

Fracton models from product codes

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We explore a deep connection between fracton order and product codes. In particular, we propose and analyze conditions on classical seed codes which lead to fracton order in the resulting quantum product codes. Depending on the properties of the input codes, product codes can realize either Type-I or Type-II fracton models, in both nonlocal and local constructions. For the nonlocal case, we show that a recently proposed model of lineons on nonlocal graphs can be obtained as a hypergraph product code. Interestingly, constrained mobility in this model arises only from energy barriers associated with the graph. For the local case, we introduce a novel type of classical LDPC code defined on a planar aperiodic tiling. By considering the specific example of the pinwheel tiling, we demonstrate the systematic construction of local Type-I and Type-II fracton models as product codes. Our work establishes product codes as a natural setting for exploring fracton order.

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Introduction. Recent years have seen ideas and techniques from quantum information drive new developments in many areas of physics. Among these are new insights around an old problem in statistical mechanics, namely, the physics of glassy dynamics. Indeed, exotic models with glassy dynamics typify both the emerging field of fracton topological order [1–13] as well as breakthrough developments of asymptotically good quantum low-density parity check (LDPC) codes implemented in nonlocal stabilizer Hamiltonians [14–16].

At first glance, these two topics seem rather distinct. On the one hand, fracton phases of matter constitute a fascinating counterexample to the lore of solid state physics by reproducing glassy dynamics even in translation-invariant lattice models [1]. Features distinguishing such phases from both symmetry breaking and conventional topological order include: (i) ground-state degeneracy (GSD) in finite systems that depends nontrivially on the volume, and (ii) low-energy excitations with constrained or no mobility. Instances with constrained mobility are referred to as Type-I fracton models, and those with no mobility as Type-II fracton models [2–5].

On the other hand, quantum LDPC codes represent a promising type of quantum error correction based on stabilizer Hamiltonians $\mathcal{H} = \sum_j \Pi_j$, where each term Π_j has bounded norm and acts on a constant number of qubits, and each qubit participates in only a constant number of terms. A systematic approach to generating such codes is the family of product constructions, which take as input multiple classical

LDPC codes and output a quantum LDPC code [14,17,18]. The canonical example is the *hypergraph product* (HGP), a prescription for generating a CSS quantum code from a pair of classical codes [19]. Crucially, product codes allow analytic calculation of the resulting code parameters including the encoding rate (the number of logical qubits per physical qubit) and the code distance (the minimal number of errors required to connect logical states).

Despite their different contexts, these topics are connected by certain individual results: for instance, Haah's code, a Type-II fracton model, is obtained via a *lifted product* [15,18]. One can also use classical codes to construct quantum *fractal spin liquids* [3], based on the polynomial formalism [20] (Fig. 1).

More broadly, an interesting observation suggests a fundamental connection between fracton order and HGP codes. Namely, for gapped phases of matter, the logarithm of the GSD determines the number of qubits that could, in principle, be encoded in the ground space. In conventional phases, this effective qubit number is independent of system size. This scaling is evidently incompatible with the number of logical qubits in typical HGP codes, which is polynomial in the system size. By contrast, such polynomial scaling is reminiscent of precisely the nontrivial dependence of the GSD on system size in fracton phases. Nevertheless, a sharp understanding of the relationship between fracton order and product codes remains lacking.

In this Letter, we provide such a framework, and demonstrate that product codes provide a natural setting for discovering new fracton phases of matter (Fig. 1). Our main results are threefold. First, we extend the definition of fracton order to stabilizer models without geometric locality. Based on this definition, we propose a set of criteria on the input classical codes of a product construction. When satisfied, the resulting quantum LDPC code realizes nonlocal fractons.

