

Revisiting Gaussian genuine entanglement witnesses with modern software

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Continuous-variable Gaussian entanglement is an attractive notion, both as a fundamental concept in quantum information theory, based on the well-established Gaussian formalism for phase-space variables, and as a practical resource in quantum technology, exploiting in particular, unconditional room-temperature squeezed-light quantum optics. The readily available high level of scalability, however, is accompanied by an increased theoretical complexity when the multipartite entanglement of a growing number of optical modes is considered. For such systems, we present several approaches to reconstruct the most probable physical covariance matrix from a measured nonphysical one and then test the reconstructed matrix for different kinds of separability (factorizability, concrete partite separability, or biseparability) even in the presence of measurement errors. All these approaches are based on formulating the desired properties (physicality or separability) as convex optimization problems, which can be efficiently solved with modern optimization solvers, even when the system grows. To every optimization problem, we construct the corresponding dual problem used to verify the optimality of the solution. Besides this numerical part of work, we derive an explicit analytical expression for the symplectic trace of a positive definite matrix, which can serve as a simple witness of an entanglement witness, and extend it for positive semidefinite matrices. In addition, we show that in some cases our optimization problems can be solved analytically. As an application of our analytical approach, we consider small instances of bound entangled or genuine multipartite entangled Gaussian states, including some examples from the literature that were treated only numerically, and a family of non-Gaussian states.

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I. INTRODUCTION

The largest entangled system ever experimentally demonstrated is that of a fully inseparable light-mode state for more than one million modes [1]. This extraordinary level of scalability of continuous-variable systems [2–5] relies upon the deterministic nature of field-quadrature squeezing for photonic “qumodes,” not available for photonic qubits encoded in single photons and using standard quantum optics sources, and on advanced methods such as time-domain multiplexing. While the photonic qumode and qubit approaches share the benefit of room-temperature operations, unlike typical solid-state platforms for quantum computing, with regard to a universal set of quantum gates, the main experimental hurdles are distinct: Deterministic non-Gaussian gates are hard to obtain for optical qumodes and deterministic entangling gates are not readily available for photonic qubits. Since building large continuous-variable cluster states, or multipartite multimode entangled states [6,7], can be done efficiently with optical qumodes, it is of high interest to theoretically characterize and experimentally determine the entanglement properties of

such states—efficiently and reliably, even on a large scale. The Gaussian entanglement can then be employed for the ultimate applications such as universal measurement-based quantum computing [2,4,6,7] or for more near-term, subuniversal quantum advantage demonstrations [8]. In this work, we shall focus on the increased complexity of Gaussian entangled states of up to almost 20 optical modes, exhibiting a rich variety of (in)separability properties, including the ultimate form of genuine multipartite multimode entanglement. Supplementing such large-scale optical systems with additional non-Gaussian elements may eventually lead to the ultimate quantum applications [9,10]. Non-Gaussian (entangled) states reach a high level of complexity even on a smaller scale [11,12].

We present convex optimization formulations of two problems for multimode Gaussian states—how to extract the best possible physical covariance matrix (CM) from the measured set of data and how to verify genuine multipartite entanglement of a physical covariance matrix. Our work is not the first one to apply methods of convex optimization for studying entanglement properties of multipartite multimode states, but we go beyond just verifying entanglement. We also compute the degree to which such a verification can be trusted in the presence of imperfect measurement data. Taking into account the recent progress in hardware and, more importantly, in software, it is the right time to reconsider application of convex optimization to such problems. In contrast with most other works where just a few states are tested for multipartite entanglement, we can perform such tests “on an industrial scale” for a sufficiently large number of parts. Our approach is not restricted to CMs only; it can be extended to a more general

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problem of finding the most likely set of data with special properties from a set of measurements, which do not have these properties. As long as the properties can be formulated in terms of convexity, the ideas presented in this work can also be applied to such a problem as well.

For every problem developed in this work, we present a detailed comparison of three solvers—Clarabel [13], Mosek, and Splitting Conic Solver (SCS) [14]. The hardware is a high-end desktop PC with 24 cores and 128 Gb of random access memory (RAM). We do not program to each solver’s Application Programming Interface (API) individually, we use the JuMP [15] framework instead. The choice of these solvers is determined by the fact that among the solvers supported by JuMP only these three can solve all kinds of problems present in this work. All solvers are used with the default settings, except Clarabel, where we changed the default single-threaded linear solver to the multithreaded mkl solver. Using nondefault settings for parameters that influence the convergence to the solution is mostly “guess and try” approach. We thus avoid guessing the parameters and use the defaults. Note that thanks to the dual problem and the Karush-Kuhn-Tucker (KKT) conditions we never have to guess whether the produced solution is correct or not—the dual solution is a witness of optimality and accuracy. Irrespective of the parameters chosen for the execution of a solver, we can always control the quality of the produced output.

Our results will be illustrated by six figures for each problem. We show the average time needed to solve a problem of a given size (averaged over ten runs) together with its standard deviation, the maximal amount of RAM required [16], and three accuracy metrics (which are not averages, but the worst cases of the ten runs). The first two metrics are the magnitudes ($\log_{10}$) of the minimal eigenvalue of the primal and dual optimal solution, the other one being the magnitude of the relative duality gap, which we define as the absolute value of the difference between the optimal values of the primal and dual objective functions divided by the optimal value of the primal function. It turns out that the primal and dual solutions are boundary, so their minimal eigenvalues are zero; the other metric characterizes the degree of optimality of the solution. This choice of the metrics is motivated by the fact that boundary conditions of the primal and dual solutions are not explicit in the problem formulation. It is a consequence of one of the KKT conditions, but solvers do not have to preserve it explicitly. These metrics are thus a good test for the quality of the produced solution. For a perfect solution, the minimal eigenvalues and the duality gap must be zero, so all these metrics must be $-\infty$. Due to limited precision of the machine arithmetic, the numerical solutions are imperfect and the metrics are finite. The more negative they are, the more accurate the solution is. SCS solver is the least accurate for the problems considered, but its truly remarkable feature is low RAM requirement by comparison with the other two solvers, so we included it in our comparison. In addition, its precision is likely to be improved by increasing the number of iterations (which would also increase the time to find a solution), but that is the topic of a more detailed comparison and is outside the scope of this work. Whether the default accuracy is enough or not can be decided on a case-by-case basis.

The figures also demonstrate that the reduced case can be solved faster and with less RAM than the full case, so our specialization of the problems for the full and reduced cases separately is *very valuable for practical applications of states with a large number of modes*. We emphasize that the exact results might depend on the test sets (on the same hardware), but the curves give a rough estimate of the complexity to solve the problems of various sizes. For Clarabel and Mosek, we show the maximal problem that is possible to solve with these solvers with 128 GB of RAM. For SCS, which typically requires far less RAM, the main limiting factor is the time to perform the tests.

A significant feature of our “modern treatment” is that we can deal with Gaussian multipartite multimode states of a larger scale compared with existing works and even treat them analytically in some cases. More specifically, we shall treat systems of up to almost 20 optical modes, thus significantly going beyond the existing schemes. With SCS, we can test 17-partite full covariance matrices and 18-partite reduced covariance matrices (when xp part is zero). This range of state sizes can be divided into three categories—trivial, simple, and difficult. States with up to ~ 5 modes are totally trivial and can be solved almost instantly, for the number of modes from ~ 6 to ~ 10 , the solver gives small but noticeable delay and if the number of modes is larger than ~ 10 then the time required to perform the test rapidly increases. These numbers give a very rough boundary between trivial, simple, and difficult sizes of states for a typical, non-high-end PC. For the next generations of hardware and new versions of the optimization software, these numbers will likely increase, so boundary for difficult state sizes will increase and more states become simple or trivial to solve.

We explicitly specialize every optimization problem to this reduced case. This specialization is insignificant for small values of the number of parts n (let us say, for $n \leq 8$), has observable effect for $n = 9, 10$, and dramatically changes the situation for $n > 10$. The optimization problem in the full case of $n = 17$ has nearly 19M variables ($1M = 10^6$), and the problem in the reduced case of $n = 18$ has more than 22M variables. Though the precision of SCS is somewhat reduced by comparison with the other two solvers, by tweaking the parameters of the solver it is likely possible to improve the precision (and increase the time to solve the problem), but this should be done for each problem individually.

Another observation is the RAM required to perform the genuine entanglement tests. It turned out that in all cases (different problems, full and reduced forms, various solvers), the RAM grows superexponentially. This effect becomes important only for the difficult category of state sizes, which currently means that the number of modes is larger than ~ 10 .

Our work is a combination of both numerical and analytical results. In the first part, we formulate all the optimization problems and show how computational resources needed to solve them are scaled as functions of the number of parts. In the second part, we demonstrate that in some cases the problems can be solved exactly and provide analytical solutions. In more details, the article is structured as follows. In Sec. II, we give a nonexhaustive list of existing works on the topic. In Sec. III, we give a short review of the convex optimization theory. Section IV is devoted to physical covariance matrix

reconstruction. After a short introduction to physicality of covariance matrices and establishing the notation, we formulate the problem of reconstructing a physical CM from measured data as optimization problems. In Sec. IV A, we use Chebyshev approximation to determine the closest physical CM to a measured nonphysical one. In Sec. IV B, we use an alternative approach to find the most probable physical CM under the assumption of a Gaussian distribution of the results of the measurements. In Sec. V, we derive an analytical expression for the symplectic trace of a symmetric, positive-defined matrix as an optimal objective value of some trace minimization problem and construct its dual problem. In Sec. VI, we review the symmetric states used later as a benchmark for our entanglement detection procedures. In Sec. VII, we review the CM-based test for entanglement and establish terminology we use later. In Sec. VII A, using the dual problem for symplectic trace obtained in Sec. V, we derive the optimization problem and its dual for the best witness of a given physical CM. Here, we also construct a parametric family of n -partite states and analytically show that these states are genuine multipartite entangled. These states are then used for benchmarking of our programs. In addition, from the optimal values of the optimization problems, we construct two entanglement measures. These measures are zero on separable states and only on them, and they are convex; i.e., mixtures can only decrease these measures, but never increase. These measures can be computed numerically and in some cases even analytically. In Sec. VII B, we give an alternative approach, based on scaling, and construct the corresponding optimization problem and its dual. These tests do not take into account the inaccuracy of the measurements. In Sec. VII C, we include the inaccuracy in the form of covariances of the measurements. The formulations of the theorems are rather long and very detailed, with lots of KKT conditions, which might look excessive at first. However, it is these conditions that allow one to construct a system of equations for the optimal solution and obtain an analytical solution for it. Without the KKT conditions, it would be impossible to construct such a system. In Sec. VIII, we analyze a condition obtained in Ref. [17] and show its performance for testing the genuine multipartite entanglement relative to the standard, “classical” approach. For testing single partitions, the performance advantage can be huge, but for multiple partitions this advantage is more modest. In Sec. IX A, we present applications of the established results by comparing them with some existing works. In Secs. IX B–IX D, we construct analytical solutions of two optimization problems for three different states—fixed bipartitions for a bound 4-partite entangled state and all bipartitions for 3- and 4-partite states. Our examples are either not known yet or they are known from the literature, but were only numerically treated before. Our solutions turn out to be algebraic numbers and we explicitly present their minimal polynomials. These solutions demonstrate highly nontrivial relations between some algebraic numbers of various orders. Explicit expression of the solution as algebraic numbers in terms of their minimal polynomials might not be the most practical way to work with this solution, but it allows one to get the result exactly, without relying on any approximation at all. In addition, it gives a very detailed illustration of how the primal and dual solutions work in harmony to deliver the common optimal value. The

KKT conditions used to obtain analytical results can thus be tested for a numerical result to verify the accuracy of the numerical solution produced by the solver, which is especially important for large-partite systems. In Sec. IX E, we construct an analytical solution for a parametric family of states, not just for one concrete state. In Sec. IX F, we give a detailed study of a family of tripartite symmetric states and obtain analytical conditions of the various kinds of entanglement. We present a method to find an analytical equation for the boundary of an entanglement region even if the problem cannot be solved analytically. In addition, we show that the boundary of the genuine entanglement region consists of two different curves. As an extra, more special example, we consider a family of non-Gaussian states and test our CM-based formalism for such states. In Sec. X, we give some conjectures and open problems related to the results presented in this work. In the conclusion, we summarize our results.

II. EXISTING WORKS

As our goal is to revisit the subject of Gaussian entanglement from a modern perspective, let us first discuss the similarity and difference to some other works on the topic. The semidefinite optimization approach to determine Gaussian entanglement properties was introduced in Ref. [18]. The condition on the witness matrix that it is really a witness was given in an Appendix and referred to a manuscript in preparation. To our knowledge, that manuscript has never been published. The KKT conditions that relate primal and dual solutions of the optimization problem were not explicitly presented in Ref. [18]. In some cases, KKT conditions allow one to solve the problem analytically. In fact, this is how we derived an explicit expression for the symplectic trace and obtained explicit analytical solutions for a few 3- and 4-partite states. Where an analytical solution is either impossible or not practical, the KKT conditions can be used to estimate the correctness and quality of the numerical solution. The page with the software that the authors of Ref. [18] used in their work does not seem to exist anymore, so it is not immediately possible to use that software and test how well it performs on modern hardware for larger problems. In the arXiv version of their work, the authors of Ref. [18] give concrete numbers—2 s to test for 3-partite entanglement and 40 min for 4-partite. It means that each additional part increases the computation time by more than one thousand. These numbers have not been included in the published journal version, which, together with the absence of the software, makes it impossible to reproduce that result. However, if the scalability of the computation time with the size of the states (number of modes) was similar to those cases mentioned, the software appeared to be no longer usable in practice for more than five parts. It gives a strong incentive to reconsider the problem and explore the possibilities of modern hardware and software. Our analytical solution also shows the progress made in hardware and software during the last 20 years—what required a nontrivial amount of time to solve numerically, now can be solved analytically in a second.

To our knowledge, the term “symplectic trace,” which is defined as the sum of the symplectic eigenvalues, was introduced in Ref. [19]. The symplectic eigenvalues were

introduced in Ref. [20] (see also Refs. [3,21]). The Gaussian state represented by the CM in Eq. (50) below, which plays an important role in deducing an expression for the symplectic trace through some minimization problem, was mentioned in Ref. [17] without relation to its minimization properties. The expression of the symplectic trace as the minimal value of some quantity [Eq. (70)] was mentioned in Ref. [22] and referred back to Ref. [19]. To our knowledge, the explicit expression for the symplectic trace and its dual characterization as a maximum of some quantity have not been known before. The dual form allows one to construct an appropriate optimization problem, and the explicit expression can be used to easily test the obtained solution.

A treatment to incorporate error analysis into entanglement detection was initiated in Ref. [23]. That approach was based on heuristic genetic programming, which generally scales rather poorly and also delivers suboptimal results. In that work, the authors dealt with only fixed partitions and our Fig. 10 below clearly demonstrates the superiority of our approach. Other than that figure, we do not consider fixed partitions numerically due to their computational simplicity and exclusively study genuine multipartite entanglement. We deal with fixed bipartitions analytically when we construct an exact solution for a bound entangled 4-partite state.

In Ref. [24], several conditions for genuine multipartite entanglement in terms of the CM are derived. Those conditions work with subsets of the elements of CM, so they require less measurements, but in general they are only sufficient for entanglement and thus somewhat incomplete. The conditions are also based on optimization, but not convex, which makes it harder to perform in practice. The authors of that work give examples for states with up to $n = 6$ parts. Because they say nothing about how they performed the optimization, how much computational resources were needed, and how much time it took, we cannot compare the efficiency of their approach to ours. With our method, it takes just a fraction of a second to test 6-partite states. The scalability of the entanglement verification (how much time and RAM it takes) has not been discussed in the papers cited, and the accuracy of the optimal solution has not been addressed as well.

III. CONVEX OPTIMIZATION

All problems in this work are instances of the following general convex optimization problem:

$$\begin{aligned} & \text{minimize} && f_0(\mathbf{x}), \\ & \text{subject to} && f_i(\mathbf{x}) \preceq_{\mathcal{K}_i} 0, \quad i = 1, \dots, m, \\ & && h_j(\mathbf{x}) = 0, \quad j = 1, \dots, p, \end{aligned} \quad (1)$$

where $f_i(\mathbf{x})$, $i = 0, \dots, m$ are convex functions, $h_j(\mathbf{x})$, $j = 1, \dots, p$ are linear functions, and $\mathcal{K}_i$ are convex cones of dimensions k_i . The problems considered in this work use three types of cones

$$\begin{aligned} \mathcal{K} &= \mathbf{R}_+^n = \{(x_1, \dots, x_n) \mid x_1 \geq 0, \dots, x_n \geq 0\}, \\ \mathcal{K} &= \mathbf{S}_+^n = \{X \in \mathbf{R}^{n \times n} \mid X = X^T, X \succcurlyeq 0\}, \\ \mathcal{K} &= \mathbf{K}^{n+1} = \{(x_1, \dots, x_n, t) \mid \|\mathbf{x}\| \leq t\}, \end{aligned} \quad (2)$$

where the norm in the last cone is the standard Euclidean norm. The dimensions of these cones are

$$\dim \mathbf{R}_+^n = n, \quad \dim \mathbf{S}_+^n = \frac{n(n+1)}{2}, \quad \dim \mathbf{K}^{n+1} = n+1.$$

The objective functions $f_0(\mathbf{x})$ are linear except the case of the most probable CM, where $f_0(\mathbf{x})$ is quadratic.

To construct the dual problem to the problem (1), we define the Lagrangian function

$$\mathcal{L}(\mathbf{x}, \lambda, \mathbf{v}) = f_0(\mathbf{x}) + \sum_{i=1}^m \lambda_i^T f_i(\mathbf{x}) + \sum_{j=1}^p v_j h_j(\mathbf{x}), \quad (3)$$

where $\mathbf{v} = (v_1, \dots, v_p)$ is a vector of numbers and $\lambda = (\lambda_1, \dots, \lambda_m)$ is a vector of vectors, where each component is of the dimension of the corresponding cone $\mathcal{K}_i$. The dual objective function is

$$g(\lambda, \mathbf{v}) = \inf_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda, \mathbf{v}). \quad (4)$$

The dual problem then reads as

$$\begin{aligned} & \text{maximize} && g(\lambda, \mathbf{v}), \\ & \text{subject to} && \lambda_i \succcurlyeq_{\mathcal{K}_i^*} 0, \quad i = 1, \dots, m, \\ & && \text{finite constraints,} \end{aligned} \quad (5)$$

where $\mathcal{K}_i^*$ is the cone dual to $\mathcal{K}_i$ and finite constraints are the conditions under which the dual objective $g(\lambda, \mathbf{v})$ is finite. All the cones in Eq. (2) are self-dual, so no new types of cones appear. The KKT conditions, which connect the primal $\mathbf{x}^*$ and dual $(\lambda^*, \mathbf{v}^*)$ optimal solutions, are

$$h_j(\mathbf{x}^*) = 0, \quad (6)$$

$$\lambda_i^{*T} f_i(\mathbf{x}^*) = 0, \quad (7)$$

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^m Df_i(\mathbf{x}^*)^T \lambda_i^* + \sum_{j=1}^p v_j^* \nabla h_j(\mathbf{x}^*) = 0, \quad (8)$$

where the $k_i \times n$ matrix $Df_i(\mathbf{x}^*)$ is the derivative of $f_i(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{x}^*$. If there is a feasible point $\mathbf{x}$, interior to the domain of all $f_i(\mathbf{x})$, $i = 1, \dots, m$, and such that the strong inequalities

$$f_i(\mathbf{x}) \prec_{\mathcal{K}_i} 0 \quad (9)$$

hold, then the strong duality between the problems (1) and (5) also holds,

$$f_0(\mathbf{x}^*) = g(\lambda^*, \mathbf{v}^*). \quad (10)$$

The problems considered in this work, when feasible, have strong duality property.

IV. PHYSICAL COVARIANCE MATRIX RECONSTRUCTION

The basic property that defines a physical n -partite CM γ (of size $2n \times 2n$) is the physicality condition

$$\mathcal{P}(\gamma) = \gamma + \frac{i}{2} \Omega_n \succcurlyeq 0, \quad (11)$$

where Ω_n is given by

$$\Omega_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}, \quad (12)$$

and E_n is the identity matrix of size n . We use the notation $A \succcurlyeq 0$ to denote a positive semidefinite matrix A and $A \geq 0$ to express that all elements of A are non-negative. Similar notation is used for strict inequalities. From the physicality condition (11), it follows that any physical CM γ is strictly positive definite, $\gamma \succ 0$ [25].

We now establish a real form of physicality of a CM γ . We group all x parts before all p parts in covariance matrices (another notation in use is to alternate xs and ps), so a general n -partite CM can be written in this notation as

$$\gamma = \begin{pmatrix} \gamma_{xx} & \gamma_{xp} \\ \gamma_{xp}^T & \gamma_{pp} \end{pmatrix}, \quad (13)$$

where γ_{xx} , γ_{pp} , and γ_{xp} are $n \times n$ blocks. Let $H = S + iA$ be a Hermitian, where S is a real symmetric matrix, $S^T = S$, and A is a real antisymmetric matrix, $A^T = -A$. If H is the matrix of a linear map in some basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ over the field of complex numbers $\mathbb{C}$, then the same map has the matrix

$$\tilde{H} = \begin{pmatrix} S & \mp A \\ \pm A & S \end{pmatrix} \quad (14)$$

in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \pm i\mathbf{e}_1, \dots, \pm i\mathbf{e}_n\}$ over the field of real numbers $\mathbb{R}$ (where we always chose either the upper sign or always the bottom sign). It follows that

$$H = S + iA \succcurlyeq 0 \quad \Leftrightarrow \quad \tilde{H} = \begin{pmatrix} S & \mp A \\ \pm A & S \end{pmatrix} \succcurlyeq 0. \quad (15)$$

In particular, we have an equivalence

$$S + iA \succcurlyeq 0 \quad \Leftrightarrow \quad S - iA \succcurlyeq 0. \quad (16)$$

It also shows that it does not matter which sign to choose in Eq. (14), so we choose the bottom one. Thus, the physicality condition $\mathcal{P}(\gamma) \succcurlyeq 0$ of the covariance matrix γ , which involves a complex Hermitian matrix, can be written in the real form as

$$\begin{pmatrix} \gamma & \frac{1}{2}\Omega \\ -\frac{1}{2}\Omega & \gamma \end{pmatrix} = \begin{pmatrix} \gamma_{xx} & \gamma_{xp} & 0 & \frac{1}{2}E \\ \gamma_{xp}^T & \gamma_{pp} & -\frac{1}{2}E & 0 \\ 0 & -\frac{1}{2}E & \gamma_{xx} & \gamma_{xp} \\ \frac{1}{2}E & 0 & \gamma_{xp}^T & \gamma_{pp} \end{pmatrix} \succcurlyeq 0. \quad (17)$$

By rearranging the rows and columns of the latter matrix, which is achieved by multiplying it with appropriate permutation matrices, we obtain the following equivalent condition:

$$\begin{pmatrix} \gamma_{xx} & \frac{1}{2}E & 0 & \gamma_{xp} \\ \frac{1}{2}E & \gamma_{pp} & \gamma_{xp}^T & 0 \\ 0 & \gamma_{xp} & \gamma_{xx} & -\frac{1}{2}E \\ \gamma_{xp}^T & 0 & -\frac{1}{2}E & \gamma_{pp} \end{pmatrix} \succcurlyeq 0. \quad (18)$$

It follows that for the reduced states, i.e., the states with zero off-diagonal part, $\gamma_{xp} = 0$, the condition (17) is equivalent to two simpler conditions,

$$\mathcal{P}(\gamma_{xx}, \gamma_{pp}) = \begin{pmatrix} \gamma_{xx} & \pm \frac{1}{2}E \\ \pm \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \succcurlyeq 0. \quad (19)$$

As we have already seen above, the choice of the sign does not matter, so it is enough to test just one of them. Below, we use this condition with the sign $+$. Additionally, if Eq. (13) is a general, full CM, then the reduced CM

$$\gamma' = \begin{pmatrix} \gamma_{xx} & 0 \\ 0 & \gamma_{pp} \end{pmatrix}, \quad (20)$$

obtained by setting $\gamma_{xp} = 0$ in γ , is also physical. It is this observation that allows us to simplify the optimization problems developed in later sections of this work in the cases where γ_{xp} does not appear in the objective functions. In such cases, it is enough to optimize for γ_{xx} and γ_{pp} only, which requires n^2 variables less for an n -partite state. If the reduced γ' , obtained from a full physical CM γ , were not always physical, we would have to optimize over the whole matrices γ in all cases. In the following, by a reduced state we mean a pair of matrices γ_{xx}, γ_{pp} that satisfies the condition (19). In all the optimization problems presented below, the physicality conditions for all the CM variables γ , with or without sub- or superscripts, are assumed. We do not explicitly state them not to repeat them all over again.

An n -mode Gaussian state with CM γ is pure iff

$$(\gamma\Omega_n)^2 = -\frac{1}{4}E_{2n}. \quad (21)$$

For reduced states, $\gamma_{xp} = 0$, this condition reads as follows:

$$\gamma_{xx}\gamma_{pp} = \frac{1}{4}E_n. \quad (22)$$

This condition will be used below to test states for purity.

A positive-semidefinite matrix $A \succcurlyeq 0$ is called boundary if it is on the boundary of the positive-semidefinite cone, i.e., if the minimal eigenvalue $\lambda_0(A)$ of A is zero. A CM γ is called boundary if $\mathcal{P}(\gamma)$ is boundary. In the reduced case, a pair γ_{xx}, γ_{pp} is boundary if $\mathcal{P}(\gamma_{xx}, \gamma_{pp})$ is boundary. Since each physical CM is also positive semidefinite, an ambiguity can arise. Because we always distinguish physical CMs and arbitrary matrices, it is always clear from the context which notion of boundary is used. Below, we need the following simple statement about arbitrary boundary matrices.

Lemma 1. If A and B are Hermitian positive-semidefinite matrices, $A, B \succcurlyeq 0$, with $\text{Tr}(AB) = 0$, then $AB = 0$. In addition, if $B \neq 0$, then A is boundary.

Proof. The proof is based on a simple fact that a positive semidefinite matrix X is zero, $X = 0$, iff its trace is zero, $\text{Tr}X = 0$. Suppose that $\text{Tr}(AB) = 0$. We cannot directly apply the fact above because AB is, in general, not Hermitian, but we can rewrite this equality as

$$\text{Tr}(\sqrt{A}B\sqrt{A}) = 0, \quad (23)$$

where $\sqrt{A}$ is a Hermitian square root of A . Square roots of matrices are not unique, so there might be non-Hermitian square roots, but a Hermitian square root of a positive-semidefinite Hermitian matrix always exists. The under trace is Hermitian, so from the fact above we derive that

$$\sqrt{A}B\sqrt{A} = 0. \quad (24)$$

This equality can be rewritten as $C^+C=0$, where $C=\sqrt{B}\sqrt{A}$. It follows that $C = 0$ and by multiplying by $\sqrt{B}$ on the left and by $\sqrt{A}$ on the right, we obtain the equality $AB = 0$.

Now, suppose that $B \neq 0$, but A is not boundary. Then, A is invertible and from the equality $AB = 0$, we derive that $B = 0$, which contradicts our assumption. Thus, A must be boundary. ■

As a consequence, if $\text{Tr}(AB) = 0$ and both A and B are nonzero, then they are both boundary.

The reduced CM given by Eq. (20) is boundary if

$$\det \mathcal{P}(\gamma') = \det \begin{pmatrix} \gamma_{xx} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \det \begin{pmatrix} \gamma_{xx} & -\frac{1}{2}E \\ -\frac{1}{2}E & \gamma_{pp} \end{pmatrix} = 0.$$

According to Ref. [26], we have the equality

$$\det \begin{pmatrix} \gamma_{xx} & \pm \frac{1}{2}E \\ \pm \frac{1}{2}E & \gamma_{pp} \end{pmatrix} = \det (\gamma_{xx}\gamma_{pp} - \frac{1}{4}E), \quad (25)$$

so the two matrices in Eq. (19) are simultaneously degenerate or nondegenerate. It follows that a reduced state γ_{xx}, γ_{pp} is boundary if $\det \mathcal{P}(\gamma_{xx}, \gamma_{pp}) = 0$.

The experimentally measured covariance matrices γ° sometimes happen to be nonphysical, i.e., they violate the physicality condition [Eq. (11)]. If performing a more precise experiment is not an option, then one needs to determine somehow the most likely physical covariance matrix that agrees with the experimental data. In addition to the matrix γ° itself, the matrix σ of standard deviations of the elements of γ° is measured as well, so the true physical covariance matrix γ should not deviate too much from the measured γ° , where the distance is measured in units of σ . Below, we present two approaches to this problem.

Given a nonphysical experimentally measured CM γ° with the matrix of corresponding standard deviations σ of measurements, what is the most likely physical CM consistent with these data? Below, we propose two approaches to answer this question and test each approach on some randomly generated data. The tests are performed on ten randomly generated matrices for the number of modes from 20 to 200 with step 10. The problems with sizes below 20 are too simple to include into the figures. The covariance test matrices are randomly generated with the elements in the range $[-1, 1]$ and the test matrices of standard deviations are randomly generated with elements in the range $[0.001, 0.1]$. In this way, our generated data are physical CMs plus some noise. And we expect that the larger the dimension of the states is, the stronger the noise in the measured data will be. In general, these covariance matrices are far away from the set of physical covariance matrices, but we do not have a better test suite and nonideal input data are a kind of stress test for the problem formulation and its implementation. Experiments with hundreds-partite states, determining and exploiting the full CM, are not commonplace, although even larger-scale experiments have been performed relying on a much smaller set of measurement data used as sufficient criteria to witness full inseparability [1,27]. These tests should be considered as a comparison of the efficiency of different solvers on some randomly chosen problems. Realistic data, which are typically much closer to the physical set, should be easier to solve, so the reported time is likely to give an upper bound for the time it takes to solve the corresponding problems in practice.

A. Closest covariance matrix

In Ref. [28], we presented a way to determine the closest physical CM to a given nonphysical one. Here, we reproduce that optimization problem in a more detailed form and specialize it to the reduced CM (where $\gamma_{xp} = 0$).

Theorem 1. For any symmetric $2n \times 2n$ matrices γ° and σ , the latter with strictly positive elements, the following equality takes place:

$$\min_{\gamma} \max_{1 \leq i, j \leq 2n} \frac{|\gamma_{ij} - \gamma_{ij}^\circ|}{\sigma_{ij}} = \max_{\Lambda \succcurlyeq 0, U^\pm \geq 0} -\text{Tr}[\Lambda \mathcal{P}(\gamma^\circ)]. \quad (26)$$

The minimization is performed over all physical covariance matrices γ . The maximization is over a Hermitian $2n \times 2n$ matrix Λ and a pair of real symmetric $2n \times 2n$ matrices U^+ and U^- that satisfy the conditions $\Lambda \succcurlyeq 0, U^\pm \geq 0$, and

$$\text{Tr}[(U^+ + U^-)\sigma] = 1, \quad U^+ - U^- = \text{Re}(\Lambda). \quad (27)$$

The primal and dual optimal solutions are related by the equalities (KKT conditions) $\mathcal{P}(\gamma^*)\Lambda^* = 0$ and

$$U^{\pm*} \cdot (\gamma^* - \gamma^\circ \mp s^*\sigma) = 0, \quad (28)$$

where s^* is the common value of Eq. (26) and where $A \cdot B$ is the Hadamard (elementwise) product of two matrices.

In the reduced case, the previous result is specialized as follows. For any $n \times n$ matrices $\gamma_{xx}^\circ, \gamma_{pp}^\circ$ and σ_{xx}, σ_{pp} , the latter two with strictly positive elements, the dual relation reads as

$$\begin{aligned} \min_{\gamma_{xx}, \gamma_{pp}} \max_{1 \leq i, j \leq n} & \left[\frac{|\gamma_{xx,ij} - \gamma_{xx,ij}^\circ|}{\sigma_{xx,ij}}, \frac{|\gamma_{pp,ij} - \gamma_{pp,ij}^\circ|}{\sigma_{pp,ij}} \right] \\ & = \max_{\Lambda \succcurlyeq 0, U^\pm, V^\pm \geq 0} -\text{Tr}[\Lambda \mathcal{P}(\gamma_{xx}^\circ, \gamma_{pp}^\circ)]. \end{aligned} \quad (29)$$

The minimization is over all reduced CMs. The maximization is over a real symmetric $2n \times 2n$ matrix Λ and real symmetric $n \times n$ matrices $U^\pm$ and $V^\pm$ subject to the conditions $\Lambda \succcurlyeq 0, U^\pm, V^\pm \geq 0$, and

$$\begin{aligned} \text{Tr}[(U^+ + U^-)\sigma_{xx} + (V^+ + V^-)\sigma_{pp}] &= 1, \\ \Lambda_{xx} &= U^+ - U^-, \quad \Lambda_{pp} = V^+ - V^-. \end{aligned} \quad (30)$$

The KKT conditions read as $\mathcal{P}(\gamma_{xx}^*, \gamma_{pp}^*)\Lambda^* = 0$ and

$$\begin{aligned} U^{\pm*} \cdot (\gamma_{xx}^* - \gamma_{xx}^\circ \mp s^*\sigma_{xx}) &= 0, \\ V^{\pm*} \cdot (\gamma_{pp}^* - \gamma_{pp}^\circ \mp s^*\sigma_{pp}) &= 0, \end{aligned} \quad (31)$$

where s^* is the common value of Eq. (29). In both cases, the optimal state γ^* or $\gamma_{xx}^*, \gamma_{pp}^*$ is a boundary state.

