

SUSY design of smooth quantum rings in graphene

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We develop a suitable technique to design zero-energy graphene models with radial electrostatic potentials capable of achieving electrostatic confinement. Using the Gaussian law for electrostatics, we derive the charge density associated with these potentials that correspond to concentric electrostatic rings. The technique is based on a modified supersymmetric transformation that allows us to design time-reversal invariant interaction terms and to find the corresponding zero-energy bound states in analytical form. Consequently, solutions with the same probability density but different angular momentum are characterized by circular probability currents flowing in opposite directions. The energies of the systems defined in two Dirac valleys (one-valley) have a fourfold (twofold) degeneracy. As an example of the technique, we construct a ring-decorated Coulomb potential that exhibits zero-energy collapse and bound states together.

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I. INTRODUCTION

Due to its remarkable physical properties, graphene is a promising material for the construction of new electronic devices. Graphene configurations that would have been considered unrealistic a decade ago are now possible due to the rapid development of experimental methods. Such systems can be p - or n -doped using electronic and chemical methods [1], or mechanically deformed [2,3], allowing the charge density on their surface to be adjusted by applying external positive and negative voltages. These techniques have been successfully used to build structures such as graphene quantum dots [4–7], circular $p - n$ junctions [8,9], and quantum rings [10] in laboratory settings. In the current scenario, new theoretical models describing the electronic properties of graphene are desirable.

An interesting example is the quantum ring (QR). It is a conducting ring-shaped region in nanoscale space with a local density of states of the same shape [11,12]. The first graphene-based QR was produced experimentally in order to investigate Aharonov-Bohm oscillations and the existence of persistent circular currents [10,13,14]. Nowadays, systems such as “whispering galleries” have been developed in the laboratory, being efficient traps for charge carriers within the material [15,16]. Analytically tractable models of electrostatic QR in graphene were studied theoretically, e.g., in [17–23]. It has been shown that these structures can provide confinement of quasiparticles of zero energy, overcoming the Klein tunneling that affects their localization [24].

So far, the mentioned models studied in the literature are relatively simple, representing a single ring with either a smooth or a rectangular profile. To the best of our

knowledge, theoretical models analyzing the physics of more complex structures, such as concentric rings, have not yet been discussed. The current article aims to address this gap by providing a robust framework for the theoretical design of a wide variety of radially symmetric electric potentials enabling quantum rings, hosted by a single layer of graphene. Our approach is based on an analytical scheme known as supersymmetric transformations.

The supersymmetric transformations play an important role in the theory of integrable systems [25] and supersymmetric quantum mechanics [26]. The transformations (also called Darboux transformations in the literature) essentially map solutions of an initial differential equation into solutions of another equation of the same type, but with a modified potential term [27]. They found wide application in quantum mechanics when applied in the context of Schrödinger [25–28] or Dirac-type equations [29–31], but also include optical and lattice systems, see, e.g., [32,33]. The transformation is represented by a differential operator uniquely defined by its kernel, which consists of a set of eigenstates of the initial Hamiltonian, usually called seed states. This operator connects the initial system with the new one in the so-called intertwining relation. In the same way, the seed states define the additional potential term of the new system. Due to its non-trivial kernel, the transformation map is not always one-to-one and, more generally, it sometimes does not map states into physical ones. Therefore, the new systems may have different spectral properties when compared to the original. For example, the new model can have additional bound states, usually called “missing” states in the literature.

The supersymmetric transformations were discussed both for stationary and nonstationary one-dimensional Dirac equations, see [30] and [31], respectively. They have been used, for example, to reveal the connection between Klein tunneling in one-dimensional carbon structures and the free particle due to supersymmetry [34], in the spectral design of twisted carbon nanotubes [35], in Dirac systems with transparent potentials

[36], and systems described by the non-Hermitian Dirac equation [37]. They have also been generalized to higher spin operators in [38], representing an effective tool for studying a wide range of Dirac materials [39,40], see also [41–44]. Although the results for higher spatial dimensions are limited [45–49], and some of them rely on the adaptation of techniques originally developed for 1D or (1 + 1)D systems, they have been useful in revealing interesting physics of Dirac fermions in graphene. To give just a few examples, some of these approaches have been used to construct models that exhibit omnidirectional Klein tunneling [46] and, more recently, to explain the confinement in Lorentzian well potentials due to shape invariance [49]. Inspired by [49], in this article, we present a modified supersymmetric transformation based on the asymmetric intertwining relation. It allows us to construct time-reversal invariant planar models of Dirac fermions in graphene with electrostatic potentials exhibiting rotational symmetry, while providing analytical expressions for their zero-energy states. The generalization of the technique from one to two dimensions allows us to study scenarios such as graphene. However, the main difference between the two cases is that the 2D case does not allow us to map known states to new ones, mainly due to the fact that the intertwining relation is no longer symmetric.

To demonstrate the applicability of the new technique, we modify the model of a Coulomb impurity in graphene. As a result, new systems with concentric rings surrounding the impurity are constructed. The new potentials do not prevent the fall to the center of the Dirac particles [50–53], but each of the rings is able to trap an electron-hole pair of zero energy with a determinate angular momentum quantum number that is intrinsically related to the geometry of the ring.

The article is organized as follows. In Sec. II, we briefly review the physics of zero-energy states for rotationally symmetric electrostatic potentials. The electrostatic confinement effect due to the presence of bound states is discussed, as well as a general perspective on atomic collapse phenomena related to singular potentials at the origin. In Sec. III, the modified supersymmetric transformation is presented, which enables the construction of the new deformed systems with QR-type electric potentials. We show how the technique is based on a specific nonunitary transformation and asymmetric intertwining relations. In Sec. IV, we use the technique to construct a Coulomb potential decorated by concentric rings. We show that each of these rings supports a pair of zero-energy bound states representing two confined carriers with opposite total angular momentum. It is also shown that the system exhibits the simultaneous existence of bound states and atomic collapse phenomena. In addition, physical aspects of the new potentials are discussed, in particular, the effective charge density corresponding to the electric potential is calculated. Finally, we leave Sec. V for discussion and outlook.

II. ELECTROSTATIC CONFINEMENT THROUGH ZERO-ENERGY STATES

Electrostatic confinement in a graphene layer is of theoretical and experimental interest and is related to the possibility of creating bound states by the action of an electric field. Numerous studies have suggested that the Klein

tunneling effect poses a challenge to achieving this phenomenon. Nevertheless, theoretical research has shown that certain electric field configurations, such as quantum dots [20], rings [21], and circular p - n junctions [54], can still achieve confinement for carriers with zero energy. The experimental evidence also supports these findings, see, for example, [4,5,7].

This section analyzes some general aspects of the continuous model of graphene when an external radially symmetric potential is considered. We first examine the symmetries of the two-valley model and their corresponding reductions to the spin- $\frac{1}{2}$ case to emphasize the importance of Kramer's theorem, which explains the energy level degeneracy. Hereafter, we explore the electrostatic confinement effect more deeply by studying the zero-energy equation and the conditions for creating zero-energy bound states. Additionally, we investigate the possibility of having collapsing states instead of bound states when the potentials are singular at the origin of coordinates. Finally, we reconsider the renowned Coulomb problem from the perspective of zero-energy states, demonstrating the impossibility of hosting bound states and the presence of the atomic collapse phenomenon.

A. Symmetries of a graphene layer in the presence of radial electric potentials

Ideal graphene is described as a two-dimensional honeycomb carbon lattice [24]. The tight-binding approximation reveals that there are two inequivalent Dirac points (denoted $\mathbf{K}$ and $\mathbf{K}' \equiv -\mathbf{K}$) located in two mutually opposite corners of the first Brillouin zone. In the low-energy regime, the charge carriers behave like Dirac fermions. They are described by the following equation:

$$\hbar v_F \mathbb{H} \Psi(\mathbf{r}, t) = i \hbar \partial_t \Psi(\mathbf{r}, t), \quad \mathbb{H} = \begin{pmatrix} H_{\mathbf{K}} & 0 \\ 0 & H_{\mathbf{K}'} \end{pmatrix},$$

$$\Psi(\mathbf{r}, t) = \begin{pmatrix} \psi_{\mathbf{K}}(\mathbf{r}, t) \\ \psi_{\mathbf{K}'}(\mathbf{r}, t) \end{pmatrix}. \quad (2.1)$$

Here, $H_{\mathbf{K}}$ (or $H_{\mathbf{K}'}$) is the Dirac Hamiltonian capturing dynamics in the vicinity of the Dirac point $\mathbf{K}$ (or $\mathbf{K}'$). Weyl spinor $\psi_{\mathbf{K}}$ (and $\psi_{\mathbf{K}'}$) represents Dirac fermion with its momentum being close to $\mathbf{K}$ (or $\mathbf{K}'$). For the external, radial symmetric potential interaction, the Hamiltonians read as

$$H_{\mathbf{K}} = -i \boldsymbol{\sigma} \cdot \nabla + V(r),$$

$$V(r) = \frac{e\phi(r)}{\hbar v_F}, \quad H_{\mathbf{K}'} = H_{\mathbf{K}}^T = \sigma_1 H_{\mathbf{K}} \sigma_1. \quad (2.2)$$

Here, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ with σ_i denoting the Pauli matrices, v_F is the Fermi velocity, and e is the electron charge. The effect of a scalar potential does not introduce intervalley interaction, but it can modify the energy spectrum of the system. This allows for the creation of bound states at zero energy for certain configurations, as we show below.