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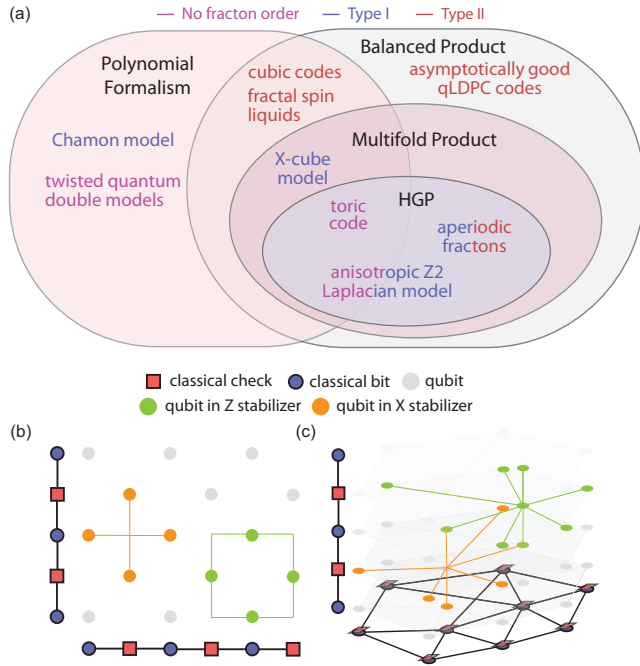


FIG. 1. (a) Set inclusions are shown for stabilizer models. Some explicit product code constructions are provided in SM [21]. (b), (c) Classical Tanner graphs are depicted, with bits represented by blue circles and checks by red squares. The HGP is represented graphically: in (b), a surface code is found as the HGP of two repetition codes, and in (c), the anisotropic $\mathbb{Z}_2$ Laplacian model [22,23] is obtained from a repetition code and a Laplacian code.

Second, we show that a recent model, purported to exhibit Type-I fracton order on nonlocal graphs, is a HGP code. Interestingly, while this case does not constitute a fracton model, we provide a generalized construction which does.

Finally, we consider *geometrically local* product constructions, introducing a new family of classical LDPC codes defined on aperiodic tilings of the plane. Specifically, we consider a local two-dimensional code on the pinwheel tiling (Fig. 3), which we term the *pinwheel code*. We show that the HGP of the pinwheel code with a repetition code produces a local Type-I fracton model in three dimensions, and that the HGP of two pinwheel codes produces a local Type-II fracton model in four dimensions. These results open the door to the systematic discovery of new *local* fracton models using product codes. The resulting models also exhibit evidence of a feature known as confinement (see discussions below), a favorable property that is lacking from most known fracton models [24,25].

Hypergraph product codes. A classical LDPC code on n bits with m terms can be associated with a κ -sparse $m \times n$ parity check matrix $H \in \mathbb{F}_2^{m \times n}$, where $\mathbb{F}_2 = \{0, 1\}$. It is characterized by a triplet $[n, k, d]$, where k is the number of encoded logical bits and d is the code distance [26]. The number of bits participating in each check is bounded above by a constant κ_c , and the number of checks acting on any given bit is bounded by κ_b ; then $\kappa = \max(\kappa_b, \kappa_c) = O(1)$. Typical instances of classical LDPC codes, known as Gallager codes, are obtained by randomly sampling parity check matrices H [27].

Similarly, a CSS quantum code is represented by two parity check matrices H_X and H_Z specifying the action of the X and Z stabilizers, respectively [28,29]. Both classical and quantum parity check matrices are maps taking a configuration of errors in $\mathbb{F}_2^n$, to a syndrome or excitation in $\mathbb{F}_2^m$.

The HGP acts on a pair of classical “seed” codes, producing a CSS quantum code which is LDPC if the inputs are LDPC [19]. The quantum code resulting from the HGP of classical seed codes $H_1 \in \mathbb{F}_2^{m_1 \times n_1}$ and $H_2 \in \mathbb{F}_2^{m_2 \times n_2}$ is given by

$$H_X = [H_1 \otimes \mathbb{I} \quad \mathbb{I} \otimes H_2^\top], \quad H_Z = [\mathbb{I} \otimes H_2 \quad H_1^\top \otimes \mathbb{I}], \quad (1)$$

with $\mathbb{I}$ being the identity matrix. Classical seed codes with parameters $[n_1, k_1, d_1]$ and $[n_2, k_2, d_2]$ produce a quantum code with parameters $[[n_q, k_q, d_q]]$, where [19]

$$n_q = n_1 n_2 + m_1 m_2, \quad d_q = \min(d_1, d_2, d_1^\top, d_2^\top), \\ k_q = k_1 k_2 + k_1^\top k_2^\top, \quad k_X^\top = k_1^\top k_2, \quad k_Z^\top = k_1 k_2^\top. \quad (2)$$