The number of variables in the primal optimization problem is

$$N_v = \begin{cases} n(2n+1)+1, & \text{full case,} \\ n(n+1)+1, & \text{reduced case.} \end{cases} \quad (32)$$

In the reduced case, the problem has n^2 variables less, so for large n it is almost half the size of the full case. Such problems, minimization of the maximal difference, are called Chebyshev approximation problems [29]. If we represent a symmetric matrix as a vector of its different components and denote by s the objective function of the primal problem of Eq. (26), then

$$s = \left\| \frac{\gamma - \gamma^\circ}{\sigma} \right\|_\infty, \quad (33)$$

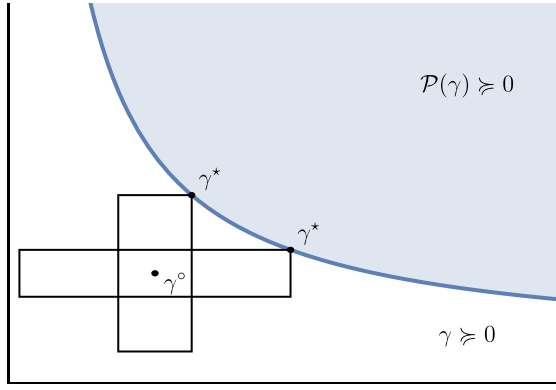


FIG. 1. The optimal solution of Eq. (26) for different σ .

where the division is performed elementwise, so γ^* is the physical covariance matrix closest to γ^0 in the ∞ norm. The unit sphere of the ∞ norm is a unit square, and the scaling by σ turns square into a general rectangle. The optimization (26) then seeks the minimal factor s^* by which we need to expand or shrink the “unit rectangle” so that it touches the boundary of the set of the physical covariance matrices. The optimal value s^* gives the minimal $s^*\sigma$ neighborhood of γ^0 , which contains a physical covariance matrix, γ^* . A couple of such minimal neighborhoods for different σ are shown on Fig. 1, where the positive orthant denotes the set of positive-semidefinite matrices, the blue region denotes the set of physical CMs, and the rectangles are the sets of γ defined by $\|(\gamma - \gamma^0)/\sigma\| \leq s$ for two σ 's. Put another way, for any physical γ at least one component γ_{ij} is s^* or more σ_{ij} intervals away from γ_{ij}^0 . The smaller the optimal s^* , the higher the probability that the experiment was performed correctly and the optimal solution γ^* is close to the true covariance matrix of the state under study. If γ^0 is physical from the very beginning, then $\gamma^* = \gamma^0$ and $s^* = 0$. According to the standards of reliability [30,31], if $s^* > 2$, then the experiment should be considered very inaccurate. If $s^* > 3$, then the measured data are erroneous. We emphasize that this conclusion is correct for any distribution of the measured data, provided that it is not a pathological one with fat tails, which it should not be in a real experiment.

The objective of the dual problem is $-\text{Tr}[\Lambda \mathcal{P}(\gamma^0)]$. If γ^0 is physical, then $\mathcal{P}(\gamma^0) \geq 0$ and $-\text{Tr}[\Lambda \mathcal{P}(\gamma^0)] \leq 0$ for all $\Lambda \geq 0$, so the minimum of the primal objective is zero, and the corresponding dual optimal solution is $\Lambda^* = 0$ and $U^{+*} = U^{-*}$, appropriately normalized so that the first condition of Eq. (27) is satisfied. If γ^0 is nonphysical, then $\mathcal{P}(\gamma^0)$ has negative eigenvalues, which can be amplified by a positive semidefinite Λ^* . Exactly how much the negative eigenvalues can be amplified is determined by the appropriate conditions on the dual variables. In addition, γ^* and Λ^* are boundary. In fact, γ^* is physical by construction and nonzero (its imaginary part is $\Omega/2$), and Λ^* is nonzero since γ^0 is nonphysical and thus $\gamma^* \neq \gamma^0$. As a consequence of Lemma 1, it follows that both γ^* and Λ^* are boundary. The minimal eigenvalues of $\mathcal{P}(\gamma^*)$ [or $\mathcal{P}(\gamma_{xx}^*, \gamma_{pp}^*)$ in the reduced case] and Λ^* can be used as metrics of the accuracy of the solution produced by a convex optimization solver. These metrics, together with the relative gap and performance characteristics are shown

in Fig. 2 for the three solvers used in this work. From this figure, one can conclude that Mosek is likely to be the most appropriate choice for this problem and the data used as tests. We see that true CM of a 100-partite state can be obtained in a reasonable amount of time on a moderate desktop hardware.

B. The most probable covariance matrix

In the approach above, we made no assumption about the exact form of the probability distributions of the elements of γ except that they have averages γ^* , standard deviations σ , and have bell shape so that the σ intervals have meaning close to that of Gaussian distributions. If we make a stronger assumption about the probability distributions of the individual elements of γ , we can define the most probable physical covariance matrix as the matrix that maximizes the probability distribution. We assume now that each measured matrix element has perfect Gaussian distribution with the average γ_{ij}^0 and standard deviation σ_{ij} :

$$p(\gamma_{ij}) = \frac{1}{\sqrt{2\pi}\sigma_{ij}} \exp \left[-\frac{(\gamma_{ij} - \gamma_{ij}^0)^2}{2\sigma_{ij}^2} \right]. \quad (34)$$

The perfectly Gaussian distribution is nonphysical for diagonal elements (they cannot be negative), but once the measured covariance matrix is nonphysical, we already work with such objects. If the experiment is not totally inaccurate, the ratios $\gamma_{ii}^0/\sigma_{ii}$ should be large enough, so that the probability of γ_{ii} being negative under the distribution [Eq. (34)] is exponentially small.

Assuming that the measurements of the matrix elements are done independently, the probability density of γ being the true physical matrix reads as the product

$$p(\gamma) = \prod_{1 \leq i \leq j \leq 2n} p(\gamma_{ij}). \quad (35)$$

In the reduced case, the product is over the elements of γ_{xx} and γ_{pp} . Now, we can state that the most likely physical matrix γ^* is the one that maximizes the probability distribution $p(\gamma)$. Expanding the product, we see that we need to minimize the quadratic form

$$\sum_{1 \leq i \leq j \leq 2n} \frac{(\gamma_{ij} - \gamma_{ij}^0)^2}{2\sigma_{ij}^2}. \quad (36)$$

This is a quadratic form in the variables γ_{ij} and we need to minimize this form subject to the semidefinite constraint $\mathcal{P}(\gamma) \geq 0$. Some components of the optimal solution might be far away from their measured averages (but others might be very close), so this approach cannot be considered completely satisfactory. A more reliable approach is to constrain all components γ_{ij} inside a reasonable interval around the average,

$$|\gamma_{ij} - \gamma_{ij}^0| \leq s_0 \sigma_{ij}, \quad (37)$$

where, for example, $s_0 = 2$. Then, the algorithm to find the most probable physical covariance matrix reads as follows: We first solve the problem (26) and find s^* . If $s^* \geq s_0$, then the experiment is considered inaccurate and further processing stops. If $s^* < s_0$, then the most likely physical covariance matrix is the solution that minimizes the quadratic form given by Eq. (36), subject to the physicality condition and the

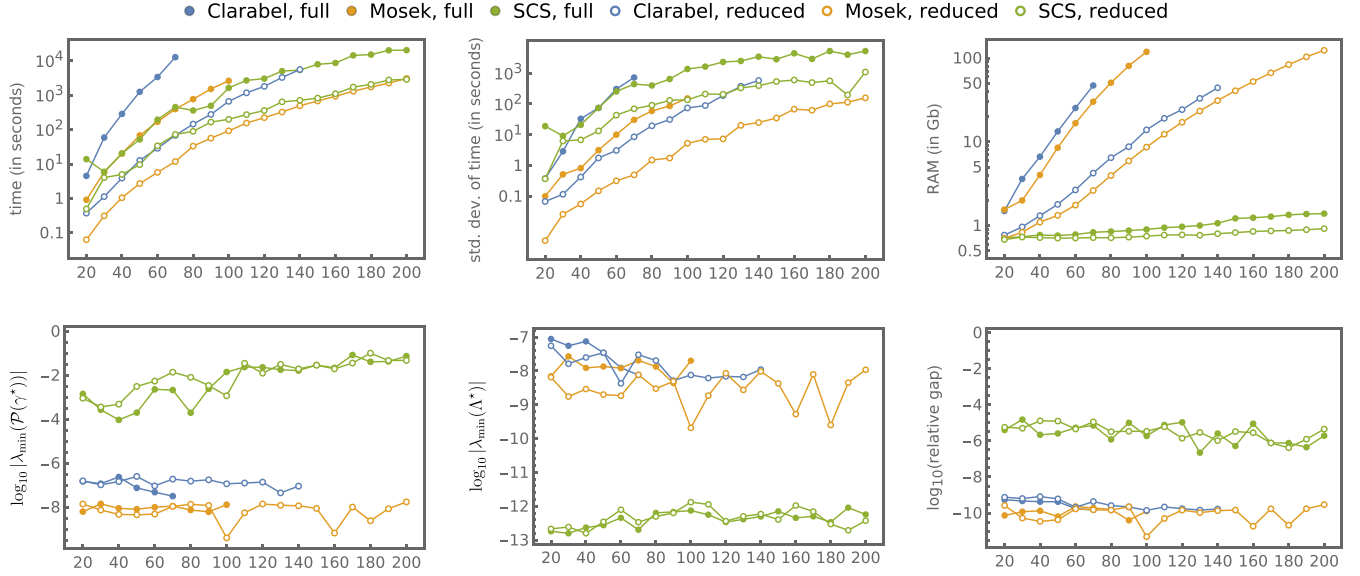


FIG. 2. Comparing performance and accuracy characteristics of different solvers for the closest physical covariance matrix problem as a function of the number of modes.

conditions expressed by Eq. (37). At least one component γ_{ij}^* of the solution of this problem will have the property $s^* \leq |\gamma_{ij}^* - \gamma_{ij}^\circ| \leq s_0$. It is this solution γ^* that we use in the applications to entanglement verification below. The dual problem is given by the following theorem.

Theorem 2. For any symmetric $2n \times 2n$ matrices γ° and σ , the latter with strictly positive elements, the following equality takes place:

$$\min_{\mathcal{P}(\gamma) \geq 0} \sum_{1 \leq i \leq j \leq 2n} \frac{(\gamma_{ij} - \gamma_{ij}^\circ)^2}{2\sigma_{ij}^2} = \max_{\Lambda \geq 0} -\text{Tr} \left[\Lambda \mathcal{P}(\gamma^\circ) + \frac{1}{2} M'^2 \sigma^{-2} \right], \quad (38)$$

where the maximization on the right-hand side is over all Hermitian positive semidefinite matrices $\Lambda \geq 0$ and M'_{ij} are the elements of $M = \text{Re}(\Lambda)$ with nondiagonal elements mul-

tiplied by $\sqrt{2}$ so that

$$M'_{ij} = \begin{cases} M_{ij}, & i = j, \\ \sqrt{2}M_{ij}, & i \neq j. \end{cases} \quad (39)$$

Note that M'^2 and σ^{-2} are Hadamard squares, but their product is an ordinary matrix product. The KKT conditions are given by $\text{Tr}[\Lambda^* \mathcal{P}(\gamma^*)] = 0$ and

$$\gamma^* = \gamma^\circ + M^{*''} \cdot \sigma^{-2}, \quad (40)$$

where M' is matrix with elements M'_{ij} (note that the prime is applied twice and the product between $M^{*''}$ and σ^{-2} is Hadamard).

With the additional constraint (37), where $s_0 > s^*$ is larger than the optimal solution of the problem (26), the following primal and dual problems are feasible and their duality reads as

$$\min_{\substack{\mathcal{P}(\gamma) \geq 0 \\ |\gamma - \gamma^\circ| \leq s_0 \sigma}} \sum_{i \leq j} \frac{(\gamma_{ij} - \gamma_{ij}^\circ)^2}{2\sigma_{ij}^2} = \max_{\Lambda, U^\pm} -\text{Tr} \left[\Lambda \mathcal{P}(\gamma^\circ) + \frac{1}{2} (M - U^+ + U^-)^2 \sigma^{-2} + s_0 (U^+ + U^-) \sigma \right], \quad (41)$$

where the maximization is over all Hermitian matrices $\Lambda \geq 0$ and a pair of real symmetric matrices $U^\pm \geq 0$ with non-negative elements. The KKT conditions are $\mathcal{P}(\gamma^*) \Lambda^* = 0$ and

$$\begin{aligned} \gamma^* &= \gamma^\circ + (M^* - U^{+*} + U^{-*})'' \cdot \sigma^{-2}, \\ U^{\pm*} (\gamma^* - \gamma^\circ \mp s_0 \sigma) &= 0. \end{aligned} \quad (42)$$

In the reduced case, the general problem above for $s_0 = +\infty$ is specialized as follows:

$$\min_{\mathcal{P}(\gamma_{xx}, \gamma_{pp}) \geq 0} \sum_{i \leq j} \left[\frac{(\gamma_{xx,ij} - \gamma_{xx,ij}^\circ)^2}{2\sigma_{xx,ij}^2} + \frac{(\gamma_{pp,ij} - \gamma_{pp,ij}^\circ)^2}{2\sigma_{pp,ij}^2} \right] = \max_{\Lambda \geq 0} \left(-\text{Tr}[\Lambda \mathcal{P}(\gamma_{xx}^\circ, \gamma_{pp}^\circ)] - \frac{1}{2} \text{Tr}[\Lambda_{xx}'^2 \sigma_{xx}^{-2} + \Lambda_{pp}'^2 \sigma_{pp}^{-2}] \right),$$

where the maximization is over all real $\Lambda \geq 0$. The KKT conditions are $\mathcal{P}(\gamma_{xx}^*, \gamma_{pp}^*) \Lambda^* = 0$ and

$$\gamma_{xx}^* = \gamma_{xx}^\circ + \Lambda_{xx}^{*''} \cdot \sigma_{xx}^{-2}, \quad \gamma_{pp}^* = \gamma_{pp}^\circ + \Lambda_{pp}^{*''} \cdot \sigma_{pp}^{-2}. \quad (43)$$

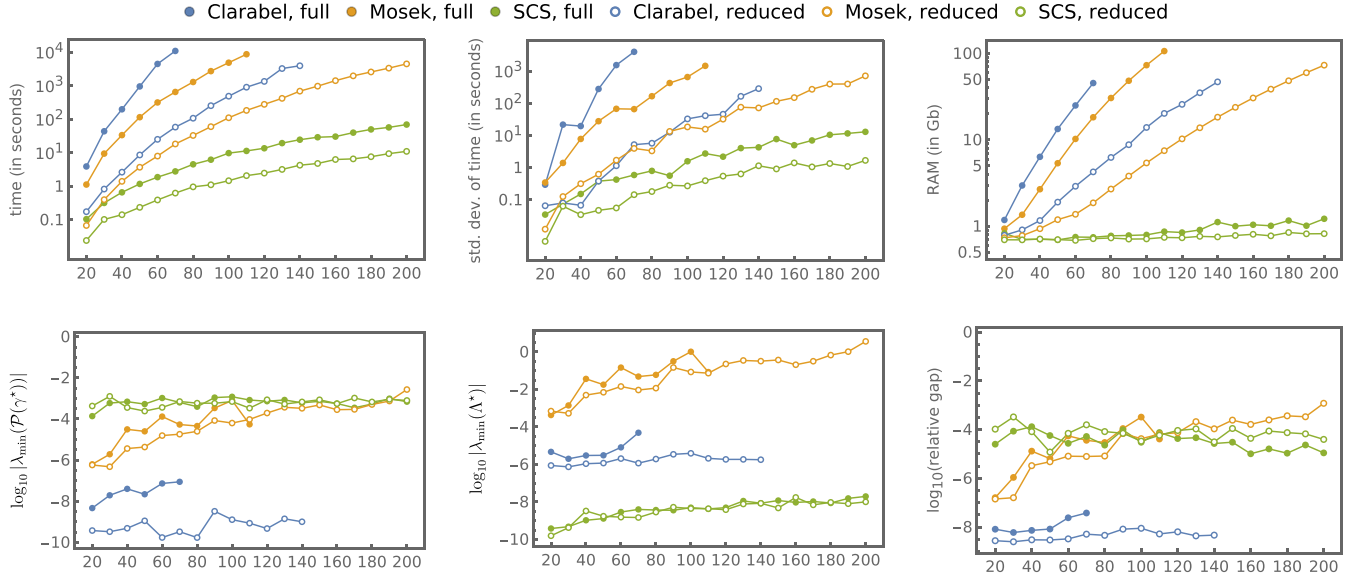


FIG. 3. Comparing performance and accuracy characteristics of different solvers for the most probable physical covariance matrix problem (with $s_0 = +\infty$) as a function of the number of modes.

For $s_0 < +\infty$, several linear constraints are added in addition to the semidefinite constraint. The problem reads as follows, where the maximization is over all real $\Lambda \geq 0$ and $U^\pm, V^\pm \geq 0$:

$$\min_{\gamma_{xx}, \gamma_{pp}} \sum_{i \leq j} \left[\frac{(\gamma_{xx,ij} - \gamma_{xx,ij}^\circ)^2}{2\sigma_{xx,ij}^2} + \frac{(\gamma_{pp,ij} - \gamma_{pp,ij}^\circ)^2}{2\sigma_{pp,ij}^2} \right] = \max_{\Lambda \geq 0, U^\pm, V^\pm \geq 0} \left(-\text{Tr}[\Lambda \mathcal{P}(\gamma_{xx}^\circ, \gamma_{pp}^\circ)] - \text{Tr} \left[\frac{1}{2} (\Lambda_{xx} - U^+ + U^-)^2 \sigma_{xx}^2 \right. \right. \\ \left. \left. + \frac{1}{2} (\Lambda_{pp} - V^+ + V^-)^2 \sigma_{pp}^2 + s_0 (U^+ + U^-) \sigma_{xx} + s_0 (V^+ + V^-) \sigma_{pp} \right] \right), \quad (44)$$

with the addition constraints of the primal problem (on the left-hand side) being

$$|\gamma_{xx} - \gamma_{xx}^\circ| \leq s_0 \sigma_{xx}, \quad |\gamma_{pp} - \gamma_{pp}^\circ| \leq s_0 \sigma_{pp}. \quad (45)$$

The KKT conditions are $\mathcal{P}(\gamma_{xx}^*, \gamma_{pp}^*) \Lambda^* = 0$,

$$\gamma_{xx}^* = \gamma_{xx}^\circ + (\Lambda_{xx}^* - U^{+\star} + U^{-\star})'' \cdot \sigma_{xx}^2, \\ \gamma_{pp}^* = \gamma_{pp}^\circ + (\Lambda_{pp}^* - V^{+\star} + V^{-\star})'' \cdot \sigma_{pp}^2, \quad (46)$$

and

$$U^{\pm\star} \cdot (\gamma_{xx}^* - \gamma_{xx}^\circ \mp s_0 \sigma_{xx}) = 0, \\ V^{\pm\star} \cdot (\gamma_{pp}^* - \gamma_{pp}^\circ \mp s_0 \sigma_{pp}) = 0. \quad (47)$$

If the original CM γ° (or the pair γ_{xx}° and γ_{pp}°) is nonphysical, then the optimal, most probable CM γ^* is boundary.

The objectives of the primal problems are positive quadratic forms of primal variables, thus bounded from below, and the minimum of the forms is finite. The objectives of dual problems are negative quadratic forms of dual variables, thus bounded from above, and the maximum is finite. The duality states that the minimum of primal quadratic form equals the maximum of the dual quadratic form. As in the previous case, if γ° is nonphysical, then both γ^* and Λ^* are boundary, and their minimal eigenvalues can be used as metrics of the accuracy of the numerical solution. In Fig. 3, we show these

metrics together with performance characteristics of the three solvers. It seems that in this case Clarabel produces more accurate solutions (in two out of three metrics) than the others.

We now have three approaches to determine the most likely physical CM that is consistent with the experimental data γ° and σ —the closest CM, the most probable CM without restriction on its elements, and the most probable CM with the restriction given by Eq. (37). The difference between the three approaches can be described as follows. The closest CM gives the best worst-case solution, the one where the component furthest away from the average is as close to the average as possible for a physical CM. The most probable CM without restriction on its elements is the best average case, where the average “quality” of all components is the best possible, but some components might be of very low “quality” (far away from the average). Finally, the most probable CM with the restriction given by Eq. (37) is the best average case with bounded worst case, where no component can be very bad. We cannot distinguish one of these approaches as the “right” one based on probability theory only, though the last approach might be more “correct” than the other two (its solution maximizes the probability distribution of the covariance matrix of which all components are inside a reasonable σ -interval around the average). However, the ultimate way to have a true physical covariance matrix is to perform the measurements with higher accuracy.

V. SYMPLECTIC TRACE

Here, we give an analytical expression for the symplectic trace of an arbitrary positive definite matrix $M \succ 0$ by showing that it is the minimal value of some optimization problem. In addition, we construct the dual problem, which will be used below to implement a test for genuine multipartite entanglement. The theorem that states these results is as follows.

Theorem 3. For any $M \succ 0$, the following relation is valid:

$$\min_{\gamma} \text{Tr}(M\gamma) = \max_{M+iK \succ 0} \frac{1}{2} \text{Tr}(K\Omega) = \frac{1}{2} \text{Tr} \sqrt{\sqrt{M}\Omega^T M \Omega \sqrt{M}}, \quad (48)$$

where the optimization variable on the right-hand side is an antisymmetric matrix K , $K^T = -K$. The KKT condition is given by

$$\mathcal{P}(\gamma^*)(M + iK^*) = 0. \quad (49)$$

For $M \succ 0$, the optimal value (48) is equal to the symplectic trace of M and the optimal covariance matrix is unique and explicitly reads as

$$\gamma_M = \frac{1}{2} \sqrt{M^{-1}} \sqrt{\sqrt{M}\Omega^T M \Omega \sqrt{M}} \sqrt{M^{-1}}. \quad (50)$$

In the reduced case, we have

$$\begin{aligned} \min_{\gamma_{xx}, \gamma_{pp}} \text{Tr}(X\gamma_{xx} + P\gamma_{pp}) \\ = \max_Z \text{Tr} Z = \text{Tr} \sqrt{\sqrt{X}P\sqrt{X}} = \text{Tr} \sqrt{\sqrt{P}X\sqrt{P}}, \end{aligned} \quad (51)$$

where the optimization is over all matrices Z such that

$$\begin{pmatrix} X & Z \\ Z^T & P \end{pmatrix} \succ 0. \quad (52)$$

The KKT condition is given by

$$\mathcal{P}(\gamma_{xx}^*, \gamma_{pp}^*) \begin{pmatrix} X & -Z^* \\ -Z^{*T} & P \end{pmatrix} = 0. \quad (53)$$

For $X, P \succ 0$, the optimal solution is unique and reads as

$$\begin{aligned} \gamma_{xx}^* &= \frac{1}{2} \sqrt{X^{-1}} \sqrt{\sqrt{X}P\sqrt{X}} \sqrt{X^{-1}}, \\ \gamma_{pp}^* &= \frac{1}{2} \sqrt{P^{-1}} \sqrt{\sqrt{P}X\sqrt{P}} \sqrt{P^{-1}}, \end{aligned} \quad (54)$$

and the optimal value is equal to the symplectic trace of the following reduced matrix:

$$M = \begin{pmatrix} X & 0 \\ 0 & P \end{pmatrix}. \quad (55)$$

The optimal states are pure and on the boundary of the physical states.

Proof. The Lagrangian of the primal problem (48) reads as

$$\begin{aligned} \mathcal{L}(\gamma, \Lambda) &= \text{Tr}(M\gamma) - \text{Tr}[\mathcal{P}(\gamma)\Lambda] \\ &= \text{Tr}[(M - \text{Re}(\Lambda))\gamma] + \frac{1}{2} \text{Tr}(\text{Im}(\Lambda)\Omega), \end{aligned} \quad (56)$$

where $\Lambda \succ 0$. The dual objective is finite if $\text{Re}(\Lambda) = M$ and is given by

$$g(\Lambda) = \inf_{\gamma} \mathcal{L}(\gamma, \Lambda) = \frac{1}{2} \text{Tr}(\text{Im}(\Lambda)\Omega). \quad (57)$$

Denoting $K = \text{Im}(\Lambda)$, we obtain the desired dual optimization problem and the KKT condition (49).

Let us first assume that $M \succ 0$ and derive an analytical expression for the solution γ^* of the primal optimization problem (48). The primal and dual optimal solutions γ^* and K^* satisfy the KKT condition

$$\text{Tr}[\mathcal{P}(\gamma^*)(M + iK^*)] = 0, \quad (58)$$

and from Lemma 1, the equality (49) follows. Explicitly, it reads as

$$\left(\gamma^* + \frac{i}{2}\Omega\right)(M + iK^*) = 0. \quad (59)$$

Separating the real and imaginary parts, we derive two equalities

$$\gamma^* M = \frac{1}{2} \Omega K^*, \quad \frac{1}{2} \Omega M + \gamma^* K^* = 0. \quad (60)$$

By multiplying both sides of the first equality by $\gamma^* \Omega$ on the left, we obtain

$$\gamma^* \Omega \gamma^* M = -\frac{1}{2} \gamma^* K^* = \frac{1}{4} \Omega M. \quad (61)$$

Since M is nondegenerate, we can cancel it from both sides and arrive at the equality

$$\gamma^* \Omega \gamma^* = \frac{1}{4} \Omega, \quad (62)$$

which is equivalent to the condition (21). It means that any optimal solution of the primal problem is pure.

Now, we derive an explicit expression for γ^* . The second equality of Eq. (60) is equivalent to

$$K^* \gamma^* = \frac{1}{2} M \Omega^T. \quad (63)$$

Multiplying the first equality of Eq. (60) by γ^* on the right, we obtain

$$\gamma^* M \gamma^* = \frac{1}{2} \Omega K^* \gamma^* = \frac{1}{4} \Omega M \Omega^T = \frac{1}{4} \Omega^T M \Omega. \quad (64)$$

According to Ref. [32], the unique solution of this equation is given by

$$\gamma^* = \frac{1}{2} \sqrt{M^{-1}} \sqrt{\sqrt{M}\Omega^T M \Omega \sqrt{M}} \sqrt{M^{-1}}. \quad (65)$$

Such states were considered in Ref. [17], but without studying their minimizing property. For the minimum of the trace $\text{Tr}(M\gamma)$, we thus have

$$\min_{\gamma} \text{Tr}(M\gamma) = \text{Tr}(M\gamma^*) = \frac{1}{2} \text{Tr} \sqrt{\sqrt{M}\Omega^T M \Omega \sqrt{M}}.$$

We show that this (obviously symmetric) matrix satisfies the physicality condition. In fact, we have

$$\mathcal{P}(\gamma^*) = \frac{1}{2} \sqrt{M^{-1}} [\sqrt{A^T A} + iA] \sqrt{M^{-1}}, \quad (66)$$

where

$$A = \sqrt{M} \Omega \sqrt{M}. \quad (67)$$

Since $AA^T = A^T A = -A^2$, according to Ref. [33], the matrix

$$\begin{pmatrix} \sqrt{A^T A} & A \\ A^T & \sqrt{A^T A} \end{pmatrix} \succ 0 \quad (68)$$

is positive semidefinite, which proves the physicality condition $\mathcal{P}(\gamma^*) \succ 0$. In addition, the matrix $\mathcal{P}(\gamma^*)$ is singular; i.e.,

the matrix γ^* is on the boundary of the physical covariance matrices set, as it must be due to the KKT condition (49) and Lemma 1. It is instructive to see that the covariance matrix given by Eq. (65) satisfies the relation (62). In fact, we have

$$\begin{aligned} 4\gamma^*\Omega\gamma^* &= -\sqrt{M^{-1}}\sqrt{-A^2}A^{-1}\sqrt{-A^2}\sqrt{M^{-1}} \\ &= \sqrt{M^{-1}}A\sqrt{M^{-1}} = \Omega. \end{aligned} \quad (69)$$

Here, we used the fact that any two functions of the same matrix A commute ($\sqrt{-A^2}$ and A^{-1}).

We now show that

$$\min_{\gamma} \text{Tr}(M\gamma) = \text{str}(M), \quad (70)$$

where $\text{str}(M)$ is the symplectic trace of M , the sum of the symplectic eigenvalues of M . The term ‘‘symplectic eigenvalue’’ was introduced in Ref. [20]. According to the Williamson theorem [34], for a positive definite M there is a symplectic matrix S such that

$$S^TMS = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \quad (71)$$

where D is a diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$. These diagonal elements are uniquely defined by M as follows: $\pm i\lambda_j$ are the eigenvalues of ΩM . The sum of these eigenvalues is referred to as the symplectic trace of M :

$$\text{str}(M) = \text{Tr}(D) = \lambda_1 + \dots + \lambda_n. \quad (72)$$

For the primal minimization problem, we have

$$\begin{aligned} \min_{\gamma} \text{Tr}(M\gamma) &= \min_{\gamma} \text{Tr}\left(S^{T-1}\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}S^{-1}\gamma\right) \\ &= \min_{\gamma} \text{Tr}\left(\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}S^{-1}\gamma S^{T-1}\right) \\ &= \min_{\gamma'} \text{Tr}\left(\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\gamma'\right) \\ &= \sum_{j=1}^n \lambda_j(\gamma_{xx,jj} + \gamma_{pp,jj}), \end{aligned}$$

where $\gamma' = S^{-1}\gamma S^{T-1}$ is another physical CM and γ' ranges over all CMs when γ does. The classical uncertainty relation states the inequality

$$\begin{aligned} \gamma'_{xx,jj} + \gamma'_{pp,jj} &\equiv \langle (\Delta x_j)^2 \rangle + \langle (\Delta p_j)^2 \rangle \\ &\geq 2\sqrt{\langle (\Delta x_j)^2 \rangle \langle (\Delta p_j)^2 \rangle} \geq 1, \end{aligned} \quad (73)$$

and thus we conclude that

$$\min_{\gamma} \text{Tr}(M\gamma) = \lambda_1 + \dots + \lambda_n = \text{str}(M). \quad (74)$$

The equality is attained for

$$\gamma'^* = S^{-1}\gamma^*S^{T-1} = \frac{1}{2}E_{2n}. \quad (75)$$

For the optimal solution, we have

$$\gamma^* = \frac{1}{2}SS^T, \quad (76)$$

where S is defined by Eq. (71). Note that such S does not have to be unique, but according to Ref. [35], the product SS^T is the same for all symplectic S that diagonalize M . An

explicit expression for this product has been obtained above [Eq. (65)].

For any symplectic matrix $S \in \text{Sp}(2n)$, the matrix γ defined via

$$\gamma = \frac{1}{2}SS^T \quad (77)$$

is a physical CM, and the minimum in Eq. (70) is attained on a CM of this form, so we obtain the following relation:

$$\min_{S \in \text{Sp}(2n)} \text{Tr}(S^TMS) = 2\text{str}(M). \quad (78)$$

This relation has been derived in Ref. [36] in a more complicated way.

All the constructions up to now were made in the assumption $M \succ 0$. If M is degenerate, then the derivations above are not applicable, but the minimum of $\text{Tr}(M\gamma)$ still exists. Any degenerate M is a limit for $\varepsilon \rightarrow 0$ of some nondegenerate M_ε with small ε -disturbance of the elements, and the continuity argument shows that the equality (48) is also valid in the degenerate case.

The results for the reduced case are straightforwardly derived from the general case, since for any full physical CM γ , its blocks γ_{xx} and γ_{pp} (i.e., setting $\gamma_{xp} = 0$ in γ) also form a physical (reduced) CM. ■

The symplectic trace is not defined for degenerate matrices, but the equality (48) allows one to extend this notion to all positive-semidefinite matrices M :

$$\text{str}(M) = \frac{1}{2}\text{Tr}\sqrt{M\Omega^T M\Omega\sqrt{M}}, \quad (79)$$

and to all positive-semidefinite pairs X and P :

$$\text{str}(X, P) = \text{Tr}\sqrt{\sqrt{X}P\sqrt{X}} = \text{Tr}\sqrt{\sqrt{P}X\sqrt{P}}. \quad (80)$$

The latter expression is the symplectic trace of the block-diagonal matrix given by Eq. (55). For degenerate matrices, the minimum given by these expressions is not necessarily achievable. Consider an extreme (but legal) case of $X = E$ and $P = 0$. Then, $\text{str}(X, P) = 0$ and

$$\text{Tr}(X\gamma_{xx} + P\gamma_{pp}) = \text{Tr}(\gamma_{xx}) > 0 \quad (81)$$

for all physical CMs γ_{xx} . In this case, the minimum is not achievable, though it can be approached arbitrarily close by the CM $\gamma_{xx} = (1/r)E$, $\gamma_{pp} = (r/4)E$ for $r \rightarrow +\infty$.

