem. So, the representation of radiation sources coherence properties in the longitudinal domain follows the same principles as in the transverse domains. Here, we present several useful approximations for describing the properties of partially coherent radiation sources, which naturally lead to these two theorems.

We begin by defining the *mutual coherence function*, also referred to as the *second-order field correlation function in the time domain*. This is defined via a *statistical ensemble averaging operation*:

$$\begin{aligned}\Gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2) &= \langle E(z, \vec{r}_1, t_1) E^*(z, \vec{r}_2, t_2) \rangle \\ &\equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N E^{[i]}(z, \vec{r}_1, t_1) E^{*[i]}(z, \vec{r}_2, t_2),\end{aligned}\quad (1)$$

where $E^{[i]}$ is an i th statistical realization of the complex amplitude representation of an electric field E . This electric field E , in fact, represents some random process. The superscript $*$ denotes complex conjugate operation. The ensemble averaging operation is denoted with $\langle \dots \rangle$, N is the number of statistical realizations, t_1 and t_2 are two instants in time, $\vec{r}_{1,2}$ are transverse coordinates, and z denotes the longitudinal position along the optical axis. There is the *time averaging* definition, which is, however, only applicable for stationary sources. We can define a normalized version of the mutual coherence function and write it as

$$\gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2) \equiv \frac{\Gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2)}{\sqrt{I(z, \vec{r}_1, t_1) I(z, \vec{r}_2, t_2)}}, \quad (2)$$

which is referred to as the *complex degree of coherence*. In Eq. (2), $I(z, \vec{r}, t) = \langle E(z, \vec{r}, t) E^*(z, \vec{r}, t) \rangle$. In principle, Eq. (2) fully defines the coherence properties of radiation if E is a Gaussian process, according to the Gaussian moment theorem. However, in practice, the general representation of source properties using Eqs. (1) and (2) can be cumbersome, and it is often advantageous to apply approximations to achieve a simplified source representation.

A. Quasistationary approximation

To begin with the simplifications, we express the mutual coherence function (Γ_t) in terms of the complex degree of coherence (γ_t):

$$\begin{aligned}\Gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2) &= \gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2) \\ &\quad \times \sqrt{I(z, \vec{r}_1, t_1) I(z, \vec{r}_2, t_2)}.\end{aligned}\quad (3)$$

We then assume that $\gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2)$ depends only on the time difference $\Delta t = t_2 - t_1$, which implies that the typical width of the radiation spikes is uniform across the intensity distribution $I(z, \vec{r}, t)$ in the time domain. Thus,

$$\begin{aligned}\Gamma_t(z, \vec{r}_1, \vec{r}_2, t_1, t_2) &= \gamma_t(z, \vec{r}_1, \vec{r}_2, t_2 - t_1) \\ &\quad \times \sqrt{I(z, \vec{r}_1, t_1) I(z, \vec{r}_2, t_2)}.\end{aligned}\quad (4)$$

In this manner, we introduce the quasistationary approximation. It can be further simplified by assuming that the width of these spikes is much smaller than the typical width of the distribution $I(\vec{r}, t)$, so that $\sqrt{I(\vec{r}_1, t_1) I(\vec{r}_2, t_2)} \approx f(\vec{r})$:

$$\Gamma_t(z, \vec{r}_1, \vec{r}_2, \vec{r}, \Delta t) \approx \gamma_t(z, \vec{r}_1, \vec{r}_2, \Delta t) f(\vec{r}), \quad (5)$$

where $\vec{r} = (t_1 + t_2)/2$. One must carefully choose the source representation in terms of the exact form of the mutual coherence function and provide appropriate justification for the chosen approximation.

In practice, it is typically not feasible to measure the electric field in the time domain, whereas measuring the spectrum on a shot-to-shot basis is a more accessible task. Therefore, it is reasonable to represent the radiation coherence properties using the *field correlation function*, but in the *frequency domain*:

$$\Gamma_\omega(z, \vec{r}_1, \vec{r}_2, \omega_1, \omega_2) = \langle \bar{E}(z, \vec{r}_1, \omega_1) \bar{E}^*(z, \vec{r}_2, \omega_2) \rangle, \quad (6)$$

where $\bar{E}(z, \vec{r}, \omega)$ is the field representation in the frequency domain. It can be shown that performing a Fourier transform of Eq. (5) in the time domain yields a version of the Wiener-Khinchin theorem applicable to a quasistationary source. This is demonstrated in Appendix A. Specifically, $\Gamma_\omega(z, \vec{r}_1, \vec{r}_2, \bar{\omega}, \Delta\omega)$ is also a factorizable quantity, as follows:

$$\Gamma_\omega(z, \vec{r}_1, \vec{r}_2, \bar{\omega}, \Delta\omega) = G(z, \vec{r}_1, \vec{r}_2, \bar{\omega}) \bar{f}(\Delta\omega), \quad (7)$$

where $\Delta\omega = \omega_2 - \omega_1$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$. $G(z, \vec{r}_1, \vec{r}_2, \bar{\omega})$ is the *cross-spectral density function* and $\bar{f}(\Delta\omega)$ is the Fourier transform of $f(\vec{r})$, as shown in Appendix A. The fact that Eq. (7) is factorizable can be deduced directly from the expression for the field in the frequency domain: $\langle \bar{E}(z, \vec{r}_1, \omega_1) \bar{E}^*(z, \vec{r}_2, \omega_2) \rangle$, which, for example, was shown in Ref. [44] for the case of synchrotron radiation.

In a similar way to how we treated the mutual coherence function (Γ_t) in Eq. (7), we introduce the intensity normalization of the cross-spectral density function (G), which is referred to as the *spectral degree of coherence* (g):

$$g(z, \vec{r}_1, \vec{r}_2, \omega) \equiv \frac{G(z, \vec{r}_1, \vec{r}_2, \omega)}{\sqrt{I(z, \vec{r}_1, \omega) I(z, \vec{r}_2, \omega)}}, \quad (8)$$

where $I(z, \vec{r}, \omega) = \langle \bar{E}(z, \vec{r}, \omega) \bar{E}^*(z, \vec{r}, \omega) \rangle$. To proceed with the introduction of the quasihomogeneous approximation, we express the cross-spectral density function (G) in terms of the spectral degree of coherence (g) and consider the source location ($z = 0$):

$$G(0, \vec{r}_1, \vec{r}_2, \omega) = g(0, \vec{r}_1, \vec{r}_2, \omega) \sqrt{I(0, \vec{r}_1, \omega) I(0, \vec{r}_2, \omega)}. \quad (9)$$

B. Quasihomogeneous approximation

A source is called quasihomogeneous if g —the spectral degree of coherence—depends only on the distance difference $\vec{r}_2 - \vec{r}_1$:

$$G(0, \vec{r}_1, \vec{r}_2, \omega) = g(0, \vec{r}_2 - \vec{r}_1, \omega) \sqrt{I(0, \vec{r}_1, \omega) I(0, \vec{r}_2, \omega)}. \quad (10)$$

This definition was introduced in Ref. [90], and we may refer to it as the more general one. However, there are certain sources for which the width of the spikes is much smaller than that of the intensity distribution I at the source, allowing us to further simplify Eq. (10):

$$G(0, \vec{r}, \Delta\vec{r}, \omega) = g(0, \Delta\vec{r}, \omega) I(0, \vec{r}, \omega), \quad (11)$$

where we again exchanged variables $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ and $\vec{r} = (\vec{r}_1 + \vec{r}_2)/2$. This is the definition of a quasihomogeneous source used in the formulation of the generalized van Cittert-Zernike theorem, as presented in textbooks [64,65], and it is highly useful in many practical applications for describing radiation coherence. If the width of the spikes is further reduced,

it can be approximated by a Dirac delta function $\delta(\Delta\vec{r})$:

$$G(0, \vec{r}, \Delta\vec{r}, \omega) = \bar{I}(0, \vec{r}, \omega) \delta(\Delta\vec{r}), \quad (12)$$

which describes a *fully incoherent* source. A source can be approximated using this formula when the point spread function of an optical system is much larger than the actual coherence spot [65]. Nevertheless, a truly incoherent source has a natural coherence length on the order of the radiation wavelength, which represents the smallest feature of the field. For simplicity, in some parts of the text, we will refer to the listed coherence functions (G , Γ_t , Γ_ω) simply as correlation functions, as all of them describe the coherence state of radiation, and a general term is needed for this reference.