Here, $k_i^\top$ and $d_i^\top$ denote the parameters of the transposed (or dual) code, with parity check matrix $H_i^\top$. The number of *superselection sectors* in the quantum code is given by $2^{k_X^\top + k_Z^\top}$ [21]. The prototypical example of a HGP is the toric code, obtained from two cyclic repetition codes having $k_1 = k_1^\top = k_2 = k_2^\top = 1$ and $d_1 = n_1, d_2 = n_2$ [19].

As shown in Fig. 1(b), the HGP admits an intuitive graphical representation based on the *Tanner graph*; a bipartite Tanner graph describes any classical code by representing both checks and bits as vertices, with edges denoting the connectivity of the terms in H [30].

Fracton order in nonlocal product codes. Product constructions do not generically respect spatial symmetry or geometric locality; however, their structure provides the advantage of reasoning about quantum codes through investigation of their classical seeds. We have in mind sampling from random ensembles of classical codes with connected Tanner graphs of arbitrary size.

Before presenting the characterization of input codes, we first interpret the signatures of fracton order in these families of product code stabilizer models. The thermodynamic and dynamic signatures—nontrivial scaling of GSD, and constrained mobility of excitations, respectively—define two categories of fractons on lattices and also apply here in a simplified form. Namely, we require that the logarithm of the GSD scales polynomially in the system size with high probability (w.h.p.) [31]. In addition, we require that excitations are pointlike and observe energetic constraints on mobility [32]. Within the product code construction mobility can be tuned from partially to fully constrained, encompassing both Type-I and Type-II fractons.

We now trace these signatures of nonlocal fracton order back to their origins within the classical codes. Accordingly, we propose three criteria for classical codes leading to nonlocal fracton order in an HGP code: these are referred to as *rank deficiency*, *confinement*, and *isolability*.

Rank deficiency denotes a property of the parity check matrix H . As codewords are zero eigenvectors, a rank-deficient H implies a large code space, with $k = n - \text{rank}(H)$, and such a seed code gives rise to superselection sectors in the resulting quantum code. A superselection sector is an equivalence class of syndromes under local errors; as the number of sectors

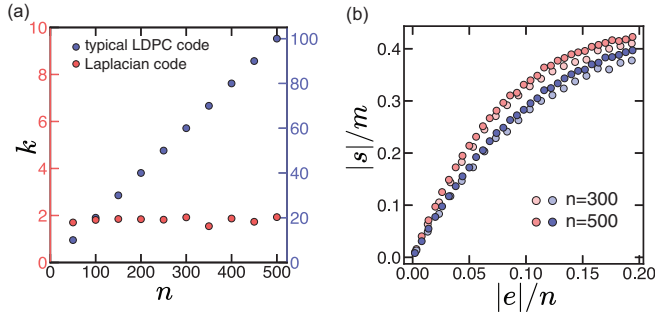


FIG. 2. (a) Rank deficiency is exhibited for Gallager codes, from the linear scaling of k with n ; by contrast, the number of logical bits in a Laplacian code is constant. (b) Confinement data is shown for both Gallager codes and Laplacian codes for $n = 300$ and 500 . Pictured is the minimum syndrome weight of randomly sampled errors. Both constructions display evidence of confinement, as expected for nonlocal codes.

grows, a diminishing number of excitation configurations may be dynamically connected by *any* physical process [33]. This constitutes a strong mechanism for immobility, and is central to both local and nonlocal fracton models [34].