It is not easy to give necessary and sufficient conditions for the existence of the optimal solution. In the full case, we need to solve the Eq. (64), which has the form

$$XAX = B. \quad (82)$$

Some general conditions for just the solvability of such equations are given in Ref. [37], but we also need that a solution be a physical CM. In the reduced case, the KKT condition (53) gives the equality

$$X\gamma_{xx}^* = \gamma_{pp}^*P, \quad (83)$$

so the matrices X and P should be of the same rank. Because we need only the optimal value of the optimization problems and not the condition for the attainability of this value, we do not explore this topic further.

VI. SYMMETRIC STATES

We now construct a class of genuine multipartite entangled states with any number of parts, which we will use as a benchmark below. Let us start with the matrices of a special form

$$G(x, y) = \begin{pmatrix} x & y & \dots & y & y \\ y & x & \dots & y & y \\ \dots & \dots & \dots & \dots & \dots \\ y & y & \dots & x & y \\ y & y & \dots & y & x \end{pmatrix}. \quad (84)$$

The eigenvalues of this matrix are

$$\lambda = x + (n-1)y, \quad \mu = x - y, \quad (85)$$

where the latter is with multiplicity $n-1$. The eigenvector corresponding λ is $(1, 1, \dots, 1)$, the eigenvectors of μ are $(-1, 1, 0, \dots, 0)$, $(-1, 0, 1, \dots, 0)$, ..., $(-1, 0, 0, \dots, 1)$. It follows that the matrix (84) is positive semidefinite iff $\lambda, \mu \geq 0$. The necessary condition $x \geq 0$ automatically follows from these two.

To compute the symplectic trace of a matrix, we need to take square roots and products (and for the minimizing state, we also need the inverse, provided that the matrix is nondegenerate). It turns out that the square root, product, and inverse of matrices of the form (84) are also matrices of the same form. For the square root, we have

$$\sqrt{G(x, y)} = G(u, v), \quad (86)$$

where u and v read as

$$\begin{aligned} u &= \frac{1}{n}[\sqrt{x + (n-1)y} + (n-1)\sqrt{x-y}], \\ v &= \frac{1}{n}[\sqrt{x + (n-1)y} - \sqrt{x-y}]. \end{aligned} \quad (87)$$

The product is given by

$$G(x, y)G(p, q) = G(xp + (n-1)yq, xq + yp + (n-2)yq),$$

and the inverse is

$$G^{-1}(x, y) = G\left(\frac{x + (n-2)y}{\lambda\mu}, -\frac{y}{\lambda\mu}\right), \quad (88)$$

where the underlined terms are on the i and $n+i$ position, respectively, for $i = 1, \dots, n$. The eigenvector with the eigenvalue $\lambda + \lambda^{-1}$ is

$$(\lambda, \dots, \lambda, 1, \dots, 1). \quad (95)$$

The eigenvectors with the eigenvalue $\mu + \mu^{-1}$ are

$$(-\mu, 0, \dots, \underline{\mu}, \dots, 0, -1, 0, \dots, \underline{1}, \dots, 0), \quad (96)$$

where the underlined terms are in the position i and $n+i$, respectively, for $i = 2, \dots, n$. ■

These CMs are diagonal if $\lambda = \mu$, and for $\lambda = \mu = 1$ it is the CM of the vacuum state, $\gamma^S(1, 1) = (1/2)E$. These states have been introduced in Ref. [38] (see also Refs. [27,39,40]).

where the denominator is nonzero when $\lambda, \mu > 0$. The relations between the parameters of the resulting matrix and the original matrices are complicated, so we construct a parametrization in which these relations become trivial. Note that we can express x and y in terms of the eigenvalues (85) as follows:

$$x = \frac{\lambda + (n-1)\mu}{n}, \quad y = \frac{\lambda - \mu}{n}. \quad (89)$$

We now change the parametrization

$$C(\lambda, \mu) = G\left(\frac{\lambda + (n-1)\mu}{n}, \frac{\lambda - \mu}{n}\right). \quad (90)$$

In this parametrization, the rules for the square root, product, and inverse have an especially simple form

$$\begin{aligned} \sqrt{C(\lambda, \mu)} &= C(\sqrt{\lambda}, \sqrt{\mu}), \\ C(\lambda, \mu)C(\lambda', \mu') &= C(\lambda\lambda', \mu\mu'), \\ C(\lambda, \mu)^{-1} &= C(\lambda^{-1}, \mu^{-1}), \end{aligned} \quad (91)$$

where the last equality is valid if $\lambda, \mu > 0$.

Theorem 4. The reduced symmetric states $\gamma^S(\lambda, \mu)$ of the form

$$\gamma_{xx}^S = \frac{1}{2}C(\lambda, \mu), \quad \gamma_{pp}^S = \frac{1}{2}C(\lambda^{-1}, \mu^{-1}) \quad (92)$$

are pure boundary quantum states for all $\lambda, \mu > 0$.

Proof. The physicality condition reads as

$$\mathcal{P}(\gamma_{xx}, \gamma_{pp}) = \frac{1}{2} \begin{pmatrix} C(\lambda, \mu) & E \\ E & C(\lambda^{-1}, \mu^{-1}) \end{pmatrix} \succcurlyeq 0. \quad (93)$$

The eigenvalues of this matrix are

$$(0, \dots, 0, \lambda + \lambda^{-1}, \mu + \mu^{-1}, \dots, \mu + \mu^{-1}), \quad (94)$$

all of which are non-negative and also contain zeros. The eigenvectors corresponding to zero eigenvalues are

$$\left(\frac{\lambda - \mu}{n\lambda\mu}, \dots, \underline{-\frac{(n-1)\lambda + \mu}{n\lambda\mu}}, \dots, \frac{\lambda - \mu}{n\lambda\mu}, 0, \dots, \underline{1}, \dots, 0 \right),$$

Below, we show that these nondiagonal states, i.e., if $\lambda \neq \mu$ or, equivalently, if $y \neq 0$ in Eq. (84), are genuine multipartite entangled. These states will be used as a benchmark for the entanglement detection optimization problems presented in the next section.

If we take a reduced matrix with $X = C(\lambda, \mu)$ and $P = C(\lambda', \mu')$, where all the parameters are nonzero, then the corresponding optimal covariance matrix [Eq. (54)] is of the same form

$$\gamma_{xx}^* = \frac{1}{2}C\left(\sqrt{\frac{\lambda'}{\lambda}}, \sqrt{\frac{\mu'}{\mu}}\right), \quad \gamma_{pp}^* = \frac{1}{2}C\left(\sqrt{\frac{\lambda}{\lambda'}}, \sqrt{\frac{\mu}{\mu'}}\right).$$

In other words, the optimal solution is the symmetric state

$$\gamma^* = \gamma^S \left(\sqrt{\frac{\lambda'}{\lambda}}, \sqrt{\frac{\mu'}{\mu}} \right). \quad (97)$$

It follows that for the special choice of $\lambda' = \lambda^{-1}$ and $\mu' = \mu^{-1}$, we have

$$\gamma_{xx}^* = \frac{1}{2}C(\lambda^{-1}, \mu^{-1}) = \frac{1}{2}P, \quad \gamma_{pp}^* = \frac{1}{2}C(\lambda, \mu) = \frac{1}{2}X,$$

so the optimal solution is the same as the reduced matrix with the blocks X and P exchanged (and scaled by the factor $1/2$).

VII. ENTANGLEMENT DETECTION

Let $\mathcal{I} = \{I_1, \dots, I_l\}$ be some partition of the set $[n] = \{1, \dots, n\}$ of n indices, i.e., $I_i \cap I_j = \emptyset$ and

$$I_1 \cup \dots \cup I_l = [n]. \quad (98)$$

An n -partite state $\hat{\rho}$ is called $\mathcal{I}$ factorizable if it is a product of l states with indices I_i , $i = 1, \dots, l$:

$$\hat{\rho} = \hat{\rho}^{(I_1)} \otimes \dots \otimes \hat{\rho}^{(I_l)}. \quad (99)$$

A state $\hat{\rho}_{\mathcal{I}}$ is called $\mathcal{I}$ separable if it is a convex combination of $\mathcal{I}$ -factorizable states:

$$\hat{\rho}_{\mathcal{I}} = \sum_i p_i \hat{\rho}_i^{(I_1)} \otimes \dots \otimes \hat{\rho}_i^{(I_l)}, \quad (100)$$

where $p_i \geq 0$ sum up to one. If $\mathcal{J} = \{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ is a set of partitions, then a state $\hat{\rho}_{\mathcal{J}}$ is called $\mathcal{J}$ separable if it is a convex combination of $\mathcal{I}_i$ -separable states, $i = 1, \dots, k$:

$$\hat{\rho}_{\mathcal{J}} = \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \hat{\rho}_{\mathcal{I}}, \quad (101)$$

where $p_{\mathcal{I}} \geq 0$ sum up to one. For $k = 1$ and $\mathcal{J} = \{\mathcal{I}\}$, this definition reduces to $\mathcal{I}$ separability. For $k = 2^{n-1} - 1$ and $\mathcal{J}_2$ being the set of all bipartitions of $[n]$, a $\mathcal{J}_2$ -separable state is called biseparable. A nonbiseparable state is called genuine multipartite entangled. Biseparability puts the loosest requirement on the state, so the set of the biseparable states is the largest among other sets of separable states. Conversely, the set of genuine entangled states is the smallest among other sets of entangled states.

Given a partition $\mathcal{I} = \{I_1, \dots, I_l\}$, the CM $\gamma_{\mathcal{I}}$ of any $\mathcal{I}$ -separable state satisfies the inequality

$$\gamma \succcurlyeq \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)} \quad (102)$$

for some physical covariance matrices $\gamma^{(I_i)}$ [41]. Using the concavity of CM,

$$\gamma \left(\sum_i p_i \hat{\rho}_i \right) \geq \sum_i p_i \gamma(\hat{\rho}_i), \quad (103)$$

we derive that the CM $\gamma_{\mathcal{J}}$ of any $\mathcal{J}$ -separable state $\hat{\rho}_{\mathcal{J}}$ satisfies the inequality

$$\gamma_{\mathcal{J}} \geq \sum_{\mathcal{I} = \{I_1, \dots, I_l\} \in \mathcal{J}} p_{\mathcal{I}} \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)}, \quad (104)$$

for some $p_{\mathcal{I}}$ and $\gamma^{(I)}$. It is this condition that we will use for the two types of tests below. We refer to a CM γ as $\mathcal{J}$ separable

if it satisfies an inequality of the form (104). The CM of any $\mathcal{J}$ -separable state is also $\mathcal{J}$ separable, but the reverse need not be the case even for Gaussian states. As has been recently shown in Ref. [42], if $|\mathcal{J}| > 1$ then a $\mathcal{J}$ -entangled Gaussian state can have the $\mathcal{J}$ -separable CM.

The two types are the additive and the multiplicative tests. If we have two positive quantities Q and Q' and we want to know which one is larger, we can either compare their difference $Q - Q'$ with zero, or compare their ratio Q/Q' with one. These tasks are trivial when Q and Q' are given explicitly, and it does not really matter which type of test to use. However, when $Q - Q'$ and Q/Q' cannot be computed directly and are results of complicated computational procedures, the situation becomes less clear. The efficiency and the precision of the outcome of the comparison might differ for the two types of tests, depending on the nature of the computational procedures used to compute the difference and the ratio. That is why we provide both additive and multiplicative versions of the entanglement test.

A. Eigenvalue test

From inequality (104), it follows that for any $M \succcurlyeq 0$ the inequality

$$\begin{aligned} \text{Tr}(M\gamma_{\mathcal{J}}) &\geq \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \min_{\gamma^{(I_1)}, \dots, \gamma^{(I_l)}} \text{Tr}(M\gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)}) \\ &\geq \min_{\mathcal{I} \in \mathcal{J}} \min_{\gamma^{(I_1)}, \dots, \gamma^{(I_l)}} \text{Tr}(M\gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)}) \end{aligned}$$

is satisfied. The minimum on the right-hand side can be easily computed:

$$\begin{aligned} &\min_{\gamma^{(I_1)}, \dots, \gamma^{(I_l)}} \text{Tr}(M\gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)}) \\ &= \sum_{I \in \mathcal{I}} \min_{\gamma^{(I)}} \text{Tr}(M[I]\gamma^{(I)}) = \sum_{I \in \mathcal{I}} \text{str}(M[I]), \end{aligned}$$

where $M[I]$ is the submatrix of M with row and column indices equal to $I \cup (I + n)$. We thus derive the condition that the CM $\gamma_{\mathcal{J}}$ of any $\mathcal{J}$ -separable state satisfies

$$\text{Tr}(M\gamma_{\mathcal{J}}) \geq \min_{\mathcal{I} \in \mathcal{J}} \sum_{I \in \mathcal{I}} \text{str}(M[I]), \quad (105)$$

for any $M \succcurlyeq 0$. If for a given CM γ we manage to find a matrix $M \succcurlyeq 0$ such that

$$\begin{aligned} \text{Tr}(M\gamma) &< \min_{\mathcal{I} \in \mathcal{J}} \sum_{I \in \mathcal{I}} \text{str}(M[I]) \\ &= \frac{1}{2} \min_{\mathcal{I} \in \mathcal{J}} \sum_{I \in \mathcal{I}} \text{Tr} \sqrt{\sqrt{M[I]} \Omega_{|I|}^T M[I] \Omega_{|I|} \sqrt{M[I]}}, \end{aligned} \quad (106)$$

using Eq. (79), then M witnesses that γ cannot be $\mathcal{J}$ separable. In the reduced case, this inequality reads as

$$\text{Tr}(X\gamma_{xx} + P\gamma_{pp}) < \min_{\mathcal{I} \in \mathcal{J}} \sum_{I \in \mathcal{I}} \text{Tr} \sqrt{\sqrt{X[I]} P[I] \sqrt{X[I]}}, \quad (107)$$

where $X[I]$ and $P[I]$ are submatrices of X and P , respectively, with row and column indices in I . We now give a simple application of this test.

Theorem 5. The states in Eq. (92) are genuine multipartite entangled provided that $\lambda \neq \mu$.

Proof. We just need to find a pair of positive semidefinite matrices X and P that satisfy the inequality (107) for $\mathcal{I} = \mathcal{I}_2$, the set of all $2^{n-1} - 1$ nontrivial bipartitions of n parts. We take $X = C(\alpha, \beta)$ and $P = C(\alpha', \beta')$ and find their parameters to satisfy that condition. We have

$$\begin{aligned} & \text{Tr}(X\gamma_{xx}^S + P\gamma_{pp}^S) \\ &= \frac{1}{2} \left(\frac{\alpha}{\lambda} + (n-1)\frac{\beta}{\mu} + \alpha'\lambda + (n-1)\beta'\mu \right). \end{aligned} \quad (108)$$

To compute the right-hand side of Eq. (107), note that due to symmetry of the matrices of the form $C(\alpha, \beta)$, both $X[I]$ and $P[I]$ depend only on $|I| = k$, so we need to take the minimum over $n-1$ values (for $k = 1, \dots, n-1$) instead of all $2^{n-1} - 1$ nontrivial bipartitions. Every submatrix $C(\alpha, \beta)[I] = C(\tilde{\alpha}, \tilde{\beta})$ is also of the same form, where the tilded parameters are related to the original ones via

$$\frac{\tilde{\alpha} + (k-1)\tilde{\beta}}{k} = \frac{\alpha + (n-k)\beta}{n}, \quad \frac{\tilde{\alpha} - \tilde{\beta}}{k} = \frac{\alpha - \beta}{n}. \quad (109)$$

Explicitly, they read as

$$\tilde{\alpha} = \frac{k\alpha + (n-k)\beta}{n}, \quad \tilde{\beta} = \beta. \quad (110)$$

We then have $X[I] = C(\tilde{\alpha}, \tilde{\beta})$, $P[I] = C(\tilde{\alpha}', \tilde{\beta}')$, and

$$\begin{aligned} & \text{Tr} \sqrt{\sqrt{X[I]}P[I]\sqrt{X[I]}} = \sqrt{\tilde{\alpha}\tilde{\alpha}'} + (k-1)\sqrt{\tilde{\beta}\tilde{\beta}'} \\ &= \frac{1}{n} \sqrt{(k\alpha + (n-k)\beta)(k\alpha' + (n-k)\beta')} + (k-1)\sqrt{\beta\beta'}. \end{aligned} \quad (111)$$

The sum of this expression for k and $n-k$, minimized over $k = 1, \dots, n-1$, is the right-hand side of Eq. (107). We thus need to find α, β, α' , and β' such that the following inequality is satisfied:

$$\begin{aligned} & \frac{1}{2} \left(\frac{\alpha}{\lambda} + (n-1)\frac{\beta}{\mu} + \alpha'\lambda + (n-1)\beta'\mu \right) \\ & < \frac{1}{n} \sqrt{(k\alpha + (n-k)\beta)(k\alpha' + (n-k)\beta')} \\ & \quad + \frac{1}{n} \sqrt{((n-k)\alpha + k\beta)((n-k)\alpha' + k\beta')} \\ & \quad + (n-2)\sqrt{\beta\beta'}, \end{aligned} \quad (112)$$

for all $k = 1, \dots, n-1$. We show that for $\alpha = \lambda^2\mu^{-1}$, $\beta = \mu$, and $\alpha' = \beta' = \mu^{-1}$, these inequalities hold. Note that $P = C(\alpha', \beta')$ is diagonal since $\alpha' = \beta'$, but strictly positive definite, as it must be. Substituting these values into the inequality (112), we can simplify it as follows:

$$\lambda + \mu < \sqrt{\frac{k}{n}\lambda^2 + \left(1 - \frac{k}{n}\right)\mu^2} + \sqrt{\left(1 - \frac{k}{n}\right)\lambda^2 + \frac{k}{n}\mu^2}.$$

Because both sides are non-negative, we can square this inequality and get an equivalent one

$$\lambda\mu < \sqrt{\frac{k}{n}\lambda^2 + \left(1 - \frac{k}{n}\right)\mu^2} \sqrt{\left(1 - \frac{k}{n}\right)\lambda^2 + \frac{k}{n}\mu^2}. \quad (113)$$

Squaring again and taking the difference of the left-hand side and the right-hand side, for a bipartition $\mathcal{I} = \{I, \bar{I}\}$ with $|I| = k$, we obtain

$$-\frac{k}{n} \left(1 - \frac{k}{n}\right) (\lambda^2 - \mu^2)^2 < 0, \quad (114)$$

for all $k = 1, \dots, n-1$ provided that $\lambda \neq \mu$. This verifies that the states given by Eq. (92) are genuine multipartite entangled if the parameters λ and μ are different. ■

The states [Eq. (92)] have a high degree of symmetry, so it was possible to verify their entanglement analytically. However, in general, it is almost impossible to find a witness just by guessing it, so we need a more systematic approach. In addition, we aim not just to find a witness for a given entangled state, but to find the best possible one, the witness for which the difference of the left-hand side and the right-hand side of Eq. (106) is the largest possible (by absolute value). Since scaling M by a positive factor also scales the difference by the same factor, we put the additional restriction on M , $\text{Tr}(M) = 1$. Given a CM γ , we thus need to solve the following optimization problem:

$$-\mathcal{E}_{\mathcal{I}}(\gamma) = \min_{\substack{M \succcurlyeq 0 \\ \text{Tr}(M)=1}} \left[\text{Tr}(M\gamma) - \min_{\mathcal{I} \in \mathcal{I}} \sum_{I \in \mathcal{I}} \text{str}(M[I]) \right]. \quad (115)$$

If the optimal value turns out to be negative or, equivalently, $\mathcal{E}_{\mathcal{I}}(\gamma) > 0$, the corresponding optimal M^* is the optimal witness for $\mathcal{I}$ -entanglement of γ (different γ are likely to have different optimal witnesses). Using the analytical expression (48) for the symplectic trace together with Eq. (79), the objective function has an analytic expression as a function of M . This expression has a complicated structure, as we have already seen in the proof of the theorem above, so no standard optimization technique can be applied straightforwardly to it. To tackle this problem, we use the dual expression instead. This approach is expressed by the theorem below.

Theorem 6. For any CM γ , the following equality takes place:

$$\begin{aligned} & \min_{M \succcurlyeq 0} \left[\text{Tr}(M\gamma) - \min_{\mathcal{I} \in \mathcal{I}} \sum_{I \in \mathcal{I}} \text{str}(M[I]) \right] \\ &= \min_{M \succcurlyeq 0, K^{(I)}} \max_{\mathcal{I} \in \mathcal{I}} \left[\text{Tr}(M\gamma) - \frac{1}{2} \sum_{I \in \mathcal{I}} \text{Tr}(K^{(I)} \Omega_{|I|}) \right] \\ &= \max_{\gamma^{(I)}, p_{\mathcal{I}}} \lambda_{\min} \left(\gamma - \sum_{\mathcal{I} \in \mathcal{I}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma^{(I)} \right), \end{aligned} \quad (116)$$

the additional constraints of the primal problem being

$$\text{Tr}(M) = 1, \quad M[I] + iK^{(I)} \succcurlyeq 0, \quad (117)$$

and the constraints of the dual problem being

$$p_{\mathcal{I}} \geq 0, \quad \sum_{\mathcal{I} \in \mathcal{I}} p_{\mathcal{I}} = 1. \quad (118)$$

The KKT conditions read as

$$\mathcal{P}(\gamma^{(I)*})(M^*[I] + iK^{(I)*}) = 0, \quad (119)$$

for all $\mathcal{I} \in \mathfrak{J}$ and $I \in \mathcal{I}$ with $p_I^* > 0$, and

$$M^* \left(\gamma - \sum_{\mathcal{I} \in \mathfrak{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma^{(I)*} \right) = \lambda_{\min}^* M^*, \quad (120)$$

where for all $\mathcal{I} \in \mathfrak{J}$ with $p_{\mathcal{I}}^* > 0$

$$\lambda_{\min}^* = \text{Tr}(M^* \gamma) - \frac{1}{2} \sum_{I \in \mathcal{I}} \text{Tr}(K^{(I)*} \Omega_{|I|}) \quad (121)$$

is the primal optimal objective value. Note that $M^*[I]$ here are submatrices of one and the same matrix M^* , and $K^{(I)*}$ are totally independent matrices, which is reflected in different notation.

In the reduced case, the duality reads as

$$\begin{aligned} & \min_{X, P \succcurlyeq 0} \text{Tr}(X \gamma_{xx} + P \gamma_{pp}) - \min_{\mathcal{I} \in \mathfrak{J}} \sum_{I \in \mathcal{I}} \text{str}(X[I], P[I]) \\ &= \min_{X, P \succcurlyeq 0, Z^{(I)}} \max_{\mathcal{I} \in \mathfrak{J}} \left[\text{Tr}(X \gamma_{xx} + P \gamma_{pp}) - \sum_{I \in \mathcal{I}} \text{Tr} Z^{(I)} \right] \\ &= \max_{\gamma_{xx}^{(I)}, \gamma_{pp}^{(I)}, P_{\mathcal{I}}} \lambda_{\min} \left(\gamma_{xx} - \sum_{\mathcal{I} \in \mathfrak{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma_{xx}^{(I)} \right) \\ &= \max_{\gamma_{xx}^{(I)}, \gamma_{pp}^{(I)}, P_{\mathcal{I}}} \lambda_{\min} \left(\gamma_{pp} - \sum_{\mathcal{I} \in \mathfrak{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma_{pp}^{(I)} \right), \end{aligned}$$

the additional constraints of the primal problem being

$$\text{Tr}(X + P) = 1, \quad \begin{pmatrix} X[I] & Z^{(I)} \\ Z^{(I)\dagger} & P[I] \end{pmatrix} \succcurlyeq 0, \quad (122)$$

and the additional constraint of the dual problem given by Eq. (118). The KKT conditions are

$$\mathcal{P}(\gamma_{xx}^{(I)*}, \gamma_{pp}^{(I)*}) \begin{pmatrix} X^*[I] & -Z^{(I)*} \\ -Z^{(I)*\dagger} & P^*[I] \end{pmatrix} = 0, \quad (123)$$

for all $I \in \mathcal{I}$ with $p_I^* > 0$, and

$$\begin{aligned} & X^* \left(\gamma_{xx} - \sum_{\mathcal{I} \in \mathfrak{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma_{xx}^{(I)*} \right) = \lambda_{\min}^* X^*, \\ & P^* \left(\gamma_{pp} - \sum_{\mathcal{I} \in \mathfrak{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma_{pp}^{(I)*} \right) = \lambda_{\min}^* P^*, \end{aligned} \quad (124)$$

where for all $\mathcal{I} \in \mathfrak{J}$ with $p_{\mathcal{I}}^* > 0$

$$\lambda_{\min}^* = \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \sum_{I \in \mathcal{I}} \text{Tr} Z^{(I)*} \quad (125)$$

is the primal optimal objective value.

The duality is directly related to the basic condition (104). If that condition holds, then there is a mixture of factorizable states such that the difference with the given state is positive semidefinite and thus has non-negative eigenvalues only. However, if the difference with these mixtures always has a negative eigenvalue, then the condition (104) cannot hold for any mixture, so the state under study is entangled. The duality is a quantitative characterization of the “best” possible minimal eigenvalue of the difference.

The number of variables is given by

$$N_v = \begin{cases} n(2n+1) + \sum_{\mathcal{I} \in \mathfrak{J}} \sum_{I \in \mathcal{I}} |I|(2|I|-1), & \text{full case,} \\ n(n+1) + \sum_{\mathcal{I} \in \mathfrak{J}} \sum_{I \in \mathcal{I}} |I|^2, & \text{reduced case.} \end{cases}$$

The first term is the number of different elements in the symmetric matrix M (or the symmetric matrices X and P) and the rest is the number of different elements in the antisymmetric matrices $K^{(I)}$ (or the general matrices $Z^{(I)}$). For the bipartite separability $\mathfrak{J} = \mathfrak{J}_2$, we have

$$N_v = \begin{cases} 2n(n2^{n-2} + 1), & \text{full case,} \\ n[(n+1)2^{n-2} + 1], & \text{reduced case.} \end{cases} \quad (126)$$

Note that from the KKT condition (119), it follows that if $p_{\mathcal{I}} > 0$, then for all $I \in \mathcal{I}$, the state $\gamma^{(I)*}$ is given by Eq. (50), $\gamma^{(I)*} = \gamma_{M[I]}$, provided that $M[I] \succ 0$. The optimal separable state in the dual problem in Eq. (116) is thus a mixture of factorizable states, where each factor is a boundary pure state. Given M^* (or X^* and P^* in the reduced case), we can easily check (without any optimizations) that it really witnesses the entanglement of the CM γ with the inequality (106) (or inequality (107), respectively). This is where we need an explicit expression for the symplectic trace. The corresponding dual solution and KKT conditions verify that this witness is optimal.

In Fig. 4, we show the results of testing the symmetric states $\gamma^S(\lambda, \mu)$ for ten different random values of λ and μ in the range $[0.5, 2.5]$. The average time to test an n -partite state grows exponentially with n , as well as the standard deviation of this time, while the memory needed grows superexponentially. The large relative gap for SCS is likely due to the exceeding the default number of iterations, so the result produced is not fully converged to the solution. However, this curve shows the largest value (the worst) over the ten runs, for some cases, the solution obtained is much more precise.

The quantity $\mathcal{E}_{\mathfrak{J}}(\gamma)$, given by Eq. (115), can be considered as an $\mathfrak{J}$ -entanglement measure. We show that it is non-negative and convex, which justifies using it as a measure of entanglement.

Lemma 2. The measure $\mathcal{E}_{\mathfrak{J}}(\gamma) \leq 0$ iff γ is $\mathfrak{J}$ -separable. In addition, it is a convex function of γ :

$$\mathcal{E}_{\mathfrak{J}}(p\gamma_1 + (1-p)\gamma_2) \leq p\mathcal{E}_{\mathfrak{J}}(\gamma_1) + (1-p)\mathcal{E}_{\mathfrak{J}}(\gamma_2), \quad (127)$$

for all CMs γ_1, γ_2 and all $0 \leq p \leq 1$.

Proof. The first statement directly follows from Theorem 6. As for the convexity, by the definition (115), we have that $-\mathcal{E}_{\mathfrak{J}}$ is the minimum of an infinite family of linear functions, and thus is concave. We conclude that $\mathcal{E}_{\mathfrak{J}}$ is convex. ■

In fact, this measure is also convex with respect to states $\hat{\rho}$, not only with respect to their CMs. To prove this, note that if $\gamma \succcurlyeq \gamma'$, then $\mathcal{E}_{\mathfrak{J}}(\gamma) \leq \mathcal{E}_{\mathfrak{J}}(\gamma')$. The dual form of Eq. (116) shows that by “increasing” γ , we can only decrease its entanglement measure. Using the concavity of CM, expressed by Eq. (103), and the lemma above we conclude that

$$\mathcal{E}_{\mathfrak{J}}(\gamma(p\hat{\rho}_1 + (1-p)\hat{\rho}_2)) \leq p\mathcal{E}_{\mathfrak{J}}(\gamma(\hat{\rho}_1)) + (1-p)\mathcal{E}_{\mathfrak{J}}(\gamma(\hat{\rho}_2)).$$

In addition, the whole set of these measures behaves in an intuitive way with respect to $\mathfrak{J}$. We say that a partition $\mathcal{I}$ is finer than a partition $\mathcal{J}$, $\mathcal{I} \prec \mathcal{J}$, if for any $I \in \mathcal{I}$ there is $J \in \mathcal{J}$ such that $I \subseteq J$. It means that a $\mathcal{I}$ -factorizable state is

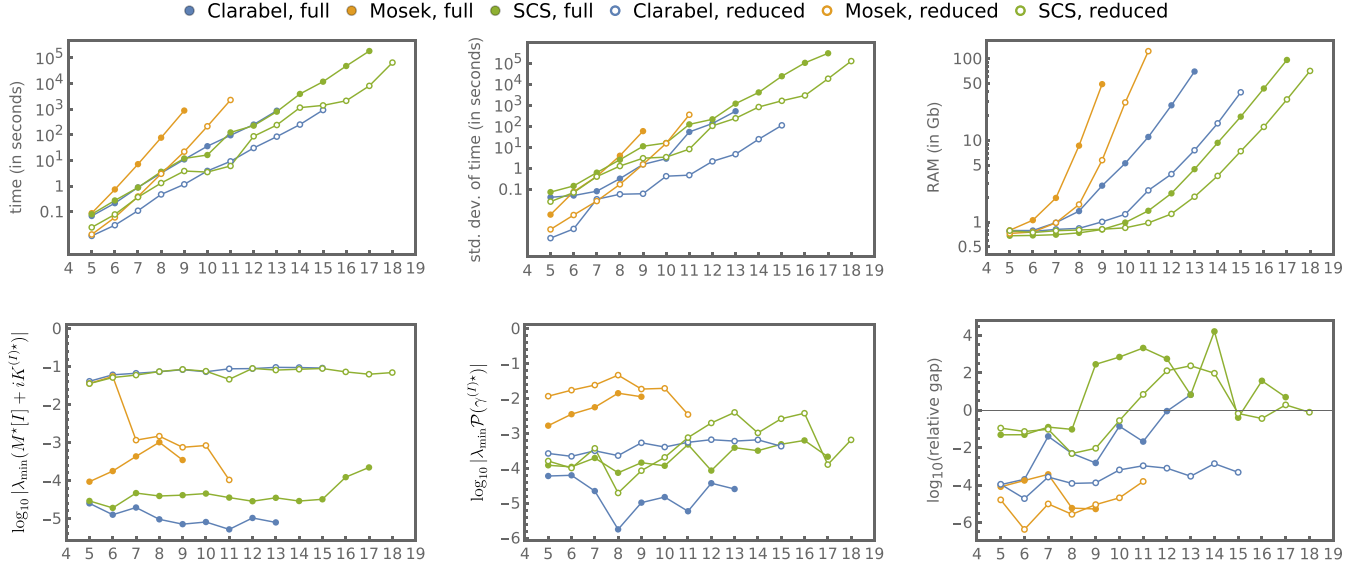


FIG. 4. Comparing performance and accuracy characteristics of different solvers for the eigenvalue entanglement test as a function of the number of modes. CMs are given by Eq. (92) for random λ and μ .

also $\mathcal{J}$ factorizable. The partition $1|2|\dots|n$ is the minimum with respect to this partial order. This order can be extended to sets of partitions. We say that $\mathcal{I} < \mathcal{J}$ if for any $\mathcal{I} \in \mathcal{J}$ there is $\mathcal{J} \in \mathcal{J}$ such that $\mathcal{I} < \mathcal{J}$. It means that a $\mathcal{I}$ -separable state is also $\mathcal{J}$ -separable. In other words, the larger $\mathcal{I}$, the larger the set of $\mathcal{I}$ -separable states (and the smaller the set of $\mathcal{I}$ -entangled states). It follows from the dual problem of Eq. (116) that by increasing the set of separable states we also increase the maximal value of the dual objective, and thus decrease the corresponding $\mathcal{E}_{\mathcal{J}}(\gamma)$ for a fixed γ . We have thus derived the following inequality:

$$\mathcal{E}_{\mathcal{I}}(\gamma) \leq \mathcal{E}_{\mathcal{J}}(\gamma), \quad (128)$$

provided that $\mathcal{I} < \mathcal{J}$. For a given CM γ , $\mathcal{E}_{1|\dots|n}(\gamma)$ is the largest and $\mathcal{E}_{\mathcal{J}_2}(\gamma)$ is the smallest.