Quasihomogeneity and quasistationarity are very useful approximations for representing partially coherent sources, but not all partially coherent sources can be described in this way. When the coherence spot (or the width of $|g|$, $|\gamma_t|$, or $|\gamma_\omega|$) varies along the source intensity distribution, a more general representation is required.

C. Wigner distribution function representation

The Wigner distribution function was initially introduced in quantum mechanics applications and used in signal analysis [62]. However, using the same notation, one can employ it as a useful representation of the state of radiation pulses. It is important to note that, for a Gaussian random process, the Wigner distribution function captures both the intensity distributions of the radiation and its coherence properties. Representing radiation using the Wigner distribution function can reveal its highly nontrivial characteristics, including chirps and nonuniform or nonfactorizable shapes, as demonstrated, for example, in Ref. [62]. From this point forward, we will consider the longitudinal (time-frequency) domains of a radiation pulse for the purpose of describing SASE FELs, which are typically energy chirped due to the energy-chirped electron bunches from which they originate.

We define the Wigner distribution function as follows:

$$\begin{aligned} W(\vec{t}, \vec{\omega}) &= \int_{-\infty}^{\infty} \Gamma_\omega(\vec{\omega}, \Delta\omega) e^{-i\Delta\omega\vec{t}} d(\Delta\omega) \\ &= \int_{-\infty}^{\infty} \Gamma_t(\vec{t}, \Delta t) e^{i\Delta t\vec{\omega}} d(\Delta t). \end{aligned} \quad (13)$$

And for example, substituting Γ_ω from Eq. (7) we obtain

$$\begin{aligned} W(\vec{t}, \vec{\omega}) &= G(\vec{\omega}) \int_{-\infty}^{\infty} \bar{f}(\Delta\omega) e^{-i\Delta\omega\vec{t}} d(\Delta\omega) \\ &= G(\vec{\omega}) f(\vec{t}), \end{aligned} \quad (14)$$

where we have omitted the dependence on transverse coordinates for brevity of notation. As a result, we observe that the factorization of Eq. (7) leads to the factorization of the Wigner function as well.

In previous works [63,85,91], it was proposed to use intensity distributions in the corresponding Fourier domains to generate partially coherent fields, specifically following the equation given below (written here for the time-frequency domain pair):

$$\bar{\phi}(\omega) := \mathcal{F}^{-1}\{\sqrt{f(t)}\mathcal{F}\{\sqrt{G(\omega')}\mathcal{N}(\omega')\}(t)\}(\omega), \quad (15)$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the direct and inverse Fourier transforms with respect to the variable pair (t, ω) , and $\mathcal{N}(\omega) = X(\omega) + iY(\omega)$, where $X(\omega)$ and $Y(\omega)$ are real-valued Gaussian random processes with zero mean and unit variance. Although the authors of Ref. [63] used input noise other than complex Gaussian noise, the correct choice of the input noise is discussed in Appendix C. Equation (15) models instances of a partially coherent field, essentially under the quasistationary approximation of the source when the Wigner function factorizes as in Eq. (14).

One can simplify Eq. (15) to model a stationary source by assuming that $f(t)$ is constant in the time domain. In this case, we obtain the following equation:

$$\bar{\phi}(\omega) := \sqrt{G(\omega)}\mathcal{N}(\omega), \quad (16)$$

where we use the assignment operator $:=$ to indicate that the form of the field $\bar{\phi}(\omega)$ depends on the simulated case: stationary or quasistationary. The similar expressions for the transverse domain can be derived for homogeneous and quasihomogeneous sources, as shown in Appendix C. These results follow from the mathematical similarity between the WK and VCZ theorems, demonstrated in Appendixes A and B.

III. WIGNER-FUNCTION-BASED GAUSSIAN RANDOM FIELD GENERATOR

Building on the notation in Eq. (15), it is natural to consider whether the approach can be generalized to cases where the radiation field is represented by a nonfactorizable Wigner function. Such a generalization would allow for modeling more complex coherence properties beyond the quasistationary approximation. Given this observation, we can see that the algorithm presented in Eq. (15) actually “contains” the Wigner distribution in a factorized form, which suggests that we may try to express the simulated field in the following, more general form:

$$\bar{\phi}(\omega) := \int_{-\infty}^{\infty} \sqrt{W(t, \omega)} \mathcal{N}(t) e^{i\omega t} dt, \quad (17)$$

assuming that W is positive definite, which holds for partially coherent radiation, as shown in Ref. [62], and $\mathcal{N}$ is the complex Gaussian noise. It is important to note that Eq. (17) is not a Fourier transform, since the Wigner distribution also depends on the frequency ω . A visual representation of this algorithm is provided in Fig. 1. Fundamentally, this algorithm is similar to those previously introduced in Eqs. (15) and (16) in that the statistical properties of the Gaussian random process must be determined in advance. These algorithms, applicable to stationary and quasistationary sources, employ radiation intensities, meaning that the coherence properties are already known. This point becomes evident, for instance, when one considers the expression for the generalized Wiener-Khinchin theorem, which we show in Appendix A. Therefore, it either requires knowledge of the intensity distribution and the assumption of the applicability of the theorem, or direct knowledge of the correlation function. Similarly, in the case of the Wigner-function-based approach, once the correlation function is known, $W(\vec{t}, \vec{\omega})$ can be calculated by definition.

We refer to this family of Gaussian random field generators as **SERVAL**, an acronym that stands for stochastic emission

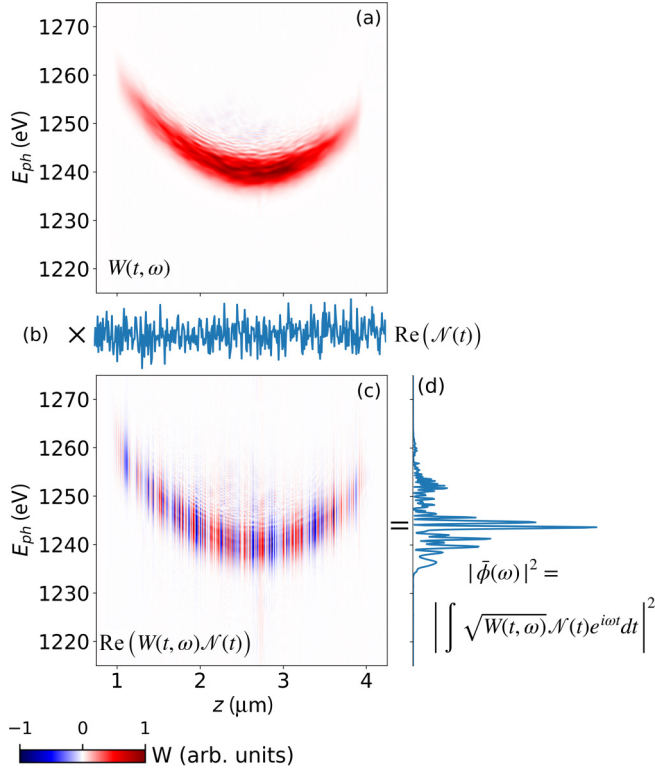


FIG. 1. Visualization of the Wigner-function-based algorithm. A known Wigner function [panel (a)] is multiplied by complex white noise [panel (b)], yielding the distribution shown in panel (c). A subsequent Fourier-like integral over this product produces a single statistical realization of the radiation spectrum [panel (d)].

rapid evaluators. Table I compares different simulation cases with SERVAL for the longitudinal domain: stationary sources, quasistationary sources, and nonstationary sources with arbitrary Wigner function distributions. For consistency in the paper, we use the term SERVAL to denote the algorithm that employs the Wigner function as a constraint, and we specify the simulated case when it differs from this default.