Another source of immobility is *confinement*, a property implying energy barriers and thus glassy dynamics [24,25]. Confinement here means that all error vectors $e \in \mathbb{F}_2^n$ with Hamming weight $|e| \leq g(n)$ satisfy $f(|s|) \geq |e|$, where $s = H \cdot e$, for some increasing functions f and g [25]. Gallager codes are confining w.h.p. due to the expansion of random graphs [27]. It can be proved that if both seed codes are confining along with their duals, then so is the resulting quantum code [21]. Confinement in the quantum code implies the lack of stringlike operators. Thus, any fully confined fracton code is Type II, and judicious use of nonconfining seed codes can also generate Type-I fractons.

One classical code satisfying the prior conditions is the nonlocal Ising model on a random graph; however, we do not define the HGP of two such codes as a fracton model because the confinement derives from the domain wall nature of excitations [24]. To exclude these cases, we impose an *isolability* condition (see End Matter for additional details) requiring all excitations to be pointlike, that is, separable from each other [35].

Isolability is a natural feature of nonlocal codes, as a simple probabilistic argument demonstrates [21]. The violation of isolability in the Ising model arises from cycles in the Tanner graph, each associated with a metacheck, or redundancy between rows of the parity check matrix. Any cycle hosts an even number of violated checks, and this directly leads to the domain-wall nature of excitations. More generally, violations of isolability require extensively many independent metachecks, which are moreover local on the Tanner graph. Such a situation is fine-tuned, and in fact implies single-shot decoding [21]. Classical metachecks implementing violations of isolability cannot be inherited by the quantum HGP code, unless the classical codes also have bounded distance [21].

Nonlocal Type-I fractons in HGP codes. We consider HGP codes, in which one classical seed is a cyclic repetition code and the other is either: case (i), a *Laplacian code*; or case (ii), a

Gallager code (see Ref. [21] for technical details). This is motivated by a recently proposed Type-I fracton model called the “anisotropic $\mathbb{Z}_2$ Laplacian model” [22,23]. While originally derived from a series of results on dipole-conserving tensor gauge theories [36–38], this model is precisely described by case (i) above [Fig. 1(c)] [21].

The parity check matrix of the classical Laplacian code used in case (i) is given by the reduction mod 2 of the Laplacian operator of a graph [39]. To study its rank deficiency, we find the number of encoded logical bits k as a function of system size n . For each system size, we average over $\sim 10^3$ random graphs. It is immediately evident that the Laplacian code on a general sparse graph does not exhibit evidence of rank deficiency [Fig. 2(a)]. As a consequence, the resulting quantum code does not support fracton order.

This seemingly contradicts results on the square lattice [22], but can be understood from the perspective of the seed code. Indeed, the Gallager codes used in case (ii) are chosen with an imbalance between bits and checks leading to $k = O(n)$ rank deficiency [Fig. 2(a)]. Because the parity check matrix of the Laplacian code is square, it is very likely to be nearly full rank. The square lattice is thus special, and exact results can in fact illuminate how translation invariance leads to rank deficiency in the Laplacian code [21].

In contrast, we find that confinement is a feature of both Gallager and Laplacian codes. We compute the minimum density of excitations $|s|/m$ induced by $\sim 10^3$ randomly sampled errors at each sparsity $|e|/n$. The monotonic growth in Fig. 2(b) suggests that both types of codes are confining. Finally, we observe that both Gallager and Laplacian codes are isolable w.h.p. [21]

Thus, we conclude that Gallager codes are rank-deficient, confining, and isolable, whereas Laplacian codes on general graphs are merely confining and isolable. Because the cyclic repetition code is also isolable, our case (ii) HGP code indeed realizes Type-I fractons. However, rather than fractons, case (i) instead realizes a phase akin to nonlocal topological order.

Local fracton models from aperiodic codes. Though the stabilizer models discussed so far are only κ local, it is natural to ask whether the framework developed above can inform the construction of geometrically local HGP codes. Whereas one well-known route to locality is to define a Tanner graph on a negatively curved manifold, which can embed expanders [40], we instead take a Euclidean approach by placing both bits and checks on vertices of a two-dimensional aperiodic tiling.