Equation (2.1) is symmetric with respect to the following set of transformations:

$$\Psi(\mathbf{r}, t) \rightarrow \mathbb{T}_1 \Psi(\mathbf{r}, -t), \quad \mathbb{T}_1 = \sigma_2 \otimes (\sigma_1 \mathcal{T}), \quad (2.3)$$

$$\Psi(\mathbf{r}, t) \rightarrow \mathbb{T}_2 \Psi(\mathbf{r}, -t), \quad \mathbb{T}_2 = \sigma_3 \otimes \mathcal{T}, \quad (2.4)$$

$$\Psi(\mathbf{r}, t) \rightarrow \mathbb{P}\Psi(\mathbf{r}, t), \quad \mathbb{P} = -\mathbb{T}_1\mathbb{T}_2 = \sigma_1 \otimes \sigma_1, \quad (2.5)$$

$$\Psi(\mathbf{r}, t) \rightarrow e^{i\theta\mathbb{J}}\Psi(\mathbf{r}, t), \quad \mathbb{J} = \begin{pmatrix} J & 0 \\ 0 & \sigma_1 J \sigma_1 \end{pmatrix}, \quad (2.6)$$

where $\mathcal{T}$ and J are the spin- $\frac{1}{2}$ versions of the time reversal transformation and the generator of rotations [24]. They are given by

$$\mathcal{T} = T\sigma_2, \quad T i = -i T, \quad J = -i(x\partial_y - y\partial_x) + \frac{1}{2}\sigma_3. \quad (2.7)$$

The transformation (2.3) represents the standard time reversal transformation for graphene, which swaps the valley index but preserves the sublattice index of the Weyl spinors ψ_K and $\psi_{K'}$ [24]. Similarly, we refer to (2.4) as the “pseudo-time reversal transformation,” as it is antilinear but preserves the valley index. Transformation (2.5) swaps both the valley and pseudo-spin indices, yet it is linear contrary to antilinear symmetries (2.3) and (2.4). Finally, (2.6) corresponds to spatial rotations by an angle θ . It is worth noting that the set of discrete transformations $\{\mathbb{I}, i\mathbb{T}_1, i\mathbb{T}_2, \mathbb{P}\}$ forms a Klein-four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [55]. Additionally, we have

$$\{\mathbb{J}, \mathbb{T}_1\} = \{\mathbb{J}, \mathbb{T}_2\} = \{\mathbb{J}, \mathbb{P}\} = 0, \quad (2.8)$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. These relations mean that $\mathbb{T}_1$ and $\mathbb{T}_2$ flip the sign of the total angular momentum of the states, while $\mathbb{P}$ does not affect this quantity. Additionally, it is clear that for a given energy eigenvalue, there exist four independent solutions:

$$\Psi(\mathbf{r}), \quad \mathbb{T}_1\Psi(\mathbf{r}), \quad \mathbb{T}_2\Psi(\mathbf{r}), \quad \mathbb{P}\Psi(\mathbf{r}). \quad (2.9)$$

In consequence, the system has a fourfold degeneracy.

As the operators H_K and $H_{K'}$ are unitary equivalent, we can consider the dynamics in the vicinity of $\mathbf{K}$ described by spin- $\frac{1}{2}$ Hamiltonian¹ H_K without loss of generality,

$$\hbar v_F H \psi(\mathbf{r}, t) = i\hbar \partial_t \psi(\mathbf{r}, t), \quad H \equiv H_K, \quad \psi(\mathbf{r}, t) \equiv \psi_K(\mathbf{r}, t). \quad (2.10)$$

Within this scheme, the relevant symmetry transformations are just (2.7).

Kramers' theorem implies that energy levels of spin- $\frac{1}{2}$ systems with time-reversal symmetry always exhibit even degeneracy [56,57]. This conclusion can be made directly, as for any solution of the stationary equation,

$$\hbar v_F H \psi(\mathbf{r}) = E \psi(\mathbf{r}), \quad (2.11)$$

there is a second linearly independent solution $\mathcal{T}\psi(\mathbf{r})$. These two solutions are usually referred to as the Kramer pair in the literature. It plays an important role in the study of the spin Hall effect [58]. The time-reversal symmetry changes the sign of the angular momentum and of the matrix vector $\boldsymbol{\sigma}$,

$$\mathcal{T}J\mathcal{T}^\dagger = -J, \quad \mathcal{T}\boldsymbol{\sigma}\mathcal{T}^\dagger = -\boldsymbol{\sigma}. \quad (2.12)$$

These transformations mean that the spinors $\psi(\mathbf{r})$ and $\mathcal{T}\psi(\mathbf{r})$ transform under rotation with opposite orientations, and additionally, the corresponding probability currents flow in

opposite directions:

$$\mathbf{j}^+ = \psi^\dagger \boldsymbol{\sigma} \psi, \quad \mathbf{j}^- = (\mathcal{T}\psi)^\dagger \boldsymbol{\sigma} (\mathcal{T}\psi) = \psi^\dagger (\mathcal{T}^\dagger \boldsymbol{\sigma} \mathcal{T}) \psi = -\mathbf{j}^+. \quad (2.13)$$

B. The zero-energy equation

In the following, we address the electrostatic confinement by examining Eq. (2.11) for $E = 0$ in polar coordinates

$$H\psi = 0, \quad H = \begin{pmatrix} V(r) & -ie^{-i\varphi}(\partial_r - \frac{i}{r}\partial_\varphi) \\ -ie^{i\varphi}(\partial_r + \frac{i}{r}\partial_\varphi) & V(r) \end{pmatrix}. \quad (2.14)$$

To solve the equation, it is convenient to diagonalize H and J simultaneously, so that we consider states of the form

$$\psi_m = e^{im\varphi} \begin{pmatrix} \chi_m(r) \\ ie^{i\varphi} \xi_m(r) \end{pmatrix}, \quad m = 0, 1, 2, \dots \quad (2.15)$$

It is sufficient to take only positive values for m , since due to (2.12) we have

$$J\psi_m = \left(m + \frac{1}{2}\right)\psi_m, \quad J(\mathcal{T}\psi_m) = -\left(m + \frac{1}{2}\right)(\mathcal{T}\psi_m), \quad (2.16)$$

i.e., the two states from the Kramer pair have opposite angular momentum. As should be expected, these states are orthogonal to each other, i.e., $\langle \psi_m | \mathcal{T}\psi_m \rangle = 0$.

Inserting the spinor ansatz (2.15) into Eq. (2.14), two coupled equations of the radial functions $\chi_m(r)$ and $\xi_m(r)$ are obtained:

$$\begin{aligned} \frac{d\chi_m(r)}{dr} &= \frac{m}{r}\chi_m(r) + V(r)\xi_m(r), \\ \frac{d\xi_m(r)}{dr} &= -\frac{m+1}{r}\xi_m(r) - V(r)\chi_m(r). \end{aligned} \quad (2.17)$$

The equations in (2.17) can also be decoupled as follows:

$$\left[\frac{d^2}{dr^2} + \left(\frac{1}{r} - \frac{1}{V(r)} \frac{dV(r)}{dr} \right) \frac{d}{dr} + V(r)^2 - \frac{m^2}{r^2} + \frac{m}{rV(r)} \frac{dV(r)}{dr} \right] \chi_m(r) = 0, \quad (2.18)$$

$$\xi_m(r) = \frac{1}{V(r)} \left(\frac{d}{dr} - \frac{m}{r} \right) \chi_m(r). \quad (2.19)$$

In order to make the discussion of the electric confinement more specific, it is necessary to provide more details on the potential and the boundary conditions. In this context, we assume that the graphene sheet is sufficiently large so that the electric field does not affect its boundaries, as it has a rather local effect. A good approximation for this situation is given by a system in the infinite plane with the potential with the following asymptotics:

$$r \rightarrow \infty, \quad V(r) \sim -\frac{\beta}{r^n}, \quad n \geq 1, \quad [\beta] = [L]^{n-1}. \quad (2.20)$$

The bound states are required to be normalizable. Therefore, the probability density

$$\rho_m(r) = \psi_m^\dagger \psi_m = (\mathcal{T}\psi_m)^\dagger (\mathcal{T}\psi_m) = |\chi_m(r)|^2 + |\xi_m(r)|^2, \quad (2.21)$$

¹We neglect the spin degree of freedom of the Dirac fermions in graphene. We use “spin” throughout the article to refer to the pseudo-spin of the quasiparticles that distinguishes the two triangular sublattices of the hexagonal crystal.

should be integrable in the entire plane, i.e.,

$$\langle \psi_m | \psi_m \rangle = \langle \mathcal{T} \psi_m | \mathcal{T} \psi_m \rangle = 2\pi \int_0^\infty r \rho(r) dr < \infty. \quad (2.22)$$

We prescribe the following boundary conditions in the origin:

$$\chi_m(0) = \xi_m(0) = 0, \quad \chi_m(r)_{r \rightarrow \infty} \sim r^{-a}, \quad \xi_m(r)_{r \rightarrow \infty} \sim r^{-b}, \quad (2.23)$$

where we define $a, b > 1$ to avoid the logarithmic divergence of ρ_m .

On the same level of generality, we can compute the probability density currents (2.13) for a state of the form (2.15) and its Kramer pair, yielding

$$\mathbf{j}_m^\pm = \pm \psi_m^\dagger \boldsymbol{\sigma} \psi_m = N^2 (|\chi_m| \xi_m + \chi_m |\xi_m|) \boldsymbol{\varphi}, \quad \boldsymbol{\varphi} = (-\sin \varphi, \cos \varphi), \quad (2.24)$$

where N is the normalization constant. The localized circular currents associated with the bound states ψ_m or $\mathcal{T} \psi_m$ can create a magnetization in the material via spin-polarization mechanisms [14,59].

Let us see whether it is possible to get solutions with the asymptotic behavior (2.23). We take the long-range approximation of Eq. (2.18), keeping in mind the asymptotic form of the potential (2.20). We get

$$\left[\frac{d^2}{dr^2} + \frac{(n+1)}{r} \frac{d}{dr} + \frac{\beta^2}{r^{2n}} - \frac{m(m+n)}{r^2} \right] \chi_m(r) = 0. \quad (2.25)$$

By solving this equation, we deduce the asymptotic behavior of the radial functions, which corresponds to

$$\chi_m(r)_{r \rightarrow \infty} \sim \begin{cases} c_1 r^m + \frac{c_2}{r^{m+n}} & n > 1 \\ c_1 r^{\gamma-1/2} + c_2 r^{-\gamma-1/2} & n = 1 \end{cases}, \quad (2.26)$$

$$\begin{aligned} \xi_m(r)_{r \rightarrow \infty} &\sim \begin{cases} \frac{c_2}{r^{m+1}} & n > 1 \\ c_1 \frac{(2m+1-2\gamma)}{2\beta} r^{\gamma-1/2} + c_2 \frac{(2m+1+2\gamma)}{2\beta} r^{-\gamma-1/2} & n = 1 \end{cases} \\ &\quad (2.27) \end{aligned}$$

Here, the dimensionless parameter γ is defined as

$$\gamma = \gamma(m, \beta) = \frac{1}{2} \sqrt{(2m+1)^2 - 4\beta^2}. \quad (2.28)$$

It can be identified with the celebrated “atomic collapse parameter,” and it will be discussed in more detail later in the text.