A. Analytical proof

Following this, we conduct a cross-check to determine whether the field emulated using Eq. (17) indeed reproduces the correct cross-spectral density function. We express the

autocorrelation as

$$\begin{aligned} \langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle &= \left\langle \iint_{-\infty}^{\infty} e^{i(\omega_1 t_1 - \omega_2 t_2)} \sqrt{W(t_1, \omega_1) W(t_2, \omega_2)} \right. \\ &\quad \times \mathcal{N}(t_1) \mathcal{N}^*(t_2) dt_1 dt_2 \Big\rangle, \end{aligned} \quad (18)$$

where we impose the condition $W(t, \omega) \geq 0$.

As before, we employ the shot noise property $\langle \mathcal{N}(t_1) \mathcal{N}^*(t_2) \rangle = \delta(t_1 - t_2)$ and integrate over t_1 :

$$\begin{aligned} \langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle &= \int_{-\infty}^{\infty} e^{it'(\omega_1 - \omega_2)} \sqrt{W(t', \bar{\omega}_1) W(t', \omega_2)} dt', \end{aligned} \quad (19)$$

where we have renamed the dummy integration variable t_2 as t' , for notational convenience.

We now assume that the Wigner distribution function can be factorized as follows:

$$W(t, \omega) \approx \sum_n \bar{I}^{(n)}(\omega) f^{(n)}(t), \quad (20)$$

where $f^{(n)}(t)$

$$f^{(n)}(t) = \begin{cases} f(t), & \text{if } t_n \leq t < t_{n+1}, \\ 0, & \text{otherwise,} \end{cases} \quad (21)$$

and $\bar{I}^{(n)}(\omega)$ is a function that approximates the cross section over the interval $t_n \leq t < t_{n+1}$. Each interval $t_{n+1} - t_n$ is chosen such that the coherence length along the pulse remains approximately constant within it.

This representation of the Wigner function dictates the form of the second-order correlation function. In the frequency domain, for example, we have

$$\begin{aligned} \Gamma_{\omega}(\bar{\omega}, \Delta\omega) &= \int_{-\infty}^{\infty} W(t', \bar{\omega}) e^{i\Delta\omega t'} dt' \\ &\approx \int_{-\infty}^{\infty} \sum_n \bar{I}^{(n)}(\bar{\omega}) f^{(n)}(t') e^{i\Delta\omega t'} dt'. \end{aligned} \quad (22)$$

We then interchange the order of integration and summation, and set the integration limits to the n th time interval:

$$\Gamma_{\omega}(\bar{\omega}, \Delta\omega) \approx \sum_n \bar{I}^{(n)}(\bar{\omega}) \int_{t_n}^{t_{n+1}} f(t') e^{i\Delta\omega t'} dt'. \quad (23)$$

TABLE I. Comparison of Gaussian random field generation algorithms in the time-frequency domain.

Stationary source	Quasistationary source (factorable Wigner function)	Nonstationary source (arbitrary Wigner function)
$\bar{\phi}(\omega) = \sqrt{G(\omega)} \mathcal{N}(\omega)$	$\bar{\phi}(\omega) = \mathcal{F}^{-1}\{\sqrt{f(t)} \mathcal{F}[\sqrt{G(\omega')} \mathcal{N}(\omega')](t)\}(\omega)$	$\bar{\phi}(\omega) = \int_{-\infty}^{\infty} \sqrt{W(t, \bar{\omega})} \mathcal{N}(t) e^{i\omega t} dt$
1. Generate complex Gaussian noise $\mathcal{N}(\omega)$	1. Generate complex Gaussian noise $\mathcal{N}(\omega')$	1. Generate complex Gaussian noise $\mathcal{N}(t)$
2. Multiply by $\sqrt{G(\omega)}$	2. Multiply by $\sqrt{G(\omega')}$	2. Multiply by $\sqrt{W(t, \bar{\omega})}$
	3. Fourier transform (to time domain)	3. Integrate over t with $e^{i\omega t}$
	4. Multiply by $\sqrt{f(t)}$	
	5. Inverse Fourier transform (to frequency domain)	

We expect to find that $\langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle$ matches the representation of $\Gamma_\omega(\bar{\omega}, \Delta\omega)$ given in Eq. (23).

Substituting Eq. (20) in Eq. (19), we obtain

$$\begin{aligned} \langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle &\approx \int_{-\infty}^{\infty} \sqrt{\sum_{m,n} \bar{I}^{(n)}(\omega_1) f^{(n)}(t') \bar{I}^{(m)}(\omega_2) f^{(m)}(t')} \\ &\times e^{it'(\omega_1 - \omega_2)} dt'. \end{aligned} \quad (24)$$

Next, we observe that the double summation reduces to a single summation, as only the terms where the product $f^{(n)} f^{(m)}$ has matching indices ($n = m$) yield nonzero contributions:

$$\begin{aligned} \langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle &\approx \int_{-\infty}^{\infty} \sqrt{\sum_n (f^{(n)}(t'))^2 \bar{I}^{(n)}(\omega_1) \bar{I}^{(n)}(\omega_2)} \\ &\times e^{it'(\omega_1 - \omega_2)} dt'. \end{aligned} \quad (25)$$

Since $f^{(n)}(t)$ vanishes outside the specified time interval, only the n th term contributes to the integral over that interval in Eq. (25), allowing us to rewrite it as a sum of integrals:

$$\begin{aligned} \langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle &\approx \sum_n \sqrt{\bar{I}^{(n)}(\omega_1) \bar{I}^{(n)}(\omega_2)} \int_{t_n}^{t_{n+1}} f(t') \\ &\times e^{it'(\omega_1 - \omega_2)} dt', \end{aligned} \quad (26)$$

where $f^{(n)}(t)$ has been replaced by the original $f(t)$, a substitution valid within the integration limits, and the square root expression has been taken outside the integral, as it does not depend on the integration variable. This expression resembles a generalized “quasi” source representation. We can further simplify by assuming that the spectral correlation function width at a given frequency is much narrower than that of the overall pulse spectral distribution $\bar{I}(\omega)$, allowing us to write

$$\begin{aligned} \langle \bar{\phi}(\omega_1) \bar{\phi}^*(\omega_2) \rangle &\approx \sum_n \bar{I}^{(n)} \left(\frac{\omega_1 + \omega_2}{2} \right) \\ &\times \int_{t_n}^{t_{n+1}} f(t') e^{it'(\omega_1 - \omega_2)} dt'. \end{aligned} \quad (27)$$

From this expression, we observe that it coincides with the expected form given in Eq. (23). Following this analytical proof, we present several numerical examples demonstrating the validity of Eq. (17) in emulating partially coherent fields with nonfactorizable Wigner functions.