The measure $\mathcal{E}_{\mathcal{J}}(\gamma)$ can be strictly negative, for example, for the scaled vacuum state $\gamma = c\gamma_0 \oplus \dots \oplus \gamma_0$ with $c > 1$ and

$$\gamma_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (129)$$

It thus makes sense to “fix” this measure by truncating its values below zero:

$$\mathcal{E}_{\mathcal{J}}^+(\gamma) = \max(\mathcal{E}_{\mathcal{J}}(\gamma), 0). \quad (130)$$

This function is non-negative, convex (as the maximum of two convex functions) and has the following addition property: $\mathcal{E}_{\mathcal{J}}^+(\gamma) = 0$ iff γ is $\mathcal{J}$ -separable.

B. Scaling test

We now use a different approach to obtain an entanglement test from the inequality (104). Instead of the difference between the left-hand side and the right-hand side, we studied above we now study their “ratio.” More concretely, given a CM γ , we will find the maximal value of scaling factors t that

satisfy the condition

$$\gamma \geq t \sum_{\mathcal{I}=\{I_1, \dots, I_l\} \in \mathcal{J}} p_{\mathcal{I}} \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)}, \quad (131)$$

for some $p_{\mathcal{I}}$ and $\gamma^{(I)}$. From Eq. (104), it follows for the maximal scaling factor t^* that $t^* \geq 1$ iff γ is $\mathcal{J}$ separable and thus the condition $t^* < 1$ is necessary and sufficient for entanglement. The dual problem is given by the theorem below.

Theorem 7. For a CM γ the following equality is valid:

$$\max_{\gamma^{(I)}, p_{\mathcal{I}}} t = \min_{M \succcurlyeq 0} \text{Tr}(M\gamma), \quad (132)$$

with the additional constraint for the primal problem (on the left-hand side) given by Eqs. (131) and (118), and the additional constraint for the dual problem (on the right-hand side) given by

$$M[I] + iK^{(I)} \succcurlyeq 0, \quad \frac{1}{2} \sum_{I \in \mathcal{I}} \text{Tr}(K^{(I)} \Omega_{|I|}) = 1, \quad (133)$$

for all $\mathcal{I} \in \mathcal{J}$. The KKT conditions are Eq. (119) and

$$M^* \left(\gamma - t^* \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma^{(I)*} \right) = 0, \quad (134)$$

where $t^* = \text{Tr}(M^*\gamma)$ is the primal optimal objective value of the primal objective function.

In the reduced case the duality reads as

$$\max_{\gamma_{xx}^{(I)}, \gamma_{pp}^{(I)}, p_{\mathcal{I}}, t} t = \min_{X, P \succcurlyeq 0, Z^{(I)}} \text{Tr}(X\gamma_{xx} + P\gamma_{pp}), \quad (135)$$

with the additional constraint for the primal problem (on the left-hand side) being Eq. (118) and

$$\gamma_{xx} \succcurlyeq t \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma_{xx}^{(I)}, \quad \gamma_{pp} \succcurlyeq t \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma_{pp}^{(I)}, \quad (136)$$

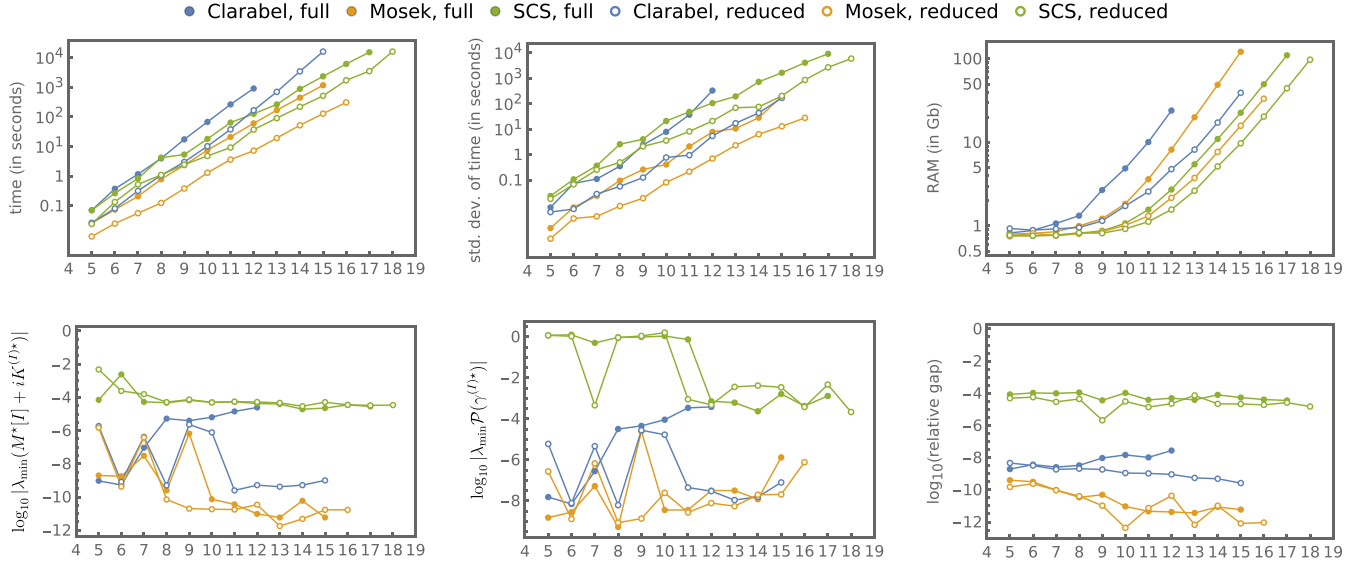


FIG. 5. Comparing performance and accuracy characteristics of different solvers for the scaling entanglement test as a function of the number of modes.

and the additional constraint for the dual problem (on the right-hand side) being

$$\begin{pmatrix} X[I] & Z^{(I)} \\ Z^{(I)\top} & P[I] \end{pmatrix} \succcurlyeq 0, \quad \sum_{I \in \mathcal{I}} \text{Tr} Z^{(I)} = 1. \quad (137)$$

The KKT conditions are given by Eq. (123) and

$$\begin{aligned} X^* \left(\gamma_{xx} - t^* \sum_{I \in \mathcal{I}} P_I^* \bigoplus_{I \in \mathcal{I}} \gamma_{xx}^{(I)*} \right) &= 0, \\ P^* \left(\gamma_{pp} - t^* \sum_{I \in \mathcal{I}} P_I^* \bigoplus_{I \in \mathcal{I}} \gamma_{pp}^{(I)*} \right) &= 0, \end{aligned} \quad (138)$$

where $t^* = \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp})$ is the primal optimal objective value.

From the first KKT condition, we see that in this case the optimal states $\gamma^{(I)*}$ are again the pure boundary states. It is easy to see that the optimal dual matrix M^* satisfies the inequality

$$\text{Tr} \left[M^* \bigoplus_{I \in \mathcal{I}} \gamma^{(I)} \right] \geq 1, \quad (139)$$

for arbitrary CMs $\gamma^{(I)}$ and for all $I \in \mathcal{I}$. In fact, the inequalities

$$\begin{aligned} \text{Tr}[\mathcal{P}(\gamma^{(I)})(M^*[I] + iK^{(I)*})] \\ = \text{Tr}(M^* \gamma^{(I)}) - \frac{1}{2} \text{Tr}(K^{(I)*} \Omega_{|I|}) \geq 0, \end{aligned} \quad (140)$$

are valid for all I since $M^*[I] + iK^{(I)*} \succcurlyeq 0$. Then, we have

$$\begin{aligned} \text{Tr} \left[M^* \bigoplus_{I \in \mathcal{I}} \gamma^{(I)} \right] &= \sum_{I \in \mathcal{I}} \text{Tr}(M^*[I] \gamma^{(I)}) \\ &\geq \frac{1}{2} \sum_{I \in \mathcal{I}} \text{Tr}(K^{(I)*} \Omega_{|I|}) = 1, \end{aligned} \quad (141)$$

where we used Eq. (133). It follows that any M^* satisfies the condition $\text{Tr}(M^* \gamma_{\mathcal{I}}) \geq 1$ for the CM $\gamma_{\mathcal{I}}$ of any $\mathcal{I}$ -separable state. Thus, if the optimal value

$$t^* = \text{Tr}(M^* \gamma) < 1, \quad (142)$$

then M^* witnesses that γ cannot be $\mathcal{I}$ -separable. We have proved that M^* satisfies the property (139) with the help of the related matrices $K^{(I)*}$ provided by the solver, but given an M^* we can always check this property directly with the explicit expression for the symplectic trace.

The results of testing the same CMs as in the eigenvalue case are shown in Fig. 5. The time and RAM requirements are basically the same, but the relative gap is much smaller, so it seems that this problem can be solved more accurately (for these test data at least).

The optimal value $t^*(\gamma)$ is a concave function of the CM γ , since it is the minimum of a family of linear functions. It follows that the quantity $\mathcal{T}_{\mathcal{I}}(\gamma)$ defined via

$$\mathcal{T}_{\mathcal{I}}(\gamma) = 1 - t^*(\gamma) \quad (143)$$

is convex and thus can be referred to as an entanglement measure. This measure has the same properties as $\mathcal{E}_{\mathcal{I}}$ —it is convex also with respect to the density operator and is ordered with respect to the order on $\mathcal{I}$. As in the previous case, this function can be negative, so we “fix” it in the same way

$$\mathcal{T}_{\mathcal{I}}^+(\gamma) = \max(\mathcal{T}_{\mathcal{I}}(\gamma), 0). \quad (144)$$

This function is non-negative, convex (as the maximum of two convex functions) and has the following addition property: $\mathcal{T}_{\mathcal{I}}^+(\gamma) = 0$ iff γ is $\mathcal{I}$ -separable.

C. Tests with uncertainty

When the CM γ is known with uncertainty expressed by the matrix of standard covariances σ , it is not enough just to

find an M that satisfies the inequality (106) to demonstrate entanglement of γ . We need to establish a stronger inequality

$$\text{Tr}(M\gamma) < \min_{\mathcal{I} \in \mathcal{J}} \sum_{I \in \mathcal{I}} \text{str}(M[I]) - s\sigma(M), \quad (145)$$

where s is a sufficiently large number, e.g., $s = 3$ and

$$\sigma^2(M) = \sum_{i,j=1}^{2n} M_{ij}^2 \sigma_{ij}^2. \quad (146)$$

We now formulate the problem of finding the maximal s such that there is an M that satisfies the inequality (145). Since the inequality (145) is invariant under the scaling of M , without loss of generality we can assume that $\sigma(M) \leq 1$ (we cannot use strict equality here because in this case the optimization domain will not be convex).

Theorem 8. For any CM γ , we have the duality

$$\max_{M \succ 0} s = \min_{\gamma^{(l)}, Q} q, \quad (147)$$

with the additional constraints of the primal problem (on the left-hand side) given by Eq. (145) and $\sigma(M) \leq 1$, and the constraints of the dual problem (on the right-hand side) are Eq. (118) and

$$\gamma \succ \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma^{(l)} + \sigma \cdot Q, \quad \|Q\| \leq q, \quad (148)$$

where the norm of the symmetric matrix Q is defined via

$$\|Q\|^2 = \text{Tr}(Q^T Q) = \text{Tr}(Q^2). \quad (149)$$

The KKT conditions read as

$$\begin{aligned} \text{Tr}[M^*(\sigma \cdot Q^*)] + q^* &= 0, \\ M^* \left(\gamma - \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma^{(l)*} - \sigma \cdot Q^* \right) &= 0. \end{aligned} \quad (150)$$

In the first equality, we have the standard matrix product of M^* with the Hadamard product $\sigma \cdot Q^*$. In addition,

$$\text{Tr}(M^* \gamma) - \sum_{I \in \mathcal{I}} \text{str}(M^*[I]) + q^* = 0, \quad (151)$$

for all $\mathcal{I} \in \mathcal{J}$ with $p_{\mathcal{I}} > 0$.

In the reduced case, the duality reads as

$$\max_{X, P \succ 0} s = \min_{\gamma_{xx}^{(l)}, \gamma_{pp}^{(l)}, Q, R} q, \quad (152)$$

with the additional constraint of the primal problem (on the left-hand side) being

$$\begin{aligned} \text{Tr}(X\gamma_{xx} + P\gamma_{pp}) - \min_{\mathcal{I} \in \mathcal{J}} \sum_{I \in \mathcal{I}} \text{str}(X[I], P[I]) \\ + s\sqrt{\sigma^2(X) + \sigma^2(P)} < 0, \end{aligned} \quad (153)$$

where

$$\sigma^2(X) = \sum_{i,j=1}^n X_{ij}^2 \sigma_{xx,ij}^2, \quad \sigma^2(P) = \sum_{i,j=1}^n P_{ij}^2 \sigma_{pp,ij}^2, \quad (154)$$

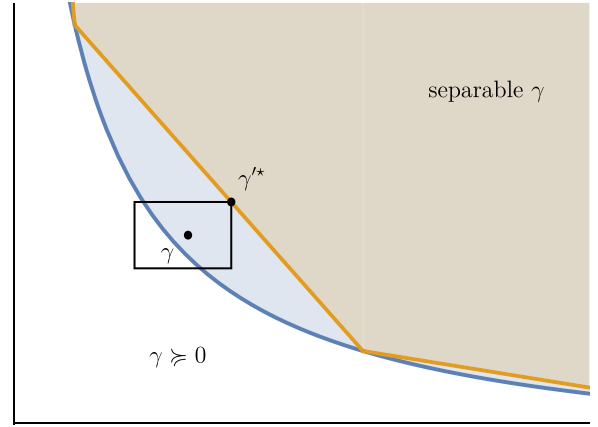


FIG. 6. Closest separable state.

and $\sigma^2(X) + \sigma^2(P) \leq 1$, and the constraints of the dual problem (on the right-hand side) are Eq. (118) and

$$\begin{aligned} \gamma_{xx} &\succ \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma_{xx}^{(l)} + \sigma_{xx} \cdot Q, \\ \gamma_{pp} &\succ \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \bigoplus_{I \in \mathcal{I}} \gamma_{pp}^{(l)} + \sigma_{pp} \cdot R, \end{aligned} \quad (155)$$

where symmetric matrices Q and R satisfy the inequality

$$\sqrt{\|Q\|^2 + \|R\|^2} \leq q. \quad (156)$$

The KKT conditions read as

$$\begin{aligned} \text{Tr}[X^*(\sigma_{xx} \cdot Q^*)] + \text{Tr}[P^*(\sigma_{pp} \cdot R^*)] + q^* &= 0, \\ X^* \left(\gamma_{xx} - \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma_{xx}^{(l)*} - \sigma_{xx} \cdot Q^* \right) &= 0, \\ P^* \left(\gamma_{pp} - \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \bigoplus_{I \in \mathcal{I}} \gamma_{pp}^{(l)*} - \sigma_{pp} \cdot R^* \right) &= 0. \end{aligned} \quad (157)$$

In addition,

$$\text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \sum_{I \in \mathcal{I}} \text{str}(X^*[I], P^*[I]) + q^* = 0, \quad (158)$$

for all $\mathcal{I} \in \mathcal{J}$ with $p_{\mathcal{I}} > 0$.

Another way to reliably test a CM γ , known with uncertainty σ , is to determine the distance (measured in units of σ) to the closest separable state (Fig. 6). If γ is entangled, then we will look for the largest σ neighborhood $|\gamma' - \gamma| \leq s\sigma$, such that all its points γ' are also entangled. An equivalent formulation is as a smallest σ neighborhood that contains a separable point γ' . Such a neighborhood is illustrated in Fig. 6. The structures in this figure are the same as those of Fig. 1, and in addition the yellow polygon denotes the set of separable states. For this kind of test, we use the scaling approach.

Theorem 9. For any CM γ and the matrix of standard deviations σ , the following duality is valid:

$$\min_{\gamma', \gamma^{(l)}} s = \max_{\Lambda, W \succ 0, U^\pm \geq 0, K^{(l)}, v} -\text{Tr}[\Lambda \mathcal{P}(\gamma)] - \text{Tr}(W\gamma) + v,$$

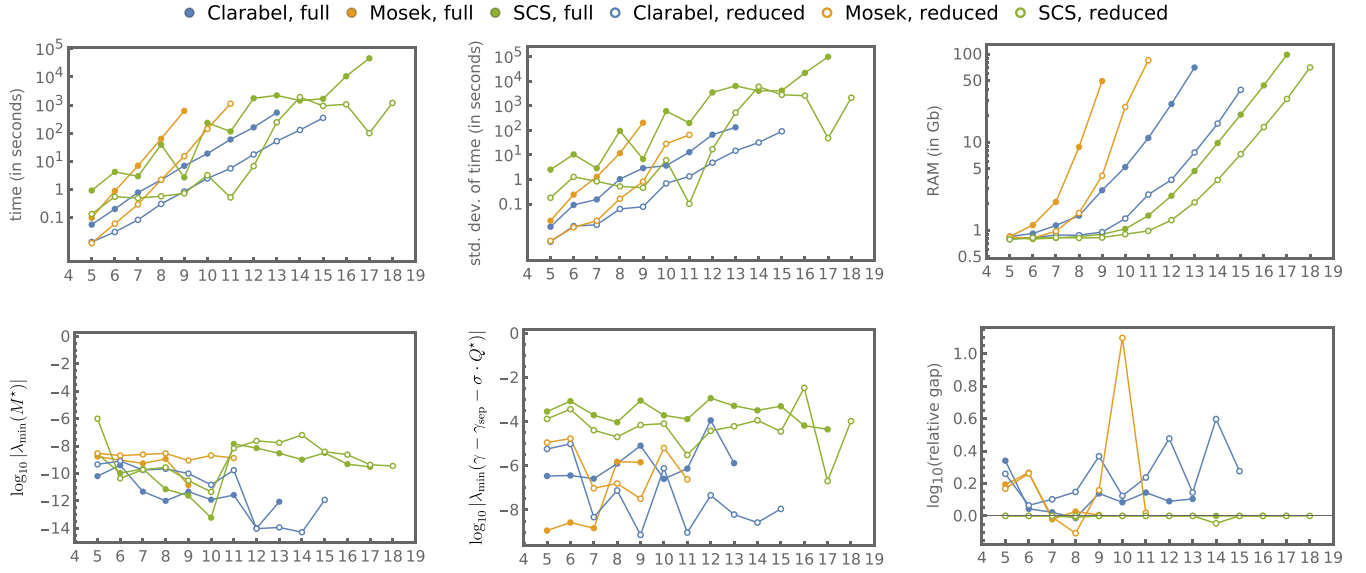


FIG. 7. Comparing performance and accuracy characteristics of different solvers for the problem expressed by Theorem 8 as a function of the number of modes.

with the additional constraints for the primal problem (on the left-hand side) being Eq. (118) and

$$|\gamma' - \gamma| \leq s\sigma, \quad \gamma' \geq \sum_{\mathcal{I}=\{I_1, \dots, I_l\} \in \mathcal{J}} p_{\mathcal{I}} \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_l)}.$$

The variables of the dual problem (on the right-hand side) are a Hermitian matrix Λ , a real symmetric W , real antisymmetric $K^{(I)}$, real $U^{\pm}$, and a real number ν . The additional constraints are $W[I] + iK^{(I)} \succcurlyeq 0$ for all $I \in \mathcal{I}$, $\mathcal{I} \in \mathcal{J}$, and

$$\begin{aligned} U^+ - U^- + \text{Re}(\Lambda) + W &= 0, \\ \text{Tr}[(U^+ + U^-)\sigma] &= 1, \\ \frac{1}{2} \sum_{I \in \mathcal{I}} \text{Tr}(K^{(I)} \Omega_{|I|}) &= \nu, \end{aligned} \quad (159)$$

for all $\mathcal{I} \in \mathcal{J}$. The KKT conditions read as

$$\begin{aligned} \mathcal{P}(\gamma'^*) \Lambda^* &= 0, \\ W^* \left(\gamma'^* - \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \gamma^{(I)*} \right) &= 0, \\ \text{Tr}[U^{\pm*} (\gamma - \gamma'^* \mp s^* \sigma)] &= 0, \end{aligned} \quad (160)$$

where s^* is the optimal prime objective value, and

$$\mathcal{P}(\gamma^{(I)*}) (W^*[I] + iK^{(I)*}) = 0, \quad (161)$$

for all $I \in \mathcal{I}$ and $\mathcal{I} \in \mathcal{J}$ with $p_{\mathcal{I}} > 0$.

In the reduced case, the duality reads as

$$\begin{aligned} \min_{\gamma'_{xx}, \gamma'_{pp}, \gamma^{(I)}_{xx}, \gamma^{(I)}_{pp}} s &= \max_{W, X, P \succcurlyeq 0, Z^{(I)}, U^{\pm}, V^{\pm} \geq 0, \nu} -\text{Tr}[W \mathcal{P}(\gamma_{xx}, \gamma_{pp})] \\ &\quad - \text{Tr}(X \gamma_{xx} + P \gamma_{pp}) + \nu, \end{aligned} \quad (162)$$

with the additional constraints of the primal problem (on the left-hand side) being Eq. (118) and

$$\gamma'_{xx} \geq \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \gamma_{xx}^{(I)}, \quad \gamma'_{pp} \geq \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}} \gamma_{pp}^{(I)}, \quad (163)$$

together with $|\gamma'_{xx} - \gamma_{xx}| \leq s\sigma_{xx}$, $|\gamma'_{pp} - \gamma_{pp}| \leq s\sigma_{pp}$. The additional constraints of the dual problem (on the right-hand side) are

$$\begin{aligned} U^+ - U^- + X + W_{xx} &= 0, \\ V^+ - V^- + P + W_{pp} &= 0, \\ \text{Tr}[(U^+ + U^-)\sigma_{xx} + (V^+ + V^-)\sigma_{pp}] &= 1, \\ \sum_{I \in \mathcal{I}} \text{Tr} Z^{(I)} &= \nu, \end{aligned} \quad (164)$$

and, in addition,

$$\begin{pmatrix} X[I] & Z^{(I)} \\ Z^{(I)\dagger} & P[I] \end{pmatrix} \succcurlyeq 0, \quad (165)$$

for all $I \in \mathcal{I}$, $\mathcal{I} \in \mathcal{J}$. The KKT conditions are given by

$$\begin{aligned} \mathcal{P}(\gamma'_{xx}, \gamma'_{pp}) W^* &= 0, \\ X^* \left(\gamma'_{xx} - \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \gamma_{xx}^{(I)*} \right) &= 0, \\ P^* \left(\gamma'_{pp} - \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \gamma_{pp}^{(I)*} \right) &= 0, \\ \text{Tr}[U^{\pm*} (\gamma_{xx} - \gamma'_{xx} \mp s^* \sigma_{xx})] &= 0, \\ \text{Tr}[V^{\pm*} (\gamma_{pp} - \gamma'_{pp} \mp s^* \sigma_{pp})] &= 0, \end{aligned} \quad (166)$$

and Eq. (119).

Figures 7 and 8 show the results of testing randomly generated covariance matrices and matrices of standard deviations according to the optimization problems given by Theorems 8

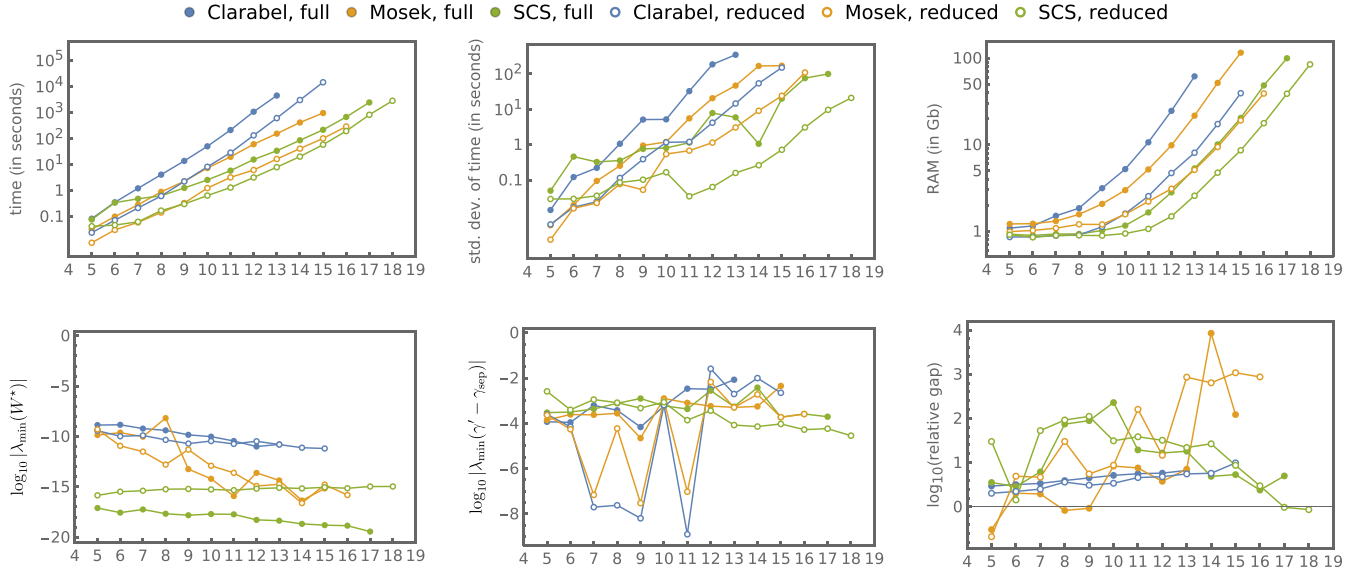


FIG. 8. Comparing performance and accuracy characteristics of different solvers for the problem expressed by Theorem 9 as a function of the number of modes.

and 9, respectively. This time the randomly generated CMs turned out not to be genuine multipartite entangled, so the primal optimal values happen to be very close to 0. The relative gap is thus a ratio of two nearly zero quantities and in this case it is not a very informative accuracy measure. It nevertheless shows that some solvers are more accurate than the others.

The optimal value s^* produced by these optimization problems gives a quantitative measure of how reliable the conclusion is with the provided experimental data. The value $s^* \geq 3$ should be considered as a reliable entanglement verification. The problem can be when the two tests, additive and multiplicative, give significantly different results, an example of which will be shown below. As with the best physical covariance matrix, which agrees with the measured nonphysical one, there is no right approach. However, from our point of view, the multiplicative test produces a more “correct” result.

VIII. REFINED CONDITION

In Ref. [17], the classical $\mathcal{I}$ -separability condition [41] given by Eq. (102), where $\mathcal{I} = \{I_1, \dots, I_l\}$ is a partition of n modes, was refined as the inequality

$$\gamma_{\mathcal{I}} \succcurlyeq \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_{l-1})} \oplus \frac{i}{2} \Omega_{|I_l|}, \quad (167)$$

which is satisfied by any $\mathcal{I}$ -separable CM $\gamma_{\mathcal{I}}$. It follows that if a CM γ is $\mathcal{J}$ separable, where $\mathcal{J} = \{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ is a collection of partitions, then the inequality

$$\gamma \succcurlyeq \sum_{\mathcal{I}=\{I_1, \dots, I_l\} \in \mathcal{J}} p_{\mathcal{I}} \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_{l-1})} \oplus \frac{i}{2} \Omega_{|I_l|}, \quad (168)$$

is satisfied for some $p_{\mathcal{I}}$ and $\gamma^{(I)}$. Unfortunately, this inequality cannot be used to construct an eigenvalue-form optimization problem: since $\text{Tr}(MK) = 0$ for any symmetric matrix M and any antisymmetric K (of compatible sizes), the information about the term $(i/2)\Omega$ on the right-hand side is lost.

To transform this inequality into the scaling-form optimization problem, we need the following simple statement: if $A \succcurlyeq tB$ for some $t > 0$, where A and B are Hermitian matrices and $A \succcurlyeq 0$, then $A \succcurlyeq t'B$ for all $0 \leq t' \leq t$. If $B \succcurlyeq 0$, as it is in the classical entanglement condition, then the proof is obvious

$$A - t'B = (A - tB) + (t - t')B \succcurlyeq 0, \quad (169)$$

since each term is positive semidefinite. In the new case, B has negative eigenvalues due to the term $(i/2)\Omega$, so this reasoning is not applicable here. We need a different approach,

$$A - t'B = (1 - t'/t)A + (t'/t)(A - tB) \succcurlyeq 0, \quad (170)$$

since both terms are positive semidefinite. We thus can look for the maximal number t such that

$$\gamma \succcurlyeq t \sum_{\mathcal{I}=\{I_1, \dots, I_l\} \in \mathcal{J}} p_{\mathcal{I}} \gamma^{(I_1)} \oplus \dots \oplus \gamma^{(I_{l-1})} \oplus \frac{i}{2} \Omega_{|I_l|}. \quad (171)$$

For any $\mathcal{J}$ -separable state, the maximal t^* must satisfy the inequality $t^* \geq 1$. If we determine that $t^* < 1$, then γ is guaranteed to be $\mathcal{J}$ -entangled. The task of finding the maximal t can be formulated as the following optimization problem.

Theorem 10. For a CM γ , the following equality is valid:

$$\max_{\gamma^{(I)}, p_{\mathcal{I}}} t = \min_{M + iK \succcurlyeq 0} \text{Tr}(M\gamma), \quad (172)$$

where $\Lambda = M + iK$ is a Hermitian matrix, with the additional constraint for the primal problem (on the left-hand side) is given by Eqs. (171) and (118), and the additional constraint for the dual problem (on the right-hand side) reads as $M[I] + iK^{(I)} \succcurlyeq 0$ and

$$\frac{1}{2} \sum_{I \in \mathcal{I}'} \text{Tr}(K^{(I)} \Omega_{|I|}) + \frac{1}{2} \text{Tr}(K^{(I_l)} \Omega_{|I_l|}) = 1, \quad (173)$$

for all $I \in \mathcal{I}'$, where $\mathcal{I}' = \{I_1, \dots, I_{l-1}\}$ for $\mathcal{I} = \{I_1, \dots, I_l\} \in \mathcal{J}$. The KKT conditions are

$$\mathcal{P}(\gamma^{(I)*})(M^*[I] + iK^{(I)*}) = 0, \quad (174)$$

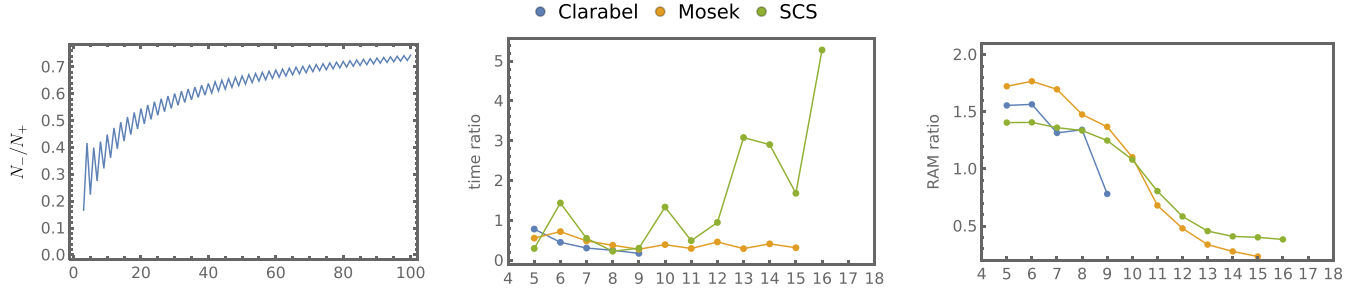


FIG. 9. The ratio N_-/N_+ (left), the ratio of times to solve the new and the standard problems (middle) and the analogous ratio of RAMs (right) as a function of the number of modes.

for all $\mathcal{I} \in \mathcal{J}$ and $I \in \mathcal{I}'$ with $p_{\mathcal{I}}^* > 0$ and

$$\bar{\Lambda}^* \left(\gamma - t^* \sum_{\mathcal{I} \in \mathcal{J}} p_{\mathcal{I}}^* \gamma^{(I_1)} \oplus \cdots \oplus \gamma^{(I_{|\mathcal{I}|})} \oplus \frac{i}{2} \Omega_{|I|} \right) = 0, \quad (175)$$

where $t^* = \text{Tr}(M^* \gamma)$ is the primal optimal objective value of the primal objective function.

As in the classical case [41], the optimal value t^* is a concave function of γ , so the difference $1 - t^*$ is convex and thus could be also called an entanglement measure. However, now this value might depend on how we arrange the parts inside the partitions. When we have a single partition, $|\mathcal{J}| = 1$, and the only partition $\mathcal{I} = \{I, J\}$ is a bipartition, we numerically verified that the optimal t^* value is the same as in the classical case, irrespective of whether we replace the variable $\gamma^{(I)}$ or $\gamma^{(J)}$ by a corresponding constant $(i/2)\Omega$. From a computation efficiency perspective, we should discard the larger component of the bipartition. For example, when n is large and $|I| = 1$ and $|J| = n - 1$, the difference in the performance of the optimization will be significant.