Revising Eqs. (2.26) and (2.27) reveals that we can achieve the requested asymptotics of bound states at infinity (2.23) as long as we fix $c_1 = 0$, $\gamma > 1/2$, and $m > 0$. When γ is a pure imaginary, the functions ξ_m and χ_m are oscillating and decay as $r^{-\frac{1}{2}}$. Therefore, they refrain from being square integrable and rather represent scattering states.

The boundary conditions at $r = 0$ can always be achieved when the potential is regular at that point. In case of singular at the origin potentials, the well-known atomic collapse phenomenon (also called the falling to the center problem) can emerge, see, e.g., [57,60]. It occurs in electrostatic potentials that decay as $1/r^q$ with $q \geq 2$ in the nonrelativistic case. It was discussed for Dirac particles in graphene in [24,50–52], and addressed from the point of view of the renormalization group approach in [61,62]. Let us review the phenomenon

here briefly, showing that relativistic particles can also fall to the force center in the potential

$$V(r) \sim -\frac{\beta}{r} \quad \text{for } r \rightarrow 0. \quad (2.29)$$

Indeed, the radial equation (2.18) near $r = 0$ and its solutions coincide with (2.25) and (2.26) for $n = 1$. The parameter γ separates the solutions into two different regimes that have qualitatively different physical implications. When $\gamma \in \mathbb{R}$, the centrifugal barrier in (2.25) is dominant and the system is in the subcritical regime. In such a case, the boundary condition for bound states can be achieved by fixing $c_2 = 0$. The case of pure imaginary γ corresponds to the critical regime, where the potential dominates over the centrifugal barrier and can attract the particle to the origin. In this case, the solutions will be singular in $r = 0$, presenting pathological oscillations as $r \rightarrow 0$. Their explicit form is

$$\begin{aligned} \chi_m(r)|_{r \rightarrow 0} &\sim c_1 r^{i\omega-\frac{1}{2}} + c_2 r^{-i\omega-\frac{1}{2}}, \\ \omega &= \frac{1}{2} \sqrt{4\beta^2 - (2m+1)^2}. \end{aligned} \quad (2.30)$$

Despite their bizarre nature, these solutions still carry crucial information about the model. For a deeper analysis, let us consider a regularized model by fixing the potential to be constant in the subregion $[0, r_0]$ with r_0 being near but different from zero. Subsequently, we study the limit $r_0 \rightarrow 0$.

After matching the two parts of the solutions by using continuity conditions, we finally get

$$\begin{aligned} \chi_m &\sim \begin{cases} \frac{1}{\sqrt{r}} \cos(\omega \log(\frac{r}{r_0}) + \theta) & r > r_0 \\ C \frac{\sinh(kr)}{r} & 0 < r < r_0 \end{cases}, \\ k &= \frac{1}{r_0} \sqrt{\omega^2 + \frac{1}{4}}, \end{aligned} \quad (2.31)$$

where we have used the parametrization

$$(c_1, c_2) = \frac{1}{2} (e^{-i(\omega \ln(r_0) - \theta)}, e^{i(\omega \ln(r_0) - \theta)}), \quad \theta \in \mathbb{R}, \quad (2.32)$$

and the constant C is fixed by the requirement of continuity. When $r_0 \rightarrow 0$, we can see how oscillations near the origin start to accumulate, increasing the number of zeros. This is the typical behavior of the solutions representing collapsing states in potentials that are too singular [60,63]. On one hand, the maximum of (2.31) is at r_0 , near the force center, indicating the most probable position of the particle. On the other hand, semi-classical trajectories indicate a fall to the center of the particle in spiral paths [52].

C. The zero-energy solutions in the Coulomb problem

The Coulomb potential is the fundamental interaction that describes atoms in relativistic quantum mechanics, where $v_F \rightarrow c$. Unlike its nonrelativistic counterpart, it allows for the possibility of atomic collapse [24]. This effect can occur when $Z > \alpha^{-1} = 137$, where α is the fine structure constant. However, the prediction has not yet been confirmed due to the difficulty of producing heavy nuclei in the laboratory. In graphene, the Coulomb interaction can be introduced as impurities in the honeycomb lattice [50,51], and the threshold for observing atomic collapse is less difficult to reach. On the

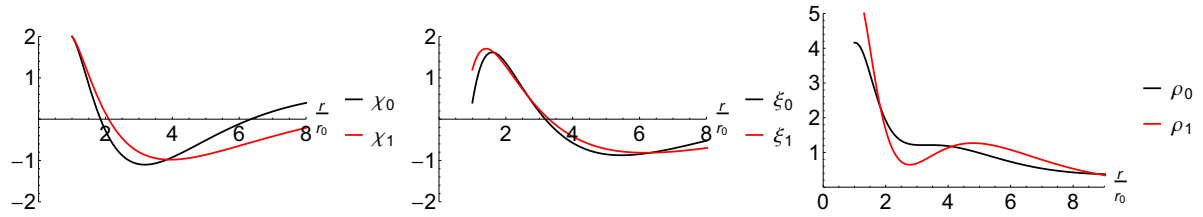


FIG. 1. The plots of the radial functions and the (non-normalized) probability densities of the two collapse states when $\beta = \frac{5}{2}$ respect to r/r_0 are presented from left to right. The constant r_0 acts as a natural scale, and as we approach its value to zero, oscillations in the radial functions begin to accumulate near the force center. This behavior is typical of states in the critical regime in general. The probability density decreases slowly as the distance increases, but diverges like $1/r$ when approaching the origin. When considering the regularized problem, one realizes that the particle is most probable to be found at $r = 0$.

experimental side, there is evidence that STM microscopy can produce this effect in the laboratory [53].

The Coulomb potential reads as

$$V(r) = -\frac{\beta}{r}, \quad \beta = \frac{1}{\hbar v_F} \frac{Ze^2}{\epsilon_{\text{ext}}}, \quad (2.33)$$

where β is the dimensionless potential strength, ϵ_{ext} is a dielectric constant related to an external substrate, and Z is the atomic number. The corresponding zero-energy solutions coincide with (2.26) and (2.27) for $n = 1$, but they are now valid for the entire space.

For the solutions in the subcritical regime, it is not possible to fix the integration constants such that the solution would fulfill the boundary conditions both at $r = 0$ and $r \rightarrow +\infty$, see (2.23). Therefore, there are no bound states in the system. On the other hand, the solutions in the critical regime imply that the particle may fall to the center. In case of graphene, the collapsing states can occur whenever $\beta > \beta_c$, where the threshold value $\beta_c = \frac{1}{2}$ is defined as the potential strength for which $\gamma = 0$ at $m = 0$. The collapsing states are also characterized by having a relatively low angular momentum with m in the interval $[0, \beta - \frac{1}{2})$. For example, fixing $\beta = 5/2$, the collapsing states are those labeled by $m = 0, 1$. The corresponding radial functions and the probability density for such states are shown in Fig. 1.

When considering energy levels other than zero, the solutions of the stationary equation should correspond to resonant quasibound states with complex energy [52,61]. These quasi-Rydberg states describe trapped electrons in a small region where the potential is regularized, acting as a barrier potential. The electrons escape from the barrier after a finite lifetime due to the Klein tunneling effect, providing partial confinement in the system. Nevertheless, the absence of the localized bound states is disappointing since in the nonrelativistic case, the system has discrete energies.

Concluding, the Coulomb potential cannot host bound states.² Nevertheless, it might not be excluded for the potentials that coincide with Coulomb at the origin and for large r , however, they differ in a rather localized region. In other

words, there arises a natural question: Can a local deformation of the Coulomb potential host the bound states?

In the following text, we provide an affirmative answer to the question. We develop an algorithm that allows for the construction of new potentials with exactly solvable zero-energy solutions. These new potentials represent radially symmetric deformations of the known potentials that can host bound states. This formalism is applied to the Coulomb problem. The model is constructed where zero-energy bound states and the collapsing states coexist.

III. DESIGNING RADIAL DEFORMATIONS OF ELECTROSTATIC POTENTIALS

In this section, the supersymmetric or Darboux transformation [25–31] is adapted to create new radially deformed systems. In the usual setup, two differential operators, h and $\tilde{h}$, are considered to be supersymmetric partners if there exists a differential mapping that transforms the eigenstates of one operator into the eigenstates of the other. This supersymmetric structure is encoded in the so-called intertwining relation

$$Lh = \tilde{h}L, \quad (3.1)$$

where L is the intertwining operator. From this equation, one automatically deduces the map

$$h\psi = \lambda\psi \quad \Rightarrow \quad \tilde{h}(L\psi) = \lambda(L\psi). \quad (3.2)$$

This method has been widely used to construct and design one-dimensional solvable models describing non-relativistic and relativistic particles. It is assumed that all solutions of the operator h are known. Some of them are used in the explicit construction of L and $\tilde{h}$. In other words, starting from the initial system, we can create new models using an algorithmic process. It is worth noticing that the Hamiltonians h and $\tilde{h}$ do not need to be Hermitian in general.

The standard supersymmetric transformation was proposed and widely used for one-dimensional systems. There are results on the extension of the transformation to higher-dimensional cases [46,47]. Nevertheless, they do not provide such a generically applicable algorithm as in one dimension. Here, we will employ a variant of the original method where the standard *symmetric* intertwining relation (3.1) will be replaced by an *asymmetric* one $\tilde{L}h = \tilde{h}L$. This type of transformation has recently been explored in [46,48,49]. On one hand, the asymmetric intertwining relation is more limited as it only provides a mapping for a fixed energy level. On

²If a mass term is inserted into the Hamiltonian, a gap can be opened in the spectrum and bound states can emerge, see [64]. However, the presence of the mass term would make the Hamiltonian no longer symmetric with respect to the operator defined in (2.7).

the other hand, it opens new possibilities, e.g., to analyze two-dimensional systems.