B. Numerical proof

We simulate a chirped pulse of SASE radiation in the linear regime using the Genesis 1.3 v2 code [56]. We remain in the linear regime because, once saturation is reached, SASE radiation begins to deviate from Gaussian statistics [88,92], whereas Eq. (17) strictly assumes a Gaussian process. We use the Wigner function output from the Genesis 1.3 v2 simulation as input to our algorithm and generate an entirely new ensemble of statistical realizations. Our expectation is that the Wigner function resulting from this emulation will match the input, thereby ensuring the correct statistical representation of every single instance of the simulated field. The relevant comparison is shown in Fig. 2. We also compare radiation

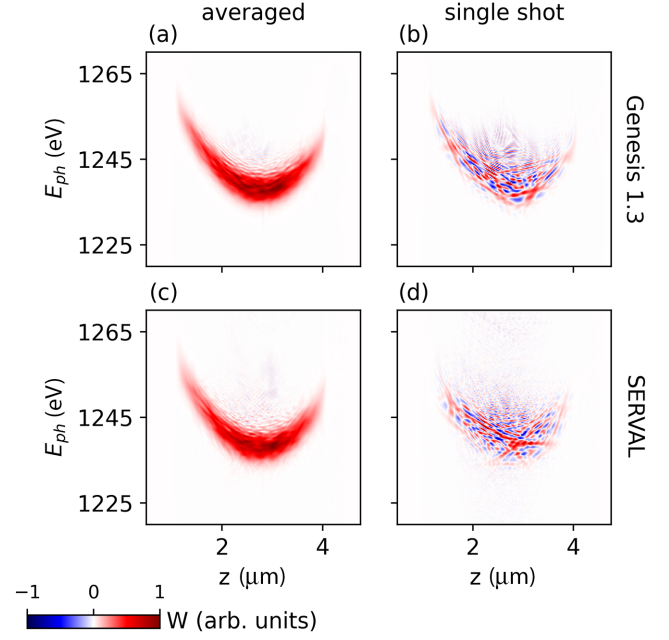


FIG. 2. Wigner distribution functions of chirped SASE FEL radiation. The top row shows results from Genesis 1.3 v2: the ensemble average over 1000 events [panel (a)] and a single-shot realization [panel (b)]. The bottom row shows the corresponding results from SERVAL: the ensemble-averaged Wigner function [panel (c)], obtained by averaging single-shot realizations generated by the algorithm, one of which is shown in panel (d). The close agreement between panels (a) and (c) confirms that the algorithm correctly reproduces the coherence properties.

intensities in Fig. 3 in both the time and frequency domains, which correspond to the marginal distributions of the Wigner distribution function along the respective axes.

Additionally, we verify the preservation of the statistical properties of the resulting field and confirm that the instantaneous intensity follows a negative exponential distribution, as expected for a Gaussian random process. This is demonstrated in Fig. 4. With this comparison, we conclude that the field emulated using Eq. (17) correctly reproduces the correlation

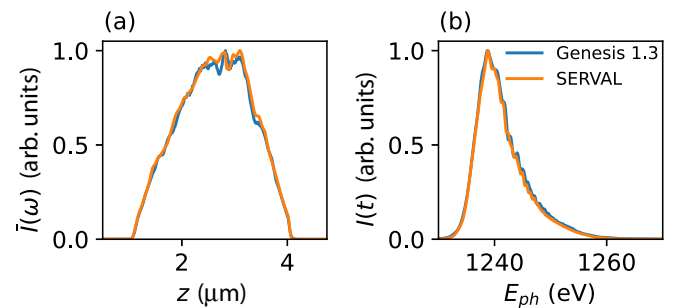


FIG. 3. Marginal intensity distributions obtained by integrating the Wigner functions over energy [panel (a)] and longitudinal position [panel (b)]. The SERVAL algorithm (orange) shows close agreement with the Genesis 1.3 v2 simulation (blue), confirming that the emulated field reproduces the correct intensity profiles in both domains.

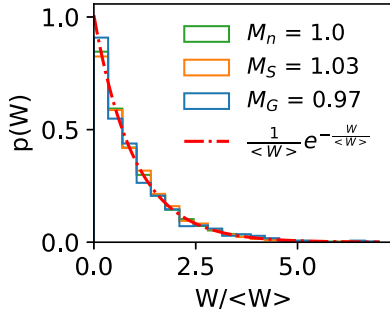


FIG. 4. Probability distribution of the normalized instantaneous intensity $W/\langle W \rangle$ observed through a perfect monochromator. The negative exponential distribution expected for a Gaussian random process (red dot-dashed line) is compared with histograms obtained from complex Gaussian noise (green, $M_n = 1.0$), the SERVAl algorithm (orange, $M_S = 1.03$), and Genesis 1.3 v2 (blue, $M_G = 0.97$). Here, $M = \langle W \rangle^2 / \sigma_W^2$ is the number of modes of the radiation, where σ_W^2 is the variance of the instantaneous intensity. For a single-mode Gaussian process, $M = 1$. All three distributions closely follow the expected statistics, confirming that the emulated field preserves the Gaussian nature of the process.

function embedded in $W(t, \omega)$ and preserves the Gaussian statistics of the process.

To showcase the ability of the algorithm to replicate various phase-space distributions, we slightly modified the input Wigner function by adding two successive radiation pulses with an energy offset, in addition to the quadratically chirped pulse presented in Fig. 2. The results of this modification are presented in Fig. 5. We see that the algorithm correctly reproduces the embedded Wigner function distribution.

One of the key advantages of this approach for generating instances of stochastic fields is its computational efficiency. For example, in the comparison presented in Fig. 2, simulations performed with Genesis 1.3 v2 required the allocation of nodes on a CPU cluster (more than 100 cores). To compute the Wigner distribution, we modeled 1000 field instances, with each simulation taking several minutes, resulting in a total runtime of several hours (and with Genesis 1.3 v4, it may take up to several days). In contrast, generating an entire ensemble of 1000 events from a known Wigner function using SERVAl takes under a minute on a regular laptop.

IV. DISCUSSION AND OUTLOOK

As was pointed out in Ref. [63], approaches for modeling Gaussian random fields are valuable for researchers less familiar with FEL theory, since these approaches enable realistic modeling of SASE FEL pulses that match experimental radiation properties without requiring complex FEL simulation programs. This is essential for accurate simulation of FEL-light-matter interactions, especially in resonant and highly nonlinear processes. In our experience, simulating realizations of SASE based on Eq. (15) has proven particularly useful in validating reconstructed short-pulse durations using machine learning in the following studies [93], replacing nonstochastic field models. It also helped simplify radiation simulation routines aimed at developing phenomenological models of x-ray pulse statistics at FELs [94]. Additionally, the SASE