When there is more than one partition, $|\mathcal{J}| > 1$, then we can discard arbitrary components from $\mathcal{I} \in \mathcal{J}$ individually. The most optimal choice is when we discard the largest component. However, the advantage will diminish with n . We are most interested in genuine multipartite entanglement, when $\mathcal{J} = \mathcal{J}_2$ is the set of all possible bipartitions of n modes. Let us compute the number of K variables when we discard the larger components (the most efficient way), with the same number when we discard the smaller ones (the least efficient way). The former number is given by

$$N_- = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} k(2k-1), \quad (176)$$

and the latter is

$$N_+ = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (n-k)(2n-2k-1). \quad (177)$$

These sums can be computed analytically:

$$N_- = \begin{cases} n^2 2^{n-2} - n(n-1) \binom{n-2}{\frac{n}{2}-1}, & n \text{ is even,} \\ n^2 2^{n-2} - n(2n-1) \binom{n-2}{\frac{n-1}{2}}, & n \text{ is odd,} \end{cases} \quad (178)$$

and

$$N_+ = \begin{cases} n^2 2^{n-2} + (3n-2)(n-1) \binom{n-2}{\frac{n}{2}-1} - n(2n-1), \\ n^2 2^{n-2} + n(2n-1) \binom{n-2}{\frac{n-1}{2}} - n(2n-1), \end{cases}$$

for even and odd n , respectively. As Fig. 9 (left) shows, the ratio N_-/N_+ grows and it can be shown that in the limit $n \rightarrow +\infty$ it tends to 1. It means that for large values of n it is irrelevant which components to discard, but these large values of n are currently intractable to deal with. For the values considered in this work, $n < 20$, the condition gives a quite noticeable saving in RAM and also in time (but not always). Since the refined condition is not necessary and sufficient when more than one partition is involved, we do not study it further and do not specialize it for the reduced case.

IX. APPLICATIONS

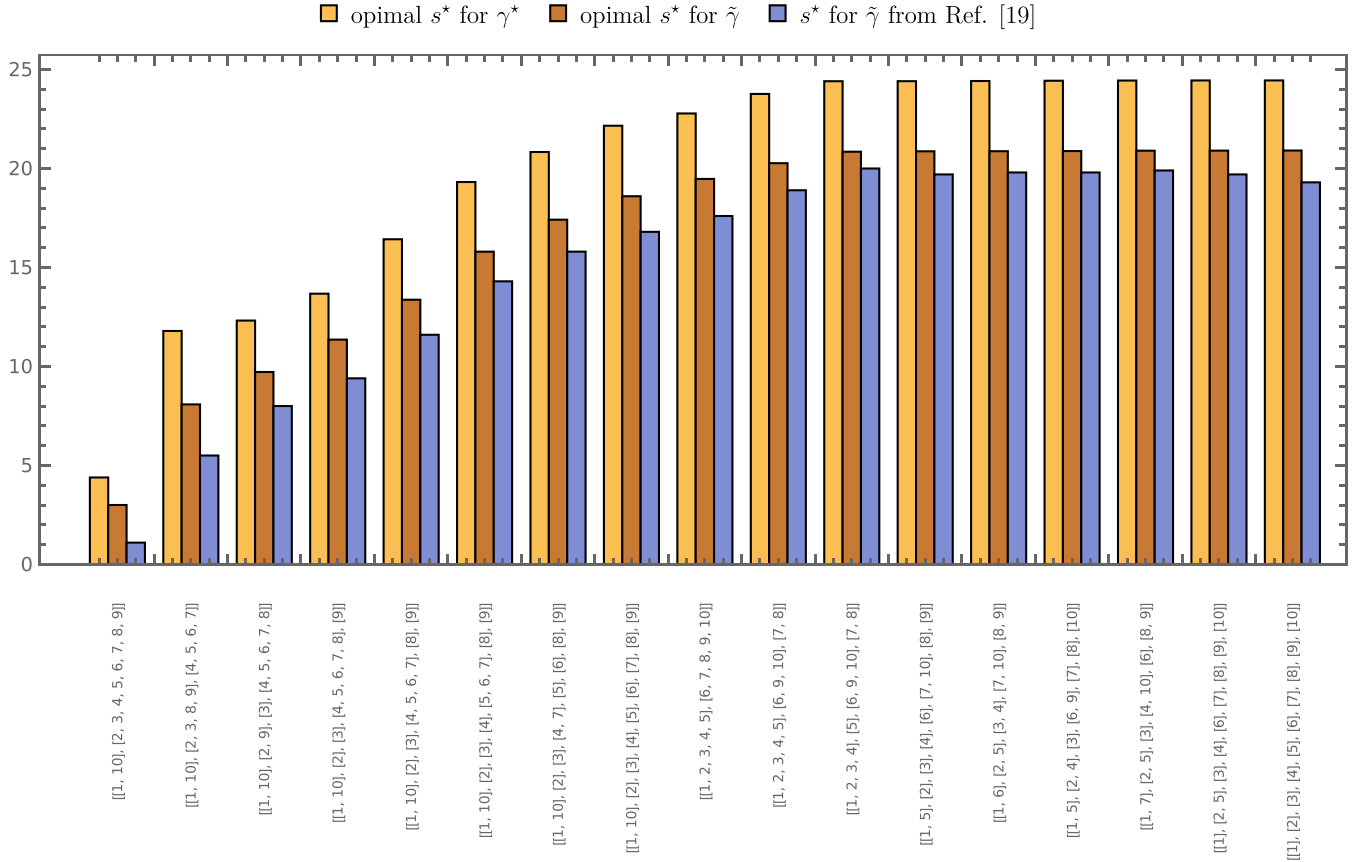
Here, we first apply our methods to some experimental data and show the advantage of our approach over another one [23]. That other approach is based on genetic programming, a heuristic method that usually produces suboptimal results. Our methods are based on convex optimization and thus guaranteed to produce the optimal result with a certificate of optimality. The computational resources needed to perform the genetic-optimization-based algorithms are not reported in that other work, so we cannot compare ours with theirs. We also show that in some cases our optimization problems admit analytical solutions.

A. Experimental data

To apply our approach to experimental measurements, we solve our optimization problems for the data presented in Ref. [23]. In that work, the physical CM was simply taken to be

$$\tilde{\gamma} = \gamma^\circ + 1.001 \lambda_{\min}(\mathcal{P}(\gamma^\circ)) E_{2n}. \quad (179)$$

The coefficient 1.001 is just a number, slightly above 1, to make the minimal eigenvalue of $\mathcal{P}(\tilde{\gamma})$ slightly positive. For the four-mode state, some diagonal elements of $\tilde{\gamma}$ are more than 10σ away from the measured averages, so $\tilde{\gamma}$ cannot

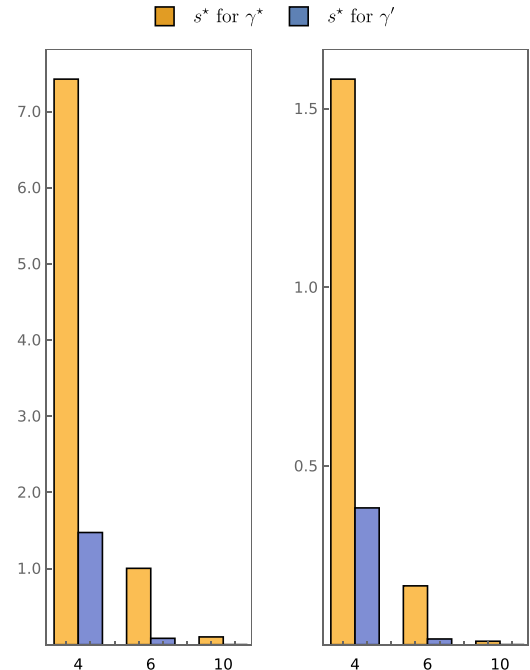
FIG. 10. The optimal s^* of the optimization problem (152) for the ten-mode state of Ref. [23].

be the true physical CM. Therefore, we choose not to fully explore this particular case.

Nevertheless, we apply the optimization problem of Theorem 6 to $\tilde{\gamma}$ for a few partitions of parts and compare the maximal s^* values with the values obtained in Ref. [23]. For the four- and six-mode states, our results are pretty close, but for the ten-mode state, the advantage of our approach can be seen in Fig. 10 (orange vs blue). For comparison, we also show the optimal s^* for the most likely physical CM γ^* given by Theorem 2 (yellow). Any value $s^* > 3$ should be enough to verify inseparability. As the first partition in Fig. 10 shows, our method clearly verifies entanglement with respect to the partition 1, 10|23456789, while that of Ref. [23] does not. The optimal values for the most probable physical covariance matrix are even higher.

Testing individual partitions is not very useful if we can establish genuine multipartite entanglement. This problem was not addressed in Ref. [23], so we can compare our own results only. We compare s^* for the naive physical CM $\tilde{\gamma}$ and the most probable CM γ^* . The results are shown in Fig. 11. The left figure shows the solution given by Theorem 8 and the right figure shows the solution given by Theorem 9. The states γ^* show larger values of s^* than that of $\tilde{\gamma}$ and only the 4-partite state is verified to be genuine multipartite entangled. The optimal solution for the 10-partite state $\tilde{\gamma}$ is zero, which means this test is inconclusive in this case. Here, we see that s^* given by Theorems 8 and 9 can be significantly different. In this case, the latter seems to be a better measure of reli-

ability of the two, since it gives the largest σ neighborhood that consists only of entangled CMs. The former only mea-

FIG. 11. Optimal s^* for genuine multipartite entanglement test.

sures the reliability of entanglement verification of the given state.

B. Bound entanglement

Here, we show that in some cases the problems given by Theorems 6 and 7 can be solved analytically. The method to solve them analytically relies on their numerical solution. Some values in the solution can be identified, some can be seen to be equal (for example, due to some symmetry in the state), or satisfy simple relations like differing by a factor of 2. Any such observations obtained from the numerical solution allow one to formulate the conjecture about the structure of the analytical solution. This conjecture is then transformed into a system of equations for the unknown parameters of the analytical solution, and, if our guess is correct, this system produces one or several analytical solutions. One solution numerically agrees with the result produced by the solver, and this solution is then the exact analytical solution of the problem. If our guess is wrong, then the system would have no solution, which agrees with the numerical result, or, more likely, it would have no solutions at all. The more relations we can correctly guess from the numerical solution, the more likely that the resulting system can be solved analytically. All the relations guessed from a numerical solution form an overspecified system of equations, i.e., the system with a larger number of equations than the number of variables. This overspecification helps to verify that our conjecture on the analytical structure of the solution is correct—if such an overspecified system has a solution, it is a strong indicator that our conjecture is correct.

We start to demonstrate this method on a well-known bound entangled state from Ref. [41] given by

$$\gamma_{xx} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

$$\gamma_{pp} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}. \quad (180)$$

According to the condition (22), the Gaussian state with this CM is a mixed state. It is easy to check that partial transposition of any two modes is positive semidefinite. We now apply Theorem 6 and show that the Gaussian state with this CM is 13|24- and 14|23-separable, but 12|34-entangled. Following the intentions of the authors of Ref. [41] to provide a fully analytical example without relying on any numerical approximations, we present an analytical solution of two of our optimization problems. The analytical solution is constructed from the numerical one produced by the solver by guessing the exact form of the solution and then verifying the conjecture by constructing the dual solution and checking that they agree. The matrix elements of the optimal matrices obtained numerically, we denote without prime, and we use prime to denote their exact analytical expressions.

1. The 13|24 partition, eigenvalue

In this case, the matrices $X_{13|24}^*$ and $P_{13|24}^*$ are easy to recognize and they are given by

$$X_{13|24}^* = \frac{1}{36} \begin{pmatrix} 5 & 0 & -3 & 0 \\ 0 & 5 & 0 & 3 \\ -3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix},$$

$$P_{13|24}^* = \frac{1}{36} \begin{pmatrix} 10 & 0 & -1 & 3 \\ 0 & 10 & 3 & 1 \\ -1 & 3 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}. \quad (181)$$

Note that $\text{rank}(X_{13|24}^*) = 4$, but $\text{rank}(P_{13|24}^*) = 2$. The matrices Z^{13*} and Z^{24*} are equally easy to determine as well,

$$Z^{13*} = \frac{1}{36} \begin{pmatrix} 7 & -1 \\ -4 & 1 \end{pmatrix}, \quad Z^{24*} = \frac{1}{36} \begin{pmatrix} 7 & 1 \\ 4 & 1 \end{pmatrix}. \quad (182)$$

The matrices satisfy the positivity condition $X_{13|24}^*, P_{13|24}^* \succcurlyeq 0$ and the normalization condition

$$\text{Tr}(X_{13|24}^* + P_{13|24}^*) = 1. \quad (183)$$

In addition, we have that the matrices

$$\begin{pmatrix} X_{13|24}^*[1, 3] & Z^{13*} \\ Z^{13*T} & P_{13|24}^*[1, 3] \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 5 & -3 & 7 & -1 \\ -3 & 2 & -4 & 1 \\ 7 & -4 & 10 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} X_{13|24}^*[2, 4] & Z^{24*} \\ Z^{24*T} & P_{13|24}^*[2, 4] \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 5 & 3 & 7 & 1 \\ 3 & 2 & 4 & 1 \\ 7 & 4 & 10 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

are positive semidefinite and satisfy the equalities

$$\text{str}(X_{13|24}^*[1, 3], P_{13|24}^*[1, 3]) = \text{Tr} Z^{13*},$$

$$\text{str}(X_{13|24}^*[2, 4], P_{13|24}^*[2, 4]) = \text{Tr} Z^{24*}. \quad (184)$$

The primal objective reads as

$$\text{Tr}(X_{13|24}^* \gamma_{xx} + P_{13|24}^* \gamma_{pp}) - \text{Tr} Z^{13*} - \text{Tr} Z^{24*} = 0. \quad (185)$$

To show that this value is really optimal, we need the dual solution.

The dual solution can also be recognized from the numbers produced by the solver:

$$\gamma_{xx}^{13*} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \gamma_{pp}^{13*} = \frac{1}{6} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

$$\gamma_{xx}^{24*} = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \gamma_{pp}^{24*} = \frac{1}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \quad (186)$$

It is easy to see that $\gamma_{xx} = \gamma_{xx}^{13*} \oplus \gamma_{xx}^{24*}$ and

$$\gamma_{pp} \succcurlyeq \gamma_{pp}^{13*} \oplus \gamma_{pp}^{24*} = \frac{1}{6} \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}. \quad (187)$$

It follows that the dual objective value, the minimal eigenvalue of the differences, is also zero, which verifies the optimality

of the solutions. One can verify that the KKT conditions (123) are satisfied

$$\begin{aligned} \mathcal{P}(\gamma_{xx}^{13*}, \gamma_{pp}^{13*}) \begin{pmatrix} X_{13|24}^{13*}[1, 3] & -Z^{13*} \\ -Z^{13*T} & P_{13|24}^{13*}[1, 3] \end{pmatrix} &= 0, \\ \mathcal{P}(\gamma_{xx}^{24*}, \gamma_{pp}^{24*}) \begin{pmatrix} X_{13|24}^{24*}[2, 4] & -Z^{24*} \\ -Z^{24*T} & P_{13|24}^{24*}[2, 4] \end{pmatrix} &= 0, \\ X_{13|24}^{13*}(\gamma_{xx} - \gamma_{xx}^{13*} \oplus \gamma_{xx}^{24*}) &= 0, \\ P_{13|24}^{13*}(\gamma_{pp} - \gamma_{pp}^{13*} \oplus \gamma_{pp}^{24*}) &= 0. \end{aligned} \quad (188)$$

In terms of the entanglement measure, this result can be simply stated as

$$\mathcal{E}_{13|24}^+(\gamma_{xx}, \gamma_{pp}) = 0, \quad (189)$$

and thus the CM given by Eq. (180) is 13|24 separable.

2. The 13|24 partition, scaling

The primal solution γ_{xx}^{13*} , γ_{pp}^{13*} , γ_{xx}^{24*} , and γ_{pp}^{24*} is given by Eq. (186) and corresponds to the optimal value $t^* = 1$. The dual solution, the matrices $X_{13|24}^*$ and $P_{13|24}^*$, are given by

$$\begin{aligned} X_{13|24}^* &= \frac{1}{16} \begin{pmatrix} 5 & 0 & -3 & 0 \\ 0 & 5 & 0 & 3 \\ -3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix}, \\ P_{13|24}^* &= \frac{1}{16} \begin{pmatrix} 10 & 0 & -1 & 3 \\ 0 & 10 & 3 & 1 \\ -1 & 3 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (190)$$

The matrices Z^{13*} and Z^{24*} read as

$$Z^{13*} = \frac{1}{16} \begin{pmatrix} 7 & -1 \\ -4 & 1 \end{pmatrix}, \quad Z^{24*} = \frac{1}{16} \begin{pmatrix} 7 & 1 \\ 4 & 1 \end{pmatrix}. \quad (191)$$

These matrices just differ by a factor from the matrices (181) and (182) of the eigenvalue version of the test. One can easily check that

$$\text{Tr} Z^{13*} + \text{Tr} Z^{24*} = 1, \quad \text{Tr}(X_{13|24}^* \gamma_{xx} + P_{13|24}^* \gamma_{pp}) = t^*.$$

The KKT conditions (123) and (138), which in this case read as Eq. (188), are obviously satisfied. In terms of the measure $\mathcal{T}_{13|24}^+$, we have

$$\mathcal{T}_{13|24}^+(\gamma_{xx}, \gamma_{pp}) = 0, \quad (192)$$

which gives another demonstration that the state is 13|24 separable.

3. The 14|23 partition, eigenvalue

The matrices $X_{14|23}^*$ and $P_{14|23}^*$ are given by

$$X_{14|23}^* = \frac{1}{36} \begin{pmatrix} 5 & 0 & -3 & 1 \\ 0 & 5 & 1 & 3 \\ -3 & 1 & 2 & 0 \\ 1 & 3 & 0 & 2 \end{pmatrix},$$

$$P_{14|23}^* = \frac{1}{36} \begin{pmatrix} 10 & 0 & 0 & 3 \\ 0 & 10 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}. \quad (193)$$

Note that $\text{rank}(X_{14|23}^*) = 2$, but $\text{rank}(P_{14|23}^*) = 4$. The matrices Z^{14*} and Z^{23*} read as

$$Z^{14*} = \frac{1}{36} \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}, \quad Z^{23*} = \frac{1}{36} \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}. \quad (194)$$

It is easy to check that $X_{14|23}^* P_{14|23}^* \succcurlyeq 0$ and

$$\text{Tr}(X_{14|23}^* + P_{14|23}^*) = 1. \quad (195)$$

In addition, we have

$$\begin{pmatrix} X_{14|23}^*[1, 4] & Z^{14*} \\ Z^{14*T} & P_{14|23}^*[1, 4] \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 5 & 1 & 7 & 2 \\ 1 & 2 & 2 & 1 \\ 7 & 2 & 10 & 3 \\ 2 & 1 & 3 & 1 \end{pmatrix},$$

$$\begin{pmatrix} X_{14|23}^*[2, 3] & Z^{23*} \\ Z^{23*T} & P_{14|23}^*[2, 3] \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 5 & 1 & 7 & 2 \\ 1 & 2 & 2 & 1 \\ 7 & 2 & 10 & 3 \\ 2 & 1 & 3 & 1 \end{pmatrix}$$

are positive semidefinite and

$$\begin{aligned} \text{str}(X_{14|23}^*[1, 4], P_{14|23}^*[1, 4]) &= \text{Tr} Z^{14*}, \\ \text{str}(X_{14|23}^*[2, 3], P_{14|23}^*[2, 3]) &= \text{Tr} Z^{23*}. \end{aligned} \quad (196)$$

The primal objective reads as

$$\text{Tr}(X_{14|23}^* \gamma_{xx} + P_{14|23}^* \gamma_{pp}) - \text{Tr} Z^{14*} - \text{Tr} Z^{23*} = 0. \quad (197)$$

To show that this value is really optimal, we need the dual solution.

The dual solution is given by

$$\begin{aligned} \gamma_{xx}^{14*} &= \frac{1}{6} \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, & \gamma_{pp}^{14*} &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}, \\ \gamma_{xx}^{23*} &= \frac{1}{6} \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, & \gamma_{pp}^{23*} &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}. \end{aligned} \quad (198)$$

It is easy to see that $\gamma_{pp} = \gamma_{pp}^{14*} \oplus \gamma_{pp}^{23*}$ and

$$\gamma_{xx} \succcurlyeq \gamma_{xx}^{14*} \oplus \gamma_{xx}^{23*} = \frac{1}{6} \begin{pmatrix} 4 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \quad (199)$$

It follows that the dual objective value, the minimal eigenvalue of the differences, is also zero, which verifies the optimality of the solutions. One can verify that the KKT conditions (123) are satisfied

$$\begin{aligned} \mathcal{P}(\gamma_{xx}^{14*}, \gamma_{pp}^{14*}) \begin{pmatrix} X_{14|23}^*[1, 4] & -Z^{14*} \\ -Z^{14*T} & P_{14|23}^*[1, 4] \end{pmatrix} &= 0, \\ \mathcal{P}(\gamma_{xx}^{23*}, \gamma_{pp}^{23*}) \begin{pmatrix} X_{14|23}^*[2, 3] & -Z^{23*} \\ -Z^{23*T} & P_{14|23}^*[2, 3] \end{pmatrix} &= 0, \\ X_{14|23}^*(\gamma_{xx} - \gamma_{xx}^{14*} \oplus \gamma_{xx}^{23*}) &= 0, \\ P_{14|23}^*(\gamma_{pp} - \gamma_{pp}^{14*} \oplus \gamma_{pp}^{23*}) &= 0. \end{aligned} \quad (200)$$

In terms of the entanglement measure, this result can be simply stated as

$$\mathcal{E}_{14|23}^+(\gamma_{xx}, \gamma_{pp}) = 0, \quad (201)$$

and thus the CM given by Eq. (180) is 14|23 separable.

4. The 14|23 partition, scaling

The primal solution γ_{xx}^{14*} , γ_{pp}^{14*} , γ_{xx}^{23*} , and γ_{pp}^{23*} is given by Eq. (198) and corresponds to the optimal value $t^* = 1$. The dual solution, the matrices $X_{14|23}^*$ and $P_{14|23}^*$ are given by

$$X_{14|23}^* = \frac{1}{16} \begin{pmatrix} 5 & 0 & -3 & 1 \\ 0 & 5 & 1 & 3 \\ -3 & 1 & 2 & 0 \\ 1 & 3 & 0 & 2 \end{pmatrix}, \quad (202)$$

$$P_{14|23}^* = \frac{1}{16} \begin{pmatrix} 10 & 0 & 0 & 3 \\ 0 & 10 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}.$$

The matrices Z^{14*} and Z^{23*} read as

$$Z^{14*} = \frac{1}{16} \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}, \quad Z^{23*} = \frac{1}{16} \begin{pmatrix} 7 & 2 \\ 2 & 1 \end{pmatrix}. \quad (203)$$

These matrices just differ by a factor from the matrices (193) and (194) of the eigenvalue version of the test. One can easily check that

$$\text{Tr} Z^{14*} + \text{Tr} Z^{23*} = 1, \quad \text{Tr}(X_{14|23}^* \gamma_{xx} + P_{14|23}^* \gamma_{pp}) = t^*.$$

The KKT conditions (123) and (138), which in this case read as Eq. (200), are obviously satisfied. In terms of the measure $\mathcal{T}_{14|23}^+$, we have

$$\mathcal{T}_{14|23}^+(\gamma_{xx}, \gamma_{pp}) = 0, \quad (204)$$

which gives another demonstration that the state is 14|23 separable.

5. The 12|34 partition, eigenvalue

The matrices $X_{12|34}^*$ and $P_{12|34}^*$ are given by

$$X_{12|34}^* = \begin{pmatrix} x & 0 & -z & 0 \\ 0 & x & 0 & z \\ -z & 0 & y & 0 \\ 0 & z & 0 & y \end{pmatrix}, \quad P_{12|34}^* = \begin{pmatrix} p & 0 & 0 & r \\ 0 & p & r & 0 \\ 0 & r & q & 0 \\ r & 0 & 0 & q \end{pmatrix}, \quad (205)$$

where the values of the parameters are given by

$$\begin{aligned} x &= 0.1359 \dots, & y &= 0.0706 \dots, & z &= 0.0979 \dots, \\ p &= 0.2578 \dots, & q &= 0.0356 \dots, & r &= 0.0958 \dots \end{aligned} \quad (206)$$

The normalization condition is given by

$$\text{Tr}(X_{12|34}^* + P_{12|34}^*) = 2(x + y + p + q) = 1. \quad (207)$$

The matrices Z^{12*} and Z^{34*} read as

$$Z^{12*} = u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z^{34*} = v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (208)$$

where $u = 0.1872 \dots$ and $v = 0.0501 \dots$. This time the numbers x, y, z, p, q , and r are not so easy to identify.

Both matrices

$$\begin{pmatrix} X^*[1, 2] & Z^{12*} \\ Z^{12*T} & P^*[1, 2] \end{pmatrix} = \begin{pmatrix} x & 0 & u & 0 \\ 0 & x & 0 & u \\ u & 0 & p & 0 \\ 0 & u & 0 & p \end{pmatrix} \quad (209)$$

and

$$\begin{pmatrix} X^*[3, 4] & Z^{34*} \\ Z^{34*T} & P^*[3, 4] \end{pmatrix} = \begin{pmatrix} y & 0 & v & 0 \\ 0 & y & 0 & v \\ v & 0 & q & 0 \\ 0 & v & 0 & q \end{pmatrix} \quad (210)$$

should have two zero eigenvalues, so we have the equations

$$xp = u^2, \quad yq = v^2. \quad (211)$$

There are only two equations, not four, since the eigenvalues have multiplicity 2.

The dual solution produced by the solver is

$$\gamma_{xx}^{12} = \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \gamma_{pp}^{12} = \frac{1}{2} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix},$$

$$\gamma_{xx}^{34} = \frac{1}{2} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad \gamma_{pp}^{34} = \frac{1}{2} \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}, \quad (212)$$

where a and b read as

$$a = 1.37716 \dots, \quad b = 0.71033 \dots \quad (213)$$

The KKT conditions (123), which read as

$$\mathcal{P}(\gamma_{xx}^{12*}, \gamma_{pp}^{12*}) \begin{pmatrix} X_{12|34}^*[1, 2] & -Z^{12*} \\ -Z^{12*T} & P_{12|34}^*[1, 2] \end{pmatrix} = 0,$$

$$\mathcal{P}(\gamma_{xx}^{34*}, \gamma_{pp}^{34*}) \begin{pmatrix} X_{12|34}^*[3, 4] & -Z^{34*} \\ -Z^{34*T} & P_{12|34}^*[3, 4] \end{pmatrix} = 0, \quad (214)$$

produce the following system of four equations:

$$u = ax, \quad p = au, \quad v = by, \quad q = bv. \quad (215)$$

The KKT conditions (124), which read as

$$X_{12|34}^*(\gamma_{xx} - \gamma_{xx}^{12*} \oplus \gamma_{xx}^{34*}) = \lambda_{\min} X_{12|34}^*,$$

$$P_{12|34}^*(\gamma_{pp} - \gamma_{pp}^{12*} \oplus \gamma_{pp}^{34*}) = \lambda_{\min} P_{12|34}^*, \quad (216)$$

where

$$\begin{aligned} \lambda_{\min} &= \text{Tr}(X_{12|34}^* \gamma_{xx} + P_{12|34}^* \gamma_{pp}) - \text{Tr}(Z^{12*} + Z^{34*}) \\ &= 1 - p + 2(q - r - u - v - z), \end{aligned}$$

produce the following system of eight equations:

$$\begin{aligned} 4x(z + u + v - q + r) &= x(a - 2p) + z, \\ 4y(z + u + v - q + r) &= y(b - 2p) + z, \\ 4z(z + u + v - q + r) &= z(b - 2p) + x, \\ 4z(z + u + v - q + r) &= z(a - 2p) + y, \\ 4p(z + u + v - q + r) &= p(1/a + 1 - 2p) + r, \\ 4q(z + u + v - q + r) &= q(1/b - 2 - 2p) + r, \\ 4r(z + u + v - q + r) &= r(1/b - 2 - 2p) + p, \\ 4r(z + u + v - q + r) &= r(1/a + 1 - 2p) + q. \end{aligned} \quad (217)$$

The equations given by Eqs. (207), (211), (215), and (217) form a system of 15 equations for ten variables $x, y, z, p, q, r, u, v, a$, and b . It is easy to derive from Eq. (215) that

$$u^* = \sqrt{x^* p^*}, \quad v^* = \sqrt{y^* q^*}. \quad (218)$$

These equalities are equivalent to the relations

$$\begin{aligned} \text{str}(X_{12|34}^*[1, 2], P_{12|34}^*[1, 2]) &= \text{Tr}(Z^{12*}), \\ \text{str}(X_{12|34}^*[3, 4], P_{12|34}^*[3, 4]) &= \text{Tr}(Z^{34*}). \end{aligned} \quad (219)$$

Less obvious equalities are given by

$$z^* = \sqrt{x^* y^*}, \quad r^* = \sqrt{p^* q^*}. \quad (220)$$

The simplest way to derive these equalities is to compute the minimal polynomials (say, in the variable ξ) of the differences

of the right- and left-hand sides. The minimal polynomials turn out to be just ξ , which means that these equalities hold true. The minimal polynomials of the other optimal values are given by:

$$\begin{aligned} 541504x^4 - 270752x^3 + 38408x^2 - 1762x + 25 &= 0, \\ 67688y^4 - 33844y^3 + 5362y^2 - 347y + 8 &= 0, \\ 33844p^4 - 16922p^3 + 2486p^2 - 100p + 1 &= 0, \\ 541504q^4 - 270752q^3 + 30008q^2 - 778q + 1 &= 0, \\ 5a^4 + 11a^3 - 8a^2 - 20a - 4 &= 0, \\ 8b^4 + 2b^3 - 26b^2 + 16b - 1 &= 0. \end{aligned}$$

It is possible to obtain the optimal values explicitly

$$\begin{aligned} 2x^* &= \frac{1}{8} \left(1 + x_0 - \sqrt{\frac{6179}{8461} - x_0^2 + \frac{1242}{8461x_0}} \right), & x_0^2 &= \frac{6179 + 4\sqrt{322\,547\,8} \cos\left(\frac{1}{3} \arccos\left(\frac{228\,366\,997\,33}{129\,019\,12\sqrt{322\,547\,8}}\right)\right)}{25383}, \\ y^* &= \frac{1}{8} \left(1 - y_0 - \sqrt{\frac{3935}{8461} - y_0^2 - \frac{1026}{8461y_0}} \right), & y_0^2 &= \frac{3935 + 16\sqrt{1093} \cos\left(\frac{1}{3} \arccos\left(\frac{121\,301}{8744\sqrt{1093}}\right)\right)}{253\,83}, \\ p^* &= \frac{1}{8} \left(1 + p_0 + \sqrt{\frac{5495}{8461} - p_0^2 + \frac{234}{8461p_0}} \right), & p_0^2 &= \frac{5495 + 16\sqrt{377\,431} \cos\left(\frac{1}{3} \arccos\left(\frac{919\,571\,683}{150\,972\,4\sqrt{377\,431}}\right)\right)}{253\,83}, \\ q^* &= \frac{1}{8} \left(1 - q_0 + \sqrt{\frac{103\,79}{8461} - q_0^2 - \frac{3474}{8461q_0}} \right), & q_0^2 &= \frac{103\,79 + 4\sqrt{429\,754\,6} \cos\left(\frac{1}{3} \arccos\left(\frac{266\,125\,739\,53}{171\,901\,84\sqrt{429\,754\,6}}\right)\right)}{253\,83}, \\ a^* &= \frac{1}{20} \left(-11 + a_0 + \sqrt{683 - a_0^2 + \frac{1818}{a_0}} \right), & a_0^2 &= \frac{683}{3} + \frac{880}{3} \cos\left[\frac{1}{3} \arccos\left(\frac{3137}{5324}\right)\right], \\ b^* &= \frac{1}{16} \left(-1 + b_0 - \sqrt{419 - b_0^2 - \frac{2466}{b_0}} \right), & b_0^2 &= \frac{419}{3} + \frac{352}{3} \cos\left[\frac{1}{3} \arccos\left(\frac{3137}{5324}\right)\right], \end{aligned} \quad (221)$$

where x_0, y_0, p_0, q_0 , and v_0 denote the positive root of the expression of the right-hand side. Their numerical values coincide with the values (206) and (213) produced by the solver. From Eq. (215), it follows that

$$a^* = \sqrt{\frac{p^*}{x^*}}, \quad b^* = \sqrt{\frac{q^*}{y^*}}, \quad (222)$$

and just by looking at the explicit expressions above, it seems like a very difficult task to validate these relations. Computing the minimal polynomials of the differences quickly verifies that these identities are valid.