Let us consider the operator (2.14). We multiply it by the factor $i\sigma_2$ and consider such a Hamiltonian as the initial one,

$$h\psi = 0, \quad h = i\sigma_2 H = \begin{pmatrix} e^{i\varphi} \left(-i\partial_r + \frac{1}{r}\partial_\varphi \right) & V(r) \\ -V(r) & e^{-i\varphi} \left(i\partial_r + \frac{1}{r}\partial_\varphi \right) \end{pmatrix}. \quad (3.3)$$

The operator h in (3.3) is not Hermitian. However, any operator such as this can be transformed into a Dirac Hamiltonian with an external electric field by multiplying it from the left by $-i\sigma_2$, as long as $V(r)$ is real. Although H and h do not commute, they share zero-energy solutions and are time reversal invariant. The advantage of working with h instead of H lies in the fact that it can be defined by following a supersymmetric transformation L and the new operator $\tilde{h}$, see [25,30,31]:

$$L = \cos \varphi \partial_r - \frac{1}{r} \sin \varphi \partial_\varphi - \Sigma, \quad \tilde{h} = h - i[\sigma_3, \Sigma], \quad \Sigma = \left(\cos \varphi \partial_r U - \frac{1}{r} \sin \varphi \partial_\varphi U \right) U^{-1}, \quad (3.4)$$

such that the intertwining relation (3.1) is satisfied. Here, the matrix U is composed of zero modes ψ_1 and ψ_2 of h , $U = (\psi_1, \psi_2)$, $hU = 0$. It is frequently called “seed matrix” in the literature and satisfies $LU = 0$ by construction. Different choices of ψ_1 and ψ_2 give rise to different L as well as $\tilde{h}$. This freedom in choice of the seed matrix can be utilized to acquire the Dirac Hamiltonian with new electric fields whose zero-energy solutions are known.

To construct the desired matrix U , we consider an arbitrary Kramer pair for the seed solutions in the following way:

$$U_m = (\psi_m, \mathcal{T}\psi_m) = \begin{pmatrix} e^{im\varphi} \chi_m & -e^{-i(m+1)\varphi} \xi_m \\ ie^{i(m+1)\varphi} \xi_m & ie^{-im\varphi} \chi_m \end{pmatrix}. \quad (3.5)$$

The radial functions $\chi_m = \chi_m(r)$ and $\xi_m = \xi_m(r)$ are fixed as real linear independent solutions of equations (2.17). A remarkable property of U_m is that it satisfies the relation $\mathcal{T}U_m\mathcal{T}^\dagger = iU_m$, which means that the matrix Σ and the operator $\tilde{h}_m$ are time-reversal invariant by construction. Explicitly, the newly constructed operator $\tilde{h}$ is given by

$$\tilde{h}_m = \begin{pmatrix} e^{i\varphi} \left(-i\partial_r + \frac{1}{r}\partial_\varphi \right) & e^{-2i\varphi} V_m(r) \\ -e^{2i\varphi} V_m(r) & e^{-i\varphi} \left(i\partial_r + \frac{1}{r}\partial_\varphi \right) \end{pmatrix}, \quad (3.6)$$

$$\mathcal{L}_m = \begin{pmatrix} e^{i\varphi} (r \cos \varphi \partial_r - \sin \varphi \partial_\varphi) + \frac{(m+1)\xi_m^2 - m\chi_m^2}{\xi_m^2 + \chi_m^2} & i(r \cos(\varphi)V(r) + \frac{e^{-i\varphi}(2m+1)\xi_m\chi_m}{\xi_m^2 + \chi_m^2}) \\ i(r \cos(\varphi)V(r) + \frac{e^{i\varphi}(2m+1)\xi_m\chi_m}{\xi_m^2 + \chi_m^2}) & e^{-i\varphi} (r \cos \varphi \partial_r - \sin \varphi \partial_\varphi) + \frac{(m+1)\xi_m^2 - m\chi_m^2}{\xi_m^2 + \chi_m^2} \end{pmatrix}. \quad (3.13)$$

The new transformation (3.13) is inspired by the recently found transformations [49] that allowed to connect the Lorentzian well with the free particle. If we now multiply (3.12) by σ_2 , we finally get to the *asymmetric* intertwining

where $V_m(r)$ is obtained via (3.4). Its final form is

$$V_m(r) = V(r) + V_m^{\text{def}}(r), \quad (3.7)$$

$$V_m^{\text{def}}(r) = \frac{\chi_m(r)\partial_r \xi_m(r) - \xi_m(r)\partial_r \chi_m(r)}{\chi_m^2 + \xi_m^2} = -2 \left(\frac{(2m+1)\chi_m \xi_m}{r(\chi_m^2 + \xi_m^2)} + V(r) \right). \quad (3.8)$$

Obviously, $V_m(r)$ represents a radially deformed $V(r)$. When compared (3.3), the potential term in (3.6) is multiplied by phase factors that would prevent making the inverse mapping of $\tilde{h}_m$ into the requested form of the Dirac operator with electric field. Fortunately, the complex phases can be eliminated by a nonunitary transformation generated by the following matrix:

$$S = \begin{pmatrix} re^{i\varphi} & 0 \\ 0 & re^{-i\varphi} \end{pmatrix}, \quad (3.9)$$

and its inverse. Thus, the transformed operator

$$h_m = S\tilde{h}_m S^{-1} = \begin{pmatrix} e^{i\varphi} \left(-i\partial_r + \frac{1}{r}\partial_\varphi \right) & V_m(r) \\ -V_m(r) & e^{-i\varphi} \left(i\partial_r + \frac{1}{r}\partial_\varphi \right) \end{pmatrix}, \quad (3.10)$$

is of the form (3.3) with the change $V(r) \rightarrow V_m(r)$. Finally, we can turn the h_m into the physical Hamiltonian by multiplying it by $-i\sigma_2$ from the left,

$$H_m = -i\sigma_2 h_m = \begin{pmatrix} V_m(r) & -ie^{-i\varphi} \left(\partial_r - \frac{i}{r}\partial_\varphi \right) \\ -ie^{i\varphi} \left(\partial_r + \frac{i}{r}\partial_\varphi \right) & V_m(r) \end{pmatrix} = -i\sigma \cdot \nabla + V_m(r). \quad (3.11)$$

In this way, we have successfully created a new pseudo-spin $\frac{1}{2}$ system with a radially deformed scalar potential. The specific features of the deformations are determined by the seed functions χ_m and ξ_m , and can vary depending on the original potential. Since χ_m and ξ_m were chosen as linearly independent functions, the term $\chi_m^2 + \xi_m^2$ never vanishes in the range $(0, \infty)$, and the deformation (3.7) is free of singularities.

Once we have constructed the Hamiltonian (3.11), we move on to the analysis of the zero-energy solutions. To do this, we include S in the intertwining relation (3.1), which results in

$$\mathcal{L}_m h = h_m \mathcal{L}_m, \quad \mathcal{L}_m = S L_m, \quad (3.12)$$

where the explicit form of the modified intertwining operator $\mathcal{L}_m$ is

relation mentioned above:

$$\sigma_2 \mathcal{L}_m \sigma_2 H = H_m \mathcal{L}_m. \quad (3.14)$$

This kind of intertwining relations allows only a mapping between the Hermitian Hamiltonians for a fixed energy level, which in this case is at $E = 0$,

$$H\psi = 0 \Rightarrow H_m(\mathcal{L}_m\psi) = 0. \quad (3.15)$$

The form of the zero-energy states for the model (3.11) is obtained explicitly in terms of the radial seed solutions by directly applying $\mathcal{L}_m$ to the states (2.15) and using equations (2.17),

$$\psi_{m,l} = \mathcal{L}_m\psi_l = e^{i\varphi l} \begin{pmatrix} \chi_{m,l} \\ ie^{i\varphi} \xi_{m,l} \end{pmatrix}, \quad l \neq m, \quad (3.16)$$

where the new functions $\chi_{m,l}$ and $\xi_{m,l}$ are given by

$$\begin{aligned} \chi_{m,l} = & \frac{(l+m+1)\xi_m^2 + (l-m)\chi_m^2}{\xi_m(r)^2 + \chi_m(r)^2} \chi_l(r) \\ & - \frac{(2m+1)\xi_m\chi_m}{\xi_m^2 + \chi_m^2} \xi_l, \end{aligned} \quad (3.17)$$

$$\xi_{m,l} = \frac{(2m+1)\xi_m\chi_m}{\xi_m^2 + \chi_m^2} \chi_l - \frac{(l-m)\xi_m^2 + (l+m+1)\chi_m^2}{\xi_m^2 + \chi_m^2} \xi_l. \quad (3.18)$$

Remarkably, the transformation preserves the total angular momentum eigenvalue of the states. It can be understood from the commutator between $\mathcal{L}_m$ and J ,

$$[\mathcal{L}_m, J] = S\sigma_2 H, \quad (3.19)$$

which vanishes in the subspace of zero-energy states. Consequently, the modified transformation allows us to obtain all eigenstates labeled $l \neq m$. The special case for $l = m$, which cannot be obtained by the described procedure, can be constructed from the Hermitian conjugation of the asymmetric intertwining relation (3.14), which gives

$$\sigma_2 \mathcal{A}_m \sigma_2 H_m = H \mathcal{A}_m, \quad \mathcal{A}_m = -\sigma_2 \mathcal{L}_m^\dagger \sigma_2. \quad (3.20)$$

By construction, the above asymmetric relation represents the transformation that maps the states of H_m into the states of H using $\mathcal{A}_m$,

$$H_m\psi_{m,l} = 0 \Rightarrow H(\mathcal{A}_m\psi_{m,l}) = 0. \quad (3.21)$$

It represents the inverse action when compared to $\mathcal{L}_m$ and its intertwining relation (3.14). The operator (3.20) also admits the representation $\mathcal{A}_m = S A_m$, where A_m is an operator of similar form to L in (3.4), but with a different matrix $U \rightarrow \tilde{U}_m$, given by

$$\begin{aligned} \tilde{U}_m = & -\sigma_2 (S^\dagger)^{-1} (U_m^{-1})^\dagger = (\psi_{m,m}, T\psi_{m,m}), \\ \psi_{m,m} = & \frac{e^{im\varphi}}{r(\chi_m^2 + \xi_m^2)} \begin{pmatrix} \xi_m(r) \\ ie^{i\varphi} \chi_m(r) \end{pmatrix}. \end{aligned} \quad (3.22)$$

The Kramer pair of states $\psi_{m,m}$ and $T\psi_{m,m}$ are obviously zero-energy solutions $H_m\psi_{m,m} = 0$. They are the missing states that complement the set of solutions (3.16). This way, the task of finding the full set of zero-energy eigenfunctions of the new model H_m is completed. Interestingly, the missing states hide relevant physics depending on the choice of $\chi_m(r)$ and $\xi_m(r)$. To explore this, we compute their probability density and the

current density:

$$\begin{aligned} \rho_{m,m} = \psi_{m,m}^\dagger \psi_{m,m} &= \frac{1}{r^2(\chi_m^2 + \xi_m^2)}, \\ j_{m,m} = \psi_{m,m}^\dagger \sigma \psi_{m,m} &= \frac{\chi_m \xi_m}{r^2(\chi_m^2 + \xi_m^2)^2} \varphi. \end{aligned} \quad (3.23)$$

These relations suggest that the missing states $\psi_{m,m}$ and $T\psi_{m,m}$ can be normalizable and, therefore, represent the bound states, provided that the radial seed functions χ_m and ξ_m diverge for large r . This can be achieved by fixing the latter functions as the diverging nonphysical eigenstates of H .