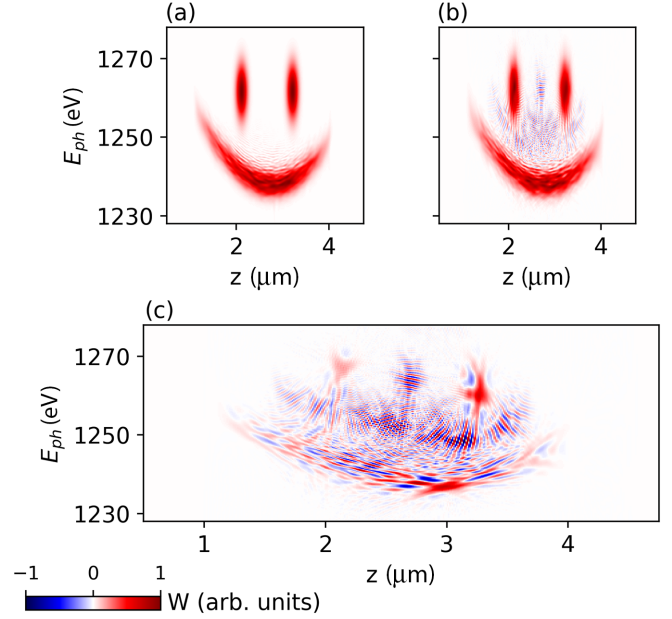


FIG. 5. Modified Wigner distribution function demonstrating the versatility of the SERVAl algorithm. The input Wigner function (a) was constructed by adding two successive radiation pulses with an energy offset to the original Genesis 1.3 output shown in Fig. 2. The ensemble-averaged Wigner function reproduced by SERVAl (b) shows close agreement with the input. Panel (c) displays a single-shot realization generated by the algorithm, illustrating the stochastic structure of the emulated field.

imitator developed by [91] based on Eq. (15) and published in the repository [95] was used to generate synthetic spectra (“dummy spectra”) for debugging diagnostic tools during FEL operations at the European XFEL. We summarize the key differences between SERVAl and Genesis 1.3 in Table II to guide the choice of approach for simulating SASE radiation.

As we show in Appendixes A and B, this type of algorithm can be extended to the transverse domain and used to simulate synchrotron radiation. Examples of such applications include simulations of beamlines with focusing [85], pinhole camera diagnostics for electron beam transverse size at PETRA IV [81], and experiments for electron beam transverse size diagnostics at the European XFEL [96,97].

With this work, we aim to generalize the applicability of Gaussian random field generators to model radiation with arbitrary Wigner distribution functions and demonstrate an example of simulating SASE with nonlinear chirps. Moreover, we hope this usage can be further extended to other stochastic radiation sources exhibiting complex coherence structures.

The true value of these noise-based approaches in simulating fields lies in their exceptionally high computational efficiency for generating ensembles of field realizations. When comparing different methods, a SERVAl-based approach offers orders-of-magnitude advantages over Monte Carlo-like methods or mode decomposition techniques, depending on the practical implementation.

However, in many cases, the resulting correlation function of FEL SASE is unknown, and obtaining it requires time-consuming numerical simulations. Depending on the task,

TABLE II. Comparison of the SERVAL algorithm and *Genesis* 1.3 v2 [56] for generating ensembles of partially coherent FEL radiation.

	SERVAL	<i>Genesis</i> 1.3
Approach	Stochastic field emulator; shaping complex Gaussian noise	First-principles FEL simulation (particle-based Monte Carlo)
Input	Wigner distribution function $W(\vec{r}, \vec{\omega})$ or correlation functions	Electron beam and undulator parameters
Output	Field realization reproducing the input correlation functions	Field realization obtained from the FEL equations
Applicability	Gaussian random fields; linear regime of SASE FEL	Arbitrary SASE FEL regimes including saturation and postsaturation
Computational cost	~ 1 min for 10^3 realizations on a laptop	$\sim$ hours to days for 10^3 realizations on a CPU cluster (> 100 cores)
Expertise required	Minimal; knowledge of radiation statistics	Detailed knowledge of accelerator physics and FEL theory

simpler models may be used to circumvent this issue. For SASE, the radiation frequency chirp replicates the electron beam energy chirp. Once the natural SASE bandwidth is known, a surrogate Wigner distribution of the expected radiation can be constructed. Alternatively, one can simulate a smaller subset ensemble, for example, with *Genesis* 1.3, and then use the obtained Wigner distribution function to significantly extend the whole ensemble with the proposed algorithm.

The proposed algorithm can also be extended to the full six-dimensional phase space to account for complex spatiotemporal coherence dependencies. Additionally, it has been demonstrated that this type of algorithm can be generalized to non-Gaussian processes [98], enabling the representation of fields with non-Gaussian statistics.

V. CONCLUSION

In this paper, we have presented a generalized algorithm for generating single-shot instances of partially coherent radiation with an arbitrary Wigner function distribution. Before introducing this method, we reviewed and analyzed two predecessor algorithms based on the same underlying concept, applied to incoherent and partially coherent sources. These approaches start with complex Gaussian noise and subsequently constrain it in both the direct and inverse domains using appropriate mathematical supports. Following the same paradigm, the proposed method employs the Wigner function distribution to shape fields with nonfactorizable coherence properties. We have analytically demonstrated that the algorithm reproduces the correct statistical properties and correlation function distributions. Furthermore, we have provided representative use cases, focusing on frequency-chirped SASE FEL radiation in the linear regime. We believe that the proposed algorithm will enable a broader community to model radiation sources with complex coherence properties, without the need for specialized first-principles-based software.

ACKNOWLEDGMENTS

We thank Serguei Molodtsov for his interest in this work. We are also grateful to Giovanni Perosa for his many helpful suggestions on the text.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: WIENER-KHINCHIN THEOREM FOR QUASISTATIONARY PROCESSES

In this section, we show that under the quasi-stationary approximation, one can derive an expression analogous to the well-known Wiener-Khinchin theorem. We begin by writing the field correlation function in the frequency domain, by definition:

$$\begin{aligned}\Gamma_{\omega}(\omega_1, \omega_2) &= \langle \bar{E}(\omega_1) \bar{E}^*(\omega_2) \rangle \\ &= \iint_{-\infty}^{\infty} \langle E(t_1) E^*(t_2) \rangle e^{i(\omega_1 t_1 - \omega_2 t_2)} dt_1 dt_2, \quad (\text{A1})\end{aligned}$$

where we bring the ensemble averaging under the integral sign. Then, using the expression in Eq. (5) and substituting it into Eq. (A1), which is equivalent to applying the quasistationary approximation, we obtain

$$\begin{aligned}\Gamma_{\omega}(\omega_1, \omega_2) &= \iint_{-\infty}^{\infty} \gamma_t(t_2 - t_1) f\left(\frac{t_1 + t_2}{2}\right) e^{i(\omega_1 t_1 - \omega_2 t_2)} dt_1 dt_2. \quad (\text{A2})\end{aligned}$$

At this point, it is useful to exchange the variables t_1, t_2 for $\Delta t, \bar{t}$ and, by analogy, ω_1, ω_2 for $\Delta\omega, \bar{\omega}$. With these variable substitutions, it becomes evident that the integral factorizes:

$$\begin{aligned}\Gamma_{\omega}(\bar{\omega}, \Delta\omega) &= \int_{-\infty}^{\infty} \gamma_t(\Delta t) e^{i\bar{\omega}\Delta t} d\Delta t \int_{-\infty}^{\infty} f(\bar{t}) e^{-i\Delta\omega\bar{t}} d\bar{t} \\ &= G(\bar{\omega}) \bar{f}(\Delta\omega). \quad (\text{A3})\end{aligned}$$

One can see that the spectral autocorrelation function $\Gamma_{\omega}(\bar{\omega}, \Delta\omega)$ consists of the spectral density function $G(\bar{\omega})$:

$$G(\bar{\omega}) = \int_{-\infty}^{\infty} \gamma_t(\Delta t) e^{i\bar{\omega}\Delta t} d\Delta t, \quad (\text{A4})$$

and the spectral correlation function $\bar{f}(\Delta\omega)$:

$$\bar{f}(\Delta\omega) = \int_{-\infty}^{\infty} f(\bar{t}) e^{-i\Delta\omega\bar{t}} d\bar{t}. \quad (\text{A5})$$

Analyzing Eqs. (A4) and (A5), one can see that Eq. (A4) corresponds to the result of the Wiener-Khinchin theorem, while Eq. (A5) represents its “anti” formulation. Overall, we can refer to this derivation and its outcome as a generalized Wiener-Khinchin theorem for quasistationary sources. As the next step, we show that the same mathematical relation can be formulated for quasihomogeneous sources.