We now derive the optimal solution in another way. From equalities (218) and (220), it follows that the primal objective value is of the form

$$\lambda_{\min} = 1 + 2q - p - 2(\sqrt{x} + \sqrt{q})(\sqrt{y} + \sqrt{p}).$$

We show that the numbers x^*, y^*, p^* , and q^* are a minimizing point of this function; i.e., they are a solution of the following

minimization problem:

$$\min_{x,y,p,q} 1 + 2q - p - 2(\sqrt{x} + \sqrt{q})(\sqrt{y} + \sqrt{p}), \quad (223)$$

subject to $x + y + p + q = 1/2$. To verify this conjecture, we find an analytical solution of this problem. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= 1 + 2q - p - 2(\sqrt{x} + \sqrt{q})(\sqrt{y} + \sqrt{p}) \\ &\quad + v(x + y + p + q - \tfrac{1}{2}). \end{aligned} \quad (224)$$

The KKT conditions produce a system of five equations for five variables:

$$\begin{aligned} \frac{\sqrt{y} + \sqrt{p}}{\sqrt{x}} &= v, & \frac{\sqrt{x} + \sqrt{q}}{\sqrt{y}} &= v, & \frac{\sqrt{x} + \sqrt{q}}{\sqrt{p}} + 1 &= v, \\ \frac{\sqrt{y} + \sqrt{p}}{\sqrt{q}} - 2 &= v, & x + y + p + q - \frac{1}{2} &= 0. \end{aligned}$$

To verify that the numbers x^* , y^* , p^* , q^* do satisfy this system, we just need to substitute them into it and check that the resulting v^* is the same in the first four equations. This can be done by computing the minimal polynomials of the differences of the left-hand sides. An explicit expression for v^* reads as

$$v^* = \frac{1}{4} \left(-1 + 2v_0 + 2\sqrt{\frac{51}{4} - v_0^2 - \frac{9}{4v_0}} \right),$$

$$v_0^2 = \frac{17}{4} + 2\sqrt{\frac{22}{3}} \cos \left(\frac{1}{3} \arccos \left(\frac{27}{22} \sqrt{\frac{3}{22}} \right) \right). \quad (225)$$

The optimal objective value $\lambda_{\min}^*$ is given by

$$\lambda_{\min}^* = \frac{1}{8} \left(9 - \lambda_0 - \sqrt{51 - \lambda_0^2 - \frac{18}{\lambda_0}} \right),$$

$$\lambda_0^2 = 17 + 8\sqrt{\frac{22}{3}} \cos \left[\frac{1}{3} \arccos \left(\frac{27}{22} \sqrt{\frac{3}{22}} \right) \right]. \quad (226)$$

The key point to efficiently work with such numbers is representing them in an explicit algebraic form, for example, we have

$$\cos \left[\frac{1}{3} \arccos \left(\frac{27}{22} \sqrt{\frac{3}{22}} \right) \right] = \frac{\text{Re} \sqrt[3]{81 + i\sqrt{25383}}}{\sqrt[3]{3}\sqrt{22}}.$$

In fact, this is how all starred quantities are dealt with here. The minimal polynomial of this optimal value is

$$8\lambda^4 - 36\lambda^3 + 48\lambda^2 - 18\lambda - 1 = 0. \quad (227)$$

Further simplification of the expression (226) does not seem to be possible. This lengthy construction shows that

$$\mathcal{E}_{12|34}^+(\gamma) = -\lambda_{\min}^* = 0.0489333 \dots, \quad (228)$$

and thus the CM given by Eq. (180) is 12|34-entangled. Note that the matrices $X_{12|34}^*$ and $P_{12|34}^*$ are degenerate, but of the same rank, $\text{rank}(X_{12|34}^*) = \text{rank}(P_{12|34}^*) = 2$.

6. The 12|34 partition, scaling

The primal solution produced by the solver is easily identified,

$$\gamma_{xx}^{12*} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{pp}^{12*} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\gamma_{xx}^{34*} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{pp}^{34*} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (229)$$

The analytical expression of the optimal value t^* can be found from the requirement that the minimums of the eigenvalues of the differences

$$\gamma_{xx} - t^* \gamma_{xx}^{12*} \oplus \gamma_{xx}^{34*}, \quad \gamma_{pp} - t^* \gamma_{pp}^{12*} \oplus \gamma_{pp}^{34*} \quad (230)$$

are zero. Equating the minimal eigenvalues of the differences to zero, we derive that t^* is the minimum of the solutions of the following two equations:

$$8 - 3\sqrt{2}t^* - \sqrt{16 + t^{*2}} = 0,$$

$$10 - 3\sqrt{2}t^* - \sqrt{52 - 12\sqrt{2}t^* + 2t^{*2}} = 0. \quad (231)$$

It turns out that the roots of these equations coincide:

$$t^* = \frac{3\sqrt{2} - \sqrt{6}}{2} = \sqrt{6 - 3\sqrt{3}}. \quad (232)$$

We could construct and solve the system of KKT equations to find the dual solution, but in this case the matrices $X_{12|34}^*$ and $P_{12|34}^*$ are easily recognizable to be in the form

$$X_{12|34}^* = \begin{pmatrix} u & 0 & -\sqrt{2uv} & 0 \\ 0 & u & 0 & \sqrt{2uv} \\ -\sqrt{2uv} & 0 & 2v & 0 \\ 0 & \sqrt{2uv} & 0 & 2v \end{pmatrix},$$

$$P_{12|34}^* = \begin{pmatrix} 2u & 0 & 0 & \sqrt{2uv} \\ 0 & 2u & \sqrt{2uv} & 0 \\ 0 & \sqrt{2uv} & v & 0 \\ \sqrt{2uv} & 0 & 0 & v \end{pmatrix}, \quad (233)$$

where the numerical values of u and v are

$$u = 0.2788 \dots, \quad v = 0.0747 \dots \quad (234)$$

The matrices Z^{12*} and Z^{34*} read as

$$Z^{12*} = \sqrt{2}u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z^{34*} = \sqrt{2}v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (235)$$

Our conjecture is that the exact values of u and v are the solution u^* and v^* of the following optimization problem:

$$\min_{\text{Tr}(Z^{12*} + Z^{34*})=1} \text{Tr}(X_{12|34}^* \gamma_{xx} + P_{12|34}^* \gamma_{pp})$$

$$= \min_{u+v=1/(2\sqrt{2})} 4(u - \sqrt{2}\sqrt{uv} + 2v). \quad (236)$$

This problem can be solved much easier than a similar problem in the eigenvalue case, and the solution reads as

$$u^* = \frac{3\sqrt{2} + \sqrt{6}}{24}, \quad v^* = \frac{3\sqrt{2} - \sqrt{6}}{24}. \quad (237)$$

The optimal value is exactly the value of Eq. (232). The normalization condition

$$\text{Tr} Z^{12*} + \text{Tr} Z^{34*} = 1, \quad (238)$$

and the KKT conditions (123) and (138) are satisfied:

$$\mathcal{P}(\gamma_{xx}^{12*}, \gamma_{pp}^{12*}) \begin{pmatrix} X_{12|34}^*[1, 2] & -Z^{12*} \\ -Z^{12*T} & P_{12|34}^*[1, 2] \end{pmatrix} = 0,$$

$$\mathcal{P}(\gamma_{xx}^{34*}, \gamma_{pp}^{34*}) \begin{pmatrix} X_{12|34}^*[3, 4] & -Z^{34*} \\ -Z^{34*T} & P_{12|34}^*[3, 4] \end{pmatrix} = 0,$$

$$X_{12|34}^*(\gamma_{xx} - t^* \gamma_{xx}^{12*} \oplus \gamma_{xx}^{34*}) = 0,$$

$$P_{12|34}^*(\gamma_{pp} - t^* \gamma_{pp}^{12*} \oplus \gamma_{pp}^{34*}) = 0. \quad (239)$$

In terms of the measure $\mathcal{T}_{12|34}^+$, we have

$$\mathcal{T}_{12|34}^+(\gamma_{xx}, \gamma_{pp}) = 1 - t^* = 1 - \sqrt{6 - 3\sqrt{3}} = 0.1034 \dots,$$

which gives another demonstration that the state is 12|34-entangled. This number was obtained in Ref. [18] in a purely numerical way, here we derived an analytical expression for

this value. Note that the matrices $X_{12|34}^*$ and $P_{12|34}^*$ are degenerate, but of the same rank, $\text{rank}(X_{12|34}^*) = \text{rank}(P_{12|34}^*) = 2$.

7. The 12|34 partition, refined condition

Here, we solve the full problem, not only the reduced one as we did up to now. The primal solution γ^{12*} produced by the solver is

$$\gamma^{12*} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (240)$$

The dual solution, the matrix M is of the form

$$M^* = \begin{pmatrix} x & 0 & -y & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & y & 0 & 0 & 0 & 0 \\ -y & 0 & 2z & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 2z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2x & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 2x & y & 0 \\ 0 & 0 & 0 & 0 & 0 & y & z & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & z \end{pmatrix}, \quad (241)$$

and matrix K is of the form

$$K^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & u & v & 0 \\ 0 & 0 & 0 & 0 & -u & 0 & 0 & -v \\ 0 & 0 & 0 & 0 & 0 & -2v & -w & 0 \\ 0 & 0 & 0 & 0 & -2v & 0 & 0 & -w \\ 0 & u & 0 & 2v & 0 & 0 & 0 & 0 \\ -u & 0 & 2v & 0 & 0 & 0 & 0 & 0 \\ -v & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & w & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (242)$$

The matrix K^{12*} is of the form

$$K^{12*} = \begin{pmatrix} 0 & 0 & -u & 0 \\ 0 & 0 & 0 & -u \\ u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \end{pmatrix}. \quad (243)$$

The numerical values of the parameters are given by

$$\begin{aligned} x &= 0.2788 \dots, & y &= 0.2041 \dots, & z &= 0.0747 \dots, \\ u &= 0.3943 \dots, & v &= 0.1443 \dots, & w &= 0.1056 \dots. \end{aligned} \quad (244)$$

The normalization condition (173) reads as

$$\frac{1}{2} \text{Tr}(K^{12*} \Omega_2) + \frac{1}{2} \text{Tr}(K^*[3] \Omega_2) = 1, \quad (245)$$

where, as in Sec. VII A, in the full case we denote by $K[I]$, the submatrix of K with row and column indices in the set $I \cup (I + n)$. In this case, the full notation would be $K^*[3, 7]$. This condition produces the equation

$$2(u + w) = 1. \quad (246)$$

The KKT condition (174) is

$$\mathcal{P}(\gamma^{12*})(M^*[1, 2] + iK^{12*}) = 0. \quad (247)$$

According to the note above, the full notation for $M[1, 2]$ would be $M[1, 2, 5, 6]$. This condition produces

a single equation

$$u = \sqrt{2}x. \quad (248)$$

The KKT condition (175) reads as

$$(M^* - iK^*) \left(\gamma - t\gamma^{12*} \oplus \frac{i}{2} \Omega_2 \right) = 0, \quad (249)$$

where

$$t = \text{Tr}(M^* \gamma) = 4(x - y + 2z). \quad (250)$$

This condition produces a system of equations, which contains

$$\begin{aligned} x - 2y + 4v(x - y + 2z) &= 0, \\ v + 2w - 4z(x - y + 2z) &= 0, \\ v + u - 2\sqrt{2}u(x - y + 2z) &= 0, \end{aligned} \quad (251)$$

and several other similar equations. Combining these equations with Eqs. (246) and (248) and solving the resulting system, we obtain two solutions:

$$\begin{aligned} x^* &= \frac{3\sqrt{2} \pm \sqrt{6}}{24}, & y^* &= \pm \frac{1}{2\sqrt{6}}, & z^* &= \frac{3\sqrt{2} \mp \sqrt{6}}{24}, \\ u^* &= \frac{3 \pm \sqrt{3}}{12}, & v^* &= \pm \frac{1}{4\sqrt{3}}, & w^* &= \frac{3 \mp \sqrt{3}}{12}. \end{aligned}$$

The solution (244) produced by the solver corresponds to the first solution, where we choose the upper sign, though both choices are valid solutions. The corresponding primal objective values are given by

$$t^* = \sqrt{6 \mp 3\sqrt{3}}, \quad (252)$$

so only the first choice gives the minimal value for t . This value coincides with the value (232) obtained with the scaling optimization problem and shows that the CM is 12|34-entangled. Note that M^* is degenerate, $\text{rank}(M^*) = 4$.

C. Genuine 3-partite entanglement

Here, we consider a more complicated case of a 3-partite genuine entangled state treated numerically in Ref. [18]. That state is also reduced with the CM given by

$$\gamma_{xx} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \gamma_{pp} = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}. \quad (253)$$

It is an instance of the symmetric states (92) with $\lambda = 2$ and $\mu = 1/2$ and thus is a pure state. It has been numerically demonstrated that this state is genuine entangled. Here, we again construct analytical solutions of the two optimization problems. These solutions are a bit more complicated than in the previous case, where only single bipartitions are involved.

1. Eigenvalue problem

The solver produces the following optimal matrices:

$$X^* = \begin{pmatrix} 2x & -x & -x \\ -x & 2x & -x \\ -x & -x & 2x \end{pmatrix}, \quad P^* = \begin{pmatrix} p & q & q \\ q & p & p \\ q & q & p \end{pmatrix}, \quad (254)$$

where the numerical values of x , p , and q are given by

$$x = 0.0905 \dots, \quad p = 0.1522 \dots, \quad q = 0.1308 \dots \quad (255)$$

Because the state is symmetric, it is enough to consider only one bipartition, for example, $12|3$. The Z matrices read as

$$Z^{12*} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}, \quad Z^{3*} = (w), \quad (256)$$

where the numerical values of u , v , and w are given by

$$u = 0.1181 \dots, \quad v = 0.0419 \dots, \quad w = 0.1660 \dots \quad (257)$$

Two eigenvalues of the matrix

$$\begin{pmatrix} X^*[1, 2] & Z^{12*} \\ Z^{12*T} & P^*[1, 2] \end{pmatrix} = \begin{pmatrix} 2x & -x & u & v \\ -x & 2x & v & u \\ u & v & p & q \\ v & u & q & p \end{pmatrix} \quad (258)$$

must be zero, so we have two equations

$$(u + v)^2 = x(p + q), \quad (u - v)^2 = 3x(p - q). \quad (259)$$

One eigenvalue of the matrix

$$\begin{pmatrix} X^*[3] & Z^{3*} \\ Z^{3*T} & P^*[3] \end{pmatrix} = \begin{pmatrix} 2x & w \\ w & p \end{pmatrix} \quad (260)$$

must be zero, so we have another equation

$$w^2 = 2xp. \quad (261)$$

Normalization condition reads as

$$\text{Tr}(X^* + P^*) = 6x + 3p = 1. \quad (262)$$

The primal objective function is given by

$$\begin{aligned} \lambda_{\min} &= \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{Tr}(Z^{12*}) - \text{Tr}(Z^{3*}) \\ &= \frac{1}{4}(9p - 6q - 8u - 4w + 6x). \end{aligned}$$

The dual solution produced by the solver is given by

$$\begin{aligned} \gamma_{xx}^{12*} &= \gamma_{xx}^{13*} = \gamma_{xx}^{23*} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \gamma_{xx}^3 = \gamma_{xx}^2 = \gamma_{xx}^1 = (c), \\ \gamma_{pp}^{12*} &= \gamma_{pp}^{13*} = \gamma_{pp}^{23*} = \begin{pmatrix} d & e \\ e & d \end{pmatrix}, \quad \gamma_{pp}^3 = \gamma_{pp}^2 = \gamma_{pp}^1 = (f), \end{aligned}$$

where the numerical values of the parameters read as

$$\begin{aligned} a &= 0.5121 \dots, \quad b = 0.3719 \dots, \quad c = 0.4584 \dots, \\ d &= 1.0327 \dots, \quad e = -3/4 \dots, \quad f = 0.5453 \dots \end{aligned} \quad (263)$$

Though all xx and pp matrices are the same, they compose in different ways, for example,

$$\gamma_{xx}^{12*} \oplus \gamma_{xx}^{3*} = \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \gamma_{xx}^{13*} \oplus \gamma_{xx}^{2*} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}.$$

The coefficients of the mixture are equal

$$p_{12|3}^* = p_{13|2}^* = p_{23|1}^* = \frac{1}{3}. \quad (264)$$

Two eigenvalues of the matrix $\mathcal{P}(\gamma_{xx}^{12*}, \gamma_{pp}^{12*})$ must be zero, which reduces to the following equations:

$$(a + b)(4d - 3) = 1, \quad (a - b)(4d + 3) = 1. \quad (265)$$

Two eigenvalues of the matrix $\mathcal{P}(\gamma_{xx}^{3*}, \gamma_{pp}^{3*})$ must be zero, which reduces to the following equation:

$$4cf = 1. \quad (266)$$

The KKT condition

$$\mathcal{P}(\gamma_{xx}^{12*}, \gamma_{pp}^{12*}) \begin{pmatrix} X^*[1, 2] & -Z^{12*} \\ -Z^{12*T} & P^*[1, 2] \end{pmatrix} = 0 \quad (267)$$

produces the following system of equations:

$$\begin{aligned} 2x(2a - b) &= u, & 2x(2b - a) &= v, \\ 2(au + bv) &= p, & 2(bu + av) &= q, \\ 3v + 4x &= 4du, & 3u - 2x &= 4dv, \\ 3q + 2u &= 4dp, & 3p + 2v &= 4dq. \end{aligned} \quad (268)$$

The KKT condition

$$\mathcal{P}(\gamma_{xx}^{3*}, \gamma_{pp}^{3*}) \begin{pmatrix} X^*[3] & -Z^{3*} \\ -Z^{3*T} & P^*[3] \end{pmatrix} = 0 \quad (269)$$

produces the following system of equations:

$$w = 4cx, \quad p = 2cw, \quad x = fw, \quad w = 2fp. \quad (270)$$

The KKT conditions yield that the matrices satisfy

$$\begin{aligned} X^*(\gamma_{xx} - \frac{1}{3}\gamma_{xx}^{12*} \oplus \gamma_{xx}^{3*} - \dots) &= \lambda_{\min} X^*, \\ P^*(\gamma_{pp} - \frac{1}{3}\gamma_{pp}^{12*} \oplus \gamma_{pp}^{3*} - \dots) &= \lambda_{\min} P^*, \end{aligned}$$

where the dots stand for the combination of the bipartitions, which produces the following system of equations:

$$\begin{aligned} 18x - 12w - 24u - 18q + 27p + 8d + 4f &= 9, \\ 8a - 4b + 4c - 8d - 4f &= -6, \\ 3p + 4ap - 2bp + 2cp - 4dp - 2fp &= 0, \\ 6q + 8aq - 4bq + 4cq - 8dq - 4fq &= 0. \end{aligned} \quad (271)$$

Combining the equations (259), (261), (262), (265), (266), (268), (270), and (271), we obtain a system of 21 equations for 11 variables x , p , q , u , v , w , a , b , c , d , and f . This system has several solutions, one of which agrees with the values produced by

the solver. This solution consists of algebraic numbers of degree 7; the minimal polynomials are given by

$$\begin{aligned}
 &2033478432x^7 - 1355652288x^6 + 360282816x^5 - 48215520x^4 + 3371688x^3 - 114528x^2 + 146x - 3 = 0, \\
 &21182067p^7 - 21182067p^6 + 7951095p^5 - 1388655p^4 + 116613p^3 - 5461p^2 + 285p - 9 = 0, \\
 &15441726843q^7 + 6291073899q^6 - 401977647q^5 - 261154287q^4 - 1979829q^3 + 565821q^2 + 448227q - 13633 = 0, \\
 &36602611776u^7 - 4066956864u^6 - 27226368u^5 + 30483216u^4 - 7018344u^3 + 38700u^2 + 35296u + 851 = 0, \\
 &36602611776v^7 + 8133913728v^6 - 272310336v^5 - 124092000v^4 + 649656v^3 + 492948v^2 - 14480v + 59 = 0, \\
 &7060689w^7 - 296802w^5 + 15552w^4 + 4138w^3 - 504w^2 - 36w + 2 = 0, \\
 &6912a^7 + 20736a^6 + 7168a^5 - 7008a^4 - 2000a^3 + 144a^2 - 56a + 51 = 0, \\
 &20736b^7 - 48384b^6 - 7296b^5 + 24736b^4 + 96b^3 - 1728b^2 - 432b + 81 = 0, \\
 &768c^7 - 2048c^6 - 4416c^5 + 160c^4 + 1104c^3 - 56c^2 - 12c + 9 = 0, \\
 &16384d^7 - 36864d^6 + 1024d^5 + 46848d^4 - 29504d^3 - 8112d^2 + 14508d - 4131 = 0, \\
 &576f^7 - 192f^6 - 224f^5 + 1104f^4 + 40f^3 - 276f^2 - 32f + 3 = 0.
 \end{aligned}$$

The indices of the roots of these polynomials read as

$$\begin{aligned}
 x^* : 4, \quad p^* : 2, \quad q^* : 4, \quad u^* : 5, \quad v^* : 4, \quad w^* : 5, \\
 a^* : 5, \quad b^* : 3, \quad c^* : 4, \quad d^* : 4, \quad f^* : 5.
 \end{aligned}$$

Different computer algebra systems might use different enumeration schemes for roots; these indices are valid in the *Mathematica* nomenclature. Standard arithmetic roots of real numbers are also definite roots of some simple polynomials, so this representation is no less analytical than an expression with explicit radicals.

It turns out that the numbers x^* , p^* , and q^* produce the minimum of the function

$$\begin{aligned}
 f(x, p, q) &= \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{str}_{12|3}(X^*, P^*), \\
 &\frac{1}{4}(1 + 6(p - q) - 4\sqrt{x}(\sqrt{2p} + \sqrt{3(p - q)} + \sqrt{p + q})).
 \end{aligned}$$

Because $p^* > q^*$, we can omit the term containing $p - q$ in the Lagrangian of this problem,

$$\mathcal{L} = f(x, p, q) + \nu(2x + p - \frac{1}{3}). \quad (272)$$

To prove the optimality of the point (x^*, p^*, q^*) , we need to show that it is a solution of the following system of equations:

$$\begin{aligned}
 2\frac{\partial \mathcal{L}}{\partial x} &= 4\nu^* - \sqrt{\frac{2p^*}{x^*}} + \sqrt{\frac{3(p^* - q^*)}{x^*}} + \sqrt{\frac{p^* + q^*}{x^*}} = 0, \\
 2\frac{\partial \mathcal{L}}{\partial p} &= 2\nu^* + 3 - \sqrt{\frac{2x^*}{p^*}} - \sqrt{\frac{3x^*}{p^* - q^*}} - \sqrt{\frac{x^*}{p^* + q^*}} = 0, \\
 2\frac{\partial \mathcal{L}}{\partial q} &= -3 + \sqrt{\frac{3x^*}{p^* - q^*}} - \sqrt{\frac{x^*}{p^* + q^*}} = 0.
 \end{aligned}$$

By computing the minimal polynomials, one can check that the numbers x^* , p^* , and q^* satisfy the equations

$$\frac{\partial \mathcal{L}}{\partial q} = 0, \quad \frac{\partial \mathcal{L}}{\partial x} - 2\frac{\partial \mathcal{L}}{\partial p} = 0, \quad (273)$$

and thus are a solution of the system above, where ν^* is the fifth root of the equation:

$$\begin{aligned}
 &576\nu^7 + 3264\nu^6 + 2368\nu^5 - 7728\nu^4 - 1744\nu^3 \\
 &+ 3060\nu^2 - 936\nu + 81 = 0.
 \end{aligned} \quad (274)$$

For $\lambda_{\min}^*$, we have

$$\lambda_{\min}^* = \frac{1}{4}(6x^* + 9p^* - 6q^* - 8u^* - 4w^*), \quad (275)$$

which is the first root of the equation

$$\begin{aligned}
 &119\,439\,36\lambda^7 - 434\,626\,56\lambda^6 + 549\,734\,40\lambda^5 - 28567296\lambda^4 \\
 &+ 571\,155\,2\lambda^3 - 119\,952\lambda^2 - 915\,56\lambda + 8163 = 0.
 \end{aligned} \quad (276)$$

We derive that

$$\mathcal{E}_{32}^+(\gamma_{xx}, \gamma_{pp}) = -\lambda_{\min}^* = 0.1202\dots, \quad (277)$$

and thus the state is genuine 3-partite entangled. Note that X and P are of different rank: $\text{rank}(X^*) = 2$, but $\text{rank}(P^*) = 3$.

2. Scaling

The primal optimal solution reads as

$$\begin{aligned}
 \gamma_{xx}^{12*} &= \gamma_{xx}^{13*} = \gamma_{xx}^{23*} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \gamma_{xx}^3 = \gamma_{xx}^2 = \gamma_{xx}^1 = (c), \\
 \gamma_{pp}^{12*} &= \gamma_{pp}^{13*} = \gamma_{pp}^{23*} = \begin{pmatrix} d & e \\ e & d \end{pmatrix}, \quad \gamma_{pp}^3 = \gamma_{pp}^2 = \gamma_{pp}^1 = (f),
 \end{aligned}$$

where the values of the parameters read as

$$\begin{aligned}
 a &= 0.5100\dots, \quad b = 0.4073\dots, \quad c = 0.4672\dots, \\
 d &= 1.3526\dots, \quad e = -1.0800\dots, \quad f = 0.5350\dots
 \end{aligned} \quad (278)$$

Two eigenvalues of the matrix $\mathcal{P}(\gamma_{xx}^{12*}, \gamma_{pp}^{12*})$ must be zero, which reduces to the following equations:

$$4(a + b)(d + e) = 1, \quad 4(a - b)(d - e) = 1. \quad (279)$$

Two eigenvalues of the matrix $\mathcal{P}(\gamma_{xx}^{3*}, \gamma_{pp}^{3*})$ must be zero, which reduces to the same equation as in the eigenvalue case

$$4cf = 1. \quad (280)$$

The dual solution has the same form as in the previous case:

$$X^* = \begin{pmatrix} 2x & -x & -x \\ -x & 2x & -x \\ -x & -x & 2x \end{pmatrix}, \quad P^* = \begin{pmatrix} p & q & q \\ q & p & p \\ q & q & p \end{pmatrix}. \quad (281)$$

The Z matrices read as

$$Z^{12*} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}, \quad Z^{3*} = (w). \quad (282)$$

The numerical values of all the parameters read as

$$\begin{aligned} x &= 0.2314 \dots, & p &= 0.4042 \dots, & q &= 0.3749 \dots, \\ u &= 0.2836 \dots, & v &= 0.1409 \dots, & w &= 0.4326 \dots \end{aligned} \quad (283)$$

Not surprisingly, we have the same equations for these parameters as in the eigenvalue case,

$$(u+v)^2 = x(p+q), \quad (u-v)^2 = 3x(p-q), \quad w^2 = 2xp. \quad (284)$$

The normalization condition is different and reads as

$$\text{Tr}(Z^{12*}) + \text{Tr}(Z^{3*}) = 2u + w = 1. \quad (285)$$

The KKT conditions (335) and (269) produce the following system of equations:

$$2x(2a-b) = u, \quad 2x(2b-a) = v,$$

$$\begin{aligned} 2(au+bv) &= p, & 2(bu+av) &= q, \\ du+ev &= x, & 2(eu+dv) &= -x, \\ 2(dp+eq) &= u, & 2(ep+dq) &= v, \end{aligned} \quad (286)$$

and

$$w = 4cx, \quad p = 2cw, \quad x = fw, \quad w = 2fp. \quad (287)$$

The KKT condition (138) in this case states that

$$\begin{aligned} X^* \left(\gamma_{xx} - \frac{t}{3} \gamma_{xx}^{12*} \oplus \gamma_{xx}^{3*} - \dots \right) &= 0, \\ P^* \left(\gamma_{pp} - \frac{t}{3} \gamma_{pp}^{12*} \oplus \gamma_{pp}^{3*} - \dots \right) &= 0, \end{aligned} \quad (288)$$

where the dots stand for the combination of the other two bipartitions and

$$t = \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) = \frac{3}{4}(2x + 3p - 2q). \quad (289)$$

These conditions produce the following additional equations:

$$\begin{aligned} (2a-b+c)(3p-2q+2x) &= 1, \\ (2q-3p-2x)(2dp+fp+2eq) &= 2q-3p, \\ (2q-3p-2x)(ep+2dq+eq+fq) &= p-2q. \end{aligned} \quad (290)$$

Combining the equations (279), (280), (284), (285), (286), (287), and (290), we obtain a system of 22 equations for 12 variables $a, b, c, d, e, f, x, p, q, u, v$, and w . This system has several solutions, one of which agrees with the values produced by the solver. The minimal polynomials of this solution read as

$$\begin{aligned} 14400a^8 - 2880a^6 - 396a^4 + 29a^2 + 4 &= 0, & 9216x^8 - 11968x^6 + 4224x^4 - 660x^2 + 25 &= 0, \\ 230400b^8 - 31680b^6 + 3024b^4 - 736b^2 + 9 &= 0, & 14400p^8 - 8635p^6 + 879p^4 + 18p^2 + 1 &= 0, \\ 384c^4 + 25600c^8 - 7360c^6 + 384c^4 - 4c^2 + 1 &= 0, & 368640q^8 - 3271360q^6 + 933744q^4 - 81300q^2 + 625 &= 0, \\ 1600d^8 - 4240d^6 + 2616d^4 - 427d^2 + 64 &= 0, & 192u^4 - 424u^3 + 324u^2 - 108u + 13 &= 0, \\ 64f^8 - 16f^6 + 96f^4 - 115f^2 + 25 &= 0, & 96v^4 + 88v^3 + 36v^2 - 1 &= 0, \\ 1600e^8 - 2640e^6 + 1056e^4 - 187e^2 + 9 &= 0, & 12w^4 + 5w^3 - 6w^2 + 3w - 1 &= 0. \end{aligned} \quad (291)$$

The indices of the roots of these polynomials read as

$$\begin{aligned} a^* : 4, & \quad b^* : 4, & c^* : 4, & \quad d^* : 4, & e^* : 1, & f^* : 3, \\ x^* : 3, & \quad p^* : 3, & q^* : 4, & \quad u^* : 1, & v^* : 2, & w^* : 2. \end{aligned}$$

It is possible to give explicit expression for these numbers in radicals; here, we show x^* , p^* , and q^* only:

$$\begin{aligned} x^{*2} &= \frac{1}{576} \left(187 + x_0 - \sqrt{28875 - x_0^2 + \frac{2281862}{x_0}} \right), & x_0^2 &= 9625 - 432(x_+ + x_-), \\ p^{*2} &= \frac{1}{11520} \left(1727 + p_0 - \sqrt{4897155 - p_0^2 + \frac{2828801662}{p_0}} \right), & p_0^2 &= 1632385 + 2304(p_+ + p_-), \\ q^{*2} &= \frac{1}{46080} \left(10223 - q_0 + \sqrt{44610915 - q_0^2 + \frac{72877799522}{q_0}} \right), & q_0^2 &= 14870305 + 13824(q_+ + q_-), \end{aligned} \quad (292)$$

where all square roots are understood in the arithmetic sense and

$$\begin{aligned} x_{\pm}^3 &= \frac{4873 \pm 395\sqrt{237}}{2}, \\ p_{\pm}^3 &= \frac{141\,323\,729 \pm 437\,411\,5\sqrt{237}}{2}, \\ q_{\pm}^3 &= \frac{321\,242\,976\,9 \pm 111\,739\,125\sqrt{237}}{2}, \end{aligned} \quad (293)$$

and here again, cubic roots are taken as real roots of real numbers. The optimal value

$$t^* = \frac{3}{4}(2x^* + 3p^* - 2q^*) \quad (294)$$

has minimal polynomial

$$1024t^8 - 11968t^6 + 38016t^4 - 53460t^2 + 18225 = 0 \quad (295)$$

and explicitly reads as

$$t^{*2} = \frac{1}{64} \left(187 + t_0 - \sqrt{28875 - t_0^2 + \frac{2281862}{t_0}} \right), \quad (296)$$

where

$$t_0^2 = 9625 - 432(t_+ + t_-), \quad t_{\pm}^3 = \frac{4873 \pm 395\sqrt{237}}{2}.$$

We derive that

$$\mathcal{T}_{32}^+(\gamma_{xx}, \gamma_{pp}) = 1 - t^* = 0.3056 \dots, \quad (297)$$

which proves that the state is genuine 3-partite entangled. Note that the values for x^* , p^* , q^* , and t^* were obtained in Ref. [18] in a purely numerical way and without any evidence of their optimality. Here, we constructed analytical expressions for these numbers together with an analytical validation of their optimality.