The whole algorithm can be repeated, fixing H_m as the initial Hamiltonian this time. Fixing a Kramer pair of zero-energy eigenstates of H_m , $\psi_{m,l}$ and $T\psi_{m,l}$, as the new seed solutions, the new system can be generated with the following deformed potential:

$$V_{m,l}(r) = V(r) + \frac{2(2m+1)\chi_m\xi_m}{r(\chi_m^2 + \xi_m^2)} - \frac{2(2l+1)\chi_{m,l}\xi_{m,l}}{r(\chi_{m,l}^2 + \xi_{m,l}^2)}. \quad (3.24)$$

Repeated application of the procedure can result in a wide variety of systems with deformed potentials and multiple zero-energy bound states. In the next section, we will illustrate the framework on the construction of radial deformations of the Coulomb potential.

IV. RING DEFORMATIONS OF THE COULOMB PROBLEM

In this section, we employ the formalism developed in Sec. III to generate the new model where both bound states and collapsing states coexist. It corresponds to local ring-type deformations of the Coulomb potential.

A. Single ring

The first step is to fix the Coulomb-Dirac Hamiltonian as our initial system. Next, we choose linearly independent seed radial functions by the appropriate selection of the integration constants in (2.26) and (2.27) for $n = 1$. In particular, we fix

$$\begin{aligned} \chi_m(r) &= r^{\gamma-\frac{1}{2}} - \frac{(2m+1-2\gamma)r_c^{2\gamma}}{2\beta} r^{-\gamma-\frac{1}{2}}, \\ \xi_m(r) &= \frac{2m+1-2\gamma}{2\beta} \left[r^{\gamma-\frac{1}{2}} - \frac{(2m+1+2\gamma)r_c^{2\gamma}}{2\beta} r^{-\gamma-\frac{1}{2}} \right], \end{aligned} \quad (4.1)$$

where r_c is a constant of length dimension whose physical meaning will be discussed below. We define $\gamma(m, \beta)$ in (2.28) to be

$$\gamma > 1 \Rightarrow m(m+1) - \frac{3}{4} > \beta^2. \quad (4.3)$$

This restriction prevents us from using complex collapsing seed states for the asymmetric Darboux transformation. In this way, inserting functions (4.1) and (4.2) into Eq. (3.7), we

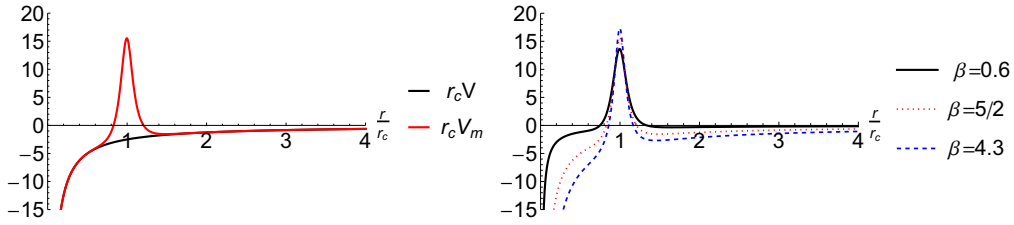


FIG. 2. Left panel: The comparison between the original and deformed potentials as a function of r/r_c . Right panel: The potential for different values of β .

obtain the deformed potential

$$V_m(r) = -\frac{1}{r} \left[\beta - \frac{8\gamma^2(r/r_c)^{2\gamma}}{(2m+1)(r/r_c)^{4\gamma} - 4\beta(r/r_c)^{2\gamma} + 2m+1} \right]. \quad (4.4)$$

A simple inspection shows that the rational deformation term is just a local perturbation so that the potential converges to that of Coulomb at the origin and at infinity. Furthermore, it can be shown that the denominator becomes zero only for the complex values $r_{\pm}^{2\gamma} = r_c^{2\gamma} (2\beta \pm i\gamma)(1+2m)^{-1}$ provided (4.3). Due to the requirement of linear independence of the seed solutions, the potential does not introduce any new singularities on the positive real line.

We can also identify some additional features that determine the nature of the deformation. For example, potential (4.4) reaches its maximum value at $r = r_c$:

$$V_m(r_c) = \frac{\beta + 1 + 2m}{r_c}. \quad (4.5)$$

Additionally, we can find the values of r where the deformation and the Coulomb term cancel each other and the potential vanishes. This occurs at the following points:

$$r_1 = \left[\frac{2(\beta^2 + 2\gamma^2 - \gamma\sqrt{3\beta^2 + 4\gamma^2})}{(1+2m)\beta} \right]^{\frac{1}{2\gamma}} r_c, \\ r_2 = \left[\frac{2(\beta^2 + 2\gamma^2 + \gamma\sqrt{3\beta^2 + 4\gamma^2})}{(1+2m)\beta} \right]^{\frac{1}{2\gamma}} r_c. \quad (4.6)$$

Based on this analysis, we can conclude that the deformation forms a positive and smooth barrier in the general case. In Fig. 2, we compare the original and the deformed potentials for cases $\beta = \frac{5}{2}$ and $m = 6$, where we can visualize the emergence of the barrier with maximum r_c .

The parameter r_c fixes the scale of the potential and determines other physical properties such as the typical width $\Delta r = r_2 - r_1$ and the maximum voltage at the top $\phi_c = \frac{\hbar v_F}{e} V_m(r_c)$. According to [65], if we choose r_c to be the radius of a typical nanoring radius $r_c = 132$ [nm], we can determine Δr and ϕ_c for different values of m and $\beta = 5/2$, as shown in Table I.

The bound states of the deformed systems correspond to the Kramer pair $\psi_{m,m}$ and $\mathcal{T}\psi_{m,m}$, see (3.22). Let us compute

the corresponding probability density and the current density:

$$\rho_{m,m} = \frac{N^2}{r_c^2} \frac{\eta^{2\gamma-1}}{(2m+1)\eta^{4\gamma} - 4\beta\eta^{2\gamma} + 2m+1}, \\ j_{m,m} = \frac{N^2}{r_c^2} \frac{\beta\eta^{4\gamma} + \beta - (2m+1)\eta^{2\gamma}}{((2m+1)\eta^{4\gamma} - 4\beta\eta^{2\gamma} + 2m+1)^2} \eta^{2\gamma-1} \varphi. \quad (4.7)$$

Here, we introduced a dimensionless parameter $\eta = r/r_c$. The function $\rho_{m,m}(r)$ is regular in the entire plane as long as $\gamma > 1$, vanishes at the origin and behaves at long distances as $\sim \eta^{-3\gamma}$. Since there are no singularities, the states composing the probability density are normalizable bound states. Here, N is a normalization constant. The density current has similar asymptotic properties. Additionally, it reflects the circular motion of the particle. At r_c , it takes the following value:

$$j_{m,m}(r_c) = -\frac{N}{r_c^2(4+8m-8\beta)} \varphi. \quad (4.8)$$

Since (4.8) is negative, see (4.3), the particle flow is clockwise. The functions (4.7) are plotted in Fig. 3 for the case $\beta = 5/2$ and $m = 6$.

To compute and analyze the remaining zero-energy solutions, it is useful to rewrite the zero-energy solutions of the Coulomb problem as follows:

$$\chi_{\beta,l}(r; \pm\gamma') = r^{\pm\gamma'-\frac{1}{2}}, \\ \xi_{\beta,l}(r; \pm\gamma') = \frac{(2l+1 \mp 2\gamma')}{2\beta} r^{\pm\gamma'-\frac{1}{2}}, \quad (4.9)$$

where $\gamma' = \gamma(l, \beta)$, see (2.28). Then the formulas (3.17) and (3.18) can be employed, providing us with the following result:

$$\chi_{\beta,m,l}(\gamma') = f_-(\eta)\chi_{\beta,l}(\gamma') - f_0(\eta)\xi_{\beta,l}(\gamma'), \\ \xi_{\beta,m,l}(\gamma') = f_0(\eta)\xi_{\beta,l}(\gamma') - f_+(\eta)\chi_{\beta,l}(\gamma'), \quad (4.10)$$

TABLE I. Typical width and top voltage of the barrier potential for $\beta = 5/2$ in dependence of m , where $v_F = 3 \times 10^6$ [m/s], $\hbar = 1.05 \times 10^{-34}$ [J/s] and $e = 1.6 \times 10^{-9}$ [C].

m	6	8	10	12	14	16	18	20
Δr [nm]	49.880	41.766	36.211	32.120	28.959	26.431	24.36	22.617
ϕ [V]	0.232	0.291	0.351	0.410	0.470	0.530	0.590	0.649

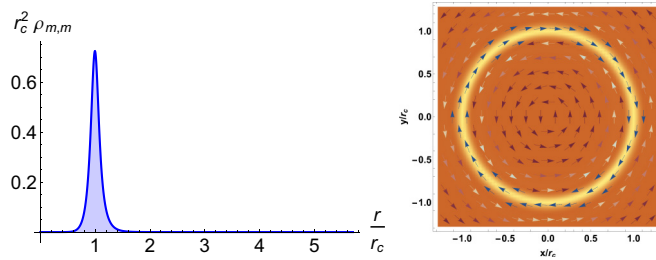


FIG. 3. Plots of the normalized probability density (left) and the current vector density times r_c^2 (right). The parameter values are $\beta = 5/2$ and $m = 6$. In this case, the numerical integration determines the normalization constant value to be $N^2 \approx 0.09$.

where the functions $f_i(\eta)$ are given by

$$f_{\pm}(\eta) = l + \frac{1}{2} \pm \frac{\gamma(2m+1)(\eta^{4\gamma} - 1)}{(2m+1)\eta^{4\gamma} - 4\beta\eta^{2\gamma} + 2m+1},$$

$$f_0(\eta) = \beta - \frac{4\gamma^2\eta^{2\gamma}}{(2m+1)\eta^{4\gamma} - 4\beta\eta^{2\gamma} + 2m+1}. \quad (4.11)$$

The asymptotic form of the functions (4.10) near zero and at infinity corresponds to

$$\chi_{\beta,m,l}|\eta \rightarrow 0 = (\gamma + \gamma')r^{\frac{1}{2}+\gamma'}, \quad \xi_{\beta,m,l}|\eta \rightarrow 0 = (\gamma - \gamma')r^{\frac{1}{2}+\gamma'}, \quad (4.12)$$

$$\chi_{\beta,m,l}|\eta \rightarrow \infty = (\gamma - \gamma')r^{\frac{1}{2}-\gamma'}, \quad \xi_{\beta,m,l}|\eta \rightarrow \infty = (\gamma + \gamma')r^{\frac{1}{2}-\gamma'}. \quad (4.13)$$

It means that the divergent states ($\gamma' \in \mathbb{R}$) remain divergent after the transformation, while the collapsing states (γ' pure imaginary) are transformed into collapsing states. In other words, the transformed solutions preserve the behavior of the original states, see Sec. II B. This result is easy to understand due to the local nature of the deformation; the form of the interaction near zero and at infinity is just the Coulomb potential and, therefore, the analysis of Sec. II C is still applicable. We show the density plots of the two collapsing states existing for the case $\beta = 5/2$ and $m = 6$ in Fig. 4.