APPENDIX B: GENERALIZED VAN CITTERT-ZERNIKE THEOREM

One can derive an expression for the cross-spectral density function in inverse space, i.e., in angular (transverse momentum) coordinates, analogous to the Wiener-Khinchin theorem for a quasistationary source, as presented above. To do this, we define the transverse-domain Fourier transform of the field $\bar{E}(\vec{r}, \omega)$ as follows:

$$\hat{E}(\vec{k}_\perp, \omega) = \iint_{-\infty}^{\infty} \bar{E}(\vec{r}, \omega) e^{i(\vec{k}_\perp \cdot \vec{r})} d\vec{r}, \quad (\text{B1})$$

where $\vec{k}_\perp$ is the transverse spatial frequency vector. The corresponding inverse Fourier transform back to real space is given by

$$\bar{E}(\vec{r}, \omega) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \hat{E}(\vec{k}_\perp, \omega) e^{-i(\vec{k}_\perp \cdot \vec{r})} d\vec{k}_\perp, \quad (\text{B2})$$

where we have defined the transverse component of the wave vector as $\vec{k}_\perp$ and the dot product as $(\vec{k}_\perp \cdot \vec{r})$. Then, by writing the cross-spectral density at the source location in the inverse-space domain:

$$\begin{aligned} \hat{G}(0, \vec{k}_{\perp 1}, \vec{k}_{\perp 2}, \omega) \\ &= \langle \hat{E}(\vec{k}_{\perp 1}, \omega) \hat{E}^*(\vec{k}_{\perp 2}, \omega) \rangle \\ &= \iiint \iiint \langle \bar{E}(\vec{r}_1, \omega) \bar{E}^*(\vec{r}_2, \omega) \rangle e^{i((\vec{k}_{\perp 1} \cdot \vec{r}_1) - (\vec{k}_{\perp 2} \cdot \vec{r}_2))} d\vec{r}_1 d\vec{r}_2, \end{aligned} \quad (\text{B3})$$

as usual, introducing a new set of variables $\vec{k}_\perp = (\vec{k}_{\perp 1} + \vec{k}_{\perp 2})/2$, $\Delta\vec{k}_\perp = \vec{k}_{\perp 1} - \vec{k}_{\perp 2}$, analogous to $\vec{r} = (\vec{r}_1 + \vec{r}_2)/2$, $\Delta\vec{r} = \vec{r}_1 - \vec{r}_2$, and factorizing the integrals accordingly, we obtain

$$\begin{aligned} \hat{G}(0, \vec{k}_\perp, \Delta\vec{k}_\perp, \omega) \\ &= \iiint \iiint g(0, \Delta\vec{r}, \omega) \bar{I}(0, \vec{r}, \omega) e^{-i((\vec{k}_\perp \cdot \Delta\vec{r}) + (\Delta\vec{k}_\perp \cdot \Delta\vec{r}))} \\ &\quad \times d\vec{r} d\Delta\vec{r} \\ &= \iint_{-\infty}^{\infty} g(0, \Delta\vec{r}, \omega) e^{-i(\vec{k}_\perp \cdot \Delta\vec{r})} d\Delta\vec{r} \\ &\quad \times \iint_{-\infty}^{\infty} \bar{I}(0, \vec{r}, \omega) e^{-i(\Delta\vec{k}_\perp \cdot \vec{r})} d\vec{r} \\ &= \hat{I}(0, \vec{k}_\perp, \omega) \hat{g}(0, \Delta\vec{k}_\perp, \omega). \end{aligned} \quad (\text{B4})$$

As one can see, this result is very similar to the formulation of the generalized van Cittert-Zernike theorem, but written for the cross-spectral density at the source in the inverse-space domain. This clear similarity arises from the fact that

the free-space propagator essentially acts as a Fourier transform, which holds in the far-zone approximation. The only difference between Eq. (B4) and the generalized van Cittert-Zernike theorem lies in the quadratic phase factor, expressed as $e^{i(k/z)\vec{r} \cdot \Delta\vec{r}}/(\lambda z)^2$, where $k = 2\pi/\lambda$.

The van Cittert-Zernike theorem and its generalized version [65,90], along with the expression for the inverse space in Eq. (B4), provide practical relations between the source distribution and the coherence spot in the far-zone/inverse space, as well as between the coherence spot at the source and the radiation size in the far-zone/inverse space. This leads to an intuitive understanding: Large features of radiation at the source (source size distribution) transform into small features in the far-zone/inverse space (coherence spot), and vice versa. Clearly, Eq. (B4) is now a full mathematical twin of the Wiener-Khinchin theorem for quasistationary sources in Eq. (A3).

APPENDIX C: GAUSSIAN RANDOM FIELD GENERATORS IN THE TRANSVERSE DOMAIN

Before presenting the field generation algorithms for the transverse domain, it is important to consider related works in the literature. There are methods for simulating Gaussian random processes with noise and a given correlation function, as presented in a general mathematical form in Ref. [99] and applied to simulations of the cosmic microwave background [80]. In terms of classical optics, this case is analogous to simulating fully incoherent sources. An algorithm for partially coherent radiation, specifically for frequency chirp-less SASE radiation, was first introduced in Ref. [63] and later developed by Serkez [91] and published in ocelot package [60,84,95]. We applied this algorithm for partially coherent radiation to model undulator radiation in Ref. [85]. We call it SERVAL, originally an acronym for synchrotron radiation rapid evaluator. In this paper, we replace “synchrotron” with “stochastic” to emphasize the capability algorithms to simulate sources beyond synchrotron radiation, and we use the plural form “evaluators” to indicate that it represents a family of algorithms. In this Appendix, we review the SERVAL algorithms for an incoherent source, which is homogeneous, and for a partially coherent source under the quasihomogeneous approximation.

1. Simulation of an incoherent source

To model a single instance of the incoherent field, we generate complex Gaussian noise $\mathcal{N}(\vec{r}) = X(\vec{r}) + iY(\vec{r})$, where $X(\vec{r})$ and $Y(\vec{r})$ are real-valued Gaussian random processes with zero mean and unit variance. We discuss the choice of noise type in Appendix C. This noise is then constrained by multiplying it by the source intensity distribution to form a field $\bar{\phi}$:

$$\bar{\phi}(\vec{r}, \omega) = \sqrt{\bar{I}(\vec{r}, \omega)} \mathcal{N}(\vec{r}). \quad (\text{C1})$$

By performing a Fourier transform of this field in inverse space, we obtain

$$\hat{\phi}(\vec{k}, \omega) = \mathcal{F}\{\sqrt{\bar{I}(\vec{r}, \omega)} \mathcal{N}(\vec{r})\}(\vec{k}). \quad (\text{C2})$$