As a specific example, we consider the pinwheel tiling, which has no exact spatial symmetries [41–44]; while the lack of spatial symmetries might seem challenging from an experimental perspective, we note the remarkable advent of recent architectures based on movable qubit degrees of freedom [45,46]. This tiling is specified by a substitution rule [Fig. 3(a)] generating a family of graphs $\{G_N\}$, $N = 1, 2, \dots$, whose volume scales as $n \sim e^N$ [Figs. 3(b) and 3(c)]. We define parity check matrices H_N based on the graph Laplacian L_N at generation N [39]. We first compute $\tilde{H}_N = (L_N - \mathbb{I}) \bmod 2$. Then H_N is obtained via a “boundary depletion” of checks, truncating the column space of $\tilde{H}_N$ by removing an evenly spaced fraction $1/p$ of checks on the boundary vertices of G_N . We dub this the *pinwheel code*, within a more general class of *aperiodic codes*.

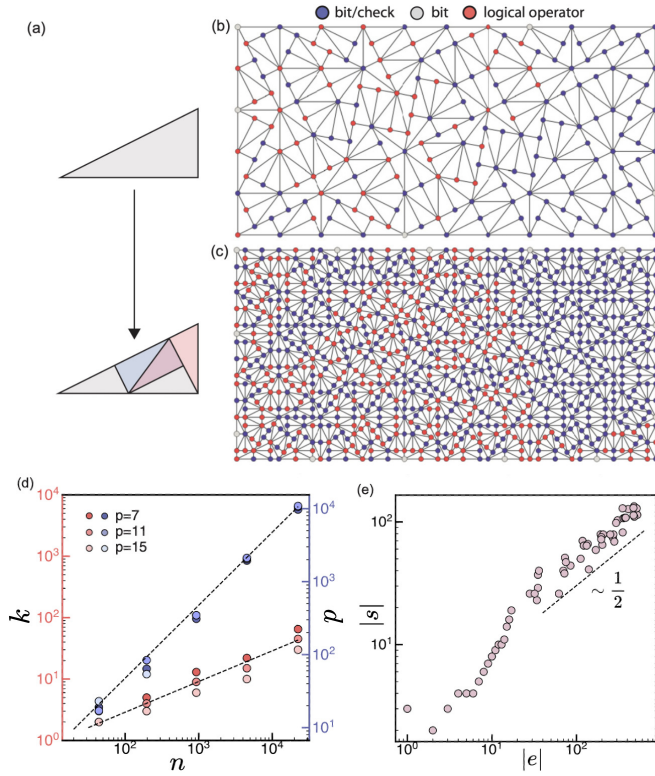


FIG. 3. (a) The pinwheel tiling substitution rule is shown. (b), (c) Graphs G_N at generations $N = 3, 4$ are shown for the pinwheel tiling on a rectangle. Each vertex hosts both a bit and a check, except for a fraction of boundary sites, whose checks are removed. The lowest-weight codeword is indicated in red. (d) The scaling of the number of logical bits $k \sim \sqrt{n}$ and distance $d \sim n$ are shown for the pinwheel code, for $p = 7, 11, 15$. Distances shown are exact, except for the largest value $k = 65$, for which an upper bound is shown. (e) The minimum syndrome weight over sampled errors at $N = 5$ provides numerical evidence for confinement. For low-weight errors, all error configurations are considered within randomly sampled balls of fixed radius. For high-weight errors, truncated logical operators illustrate the naive square-root upper bound. Though not exhaustive, such sampling can diagnose lack of confinement in practice [21].