To conclude this paragraph, it is essential to note that if we replace the seed functions (4.1)–(4.2) by the same combination but with the change $r^{2\gamma} \rightarrow -r^{2\gamma}$, the resulting potential deformation from the transformation will be a well instead of a barrier. Despite this difference, the analyses conducted for

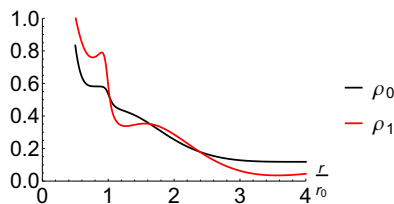


FIG. 4. Plots for the (not normalized) probability densities of the collapsing states in the case $\beta = 5/2$ and $m = 6$. For simplicity, we take $r_c = 1$, while r_0 has the same role as in Fig. 1. They correspond to the real combination of the mapping of the collapsing states for the original model.

the barrier deformation can be applied, obtaining analogous general results.

B. Repeated application of the algorithm

In this section, we shall illustrate how the algorithm for the construction of the radial deformations can be applied repeatedly. We fix the system described by (4.4) as the initial one. We take as the new seed radial functions the following combinations of (4.10):

$$\chi_{m,l} = \chi_{\beta,m,l}(\gamma') + B r_c^{2\gamma'} \chi_{\beta,m,l}(-\gamma'),$$

$$\xi_{m,l} = \xi_{\beta,m,l}(\gamma') + B r_c^{2\gamma'} \xi_{\beta,m,l}(-\gamma'), \quad (4.14)$$

where B is some real constant. The transformation then gives the potential (3.24). Instead of showing its rather complicated analytic form, we can compare its plots in Fig. 5 with the Coulomb potential and with our initial barrier. Obviously, the sign of B determines whether the transformation introduces a barrier or a well potential. On the other hand, the magnitude of this parameter controls the distance where the new deformation is located.

The bound states forming two Kramer pairs, corresponding to the angular momentum m and l , can be found. The first pair comes from the direct mapping of the bound states of the initial systems. The second one can be found via the formula (3.22) for the missing states. Figure 6 shows the position where the particles are most likely to be found, which somehow coincides with the positions of the maximal or minimal values of the potential.

The full scheme of these results can be generalized as follows: Starting from the Coulomb potential, we choose N seed radial functions of the form

$$\chi_m = \chi_{\beta,m,l}(\gamma') + B_m r_c^{2\gamma'} \chi_{\beta,m}(-\gamma'),$$

$$\xi_m = \xi_{\beta,m}(\gamma') + B_m r_c^{2\gamma'} \xi_{\beta,m}(-\gamma'), \quad (4.15)$$

$$m \in \mathfrak{m} = \{m_1, \dots, m_N\}. \quad (4.16)$$

In each of them, we can fix the multiplication constants B_m that define the shape of the rings, their relative position, or whether they are attractive or repulsive.

We start the process by using those with m_1 to produce the first transformation, resulting in the single barrier studied in this section. The next step is to map the second seed radial functions and use them to perform the second transformation. We repeat this procedure iteratively with all the seed functions in our set. This will result in a deformed potential $V_m(r)$ that has N local deformations with a barrier or well shape, which forms and positions will be determined by the values and signs of constants B_m , see, for instance, Fig. 7. Each deformation has a pair of bound states with opposite total angular momentum trapped on it, denoted by $m \in \mathfrak{m}$.

C. Physical interpretations of the deformed potentials

An effective electrostatic potential can be created in graphene in various ways. To mention a few, it can be produced by local deformations [2,3], electrical and chemical doping [1], by applying electrodes, or just by putting the

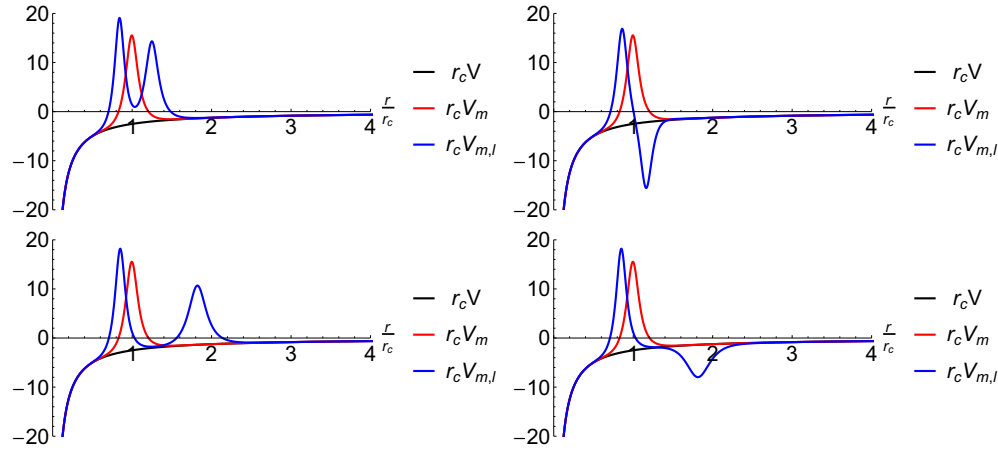


FIG. 5. Different shapes of the potential $V_{l,m}$ depending on the values of B for $\beta = 5/2$, $m = 6$ and $l = 8$. Left panel: $B = 1/2$ (top) and $B = 400$ (bottom). Right panel: $B = -1/2$ (top) and $B = -400$ (bottom).

sample near an STM [4,7]. In each of these scenarios, the electric field can be associated with an effective charge density in the layer. Accordingly, the potentials constructed with the formalism explained in Sec. III, which have the form $\mathcal{V}(r) = V(r) + V^{\text{def}}(r)$, must satisfy the Poisson equation for electrostatics

$$\nabla^2 \left(\frac{\hbar v_F}{e} \mathcal{V} \right) = \frac{\hbar v_F}{e} (\nabla^2 V + \nabla^2 V^{\text{def}}) = -\frac{1}{\epsilon} (n(\mathbf{r}) + n^{\text{def}}(\mathbf{r})), \quad (4.17)$$

where $n(\mathbf{r})$ and $n^{\text{def}}(\mathbf{r})$ are the charge densities associated to the original potentials $V(r)$ and the deformed term $V^{\text{def}}(r)$, respectively. Since the problem is strictly two-dimensional, we have

$$n^{\text{def}}(\mathbf{r}) = n^{\text{def}}(r) \delta(z),$$

$$n^{\text{def}}(r) = -\frac{\epsilon \hbar v_F}{e} \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} V^{\text{def}}(r) \right). \quad (4.18)$$

The distorted electric potential arises from an extra charge distribution on the crystal that can be computed directly from the potential itself. In particular, for the potential (4.4), one gets

$$n_m^{\text{def}}(\mathbf{r}) = -\frac{\epsilon \hbar v_F}{e r^3} \frac{8\gamma^2 \eta^{2\gamma} (a_1 + a_2 \eta^{2\gamma} + a_3 \eta^{4\gamma} + a_4 \eta^{6\gamma} + a_5 \eta^{8\gamma})}{((2m+1)\eta^{4\gamma} - 4\beta \eta^{2\gamma} + 2m+1)^3} \delta(z), \quad (4.19)$$

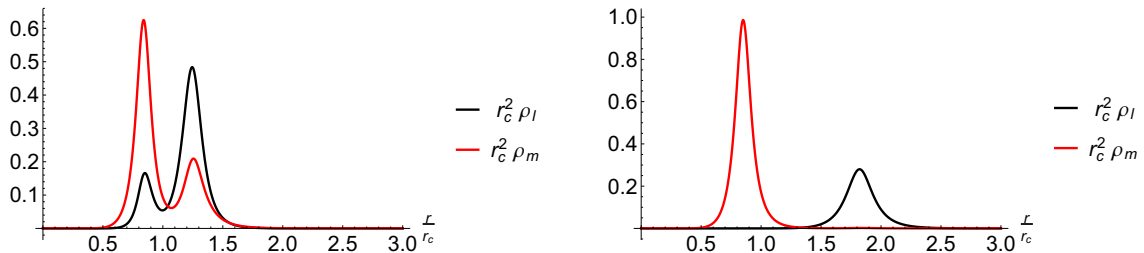


FIG. 6. The figures correspond to the density plots for the cases $B = 1/2$ (left) and $B = 400$ (right), respectively. For small B , the potentials are close to each other and the particle can be found in both deformations. For large B , the distance between the potentials is larger and the particles are localized in their respective traps.

where the constant coefficients are given by

$$a_1 = (1 - 2\gamma)^2 (2m + 1)^2,$$

$$a_2 = 8\beta(1 + 2m)(2\gamma(\gamma + 1) - 1), \quad (4.20)$$

$$a_3 = 8(3\beta^2(1 - 4\gamma^2) + \gamma^2(1 - 12\gamma^2)),$$

$$a_4 = 8\beta(2(\gamma - 1)\gamma - 1)(2m + 1), \quad (4.21)$$

$$a_5 = (2\gamma + 1)^2 (2m + 1)^2. \quad (4.22)$$

The charge density plots corresponding to the potentials from the previous paragraph are shown in Fig. 8. From them, it is explicitly seen that they correspond to ring-shaped charge accumulations. From here, we interpret the barrier and well potentials appearing at each deformation as quantum rings with positive and negative charges, respectively. These rings work as effective traps for charge carriers in the presence of the impurity at the center. From Eqs. (4.5) and (4.6), we conclude that the geometry of the rings can be tuned to trap particles in them with a desired angular momentum quantum number.