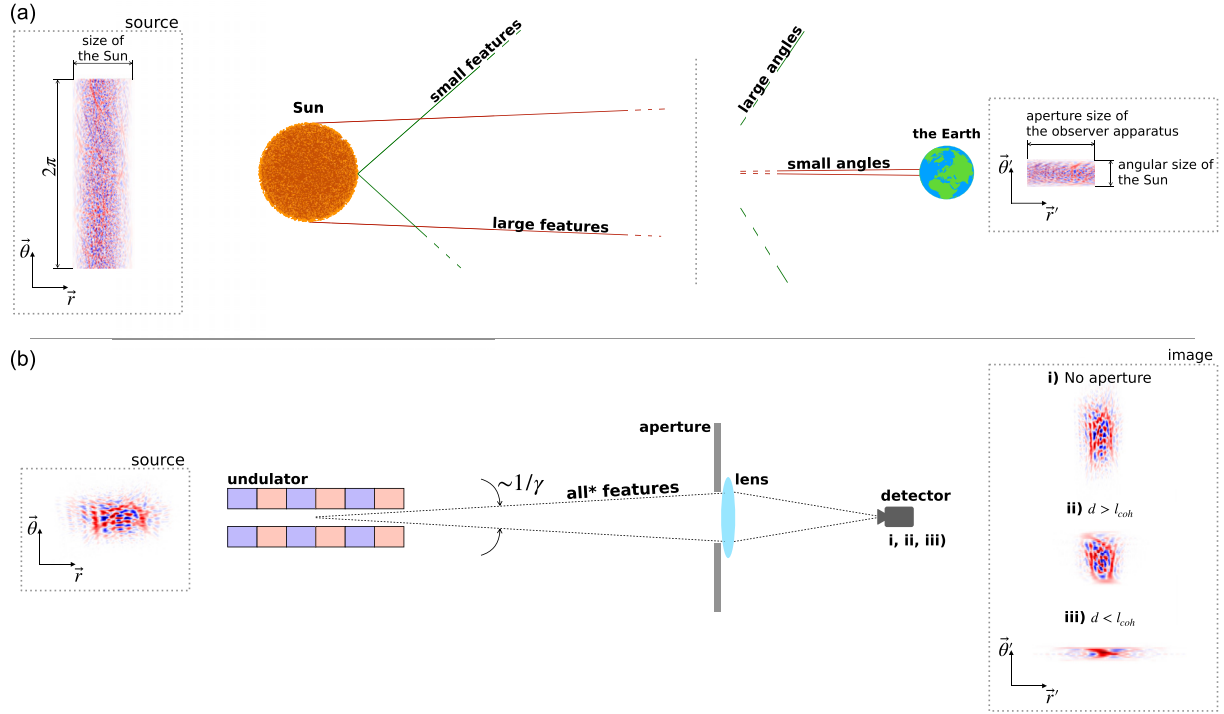


FIG. 6. Schematic comparison of an incoherent and a partially coherent radiation source, illustrating the relationship between source properties and far-field coherence. (a) An incoherent source—the Sun. The coherence length at the source is comparable to the radiation wavelength, and the source exhibits features ranging in size from the wavelength scale to the full solar disk. Smaller features diverge at larger angles, while larger features concentrate at smaller angles; the latter are captured by the correlation function in the far field. Resolving smaller features requires a correspondingly larger aperture in the observation plane. (b) A partially coherent source—undulator radiation as an example of a quasihomogeneous source. All radiation features are confined within an angular cone of $\sim 1/\gamma$, where γ is the Lorentz factor of the electron bunch emitted this radiation. The smallest features at the source are set by the coherence length, which determines the radiation size in the far field upon free-space propagation; conversely, the effective source size maps onto the correlation length in the far field, in accordance with the generalized van Cittert-Zernike theorem. Subfigures (i)–(iii) illustrate how the aperture size d relative to the coherence length l_{coh} controls the degree of coherence in the focused image.

To prove that Eqs. (C1) and (C2) correctly represent a fully incoherent electric field at the source location, we write its cross-spectral density function as

$$\langle \hat{\phi}(\vec{k}_1, \omega) \hat{\phi}^*(\vec{k}_2, \omega) \rangle = \left\langle \iint_{-\infty}^{\infty} \sqrt{\bar{I}(\vec{r}_1, \omega) \bar{I}(\vec{r}_2, \omega)} \mathcal{N}(\vec{r}_1) \times \mathcal{N}^*(\vec{r}_2) e^{i(\vec{r}_1 \vec{k}_1 - \vec{r}_2 \vec{k}_2)} d\vec{r}_1 d\vec{r}_2 \right\rangle. \quad (\text{C3})$$

Exchanging the order of ensemble averaging and integration, and accounting for the properties of Gaussian white noise, $\langle \mathcal{N}(\vec{r}_1) \mathcal{N}^*(\vec{r}_2) \rangle = \delta(\vec{r}_2 - \vec{r}_1)$, and taking the integral, we write

$$\begin{aligned} \langle \hat{\phi}(\vec{k}_1, \omega) \hat{\phi}^*(\vec{k}_2, \omega) \rangle &= \iint_{-\infty}^{\infty} \sqrt{\bar{I}(\vec{r}_1, \omega) \bar{I}(\vec{r}_2, \omega)} \delta(\vec{r}_1 - \vec{r}_2) e^{i(\vec{r}_1 \vec{k}_1 - \vec{r}_2 \vec{k}_2)} d\vec{r}_1 d\vec{r}_2 \\ &= \int_{-\infty}^{\infty} \bar{I}(\vec{r}', \omega) e^{i\vec{r}'(\vec{k}_1 - \vec{k}_2)} d\vec{r}', \end{aligned} \quad (\text{C4})$$

where $\vec{r}'$ is the integration variable and follows the definition of $\vec{r}_{1,2}$.

The last equality is the final result of the van Cittert-Zernike theorem, but without the quadratic phase factor, which we discuss in Appendix B. Therefore, we conclude that the field written in Eq. (C1) exhibits the correct coherence

properties. This result is not surprising, since by generating the appropriate noise and carefully constraining the radiation in both domains, one imposes the necessary and sufficient statistical and coherence properties to fully specify the Gaussian random process from the outset in Eq. (C1).

Equation (C1) is formulated for radiation at the source location, which makes it virtually impossible to numerically propagate this radiation to the observation plane. This difficulty arises because fully incoherent radiation diverges into the full solid angle of 4π , as depicted in Fig. 6(a), leading to numerical artifacts. Instead, we can simulate the radiation directly at the observation position and substitute the visible angular source size from the observer's perspective into Eq. (C1). This means that the field shaping in Eq. (C1) should be performed in the inverse-space domain.

This algorithm is highly efficient when simulating an incoherent source on a realization-by-realization basis. More importantly, it is applicable for simulating a variety of incoherent sources. For example, using this approach, one can simulate the cosmic microwave background [80], the spatial distribution of radiation from a bending magnet [81], or even replicate Thomas Young's interference experiment.

The original Young's experiment was reported by Young himself in the Bakerian Lecture delivered on November 24,

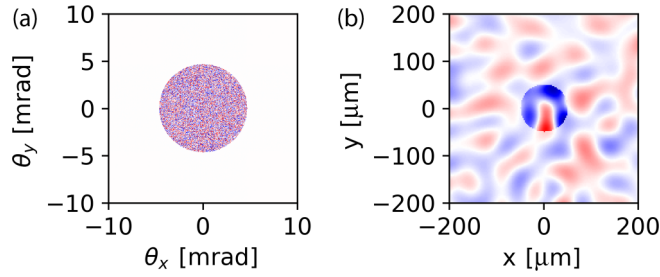


FIG. 7. Simulated field distributions of solar radiation at a wavelength of 500 μm , generated using the SERVAL algorithm for an incoherent source. (a) The angular field distribution as seen from the Earth. (b) The corresponding spatial field distribution at the observation position. The hole made by a needle (100 μm) is represented as a mask over the subplot in panel (b) to illustrate Young's experiment, as shown in Fig. 8. The color scale represents positive and negative values of the real part of the field.