3. Refined condition

We first consider the case of $\mathcal{J}_2 = \{1|23, 2|13, 3|12\}$, so the second, larger, components are discarded and the smaller ones remain. The primal solution in this case is given by

$$\gamma^{1*} = \gamma^{2*} = \gamma^{3*} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (298)$$

The optimal M^* reads as

$$M^* = \begin{pmatrix} x & -y & -y & 0 & 0 & 0 \\ -y & x & -y & 0 & 0 & 0 \\ -y & -y & x & 0 & 0 & 0 \\ 0 & 0 & 0 & p & q & q \\ 0 & 0 & 0 & q & p & q \\ 0 & 0 & 0 & q & q & p \end{pmatrix}, \quad (299)$$

and the optimal K^* is given by

$$K^* = \begin{pmatrix} 0 & 0 & 0 & -u & -v & -v \\ 0 & 0 & 0 & -v & -u & -v \\ 0 & 0 & 0 & -v & -v & -u \\ u & v & v & 0 & 0 & 0 \\ v & u & v & 0 & 0 & 0 \\ v & v & u & 0 & 0 & 0 \end{pmatrix}. \quad (300)$$

The matrices K^{j*} read as

$$K^{1*} = K^{2*} = K^{3*} = x \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (301)$$

The numerical values of the parameters produced by the solver are where the numerical values of the parameters read as

$$\begin{aligned} x &= 0.4469 \dots, & y &= 0.1516 \dots, & p &= 0.4469 \dots, \\ q &= 0.3750 \dots, & u &= 0.2765 \dots, & v &= 0.0691 \dots \end{aligned} \quad (302)$$

The normalization condition (173) reads as

$$\frac{1}{2} \text{Tr}(K^{1*} \Omega_1) + \frac{1}{2} \text{Tr}(K^*[2, 3] \Omega_2) = 1, \quad (303)$$

and produces the equation

$$2u + w = 1. \quad (304)$$

The KKT condition (174) is

$$\mathcal{P}(\gamma^{1*})(M^*[1] + iK^{1*}) = 0, \quad (305)$$

which produces a single equation

$$x = p. \quad (306)$$

The KKT condition (175) reads as

$$\begin{aligned} (M^* - iK^*) \left(\gamma - \frac{t}{3} \gamma^{1*} \oplus \frac{i}{2} \Omega_2^{23} \right. \\ \left. - \frac{t}{3} \gamma^{2*} \oplus \frac{i}{2} \Omega_2^{13} - \frac{t}{3} \gamma^{3*} \oplus \frac{i}{2} \Omega_2^{12} \right) = 0, \end{aligned} \quad (307)$$

where

$$t = \text{Tr}(M^* \gamma) = \frac{3}{4}(2x - 2y + 3p - 2q). \quad (308)$$

This condition produces a system of equations, which contains

$$\begin{aligned} (2u + x)(2x - 2y + 3p - 2q) &= 4(x - y), \\ (2v - y)(2x - 2y + 3p - 2q) &= 2(x - 3y), \end{aligned} \quad (309)$$

and several others. Combining these equations with Eqs. (304) and (306) and solving the resulting system, we obtain the following solution:

$$\begin{aligned} x^* &= \frac{-73 + 20\sqrt{73}}{219}, & y^* &= \frac{73 + 7\sqrt{73}}{876}, \\ p^* &= \frac{-73 + 20\sqrt{73}}{219}, & q^* &= \frac{-73 + 47\sqrt{73}}{876}, \\ u^* &= \frac{2(73 - 5\sqrt{73})}{219}, & v^* &= \frac{73 - 5\sqrt{73}}{438}. \end{aligned}$$

The optimal t^* reads as

$$t^* = \frac{-5 + \sqrt{73}}{4} = 0.8860 \dots \quad (310)$$

Since $t^* < 1$, it shows that the state is genuine 3-partite entangled. Note that the matrix M^* is degenerate and $\text{rank}(M^*) = 3$.

Now, we consider the case of $\mathcal{I}_2 = \{23|1, 13|2, 12|3\}$, so the smaller components are discarded and the larger ones remain. The primal solution is given by

$$\gamma^{12*} = \gamma^{13*} = \gamma^{23*} = \begin{pmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & -d & c \end{pmatrix}. \quad (311)$$

The dual solution is of the form given by Eq. (299), and K^* reads as

$$K^* = \begin{pmatrix} 0 & 0 & 0 & -u & v & v \\ 0 & 0 & 0 & v & -u & v \\ 0 & 0 & 0 & v & v & -u \\ u & -v & -v & 0 & 0 & 0 \\ -v & u & -v & 0 & 0 & 0 \\ -v & -v & u & 0 & 0 & 0 \end{pmatrix}. \quad (312)$$

The matrices K^{jk*} are given by

$$K^{12*} = K^{13*} = K^{23*} = \begin{pmatrix} 0 & 0 & -z & -w \\ 0 & 0 & -w & -z \\ z & w & 0 & 0 \\ w & z & 0 & 0 \end{pmatrix}. \quad (313)$$

The numerical values of the parameters produced by the solver are

$$\begin{aligned} a &= 0.4484 \dots, & b &= 0.2522 \dots, & c &= 0.8154 \dots, \\ d &= 0.4586 \dots, & x &= 0.5416 \dots, & y &= 0.2443 \dots, \\ p &= 0.3525 \dots, & q &= 0.2314 \dots, & u &= 0.2748 \dots, \\ v &= 0.0335 \dots, & z &= 0.3625 \dots, & w &= 0.0541 \dots \end{aligned} \quad (314)$$

The normalization condition reads as

$$\frac{1}{2} \text{Tr}(K^{12*} \Omega_2) + \frac{1}{2} \text{Tr}(K^*[3] \Omega_1) = 1 \quad (315)$$

and produces the equation

$$u + 2z = 1. \quad (316)$$

The KKT condition (174) is

$$\mathcal{P}(\gamma^{12*})(M^*[1, 2] + iK^{12*}) = 0, \quad (317)$$

which produces the equations

$$\begin{aligned} 2(ax - by) &= z, & 2(bx - ay) &= w, \\ 2(az + bw) &= p, & 2(aw + bz) &= q, \\ 2(cz - dw) &= x, & 2(dz - cw) &= y, \\ 2(cp - dq) &= z, & 2(cq - dp) &= w. \end{aligned} \quad (318)$$

The KKT condition (175) reads as

$$\begin{aligned} (M^* - iK^*) \left(\gamma - \frac{t}{3} \gamma^{12*} \oplus \frac{i}{2} \Omega_1^3 \right. \\ \left. - \frac{t}{3} \gamma^{13*} \oplus \frac{i}{2} \Omega_1^2 - \frac{t}{3} \gamma^{23*} \oplus \frac{i}{2} \Omega_1^1 \right) = 0, \end{aligned} \quad (319)$$

where

$$t = \text{Tr}(M^* \gamma) = \frac{3}{4}(2x - 2y + 3p - 2q). \quad (320)$$

This relation produces several larger equations, not presented here. Combining these larger equations with Eqs. (316) and (318) and solving the resulting system, we obtain the following analytical expressions for the parameters:

$$\begin{aligned} a^* &= \frac{4}{5} \sqrt{\frac{11}{35}}, & b^* &= \frac{9}{20} \sqrt{\frac{11}{35}}, & c^* &= \frac{16}{\sqrt{385}}, \\ d^* &= \frac{9}{\sqrt{385}}, & x^* &= \frac{491}{231} \sqrt{\frac{5}{77}}, & y^* &= \frac{443}{462} \sqrt{\frac{5}{77}}, \\ p^* &= \frac{83}{12\sqrt{385}}, & q^* &= \frac{109}{24\sqrt{385}}, & u^* &= \frac{127}{462}, \\ v^* &= \frac{31}{924}, & z^* &= \frac{335}{924}, & w^* &= \frac{25}{462}. \end{aligned}$$

The optimal value t^* is given by

$$t^* = \frac{1}{2} \sqrt{\frac{35}{11}} = 0.8918 \dots \quad (321)$$

Since $t^* < 1$, it shows the state is genuine 3-partite entangled. Note that in this case M^* is also degenerate and $\text{rank}(M^*) = 3$.

D. Genuine 4-partite entanglement

Our last example is a little bit more complicated case of a 4-partite reduced state with the following CM:

$$\begin{aligned} \gamma_{xx} &= \begin{pmatrix} \alpha & \beta & \beta & 0 \\ \beta & \alpha & 0 & \beta \\ \beta & 0 & \alpha & -\beta \\ 0 & \beta & -\beta & \alpha \end{pmatrix}, \\ \gamma_{pp} &= \begin{pmatrix} \alpha & -\beta & -\beta & 0 \\ -\beta & \alpha & 0 & -\beta \\ -\beta & 0 & \alpha & \beta \\ 0 & -\beta & \beta & \alpha \end{pmatrix}, \end{aligned} \quad (322)$$

where the parameters α and β are given by

$$\begin{aligned} \alpha &= \frac{1}{4} \left(\frac{1}{2\sqrt[3]{2}} + 10\sqrt[3]{2} \right) = 3.2490 \dots, \\ \beta &= \frac{1}{4\sqrt[3]{2}} \left(\frac{1}{2\sqrt[3]{2}} - 10\sqrt[3]{2} \right) = -2.1570 \dots \end{aligned} \quad (323)$$

This is a mixed state. Such states occur in an intermediate step of a continuous-variable entanglement swapping, as introduced in Refs. [43,44]. This somewhat uncommon choice of parameters was proposed in Ref. [18]. As usual, we construct the solutions of the two optimization problems.

1. Eigenvalue

The optimal matrices produced by the solver have the following form:

$$X^* = \frac{1}{8} \begin{pmatrix} 1 & 8x & 8x & 0 \\ 8x & 1 & 0 & 8x \\ 8x & 0 & 1 & -8x \\ 0 & 8x & -8x & 1 \end{pmatrix},$$

$$P^* = \frac{1}{8} \begin{pmatrix} 1 & -8x & -8x & 0 \\ -8x & 1 & 0 & -8x \\ -8x & 0 & 1 & 8x \\ 0 & -8x & 8x & 1 \end{pmatrix}. \quad (324)$$

The primal numerical solution shows that all 2×2 partitions are inactive, and all 1×3 partitions are active, so we show only Z matrices of the latter partitions:

$$Z^{123*} = \begin{pmatrix} u & 0 & 0 \\ 0 & v & -w \\ 0 & -w & v \end{pmatrix}, \quad Z^{124*} = \begin{pmatrix} v & 0 & -w \\ 0 & u & 0 \\ -w & 0 & v \end{pmatrix},$$

$$Z^{134*} = \begin{pmatrix} v & 0 & w \\ 0 & u & 0 \\ w & 0 & v \end{pmatrix}, \quad Z^{234*} = \begin{pmatrix} v & w & 0 \\ w & v & 0 \\ 0 & 0 & u \end{pmatrix}.$$

All single-partite Z matrices are equal:

$$Z^{1*} = Z^{2*} = Z^{3*} = Z^{4*} = \frac{1}{8}(1). \quad (325)$$

The numerical values of the parameters of these matrices read as

$$\begin{aligned} x &= 0.0880 \dots, & u &= 0.0102 \dots, \\ v &= 0.0676 \dots, & w &= 0.0573 \dots \end{aligned} \quad (326)$$

Three eigenvalues of the matrix

$$\begin{pmatrix} X^*[1, 2, 3] & Z^{123*} \\ Z^{123*T} & P^*[1, 2, 3] \end{pmatrix} \quad (327)$$

must be zero, so we have three equations

$$8(v + w) = 1, \quad u + w = v, \quad 64(2x^2 + u^2) = 1. \quad (328)$$

The primal objective value reads as

$$\lambda_{\min} = 16x\beta + \alpha - u - 2v - \frac{1}{8}. \quad (329)$$

The dual solution is given by

$$\gamma_{xx}^{123*} = \begin{pmatrix} b & -c & -c \\ -c & a & d \\ -c & d & a \end{pmatrix}, \quad \gamma_{pp}^{123*} = \begin{pmatrix} b & c & c \\ c & a & d \\ c & d & a \end{pmatrix},$$

$$\gamma_{xx}^{124*} = \begin{pmatrix} a & -c & d \\ -c & b & -c \\ d & -c & a \end{pmatrix}, \quad \gamma_{pp}^{124*} = \begin{pmatrix} a & c & d \\ c & b & c \\ d & c & a \end{pmatrix},$$

$$\gamma_{xx}^{134*} = \begin{pmatrix} a & -c & -d \\ -c & b & c \\ -d & c & a \end{pmatrix}, \quad \gamma_{pp}^{134*} = \begin{pmatrix} a & c & -d \\ c & b & -c \\ -d & -c & a \end{pmatrix},$$

$$\gamma_{xx}^{234*} = \begin{pmatrix} a & -d & -c \\ -d & a & c \\ -c & c & b \end{pmatrix}, \quad \gamma_{pp}^{234*} = \begin{pmatrix} a & -d & c \\ -d & a & -c \\ c & -c & b \end{pmatrix}, \quad (330)$$

where the numerical values of the parameters read as

$$\begin{aligned} a &= 3.3107 \dots, & b &= 6.1215 \dots, \\ c &= 4.3141 \dots, & d &= 2.81073 \dots \end{aligned} \quad (331)$$

The single-partite matrices γ_{xx}^{i*} and γ_{pp}^{i*} are all equal to the CM of the vacuum state, $(1/2)$, for $i = 1, \dots, 4$. The 2×2 partitions do not participate in the optimal solution and the 1×3 partitions have equal contribution to it,

$$\begin{aligned} P_{12|34}^* &= P_{13|24}^* = P_{14|23}^* = 0, \\ P_{1|234}^* &= P_{2|134}^* = P_{3|124}^* = P_{4|123}^* = \frac{1}{4}. \end{aligned} \quad (332)$$

The KKT conditions that the matrices

$$\mathcal{P}(\gamma_{xx}^{123*}, \gamma_{pp}^{123*}), \quad \mathcal{P}(\gamma_{xx}^{4*}, \gamma_{pp}^{4*}) \quad (333)$$

have 3 and 1 zero eigenvalues, respectively, give the following equations:

$$2(a - d) = 1, \quad a + d = b, \quad 4(b^2 - 2c^2) = 1. \quad (334)$$

The KKT condition

$$\mathcal{P}(\gamma_{xx}^{123*}, \gamma_{pp}^{123*}) \begin{pmatrix} X^*[1, 2, 3] & -Z^{123*} \\ -Z^{123*T} & P^*[1, 2, 3] \end{pmatrix} = 0 \quad (335)$$

produces the equations

$$\begin{aligned} 4(4cx + u) &= b, & 8bx &= c, & 16bu &= 1, \\ 2c(v - w) &= x, & 4(2cx + v) &= a, & d + 4w &= 8cx, \\ 2cu &= x, & 16(av - dw) &= 1, & dv - aw &= 0. \end{aligned} \quad (336)$$

The KKT condition

$$\mathcal{P}(\gamma_{xx}^{4*}, \gamma_{pp}^{4*}) \begin{pmatrix} X^*[4] & -Z^{4*} \\ -Z^{4*T} & P^*[4] \end{pmatrix} = 0 \quad (337)$$

is already satisfied, so it gives no extra equations. Due to some kind of symmetry, the other three states give exactly the same equations. The KKT condition

$$X^*(\gamma_{xx} - \frac{1}{4}\gamma_{xx}^{123*} \oplus \gamma_{xx}^{4*} - \dots) = \lambda_{\min} X^*, \quad (338)$$

where dots denote the sum of the other three 1×3 combinations, produces the following two equations:

$$\begin{aligned} 4(u + 2v + 8cx) &= 2a + b, \\ 4x(2a + b + 64x\beta) &= c + 2(8ux + 16vx + \beta). \end{aligned} \quad (339)$$

The similar condition with P^* and pp parts gives exactly the same two equations again. The combined system of equations given by Eqs. (328), (334), (336), and (339) has two solutions, one of which numerically coincides with the numbers produced by the solver. The explicit expressions are simpler than in the previous examples, the primal solution reads as

$$\begin{aligned} x^{*2} &= \frac{1023971841 - 5120288\sqrt{2} - 230404\sqrt{4}}{64 \times 2047972354}, \\ u^* &= \frac{2\sqrt{2}}{31999}(40 + \sqrt{2} + 800\sqrt{4})x^*, \\ v^* &= \frac{\sqrt{2}}{31999}(40 + \sqrt{2} + 800\sqrt{4})x^* + \frac{1}{16}. \end{aligned} \quad (340)$$

The dual solution is given by

$$\begin{aligned} b^{*2} &= \frac{1}{16} \left(400\sqrt[3]{4} + \frac{1}{\sqrt[3]{4}} - 36 \right), \\ a^* &= \frac{2b^* + 1}{4}, \quad d^* = \frac{2b^* - 1}{4}, \\ c^* &= \frac{1}{\sqrt[6]{2}} \left(5 - \frac{1}{4\sqrt[3]{4}} \right). \end{aligned} \quad (341)$$

The optimal value reads as

$$-16\lambda_{\min}^* = \sqrt{2} \sqrt{800\sqrt[3]{4} + \sqrt[3]{2} - 72} - \sqrt[3]{4} - 40\sqrt[3]{2} + 4.$$

We also need to check that the 2×2 partitions are indeed inactive. Due to the equality

$$\text{str}_{12|34}(X^*, P^*) = \text{str}_{13|24}(X^*, P^*), \quad (342)$$

we need to check only two of the three quantities. We have

$$\begin{aligned} \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{str}_{12|34}(X^*, P^*) \\ = \alpha + 16\beta x^* - \frac{1}{2} \sqrt{1 - 64x^{*2}}, \\ \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{str}_{14|23}(X^*, P^*) \\ = \alpha + 16\beta x^* - \frac{1}{2}. \end{aligned} \quad (343)$$

Mathematica can symbolically prove that these two quantities are smaller than $\lambda_{\min}^*$, and thus the 2×2 partitions are indeed inactive. One can prove this fact “by hand” by squaring and subtracting the left-hand side from the right-hand side several times, but this task is what Computer Algebra System (CASs) are good at. We arrive at the following result:

$$\mathcal{E}_{32}^+(\gamma_{xx}, \gamma_{pp}) = -\lambda_{\min}^* = 0.0618 \dots, \quad (344)$$

which verifies that the state is genuine multipartite entangled. Note that the matrices X^* and P^* are nondegenerate and thus have rank 4.

2. Scaling

The primal solution has exactly the same form as given by Eq. (330), but with different values for the parameters

$$\begin{aligned} a &= 4.2229 \dots, \quad b = 7.9458 \dots, \\ c &= 5.6074 \dots, \quad d = 3.7229 \dots, \end{aligned} \quad (345)$$

so the first set of equations is given by Eq. (334). The dual solutions read as

$$\begin{aligned} X^* &= \begin{pmatrix} x & y & y & 0 \\ y & x & 0 & y \\ y & 0 & x & -y \\ 0 & y & -y & x \end{pmatrix}, \\ P^* &= \begin{pmatrix} x & -y & -y & 0 \\ -y & x & 0 & -y \\ -y & 0 & x & y \\ 0 & -y & y & x \end{pmatrix}, \end{aligned} \quad (346)$$

where the parameters are given by

$$x = 0.4704 \dots, \quad y = 0.3319 \dots \quad (347)$$

The Z matrices read as

$$\begin{aligned} Z^{123*} &= \frac{1}{4} \begin{pmatrix} 4u & 0 & 0 \\ 0 & 1 & -4v \\ 0 & -4v & 1 \end{pmatrix}, \\ Z^{124*} &= \frac{1}{4} \begin{pmatrix} 1 & 0 & -4v \\ 0 & 4u & 0 \\ -4v & 0 & 1 \end{pmatrix}, \\ Z^{134*} &= \frac{1}{4} \begin{pmatrix} 1 & 0 & 4v \\ 0 & 4u & 0 \\ 4v & 0 & 1 \end{pmatrix}, \\ Z^{123*} &= \frac{1}{4} \begin{pmatrix} 1 & 4v & 0 \\ 4v & 1 & 0 \\ 0 & 0 & 4u \end{pmatrix}, \end{aligned} \quad (348)$$

and

$$Z^{1*} = Z^{2*} = Z^{3*} = Z^{4*} = (x), \quad (349)$$

where the values of the parameters are

$$u = 0.0295 \dots, \quad v = 0.2204 \dots \quad (350)$$

The matrices Z^{ij*} are all the same and read as

$$Z^{ij*} = \frac{1}{4} E_2. \quad (351)$$

Clearly, we have

$$\text{Tr} Z^{12*} + \text{Tr} Z^{34*} = 1, \quad (352)$$

and two other similar equalities for $2 + 2$ bipartitions. The condition

$$\text{Tr} Z^{123*} + \text{Tr} Z^{4*} = 1 \quad (353)$$

and similar for $1 + 3$ bipartitions produce the equation

$$u + x + \frac{1}{2} = 1. \quad (354)$$

The condition that the matrix

$$\begin{pmatrix} X^*[1, 2, 3] & Z^{123*} \\ Z^{123*T} & P^*[1, 2, 3] \end{pmatrix} \quad (355)$$

has three zero eigenvalues produces the equations

$$\begin{aligned} 4(x - v) &= 1, \quad 4(u + v) = 1, \\ x - 2u - 4vx + 4x^2 - 8y^2 &= 0. \end{aligned} \quad (356)$$

The other three 1×3 partitions produce identical equations. The KKT condition (335) gives the following equations:

$$\begin{aligned} 2(bx - 2cy) &= u, \quad cx - by = 0, \\ 2(2cv + y) &= c, \quad 2bu - x = 0, \\ 8(ax - cy) &= 1, \quad (a + d)y = cx, \\ 2(cy - dx) &= v, \quad 2cu - y = 0, \\ 2(2dv + x) &= a, \quad 4av - d = 0. \end{aligned} \quad (357)$$

The KKT condition (269) is already satisfied. The KKT condition (138) in this case reads as

$$\begin{aligned} X^* \left(\gamma_{xx} - \frac{t}{4} \gamma_{xx}^{123*} \oplus \gamma_{xx}^{4*} - \dots \right) &= 0, \\ P^* \left(\gamma_{pp} - \frac{t}{4} \gamma_{pp}^{123*} \oplus \gamma_{pp}^{4*} - \dots \right) &= 0, \end{aligned}$$

where the dots stand for the other three 1×3 partitions and

$$t = \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) = 8(\alpha x + 2\beta y). \quad (358)$$

These conditions produce the following additional equations:

$$\begin{aligned} 4ax + 2bx - 8cy + x - 1 &= 0, \\ (2(1 + 4a + 2b)y^2 - x - 8cxy)\beta &= (8cy^2 - 4cx^2)\alpha. \end{aligned}$$

These equations, together with equations (334), (354), (356), and (357), form a system that has a solution which numerically agrees with the values produced by the solver. The primal solution reads as

$$\begin{aligned} x^* &= \frac{341290667 - 17065600\sqrt[3]{2} + 853320\sqrt[3]{4}}{682709334}, \\ y^* &= \sqrt[6]{2} \frac{80 + 31999\sqrt[3]{2} - 1600\sqrt[3]{4}}{128004}, \\ u^* &= 20 \frac{1600 + 426640\sqrt[3]{2} - 21333\sqrt[3]{4}}{341354667}, \\ v^* &= \frac{341226667 - 34131200\sqrt[3]{2} + 1706640\sqrt[3]{4}}{1365418668}. \end{aligned} \quad (359)$$

The dual solution is simpler

$$\begin{aligned} a^* &= \frac{80 + \sqrt[3]{2} + 800\sqrt[3]{4}}{320}, \\ b^* &= 5\sqrt[3]{2} + \frac{1}{80\sqrt[3]{2}}, \quad c^* = 5\sqrt[3]{2} - \frac{1}{160\sqrt[3]{2}}, \\ d^* &= \frac{-80 + \sqrt[3]{2} + 800\sqrt[3]{4}}{320}. \end{aligned} \quad (360)$$

The optimal scaling factor t^* is equal to

$$t^* = u^* + x^* + \frac{1}{2} = 20 \frac{-40 + \sqrt[3]{2} + 800\sqrt[3]{4}}{32001} = 0.7694 \dots$$

We arrive at the following result:

$$\mathcal{T}_{32}^+(\gamma_{xx}, \gamma_{pp}) = 1 - t_{\min}^* = 0.2305 \dots, \quad (361)$$

which verifies that the state is genuine multipartite entangled. The values for x^* , y^* , and t^* were obtained in Ref. [18] in a purely numerical way and without any evidence of their optimality. Here, we again constructed analytical expressions for these numbers together with an analytical validation of their optimality. Note that the matrices X^* and P^* are non-degenerate and thus have rank 4.

E. Parametric solution

The state with the CM given by Eq. (322) can, in fact, be solved for arbitrary α and β , not only for the values given by Eq. (323). The eigenvalues of $\mathcal{P}(\gamma_{xx}, \gamma_{pp})$ are given by

$$\alpha \pm \frac{1}{2} \sqrt{1 + 8\beta^2}, \quad (362)$$

with multiplication 4, so this CM is physical iff

$$\beta^2 \leq \frac{1}{2} \left(\alpha^2 - \frac{1}{4} \right). \quad (363)$$

The condition $\alpha \geq 1/2$ is then automatically satisfied. This state is pure exactly when the inequality (363) becomes equality. For $\alpha = 1/2$, the only possible value for β is zero and this CM is then the CM of the four-mode vacuum state. We derive an entanglement condition on α and β for the general state. As in the previous parts for concrete states, we construct an analytical solution of both problems.

1. Eigenvalues

All the machinery of constructing and solving the system of KKT equations has already been developed before, so here we just present the final result. The primal solution reads as

$$\begin{aligned} X^* &= \frac{1}{8} \begin{pmatrix} 1 & 8x^* & 8x^* & 0 \\ 8x^* & 1 & 0 & 8x^* \\ 8x^* & 0 & 1 & -8x^* \\ 0 & 8x^* & -8x^* & 1 \end{pmatrix}, \\ P^* &= \frac{1}{8} \begin{pmatrix} 1 & -8x^* & -8x^* & 0 \\ -8x^* & 1 & 0 & -8x^* \\ -8x^* & 0 & 1 & 8x^* \\ 0 & -8x^* & 8x^* & 1 \end{pmatrix}, \end{aligned} \quad (364)$$

where x^* is given by

$$x^* = -\frac{\beta}{2\sqrt{1 + 32\beta^2}}. \quad (365)$$

The eigenvalues of these matrices are the same and are given by

$$\frac{1}{8} \pm \frac{\beta}{\sqrt{2(1 + 32\beta^2)}} > 0, \quad (366)$$

with multiplicity 2, so X^* and P^* are positive definite and obviously satisfy the normalization condition $\text{Tr}(X^* + P^*) = 1$.

In the dual solution, only the 1×3 bipartitions are active, so that

$$P_{123|4}^* = P_{124|3}^* = P_{134|2}^* = P_{234|1}^* = \frac{1}{4}, \quad (367)$$

and coefficients of the other bipartitions are zero. The states corresponding to nonzero coefficients read as

$$\gamma_{xx}^{123*} = \begin{pmatrix} b^* & -c^* & -c^* \\ -c^* & a^* & d^* \\ -c^* & d^* & a^* \end{pmatrix},$$

$$\gamma_{pp}^{123*} = \begin{pmatrix} b^* & c^* & c^* \\ c^* & a^* & d^* \\ c^* & d^* & a^* \end{pmatrix},$$

$$\begin{aligned}
 \gamma_{xx}^{124*} &= \begin{pmatrix} a^* & -c^* & d^* \\ -c^* & b^* & -c^* \\ d^* & -c^* & a^* \end{pmatrix}, & \gamma_{pp}^{124*} &= \begin{pmatrix} a^* & c^* & d^* \\ c^* & b^* & c^* \\ d^* & c^* & a^* \end{pmatrix}, \\
 \gamma_{xx}^{134*} &= \begin{pmatrix} a^* & -c^* & -d^* \\ -c^* & b^* & c^* \\ -d^* & c^* & a^* \end{pmatrix}, & \gamma_{pp}^{134*} &= \begin{pmatrix} a^* & c^* & -d^* \\ c^* & b^* & -c^* \\ -d^* & -c^* & a^* \end{pmatrix}, \\
 \gamma_{xx}^{234*} &= \begin{pmatrix} a^* & -d^* & -c^* \\ -d^* & a^* & c^* \\ -c^* & c^* & b^* \end{pmatrix}, & \gamma_{pp}^{234*} &= \begin{pmatrix} a^* & -d^* & c^* \\ -d^* & a^* & -c^* \\ c^* & -c^* & b^* \end{pmatrix},
 \end{aligned} \tag{368}$$

where the parameters are

$$\begin{aligned}
 a^* &= \frac{1}{4}(\sqrt{1+32\beta^2} + 1), & b^* &= \frac{1}{2}\sqrt{1+32\beta^2}, \\
 c^* &= -2\beta, & d^* &= \frac{1}{4}(\sqrt{1+32\beta^2} - 1).
 \end{aligned} \tag{369}$$

The single-partite matrices γ_{xx}^{i*} and γ_{pp}^{i*} are all equal to the CM of the vacuum state. Eigenvalues of $\mathcal{P}(\gamma_{xx}^{123*}, \gamma_{pp}^{123*})$ and of the other three 1×3 partitions read as

$$(0, 0, 0, 1, \sqrt{1+32\beta^2}, \sqrt{1+32\beta^2}), \tag{370}$$

so these are physical CMs.

It is easy to see that

$$\begin{aligned}
 \lambda_{\min}^* &\equiv \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{str}(X^*[1, 2, 3], P^*[1, 2, 3]) \\
 &\quad - \text{str}(X^*[4], P^*[4]) = -\frac{1}{4}(1 - 4\alpha + \sqrt{1+32\beta^2}),
 \end{aligned}$$

and the other three 1×3 bipartitions give the same value. For the 2×2 bipartitions, we have different values. For 12|34, we have

$$\begin{aligned}
 &\text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{str}(X^*[1, 2], P^*[1, 2]) \\
 &\quad - \text{str}(X^*[3, 4], P^*[3, 4]) \\
 &= \alpha - \frac{16\beta^2 + \sqrt{1+16\beta^2}}{2\sqrt{1+32\beta^2}} < \lambda_{\min}^*.
 \end{aligned}$$

The case of partition 13|24 is identical to 12|34, and for 14|23 we have

$$\begin{aligned}
 &\text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - \text{str}(X^*[1, 4], P^*[1, 4]) \\
 &\quad - \text{str}(X^*[2, 3], P^*[2, 3]) \\
 &= -\frac{1}{2} \left(1 - 2\alpha + \frac{16\beta^2}{\sqrt{1+32\beta^2}} \right) < \lambda_{\min}^*.
 \end{aligned}$$

We see that $\lambda_{\min}^*$ is the maximal value among the values obtained with all bipartitions. On the other hand, for the dual solution we have

$$\begin{aligned}
 \gamma_{xx} &- \frac{1}{4} \gamma_{xx}^{123*} \oplus \gamma_{xx}^{4*} - \dots = \lambda_{\min}^* E_4, \\
 \gamma_{pp} &- \frac{1}{4} \gamma_{pp}^{123*} \oplus \gamma_{pp}^{4*} - \dots = \lambda_{\min}^* E_4,
 \end{aligned} \tag{371}$$

where the dots stand for the combinations corresponding to the rest of 1×3 bipartitions. These equalities demonstrate strong duality between the constructed primal and dual solutions, verifying that they are indeed optimal.

For the entanglement measure, we have

$$\mathcal{E}_{\mathcal{J}_2}(\gamma) = -\lambda_{\min}^* = \frac{1}{4}(1 - 4\alpha + \sqrt{1+32\beta^2}). \tag{372}$$

Combining the condition $\mathcal{E}^+(\gamma) > 0$ with the physicality condition (363), we derive the following condition for the entanglement:

$$\frac{1}{4} \leq \alpha^2 - 2\beta^2 < \frac{\alpha}{2}. \tag{373}$$

The larger β^2 , the larger the entanglement measure, and for the maximal allowed value given by Eq. (363) we have

$$\mathcal{E}_{\mathcal{J}_2}(\gamma) = \frac{1}{4}(1 - 4\alpha + \sqrt{16\alpha^2 - 3}). \tag{374}$$

The plot of this quantity as a function of α is shown in Fig. 12. We see that this pure state is entangled for all $\alpha > 1/2$ and its entanglement measure tends to $1/4$ when $\alpha \rightarrow +\infty$. Note that the matrices X^* and P^* are always nondegenerate and thus have rank 4.

2. Scaling

The primal solution in this case has the form of Eq. (368), but the parameters are different and read as

$$\begin{aligned}
 a^* &= \frac{\alpha^2}{2(\alpha^2 - 2\beta^2)}, & b^* &= \frac{\alpha^2 + 2\beta^2}{2(\alpha^2 - 2\beta^2)}, \\
 c^* &= -\frac{\alpha\beta}{\alpha^2 - 2\beta^2}, & d^* &= \frac{\beta^2}{\alpha^2 - 2\beta^2}.
 \end{aligned} \tag{375}$$

Note that from the physicality of the state, the first inequality of Eq. (373), it follows that the denominators in these expressions are strictly positive. The dual solution is of the form

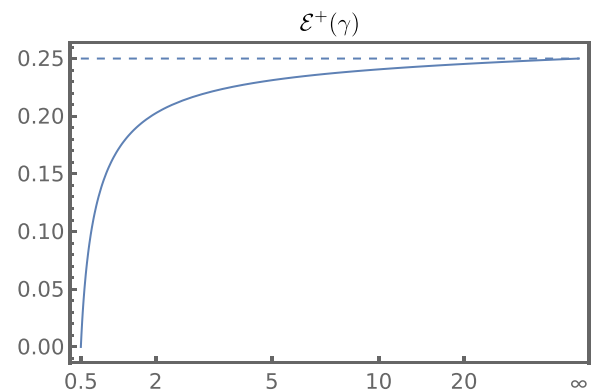


FIG. 12. $\mathcal{E}^+$ of the pure state (322) as a function of α with β given by the right-hand side of Eq. (363).