V. DISCUSSION AND OUTLOOK

In this article, we presented a method for constructing partially solvable models of Dirac fermions in graphene in the presence of radially symmetric electrostatic fields.

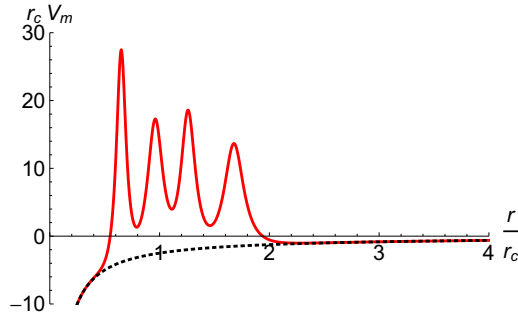


FIG. 7. Deformation with four barrier potentials. Here, we have used $m = \{6, 8, 10, 12\}$ with $\beta = 5/2$.

Our approach was based on a modification of the standard supersymmetric (Darboux) transformation applied to an intermediate, non-Hermitian operator (3.3), which is finally mapped to a physical Dirac Hamiltonian with a radially symmetric electric field (3.11). As a result, the method provides a mechanism for constructing the zero-energy eigenstates of the new system. In order to test the method developed here, we have created ring-like deformations of the initial potential. Each of these deformations preserves time-reversal symmetry, so they can host a pair of zero-energy bound states per Dirac valley. In addition, the trapped particles are characterized by circular currents flowing in opposite directions.

We focused the analysis on the construction of ring-decorated Coulomb potentials. The number of deformations in the form of concentric rings and their location in space can be deliberately adjusted by choosing specific combinations of the initial wave functions. This allows us to construct analytical potentials under a controlled scenario in the same way as the current experimental setups. Thus, the geometry of the rings is strongly related to the angular momentum quantum number of the particles they can trap, which can be chosen at will. It is worth noting that these deformations are strongly localized, i.e., they do not change the asymptotic behavior of the initial interaction at the origin and at infinity. This means that if we want to introduce controlled deformations of a specific phenomenon, we could just start from a known one. In the

particular case of the Coulomb potential, this implies that the atomic collapse states for lower angular momentum states are preserved by the transformation.

The existence of bound states was demonstrated for the models presented in Sec. IV. The question arises as to the extent to which their presence is robust with respect to changes in the potential parameters. This is important for an experimental realization, as it is rather impossible to fix the parameters with absolute precision in real settings. The zero modes confined by the deformed Coulomb potential correspond to the bound states within the continuum (BICs). These BICs are sensitive to perturbations and can transform into quasibound states (long-lived resonances) [66]. Consequently, it is plausible to anticipate quasibound states that approximate these BICs when the setup parameters approach those capable of hosting BICs. In general, the experimental detection of BICs is challenging because they do not interact with scattering states. However, some studies on graphene nanoribbons [66–68] suggest that it might be feasible through combined measurements of LDOS and conductance. Experimentally, variations in the setup, such as imperfections in the graphene sheet or the electric field, have reportedly facilitated the observation of quasibound states in the laboratory [4–7].

It is worth noting again that the method presented here only produces analytical solutions for a single energy level, which in this case is the zero energy. To understand how the constructed systems behave at nonvanishing energies, we must look beyond analytical methods such as tight binding [69] and Green function simulations [70] or the phase equation computation [71]. In this sense, it would also be interesting to investigate how the mass term can be included in the constructed Dirac operator (breaking, in this way, time-reversal symmetry), either by modifying the asymmetric transformation or following numerical techniques.

Since the presented method can be applied to any initial model, we can take a wide variety of well-known systems as a starting point. A particular example was analyzed in [46], where a connection from the free particle model to the Lorentzian well via the supersymmetric transformation was established, which also shows the shape invariance of such a system. Considering that the freedom is limited only

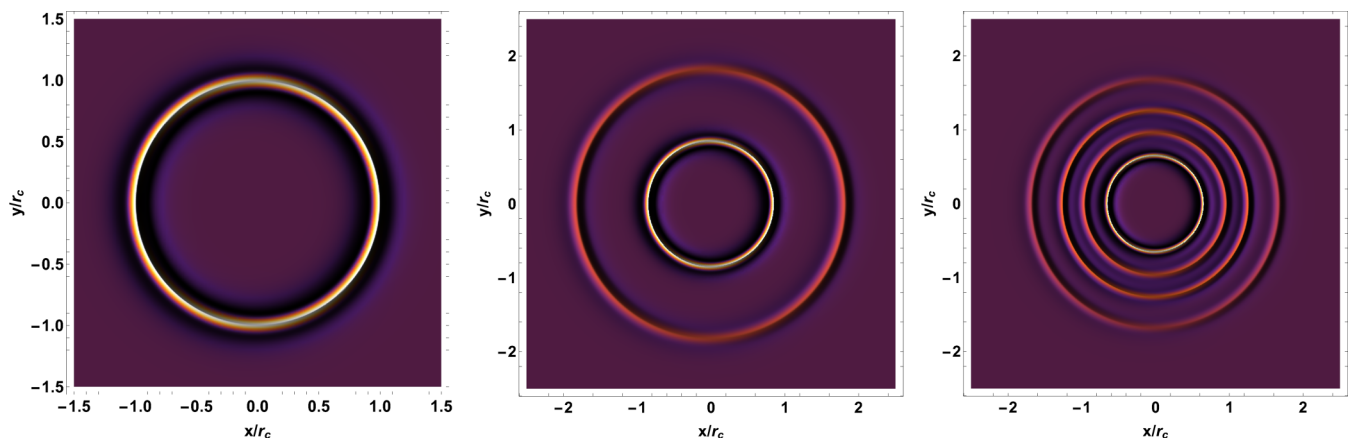


FIG. 8. Plot of the dimensionless charge $\frac{R^3 e}{\epsilon \hbar v_F} n^{\text{def}}(r)$ for the potentials in Fig. 2 (left panel), Fig. 5 (bottom left panel), and Fig. 7, respectively. As the parameter r_c can be set freely at the nanoscale, the width and radii of the rings are comparable to each other.

to the initial systems, these results open several possible research directions. It will be interesting to consider the electron-electron interaction and the screening effect associated with the deformations created with our approach in the framework of density functional and Hartree-Fock theories [24,63,72]. Studying these deformed systems in the context of QED would also be interesting. For example, one could check whether some states in the subcritical regime enter the Dirac sea and charge the QED vacuum, as in the undeformed case [73]. The method can also be used to study the existence of zero-energy bound states and their impact on the physics of the system, dictated by the Levinson theorem [71], the fermionic cloaking [74], or the Klein tunneling effect [49] in new scenarios. Additionally, another natural question is whether the presented approach is applicable in systems with curved geometric backgrounds. In this context, there could be studied effects at the level of probability and current densities when the background geometry is changed, see, e.g., [75–77]. Finally, it is possible to look for the generalization of the

present method to higher spin equations or to systems without rotational symmetry.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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- [1] M. Dragoman and D. Dragoman, Graphene-based quantum electronics, *Prog. Quantum Electron.* **33**, 165 (2009).
 - [2] J. L. Mañes, Symmetry-based approach to electron-phonon interactions in graphene, *Phys. Rev. B* **76**, 045430 (2007).
 - [3] A. Iorio and P. Pais, Revisiting the gauge fields of strained graphene, *Phys. Rev. D* **92**, 125005 (2015).
 - [4] J. Lee *et al.*, Imaging electrostatically confined Dirac fermions in graphene quantum dots, *Nat. Phys.* **12**, 1032 (2016).
 - [5] K. C. Yung, W. M. Wu, M. P. Pierpoint, and F. V. Kusmartsev, Introduction to graphene electronics – A new era of digital transistors and devices, *Contemp. Phys.* **54**, 233 (2013).
 - [6] N. M. Freitag *et al.*, Electrostatically confined monolayer graphene quantum dots with orbital and valley splittings, *Nano Lett.* **16**, 5798 (2016).
 - [7] S.-Y. Li and L. He, Recent progresses of quantum confinement in graphene quantum dots, *Front. Phys.* **17**, 33201 (2022).
 - [8] J. Yuhang *et al.*, Tuning a circular p-n junction in graphene from quantum confinement to optical guiding, *Nat. Nanotechnol.* **12**, 1045 (2017).
 - [9] R. Frisenda *et al.*, Atomically thin p-n junctions based on two-dimensional materials, *Chem. Soc. Rev.* **47**, 3339 (2018).
 - [10] S. Russo *et al.*, Observation of Aharonov-Bohm conductance oscillations in a graphene ring, *Phys. Rev. B* **77**, 085413 (2008).
 - [11] V. M. Fomin, *Physics of Quantum Rings* (Springer-Verlag, Berlin, Heidelberg, 2014).
 - [12] S. Viefers, P. Koskinen, P. Singha Deo, and M. Manninen, Quantum rings for beginners: Energy spectra and persistent currents, *Physica E* **21**, 1 (2004).
 - [13] J. Amaro Neto, M. J. Bueno, C. Furtado, Two-dimensional quantum ring in a graphene layer in the presence of an Aharonov-Bohm flux, *Ann. Phys.* **373**, 273 (2016).
 - [14] G. M. M. Wakker, R. P. Tiwari, and M. Blaauboer, Localization and circulating currents in curved graphene devices, *Phys. Rev. B* **84**, 195427 (2011).
 - [15] Y. Zhao *et al.*, Creating and probing electron whispering-gallery modes in graphene, *Science* **348**, 672 (2015).
 - [16] B. Brun *et al.*, Graphene whisperitronics: Transducing whispering gallery modes into electronic transport, *Nano Lett.* **22**, 128 (2022).
 - [17] M. Zarenia, J. M. Pereira, Jr., F. M. Peeters, and G. A. Farias, Electrostatically confined quantum rings in bilayer graphene, *Nano Lett.* **9**, 4088 (2009).
 - [18] M. Zarenia, J. M. Pereira, Jr., A. Chaves, F. M. Peeters, and G. A. Farias, Simplified model for the energy levels of quantum rings in single layer and bilayer graphene, *Phys. Rev. B* **82**, 119906 (2010).
 - [19] P. Hewageegana and V. Apalkov, Electron localization in graphene quantum dots, *Phys. Rev. B* **77**, 245426 (2008).
 - [20] C. A. Downing, D. A. Stone, M. E. Portnoi, Zero-energy states in graphene quantum dots and rings, *Phys. Rev. B* **84**, 155437 (2011).
 - [21] D. R. da Costa, A. Chaves, M. Zarenia, J. M. Pereira, Jr., G. A. Farias, and F. M. Peeters, Geometry and edge effects on the energy levels of graphene quantum rings: A comparison between tight-binding and simplified Dirac models, *Phys. Rev. B* **89**, 075418 (2014).
 - [22] S. Bellucci, A. A. Saharian, and A. K. Grigoryan, Induced fermionic charge and current densities in two-dimensional rings, *Phys. Rev. D* **94**, 105007 (2016).
 - [23] C. A. Downing and M. E. Portnoi, Zero-energy vortices in Dirac materials, *Phys. Status Solidi B* **256**, 1800584 (2019).
 - [24] M. I. Katsnelson, *The Physics of Graphene* (Cambridge University Press, United Kingdom, 2020).
 - [25] V. B. Matveev and M. A. Sale, *Darboux Transformations and Solitons* (Springer-Verlag, Berlin, 1991).
 - [26] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry in Quantum Mechanics* (World Scientific, Singapore, 2001).
 - [27] G. Darboux, Sur une proposition relative aux équations linéaires, *C. R. Acad. Sci. (Paris)* **94**, 1456 (1882).
 - [28] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics* (Springer, Berlin, 1996).