At issue are thus the rank deficiency and confinement of the pinwheel code; isolability is guaranteed by the absence of an extensive basis of local metachecks. As shown in Fig. 3(d), this family exhibits $k \approx \sqrt{n}/p$ and is thus rank deficient, with the parameter p tuning the number of logical bits [47]. Interestingly, although the depletion of checks relative to bits is a boundary effect, this does not imply that logical bits are localized on the boundary [21]. Indeed, we find that this is not the case, measuring a code distance that scales linearly with n , as shown in Fig. 3(d). The pinwheel code introduced here thus saturates the classical BPT bound $k\sqrt{d} = O(n)$ [48].

In contrast to Gallager codes, a D -dimensional local code cannot exhibit linear confinement. An upper bound is realized by the Ising model, wherein domain walls have polynomial energy cost with exponent $\frac{D-1}{D}$; more generally, in any local code, a codeword can be truncated within a D ball at the cost of only a $(D - 1)$ -dimensional boundary syndrome. Such

scaling $|s| \sim |e|^{1/2}$ is exhibited in the limit of large error weights in Fig. 3(e). Though sampling is not guaranteed to capture the tails of the distribution, it is sufficiently thorough to distinguish the present cases from the Laplacian code on the square lattice, which, though rank deficient, is not confining [21]. We thus conjecture that the pinwheel code is confining, and saturates the upper bound [49].

Having demonstrated rank deficiency, confinement, and isolability, we now consider the HGP of pinwheel codes as local fracton models. First, a pinwheel code and a cyclic repetition code yield a local Type-I fracton model in three dimensions, with translation symmetry perpendicular to the tiling only. The immobilization of particles within the plane is established by the confinement of the classical pinwheel code. Second, two instances of pinwheel codes yield a local Type-II fracton model in four dimensions, with parameters $k_q \sim n_q^{1/4}$, $d_q \sim \sqrt{n_q}$ [50]. We therefore largely capture the physics of the Gallager code [case (ii)] without needing to appeal to nonlocal interactions or disorder. We conjecture that, similar to the four-dimensional toric code, the Type-II pinwheel stabilizer model is confining without disorder, but, crucially, achieves this for pointlike rather than extended excitations.

Discussion and outlook. From the perspective of the present work, the study of quantum glassiness [1] leading to fracton order can be understood as an exchange of confinement for superselection, each providing a complementary approach to constraining mobility [34]. Here, we consider neither to be fundamentally classical or quantum, and instead characterize both as aspects of product codes.

Looking forward, our work opens the door to a number of intriguing directions. First, more sophisticated product codes include the lifted and balanced products, and achieve a dimensional reduction by making use of lattice symmetries of the seed codes [14,17,18]. It is not immediately evident how to extend such an operation to aperiodic codes; however, this could in principle reduce a code similar to our local Type-II model to three dimensions, which could be compared to other recent constructions [51,52]. Second, it would be interesting to characterize the conditions on aperiodic tilings that lead to local Type-II fractons, and whether such tilings all flow to the same fixed point under entanglement renormalization [53–56]. One possible avenue is an understanding in terms of condensation of certain topological lattice defects [57]. Finally, it would be useful to quantitatively investigate the spin-glass contribution to immobility and the energy barrier in product code fracton models.

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Appendix: Isolability property. Isolability ensures that excitations are fundamentally pointlike, rather than comprising

extended objects like the domain walls of the Ising model in two or more dimensions. As the technique is to ensure that the distance between excitations can be made large, a metric is required, for which we employ the graph distance. Consider an excitation $\sigma \in C$ which satisfies the same conditions described for the syndromes in the no-strings condition above. Denote by Λ the subgraph consisting of all R neighbors of B_r , where $|\Lambda| \leq \kappa^R$. The distance $R = w(n)$ should be taken to be some increasing function of n . Isolability requires that $\text{img}(H)$ contains a vector $\sigma + \bar{\sigma}$, where $\bar{\sigma}$ has support only on the complement of Λ .