1803 [100]. Here, we provide a quote from the lecture describing his experimental setup and results:

"I made a small hole in a window-shutter, and covered it with a piece of thick paper, which I perforated with a fine needle... I brought into the sunbeam a slip of card, about one-thirtieth of an inch in breadth, and observed its shadow, either on the wall, or on other cards held at different distances. Besides the fringes of colours on each side of the shadow, the shadow itself was divided by similar parallel fringes, of smaller dimensions, differing in number, according to the distance at which the shadow was observed, but leaving the middle of the shadow always white."

In our simulation, we divided the visible spectrum into ten colors and simulated the propagation and diffraction of each color on a "one-thirtieth-inch" obstacle separately. We assumed the distance from the window shutter to the obstacle (a slip of card) to be 0.2 m, and the distance from the obstacle to the screen to be 1 m. We present the field distribution at the source and at the pinhole in the window shutter in Fig. 7. We then performed a numerical replica of the experiment and obtained the results shown in Fig. 8. In this image, we converted the resulting intensities into a red-green-blue representation and adjusted the color appearance using a graphics editor (white balance correction, slight contrast adjustments, and saturation enhancement) to mimic what the human eye would perceive.

With this example, we present an algorithm for simulating the stochastic properties of incoherent light sources. The algorithm outlines the general principle: If the source follows Gaussian statistics, the initial noise for the simulated field is Gaussian white noise, which is then constrained by the average intensity of the source distribution. This intensity distribution is intrinsically related to the correlation function in the conjugate domain, as explained by the van Cittert-Zernike theorem. This approach is highly advantageous, as it does not require a priori knowledge of the correlation function, only the source intensity distribution. A natural next step in developing this algorithm is to account for the finite divergence of the source, which *inevitably* leads to partial coherence.

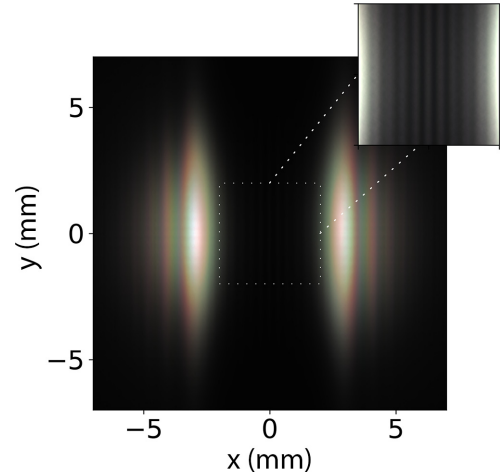


FIG. 8. Result of the original Young's interference experiment with a slip of card. We observe colorful fringes and the shadow cast by the obstacle, which is divided by white parallel lines (shown in the callout). To distinguish these lines, we adjusted the colormap in the callout to correspond to a sensitivity tentimes greater than that of the main figure.

2. Simulation of partially coherent sources

In the final results of the generalized van Cittert-Zernike theorem [65,90] (see also Appendix B), one observes that $g(\Delta\vec{r})$ at the source is related to the angular radiation distribution via a Fourier transform, and the source intensity distribution is related to the cross-spectral density in the far zone (as shown in Appendix B). We can apply this result to the simulation of a partially coherent source, similar to what we did in Eq. (C1). By restricting the inverse-space distribution, Eq. (C2), with the effective radiation divergence, we can shape a partially coherent radiation pulse. This approach yields the following expression:

$$\phi(\vec{r}, \omega) = \mathcal{F}^{-1}\{\sqrt{\hat{I}(\vec{k}, \omega)}\mathcal{F}\{\sqrt{\bar{I}(\vec{r}, \omega)}\mathcal{N}(\vec{r}')\}(\vec{k})\}(\vec{r}), \quad (\text{C5})$$

where $\hat{I}(\vec{k})$ is the radiation angular distribution. Here, we provide a visual comparison of radiation from an incoherent source—the Sun—and a partially coherent source—an undulator—in Fig. 6.

The algorithm for quasihomogeneous sources was originally called SERVAL and presented in Ref. [85]. There, we demonstrated that Eq. (C5) satisfies the correct first-order correlation function, which is sufficient to fully describe a Gaussian stochastic process. For completeness, we note that the algorithm can be formulated not only for undulator radiation but also for any partially coherent radiation characterized by an effective source intensity distribution $\bar{I}(\vec{r})$ and angular divergence $\hat{I}(\vec{k})$ at the source location. Furthermore, as we show in the main body of the paper, due to the mathematical equivalence between the VCZ and WK theorems, both algorithms can be straightforwardly adapted for simulations in the longitudinal domain.

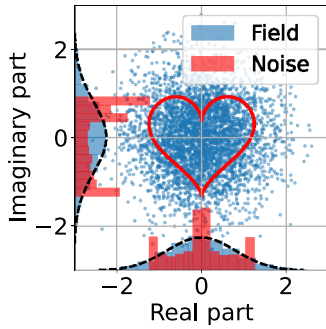


FIG. 9. Transformation of the “heart” shape to a Gaussian distribution in the complex plane. Data are presented for a central pixel of the field distribution, consisting of a mesh of 31×31 points, with the field comprising 4001 realizations. The data shown in red correspond to the initial noise Z generated using Eq. (D1), while the blue data represent samples of the resulting field produced by SERVAl following Eq. (C5) or Eq. (15). For convenience, the samples were normalized to the mean amplitude of the corresponding signal.

APPENDIX D: CHOICE OF THE NOISE FOR SERVAl ALGORITHMS

We use complex Gaussian white noise for the SERVAl algorithms, which is justified by the fact that we simulate complex Gaussian random fields, and the statistics should be preserved. However, one might observe that any noise $Z = X + iY$ would suffice for SERVAl when using Eq. (C5) or Eq. (15). For example, one can define X and Y as follows:

$$\begin{aligned} X &= -13 \cos(t) + 5 \cos(2t) + 2 \cos(3t) + \cos(4t), \\ Y &= 16 \sin(t)^3, \end{aligned} \quad (\text{D1})$$

and t is a random variable uniformly distributed within $[-\pi, \pi]$. For example, Eq. (D1) describes the method to sample one instance of the noise Z . We illustrate the shape formed by the samples of Z in the complex plane in Fig. 9 (red dots and corresponding histograms).

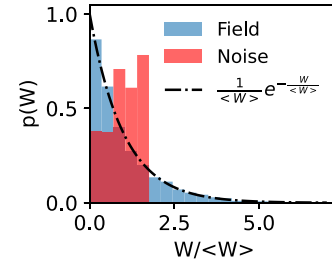


FIG. 10. Instantaneous intensity probability distribution: The red line represents the initial noise, and the blue line represents the instantaneous intensity distribution of the field generated after applying the SERVAl algorithm using this noise.

After simulating a field with Eq. (C5) using the noise defined here, one observes that the statistics of the samples of the random variable $Z(\vec{r}_n)$ at any position $\vec{r}_n$ transform into complex Gaussian noise statistics, as illustrated in Fig. 9 with blue dots and corresponding histograms. Furthermore, the instantaneous intensity of the resulting field follows the expected negative exponential distribution, as shown in Fig. 10. This effect can be explained by the fact that, according to Eq. (C5), Fourier transforms are performed. During the transform, the random variables along the given axis are summed. For a reasonably large mesh grid (although, in this case, Figs. 9 and 10 use only 31 points), the statistics of any initial noise Z will be transformed into Gaussian noise according to the central limit theorem.

However, the SERVAl algorithm for incoherent sources [Eq. (C1)] does not involve Fourier transforms. As a result, the resulting field distribution does not adhere to the correct statistics, and the aforementioned statistical effect does not occur. Therefore, we use complex Gaussian white noise at the start to ensure the accuracy of the final statistics of the simulated field and overall consistency.

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