- [29] A. Schulze-Halberg and P. Roy, Construction of zero-energy states in graphene through the supersymmetry formalism, *J. Phys. A: Math. Theor.* **50**, 365205 (2017).
- [30] L. M. Nieto, A. A. Pecheritsyn, and B. F. Samsonov, Intertwining technique for the one-dimensional stationary Dirac equation, *Ann. Phys.* **305**, 151 (2003).
- [31] A. A. Pecheritsyn, E. E. Pozdeeva, and B. F. Samsonov, Darboux transformation of the nonstationary Dirac equation, *Russ. Phys. J.* **48**, 365 (2005).
- [32] I. A. Bocanegra-Garay, M. Castillo-Celeita, J. Negro, L. M. Nieto, and F. J. Gómez-Ruiz, Exploring supersymmetry: Interchangeability between Jaynes-Cummings and anti-Jaynes-Cummings models, *Phys. Rev. Res.* **6**, 043218 (2024).
- [33] K. Roychowdhury, J. Attig, S. Trebst, and M. J. Lawler, Supersymmetry on the lattice: Geometry, topology, and flat bands, *Phys. Rev. Res.* **6**, 043273 (2024).
- [34] V. Jakubsky, L. M. Nieto, and M. S. Plyushchay, Klein tunneling in carbon nanostructures: A free particle dynamics in disguise, *Phys. Rev. D* **83**, 047702 (2011).
- [35] V. Jakubský and M. Plyushchay, Supersymmetric twisting of carbon nanotubes, *Phys. Rev. D* **85**, 045035 (2012).
- [36] F. Correa and V. Jakubský, Twisted kinks, Dirac transparent systems and Darboux transformations, *Phys. Rev. D* **90**, 125003 (2014).
- [37] F. Correa and V. Jakubský, Confluent Crum-Darboux transformations in Dirac Hamiltonians with PT -symmetric Bragg gratings, *Phys. Rev. A* **95**, 033807 (2017).
- [38] E. Pozdeeva and A. Schulze-Halberg, Darboux transformations for a generalized Dirac equation in two dimensions, *J. Math. Phys.* **51**, 113501 (2010).
- [39] V. Jakubský and K. Zelaya, Reflectionless pseudospin-1 Dirac systems via Darboux transformation and flat band solutions, *Phys. Scr.* **99**, 035220 (2024).
- [40] V. Jakubský and K. Zelaya, Flat-band engineering of quasi-one-dimensional systems via supersymmetric transformations, *Phys. Rev. B* **109**, 245406 (2024).
- [41] A. Iurov, L. Zhemchuzhna, P. Fekete, G. Gumbs, and D. Huang, Klein tunneling of optically tunable Dirac particles with elliptical dispersions, *Phys. Rev. Res.* **2**, 043245 (2020).
- [42] C.-D. Han, H.-Y. Xu, and Y.-C. Lai, Electrical confinement in a spectrum of two-dimensional Dirac materials with classically integrable, mixed, and chaotic dynamics, *Phys. Rev. Res.* **2**, 013116 (2020).
- [43] H.-Y. Xu, L. Huang, and Y.-C. Lai, Klein scattering of spin-1 Dirac-Weyl wave and localized surface plasmon, *Phys. Rev. Res.* **3**, 013284 (2021).
- [44] Y. Betancur-Ocampo, G. Cordourier-Maruri, V. Gupta, and R. de Coss, Super-Klein tunneling of massive pseudospin-one particles, *Phys. Rev. B* **96**, 024304 (2017).
- [45] M. V. Ioffe, D. N. Nishnianidze, and E. V. Prokhvatilov, New solutions for graphene with scalar potentials by means of generalized intertwining, *Eur. Phys. J. Plus* **134**, 450 (2019).
- [46] A. Contreras-Astorga, F. Correa, and V. Jakubský, Super-Klein tunneling of Dirac fermions through electrostatic gratings in graphene, *Phys. Rev. B* **102**, 115429 (2020).
- [47] A. Schulze-Halberg, Darboux algorithms for two-dimensional Dirac equations with upper triangular potential matrix, *J. Comput. Appl. Math.* **427**, 115143 (2023).
- [48] M. V. Ioffe and D. N. Nishnianidze, Solvable two-dimensional Dirac equation with matrix potential: Graphene in external electromagnetic field, *Symmetry* **16**, 126 (2024).
- [49] F. Correa, L. Inzunza, and V. Jakubský, Lorentzian quantum wells in graphene: The role of shape invariance in zero-energy states trapping, *Phys. Rev. B* **110**, 075120 (2024).
- [50] V. M. Pereira, J. Nilsson, and A. H. Castro Neto, Coulomb impurity problem in graphene, *Phys. Rev. Lett.* **99**, 166802 (2007).
- [51] A. V. Shytov, M. I. Katsnelson, and L. S. Levitov, Vacuum polarization and screening of supercritical impurities in graphene, *Phys. Rev. Lett.* **99**, 236801 (2007).
- [52] A. V. Shytov, M. I. Katsnelson, and L. S. Levitov, Atomic collapse and Quasi-Rydberg states in graphene, *Phys. Rev. Lett.* **99**, 246802 (2007).
- [53] Y. Wang *et al.*, Observing atomic collapse resonances in artificial nuclei on graphene, *Science* **340**, 734 (2013).
- [54] J. S. Wu and M. M. Fogler, Scattering of two-dimensional massless Dirac electrons by a circular potential barrier, *Phys. Rev. B* **90**, 235402 (2014).
- [55] L. Inzunza and M. S. Plyushchay, Klein four-group and Darboux duality in conformal mechanics, *Phys. Rev. D* **99**, 125016 (2019).
- [56] H. Tasaki, *Physics and Mathematics of Quantum Many-Body Systems* (Springer, Cham, Switzerland, 2020).
- [57] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon, New York, 1981).
- [58] J. E. Moore and L. Balents, Topological invariants of time-reversal-invariant band structures, *Phys. Rev. B* **75**, 121306(R) (2007).
- [59] W. Liu and Y. Xu, *Spintronic 2D Materials: Fundamentals and Applications* (Woodhead Publishing, Cambridge, United Kingdom, 2019).
- [60] K. M. Case, Singular potentials, *Phys. Rev.* **80**, 797 (1950).
- [61] A. Gorsky and F. Popov, Atomic collapse in graphene and cyclic renormalization group flow, *Phys. Rev. D* **89**, 061702(R) (2014).
- [62] K. S. Gupta and S. Sen, Bound states in graphene, *Mod. Phys. Lett. A* **24**, 99 (2009).
- [63] V. M. Pereira, V. N. Kotov, and A. H. Castro Neto, Supercritical Coulomb impurities in gapped graphene, *Phys. Rev. B* **78**, 085101 (2008).
- [64] D. S. Novikov, Elastic scattering theory and transport in graphene, *Phys. Rev. B* **76**, 245435 (2007).
- [65] T. Ihn, A. Fuhrer, M. Sigrist, K. Ensslin, W. Wegscheider, and M. Bichler, Quantum mechanics in quantum rings, *Adv. Solid State Phys.* **43**, 139 (2003).
- [66] C. W. Hsu, B. Zhen, A. D. Stone, J. D. Joannopoulos, and M. Soljačić, Bound states in the continuum, *Nat. Rev. Mater.* **1**, 16048 (2016).
- [67] J. W. González, M. Pacheco, L. Rosales, and P. A. Orellana, Bound states in the continuum in graphene quantum dot structures, *Europhys. Lett.* **91**, 66001 (2010).
- [68] N. Cortés, L. Chico, M. Pacheco, L. Rosales, and P. A. Orellana, Bound states in the continuum: Localization of Dirac-like fermions, *Europhys. Lett.* **108**, 46008 (2014).
- [69] W. Hergert and R. M. Geilhufo, *Group Theory in Solid State Physics and Photonics: Problem Solving with Mathematica* (John Wiley & Sons, Hoboken, New Jersey, 2018).

- [70] S. Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, Cambridge, UK, 1997).
- [71] C. A. Downing, A. R. Pearce, R. J. Churchill, and M. E. Portnoi, Optimal traps in graphene, [Phys. Rev. B **92**, 165401 \(2015\)](#).
- [72] M. Polini, A. Tomadin, R. Asgari, and A. H. MacDonald, Density-functional theory of graphene sheets, [Phys. Rev. B **78**, 115426 \(2008\)](#).
- [73] J. Reinhardt and W. Greiner, Quantum electrodynamics of strong fields, [Rep. Prog. Phys. **40**, 219 \(1977\)](#).
- [74] B. Liao, M. Zebarjadi, K. Esfarjani, and G. Chen, Isotropic and energy-selective electron cloaks on graphene, [Phys. Rev. B **88**, 155432 \(2013\)](#).
- [75] P. I. C. Caneda and G. Menezes, Reduced quantum electrodynamics in curved space, [Phys. Rev. D **103**, 065010 \(2021\)](#).
- [76] A. A. Saharian, Vacuum currents in curved tubes, [Phys. Rev. D **110**, 065020 \(2024\)](#).
- [77] D. N. Le, A. L. Phan, V. H. Le, and P. Roy, Relativistic Coulomb problem in curved spaces, [Europhys. Lett. **127**, 10005 \(2019\)](#).