This condition can be understood in several limits: for instance, almost all random bipartite graphs without sparsity conditions (i.e., not LDPC) have diameter bounded by 4, so are not candidates for isolability [60]. Conversely, a tree Tanner graph exhibits $R \sim \log n$ isolability. As a coarse measure of the naturalness of isolability for Gallager codes, the dimension of the error space n can be compared to the number of free parameters in such an isolation vector, namely $m - \kappa^R$. As m scales linearly with n , one naively expects $R \sim \log n$ isolability, as for a tree. In fact, because the diameter of a sparse random graph is logarithmic in n with high probability, this scaling is optimal [61].

Given the preceding discussion, one may ask which conditions allow a code to violate isolability, by which we refer to enforcing $R = O(1)$. Taking the D -dimensional Ising model as an example, one observes that for $D > 1$ the Tanner graph is composed of densely packed cycles, each supporting a geometrically local metacheck generator; see Fig. 4. These generators enforce that excitations must arrange themselves in such a way as to form a $(D - 1)$ -dimensional boundary which itself has no boundary, providing a canonical example of isolability violation.

Ising metachecks are but one instance of a general structure, whereby an extensive number of κ' -local metacheck generators enforce the violation of isolability by constraining the allowed patterns of neighboring excitations. Evidently such an extensive set of local generators is necessary to enforce $R = O(1)$ throughout the system. Each of these generators can be thought of as a constraint in a second classical linear code applied to the syndromes of the original code. In this metacheck code, syndrome measurement errors play the role of bit flips. One can define a two-stage single-shot decoder which first decodes the metacheck code, performing a recovery operation, then decodes the original code. Now the requirement of an extensive set of κ' -local generators specifies that the metacheck code is LDPC, and as a result

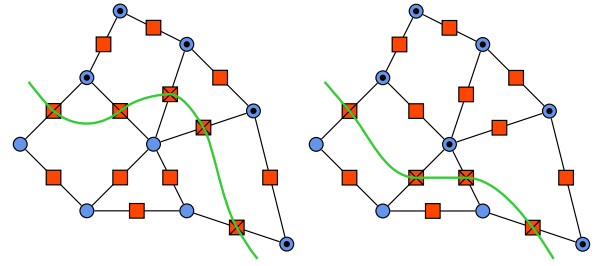


FIG. 4. Lack of isolability is illustrated as a consequence of local cyclic metachecks. Errors (denoted by black dots) preserve the domain-wall nature of the excitation (denoted by crosses) by implementing simple deformations pulling the green surface through a bit node. Equivalently, the metacheck constraining the product of checks around any cycle to be 1 enforces an even number of intersections with the domain wall, causing its boundary to vanish.

has a threshold [62]. Therefore we say that if the underlying code has a threshold, violation of isolability implies that the code is single-shot decodable.

Because the number of metacheck generators $k^\top$ is the dimension of the dual code—both of which appear in the HGP—the requirements of isolability and rank deficiency may appear to be in tension: are the duals of linearly rank-deficient codes generically isolability violating? The resolution is provided by the requirement that the metachecks admit a κ' -local basis. This implies that the distance of the dual code is bounded by a constant, which is not the case for the constructions presented in this work. By itself, an extensive $k^\top$ is therefore not sufficient to violate isolability in the dual code, and moreover this requirement can coexist with superselection sectors in HGP codes.

Inheritance of isolability in HGP codes. Recall that in order to violate isolability requires an extensive set of local metacheck generators, so $k_X^\top = k_1^\top k_2^\top \sim n_q$ and $k_Z^\top = k_1 k_2^\top \sim n_q$ for isolability of X and Z syndromes, respectively. That is, one requires linear dimension for both primal and dual codes for both seeds. The space of metachecks applicable to (say) Z stabilizers is given by $\ker H_Z^\top = \ker H_1 \otimes \ker H_2^\top$, a basis for which is immediate from the classical codes. But if these metachecks admit a local basis, both $\ker H_1$ and $\ker H_2^\top$ must also admit a local basis, implying that $d_1 = O(1)$. Similarly, for X syndromes to violate isolability requires $d_2 = O(1)$. Thus we conclude that no pair of classical codes with growing distance can produce a quantum HGP code which violates isolability.

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