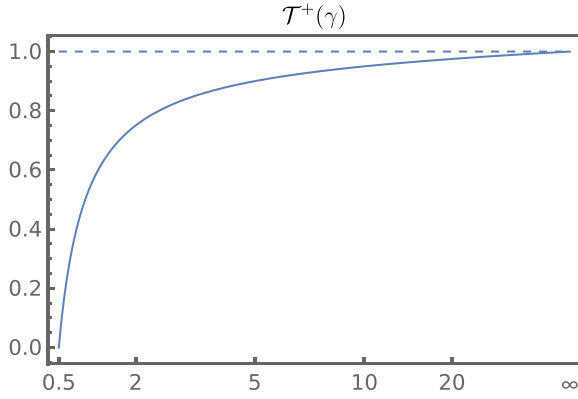


FIG. 13. $\mathcal{T}^+$ of the pure state (322) as a function of α with β given by the right-hand side of Eq. (363).

given by Eq. (346) where the parameters read as

$$x^* = \frac{\beta^2}{2\alpha^2} + \frac{1}{4}, \quad y^* = -\frac{\beta}{2\alpha}. \quad (376)$$

The dual solution is given by Eqs. (348), (349), and (351), where

$$u^* = \frac{1}{4} - \frac{\beta^2}{2\alpha^2}, \quad v^* = \frac{\beta^2}{2\alpha^2}. \quad (377)$$

The optimal scaling parameter is given by

$$t^* = \text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) = 2 \frac{\alpha^2 - 2\beta^2}{\alpha}. \quad (378)$$

Simple algebraic manipulations show that the entanglement condition $t^* < 1$, together with the physicality condition, again produce Eq. (373). For the measure $\mathcal{T}^+$, we have

$$\mathcal{T}_{\mathcal{J}_2}(\gamma) = 1 - t^* = 1 - 2\alpha + 4 \frac{\beta^2}{\alpha}. \quad (379)$$

This measure is maximal when the state is pure and is given by

$$\mathcal{T}_{\mathcal{J}_2}(\gamma) = 1 - \frac{1}{2\alpha}. \quad (380)$$

The plot of this quantity is shown in Fig. 13 and looks very similar to the plot of $\mathcal{E}_{\mathcal{J}_2}(\gamma)$ for the same state (Fig. 12).

We now show that the refined condition (168) is not necessary and sufficient when $|\mathcal{J}| > 1$. As a counter example, we take the CM (322) and $\mathcal{J} = \{1|234, 2|134, 3|124, 4|123\}$. As single-mode states, we take the vacuum

$$\gamma^1 = \gamma^2 = \gamma^3 = \gamma^4 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (381)$$

and as the coefficients we take $p_1 = p_2 = p_3 = p_4 = 1/4$. We then have

$$\gamma_0 = p_1 \gamma^1 \oplus \frac{i}{2} \Omega_3^{234} + \dots + p_4 \gamma^4 \oplus \frac{i}{2} \Omega_3^{123}$$

$$= \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 & 0 & 3i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3i \\ -3i & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3i & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3i & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3i & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (382)$$

The eigenvalues of the difference are

$$\lambda_{\pm} = \frac{1}{8}(8\alpha - 1 \pm \sqrt{128\beta^2 + 9}), \quad (383)$$

with multiplicity 4. It follows that

$$\gamma \succcurlyeq \gamma_0 \Leftrightarrow \lambda_- \geq 0 \Leftrightarrow \beta^2 \leq \frac{\alpha^2}{2} - \frac{2\alpha + 1}{16}. \quad (384)$$

We thus derived three conditions: The condition for physicality of γ reads as

$$\beta^2 \leq \frac{\alpha^2}{2} - \frac{1}{8}, \quad (385)$$

the condition for biseparability of γ is given by

$$\beta^2 \leq \frac{\alpha^2}{2} - \frac{\alpha}{4}, \quad (386)$$

and the condition for the inequality $\gamma \succcurlyeq \gamma_0$ is

$$\beta^2 \leq \frac{\alpha^2}{2} - \frac{2\alpha + 1}{16}. \quad (387)$$

In addition, one can also derive the conditions for full 1|2|3|4 separability and for 12|34 separability. We present here only the final results: the state is fully separable iff

$$|\beta| \leq \frac{1}{\sqrt{2}} \left(\alpha - \frac{1}{2} \right), \quad (388)$$

and it is 12|34-separable iff

$$|\beta| \leq \frac{1}{4}(\sqrt{8\alpha^2 - 1} - 1). \quad (389)$$

We also add the condition $\gamma \succcurlyeq 0$ here:

$$|\beta| \leq \frac{\alpha}{\sqrt{2}}. \quad (390)$$

These six regions are shown in Fig. 14 (a). The “strip” between the blue and the green lines is the domain where γ is genuine multipartite entangled. We emphasize that this figure is not schematic like a Venn diagram, which shows different sets for pure illustrative purpose only; it shows real sets of different kinds of separability. Figures 1 and 6 are also not purely abstract and correspond to a realistic situation. If we work with states of the form given by Eq. (322), then the former figure illustrates finding the best physical point

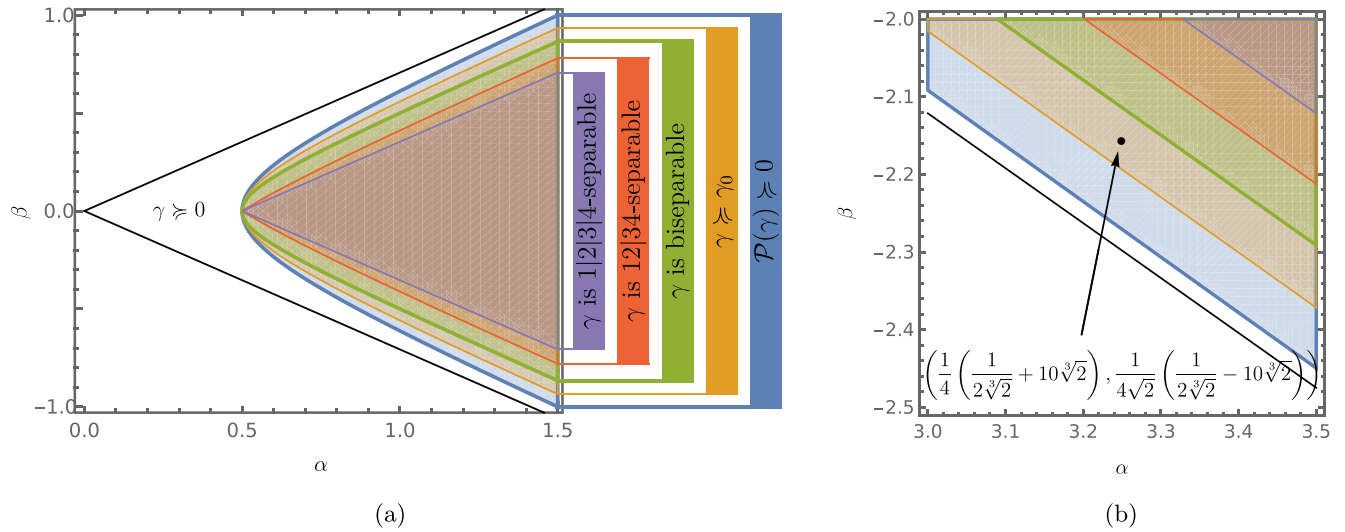


FIG. 14. The physicality region and the regions of various separability properties of the state given by Eq. (322). (a) The regions defined by Eqs. (385)–(390). (b) The location of the point given by Eq. (323).

(α^*, β^*) corresponding to a given nonphysical one $(\alpha^\circ, \beta^\circ)$, and the latter figure illustrates the fully separable state closest to a given state.

As Fig. 14 (a) illustrates, this domain intersects with the region where the inequality $\gamma \geq \gamma_0$ holds true, the intersection being the “strip” between the yellow and the green lines. The point given by Eq. (323) is located exactly in this “gray” zone. Consequently, this optimization problem does not say that the corresponding CM is entangled:

$$t^* = 1.2847 \dots > 1. \quad (391)$$

In the domain where this problem shows a violation of this separability condition, the “strip” between the blue and the yellow lines, the state is definitely genuine entangled.

F. Tripartite symmetric Gaussian and non-Gaussian states

Here, we consider a reduced tripartite state with the following CM:

$$\gamma_{xx} = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}, \quad \gamma_{pp} = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}. \quad (392)$$

This matrix is positive semidefinite, $\gamma \geq 0$, iff

$$|\beta| \leq \frac{\alpha}{2}, \quad (393)$$

and it is a physical CM, provided that

$$|\beta| \leq \frac{1}{4} \sqrt{4\alpha^2 - 1}. \quad (394)$$

This covariance matrix is symmetric, so it is enough to consider only one bipartition, for example, 1|23, and any result will be equally valid for the other two bipartitions. Below, we will also consider combinations of two bipartitions, and it is enough to study only one combination, say {1|23, 2|13}. Here, we present just the final results obtained by solving the eigenvalue optimization problem with the approach presented in details in the previous sections. Changing $\beta \rightarrow -\beta$ just

exchanges $\gamma_{xx} \leftrightarrow \gamma_{pp}$, so it suffices to consider just the case of $\beta \geq 0$. Because the presentation of the results is too short to put them into separate subsections, we use lemmas to clearly separate where each individual result starts and ends.

Our first result is as follows.

Lemma 3. The Gaussian state with the CM given by Eq. (392) is 1|2|3-separable iff the parameters α and β satisfy the relation

$$3|\beta| + \sqrt{\beta^2 + 1} \leq 2\alpha. \quad (395)$$

Proof. If $\beta \geq 0$ then the optimal matrices X^* and P^* read as

$$X^* = \begin{pmatrix} x^* & y^* & y^* \\ y^* & x^* & y^* \\ y^* & y^* & x^* \end{pmatrix}, \quad P^* = \begin{pmatrix} p^* & p^* & p^* \\ p^* & p^* & p^* \\ p^* & p^* & p^* \end{pmatrix}, \quad (396)$$

where

$$\begin{aligned} x^* &= \frac{1}{6} \left(1 - \frac{|\beta|}{\sqrt{\beta^2 + 1}} \right), \\ y^* &= -\frac{1}{12} \left(1 - \frac{|\beta|}{\sqrt{\beta^2 + 1}} \right), \\ p^* &= \frac{1}{6} \left(1 + \frac{|\beta|}{\sqrt{\beta^2 + 1}} \right). \end{aligned} \quad (397)$$

If $\beta < 0$, then X^* and P^* are swapped. The eigenvalues of X^* are 0 and

$$\frac{1}{4} \left(1 - \frac{|\beta|}{\sqrt{\beta^2 + 1}} \right) > 0, \quad (398)$$

with multiplicity 2. The eigenvalues of P^* are 0 with multiplicity 2 and

$$\frac{1}{2} \left(1 + \frac{|\beta|}{\sqrt{\beta^2 + 1}} \right) > 0, \quad (399)$$

so $X^*, P^* \geq 0$.

The optimal dual solution reads as

$$\begin{aligned}\gamma_{xx}^{1*} &= \gamma_{xx}^{2*} = \gamma_{xx}^{3*} = (a^*), \\ \gamma_{pp}^{1*} &= \gamma_{pp}^{2*} = \gamma_{pp}^{3*} = (b^*),\end{aligned}\quad (400)$$

where

$$a^* = \frac{1}{2}(\sqrt{\beta^2 + 1} + \beta), \quad b^* = \frac{1}{2}(\sqrt{\beta^2 + 1} - \beta). \quad (401)$$

These states are single-mode squeezed vacuum states with the degree of squeezing r given by

$$r = \frac{1}{2} \ln(\sqrt{\beta^2 + 1} + \beta). \quad (402)$$

It is straightforward to verify that

$$\begin{aligned}\text{Tr}(X^* \gamma_{xx} + P^* \gamma_{pp}) - 3\text{str}(X^*[1], P^*[1]) \\ = \alpha - \frac{1}{2}(3|\beta| + \sqrt{\beta^2 + 1}).\end{aligned}\quad (403)$$

On the other hand, we have the distinct eigenvalues of both differences

$$\gamma_{xx} - \gamma_{xx}^{1*} \oplus \gamma_{xx}^{2*} \oplus \gamma_{xx}^{3*}, \quad \gamma_{pp} - \gamma_{pp}^{1*} \oplus \gamma_{pp}^{2*} \oplus \gamma_{pp}^{3*}, \quad (404)$$

are given by

$$\alpha - \frac{1}{2}(\pm 3\beta + \sqrt{\beta^2 + 1}). \quad (405)$$

It follows that the minimal eigenvalue is given by the right-hand side of Eq. (403), which proves the optimality of the presented solutions.

We have just derived that

$$\mathcal{E}_{1|2|3}(\gamma_{xx}, \gamma_{pp}) = \frac{1}{2}(3|\beta| + \sqrt{\beta^2 + 1}) - \alpha. \quad (406)$$

The $1|2|3$ -separability is expressed by the inequality $\mathcal{E}_{1|2|3}(\gamma_{xx}, \gamma_{pp}) \leq 0$, which is exactly the statement of the lemma [Eq. (395)]. ■

Lemma 4. The Gaussian state with the CM given by Eq. (392) is $1|23$ -separable iff its partial transposition with respect to the first mode is positive semidefinite. This happens provided that the condition

$$10\beta^2 + 2|\beta|\sqrt{9\beta^2 + 8} + 1 \leq 4\alpha^2 \quad (407)$$

holds true.

This expression coincides with $\lambda_{\min}^*$ since

$$\sqrt{10\beta^2 + 2|\beta|\sqrt{9\beta^2 + 8} + 1} = \frac{32\beta^3 - 11\beta - \sqrt{9\beta^2 + 8}}{4\sqrt{2}(2\beta^2 - 1)} \sqrt{\frac{\beta(16\beta^2 - 3)\sqrt{9\beta^2 + 8} + 80\beta^4 - 23\beta^2 + 4}{64\beta^4 - 12\beta^2 + 1}},$$

for $\beta \geq 0$. The entanglement measure reads as

$$\mathcal{E}_{1|23}(\gamma_{xx}, \gamma_{pp}) = \frac{1}{2}\sqrt{10\beta^2 + 2|\beta|\sqrt{9\beta^2 + 8} + 1} - \alpha.$$

We conclude that $1|23$ -separability coincides with the positivity of partial transposition, which is expressed by the inequality $\mathcal{E}_{1|23}(\gamma_{xx}, \gamma_{pp}) \leq 0$ or Eq. (412), which is the same as the inequality (407). ■

Proof. The optimal witness matrices read as

$$X^* = \begin{pmatrix} x^* & y^* & y^* \\ y^* & z^* & z^* \\ y^* & z^* & z^* \end{pmatrix}, \quad P^* = \begin{pmatrix} p^* & q^* & q^* \\ q^* & r^* & r^* \\ q^* & r^* & r^* \end{pmatrix}, \quad (408)$$

and the dual solution is given by

$$\begin{aligned}\gamma_{xx}^{1*} &= (a^*), & \gamma_{xx}^{23*} &= \begin{pmatrix} b^* & c^* \\ c^* & b^* \end{pmatrix}, \\ \gamma_{pp}^{1*} &= (d^*), & \gamma_{pp}^{23*} &= \begin{pmatrix} e^* & f^* \\ f^* & e^* \end{pmatrix}.\end{aligned}\quad (409)$$

The matrix elements are now given by much more complicated expressions. For the single-mode part, we have $a^*d^* = 1/4$, so it is a squeezed vacuum state. With respect to the two-mode part, the problem is degenerate and multiple solutions exist. It is possible to find a pure state among them, i.e., a solution that satisfies the relation

$$\gamma_{xx}^{23*} \gamma_{pp}^{23*} = \frac{1}{4} \Omega_2. \quad (410)$$

The minimal eigenvalue, as produced by *Mathematica* for $\beta \geq 0$, reads as

$$\begin{aligned}\lambda_{\min}^* &= \alpha - \frac{32\beta^3 - 11\beta - \sqrt{9\beta^2 + 8}}{8\sqrt{2}(2\beta^2 - 1)} \\ &\times \sqrt{\frac{\beta(16\beta^2 - 3)\sqrt{9\beta^2 + 8} + 80\beta^4 - 23\beta^2 + 4}{64\beta^4 - 12\beta^2 + 1}}.\end{aligned}$$

Though *Mathematica* does not further simplify it, it turns out that it is possible to simplify the last term on the right-hand side (which depends on β only).

The CM of the state partially transposed with respect to mode 1 is given by

$$\gamma_{xx}^{\text{PT}} = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}, \quad \gamma_{pp}^{\text{PT}} = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & -\beta \\ \beta & -\beta & \alpha \end{pmatrix}. \quad (411)$$

The eigenvalues of $\mathcal{P}(\gamma_{xx}^{\text{PT}}, \gamma_{pp}^{\text{PT}})$ are

$$\alpha \pm \frac{1}{2}\sqrt{4\beta^2 + 1}, \quad \alpha \pm \frac{1}{2}\sqrt{10\beta^2 \pm 2|\beta|\sqrt{9\beta^2 + 8} + 1}.$$

We see that the partially transposed state is a quantum state iff

$$\alpha - \frac{1}{2}\sqrt{10\beta^2 + 2|\beta|\sqrt{9\beta^2 + 8} + 1} \geq 0. \quad (412)$$

Lemma 5. A state with the CM given by Eq. (392) is $\{1|23, 2|13\}$ -entangled if the parameters α and β satisfy the relation

$$\alpha < \frac{2}{3}|\beta| + \frac{\sqrt{28\beta^2 + 3}}{3} \cos \left[\frac{1}{3} \arccos \left(\frac{18|\beta| - 80|\beta|^3}{\sqrt{(28\beta^2 + 3)^3}} \right) \right]. \quad (413)$$

Proof. The optimal solution for $\beta \geq 0$ reads as

$$X^* = \begin{pmatrix} x^* & -x^* & 0 \\ -x^* & x^* & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P^* = \begin{pmatrix} p^* & p^* & q^* \\ p^* & p^* & q^* \\ q^* & q^* & r^* \end{pmatrix}, \quad (414)$$

and the dual solution is given by

$$p_{1|23}^* = p_{2|13}^* = \frac{1}{2}, \quad (415)$$

and

$$\gamma_{xx}^{1*} = \gamma_{xx}^{2*} = (a^*), \quad \gamma_{xx}^{23*} = \gamma_{xx}^{13*} = \begin{pmatrix} a^* & b^* \\ b^* & c^* \end{pmatrix},$$

$$\gamma_{pp}^{1*} = \gamma_{pp}^{2*} = (d^*), \quad \gamma_{pp}^{23*} = \gamma_{pp}^{13*} = \begin{pmatrix} e^* & f^* \\ f^* & g^* \end{pmatrix}, \quad (416)$$

where the matrix elements are defined by even more complicated expressions than in the previous case. The single-mode parts of the dual solution are squeezed vacuum and the two-mode parts are pure.

If we substitute these large expressions to $\lambda_{\min}^*$, we obtain

$$\lambda_{\min}^* = \alpha - \dots, \quad (417)$$

where $\dots$ stands for another large expression of β only. There is a way to represent this expression more compactly. If we add the equation $\lambda_{\min}^* = 0$ to the system of KKT equations and eliminate all other variables except α and β , we obtain the following relation between α and β when $\lambda_{\min}^*$ is zero:

$$4\alpha^3 - 8\alpha^2\beta - (4\beta^2 + 1)\alpha + 8\beta^3 = 0. \quad (418)$$

It follows that $\dots$ must be one of the roots of this polynomial of α for a fixed β . This root can be easily identified and for $\beta \geq 0$, we have

$$\lambda_{\min}^* = \alpha - \frac{2}{3}\beta - \frac{1}{3} \times \text{Re} \sqrt{18\beta - 80\beta^3 + 3i\sqrt{48(36\beta^6 + 23\beta^4 + \beta^2) + 3}}.$$

The latter term can be rewritten in a manifestly real form as it has been done in a previous section, and we derive

$$\mathcal{E}_{\{1|23, 2|13\}}(\gamma_{xx}, \gamma_{pp}) = -\alpha + \frac{2}{3}|\beta| + \frac{\sqrt{28\beta^2 + 3}}{3} \times \cos \left[\frac{1}{3} \arccos \left(\frac{18|\beta| - 80|\beta|^3}{\sqrt{(28\beta^2 + 3)^3}} \right) \right].$$

The entanglement condition $\mathcal{E}_{\{1|23, 2|13\}}(\gamma_{xx}, \gamma_{pp}) > 0$ then becomes the inequality (413). ■

All the entanglement measures above were expressed by a single condition. As we will see below, this does not have to be always the case.

Lemma 6. A state with the CM given by Eq. (392) is $\{1|23, 2|13, 3|12\}$ entangled (genuine entangled) if the parameters α and β satisfy the relation

$$\alpha < \begin{cases} \frac{1}{3}\sqrt{36\beta^2 + 1} + \frac{1}{6}, & |\beta| \leq \frac{1}{6\sqrt{3}}, \\ R_1(\beta), & |\beta| \geq \frac{1}{6\sqrt{3}}, \end{cases} \quad (419)$$

where $R_1(\beta)$ is the largest root of the following equation of the variable α :

$$186624\alpha^7\beta^2 - 1119744\alpha^5\beta^4 - 57024\alpha^5\beta^2 + 144\alpha^5 \\ + 186624\alpha^3\beta^6 + 98496\alpha^3\beta^4 + 1584\alpha^3\beta^2 - 40\alpha^3 \\ + 5225472\alpha\beta^8 + 435456\alpha\beta^6 + 1728\alpha\beta^4 + 76\alpha\beta^2 \\ + \alpha - 373248\alpha^6|\beta|^3 - 5184\alpha^6|\beta| + 2985984\alpha^4|\beta|^5 \\ + 139968\alpha^4|\beta|^3 + 1584\alpha^4|\beta| - 5598720\alpha^2|\beta|^7 \\ - 487296\alpha^2|\beta|^5 - 6768\alpha^2|\beta|^3 - 76\alpha^2|\beta| + 40|\beta|^3 \\ - 1492992|\beta|^9 - 124416|\beta|^7 + 1728|\beta|^5 + |\beta| = 0. \quad (420)$$

Proof. The optimal solution for $\beta \geq 0$ is of the form

$$X^* = \begin{pmatrix} x^* & y^* & y^* \\ y^* & x^* & y^* \\ y^* & y^* & x^* \end{pmatrix}, \quad P^* = \begin{pmatrix} p^* & q^* & q^* \\ q^* & p^* & q^* \\ p^* & q^* & q^* \end{pmatrix}, \quad (421)$$

and the dual solution reads as

$$p_{1|23}^* = p_{2|13}^* = p_{3|12}^* = \frac{1}{3}, \quad (422)$$

and

$$\gamma_{xx}^{1*} = \gamma_{xx}^{2*} = \gamma_{xx}^{3*} = (a^*),$$

$$\gamma_{xx}^{23*} = \gamma_{xx}^{13*} = \gamma_{xx}^{12*} = \begin{pmatrix} b^* & c^* \\ c^* & b^* \end{pmatrix},$$

$$\gamma_{pp}^{1*} = \gamma_{pp}^{2*} = \gamma_{pp}^{3*} = (d^*),$$

$$\gamma_{pp}^{23*} = \gamma_{pp}^{13*} = \gamma_{pp}^{12*} = \begin{pmatrix} e^* & f^* \\ f^* & e^* \end{pmatrix}. \quad (423)$$

This time, the values of the optimal parameters cannot be found analytically, but we can find an equation for α and β when $\lambda_{\min}^*$ becomes zero and thus we can determine the region of biseparability. This can be done as in the previous case by adding the equation $\lambda_{\min}^* = 0$ to the system of KKT conditions and eliminating the elements of the primal and dual solutions, but this way we do not get the expression for $\lambda_{\min}^*$. So, we write

$$\lambda_{\min}^* = \alpha - R, \quad (424)$$

and try to determine R . Adding this equation to the KKT conditions and eliminating all variables except α , β , and R , we obtain that R is a zero of a product of several terms:

$$(4R^2 - 4\beta^2 - 1)(4R^2 - 16\beta^2 - 1)(9R^2 - 9\beta^2 - 1),$$

$$(36R^2 - 36\beta^2 - 1)(36R^2 - 144\beta^2 - 1), \quad (425)$$

$$12R^2 - 48\beta^2 \pm 4R - 1, \quad (426)$$

and

$$186624R^7\beta^2 - 1119744R^5\beta^4 - 57024R^5\beta^2 + 144R^5 \\ + 186624R^3\beta^6 + 98496R^3\beta^4 + 1584R^3\beta^2 - 40R^3 \\ + 5225472R\beta^8 + 435456R\beta^6 + 1728R\beta^4 + 76R\beta^2 \\ + R \mp 373248R^6\beta^3 \mp 5184R^6\beta \pm 2985984R^4\beta^5 \\ \pm 139968R^4\beta^3 \pm 1584R^4\beta \mp 5598720R^2\beta^7$$

$$\begin{aligned} &\mp 487296R^2\beta^5 \mp 6768R^2\beta^3 \mp 76R^2\beta \pm 40\beta^3 \\ &\mp 1492992\beta^9 \mp 124416\beta^7 \pm 1728\beta^5 \pm \beta, \end{aligned} \quad (427)$$

where either the upper or lower sign is taken in all ambiguous terms. We see that $R = R(\beta)$ depends only on β , but we need to determine which of these terms R is a root. It might even be that this term is different for different values of β .

The difference $g = g(x, y, p, q)$ defined via

$$g = \text{Tr}(X\gamma_{xx} + P\gamma_{pp}) - \text{str}(X[1], P[1]) - \text{str}(X[2, 3], P[2, 3]) \quad (428)$$

has already a familiar form

$$g = 3(x + p)\alpha - 6(q + y)\beta - \sqrt{xp} - \sqrt{(x + y)(p - q)} - \sqrt{(x - y)(p + q)}, \quad (429)$$

so our conjecture is that the optimal x^* , y^* , p^* , and q^* are also optimal for this function. The optimization of g has less variables, just four, and thus is simpler than the original problem. Setting the derivatives of the Lagrangian

$$\mathcal{L} = g(x, y, p, q) + \nu[3(x + p) - 1], \quad (430)$$

to zero together with the trace normalization condition

$$\text{Tr}(X + P) = 3(x + p) - 1 = 0, \quad (431)$$

produces the following solution:

$$x^* = p^* = \frac{1}{6}, \quad y^* = -q^* = -\frac{\beta}{\sqrt{36\beta^2 + 1}}. \quad (432)$$

We then substitute these expressions into the original problem to find the dual solution:

$$a^* = d^* = \frac{1}{2}, \quad b^* = e^* = \frac{1}{2}\sqrt{36\beta^2 + 1}, \quad c^* = f^* = 3\beta. \quad (433)$$

It follows that γ^{1*} is the CM of the vacuum state and γ^{23*} is the CM of the two-mode squeezed vacuum state. The two-mode squeezed vacuum is defined as

$$|\zeta\rangle = \hat{S}(\zeta)|00\rangle = \frac{1}{\cosh(r)} \sum_{n=0}^{+\infty} \tanh^n(r) e^{-i\varphi n} |nn\rangle, \quad (434)$$

where the squeezing operator $\hat{S}(\zeta)$ reads as

$$\hat{S}(\zeta) = \exp(-\zeta \hat{a}^\dagger \hat{b}^\dagger + \zeta^* \hat{a} \hat{b}). \quad (435)$$

In general, the squeezing parameter can be an arbitrary complex number, $\zeta = re^{i\varphi}$, but here $\zeta = r$ is a real number determined from β according to the relation

$$r = \frac{1}{2} \arcsin(6\beta). \quad (436)$$

The eigenvalues of X^* are

$$\sqrt{36\beta^2 + 1} + 6\beta, \quad \sqrt{36\beta^2 + 1} - 12\beta, \quad (437)$$

the first with multiplicity 2. The eigenvalues of P^* are

$$\sqrt{36\beta^2 + 1} - 6\beta, \quad \sqrt{36\beta^2 + 1} + 12\beta, \quad (438)$$

the first with multiplicity 2 (we consider the case of $\beta \geq 0$, for $\beta \leq 0$ these matrices should be swapped). We see that P^* is

always positive semidefinite, and X^* is positive semidefinite iff

$$\beta \leq \beta^* = \frac{1}{6\sqrt{3}}. \quad (439)$$

This value corresponds to the squeezing parameter given by the following simple expression:

$$r^* = \frac{\ln(3)}{4}. \quad (440)$$

For these parameters, we have

$$g^* = g(x^*, y^*, p^*, q^*) = \alpha - \frac{1}{3}\sqrt{36\beta^2 + 1} - \frac{1}{6}. \quad (441)$$

On the other hand, we also have

$$\begin{aligned} &\gamma_{xx} - \frac{1}{3}(\gamma_{xx}^{1*} \oplus \gamma_{xx}^{23*} + \gamma_{xx}^{2*} \oplus \gamma_{xx}^{13*} + \gamma_{xx}^{3*} \oplus \gamma_{xx}^{12*}) \\ &= \gamma_{pp} - \frac{1}{3}(\gamma_{pp}^{1*} \oplus \gamma_{pp}^{23*} + \gamma_{pp}^{2*} \oplus \gamma_{pp}^{13*} + \gamma_{pp}^{3*} \oplus \gamma_{pp}^{12*}) \\ &= \lambda_{\min}^* E, \end{aligned} \quad (442)$$

where

$$\lambda_{\min}^* = \alpha - R \equiv \alpha - \frac{1}{3}\sqrt{36\beta^2 + 1} - \frac{1}{6}. \quad (443)$$

This result shows that these parameters are valid primal and dual solutions, but this construction works only provided that $|\beta| \leq \beta^*$, for larger β either X^* or P^* is not positive semidefinite. This proves the first part of the lemma.

Note that for $|\beta| \leq \beta^*$, the equality $\lambda_{\min}^* = 0$ holds iff

$$12\alpha^2 - 48\beta^2 - 4\alpha - 1 = 0, \quad (444)$$

which means that α is a root of the factor of Eq. (426) when the upper sign is taken. For $|\beta| \leq \beta^*$, the condition $\lambda_{\min}^* = 0$ must produce a root of some other term. By substitution $\beta \rightarrow \beta^*$ to all the terms, we find that the only term that is also zero is given by Eq. (427) with all upper signs, so R in this case is a root of that term. By plotting the roots of the term, we conclude that R must be the largest root, which proves the second part of the lemma. The explicit expressions for the primal and dual solutions do not seem to exist. We thus obtain the following expression for the entanglement measure:

$$\mathcal{E}_{\mathcal{J}_2}(\gamma_{xx}, \gamma_{pp}) = R(\beta) - \alpha, \quad (445)$$

where $R(\beta)$ is defined by different expressions depending on β . ■

All the obtained regions are shown in Fig. 15. The very thin region between the red and purple lines corresponds to CMs that are simultaneously 1|23-, 2|13-, and 3|12-separable, but not 1|2|3-separable. The region between the yellow and red lines is the {1|23, 2|13}-separable CMs that are not 1|23-separable. The region between the green and yellow lines is biseparable, but neither {1|23, 2|13}-, nor {1|23, 3|12}-, nor {2|13, 3|12}-separable CMs.

We now apply the obtained results to some simple, but interesting states. As a first example, which is a non-Gaussian state, we take an equally weighted mixture of products of single-mode vacuum with two-mode squeezed vacuum states:

$$\begin{aligned} \hat{\rho} = &\frac{1}{3}(|0\rangle_1\langle 0| \otimes |\zeta\rangle_{23}\langle \zeta| + |0\rangle_2\langle 0| \otimes |\zeta\rangle_{13}\langle \zeta| \\ &+ |0\rangle_3\langle 0| \otimes |\zeta\rangle_{12}\langle \zeta|). \end{aligned} \quad (446)$$

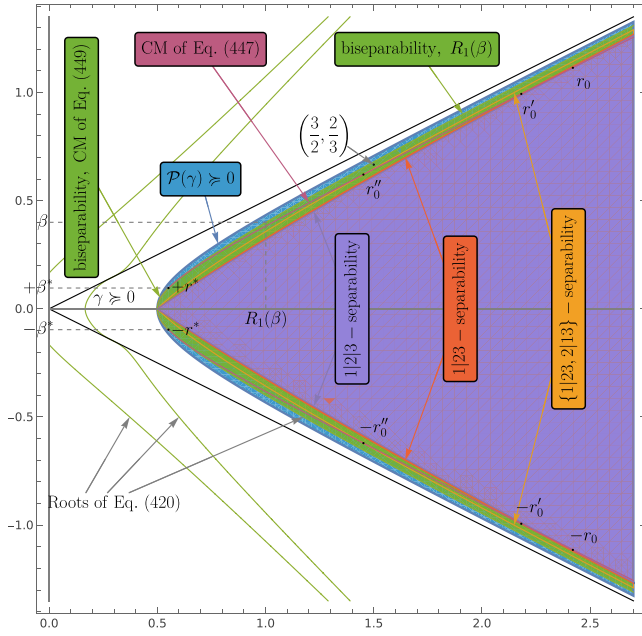


FIG. 15. The regions of different kinds of separability of the state with the CM given by Eq. (392).

This state has the same CM as the optimal dual solution from the first part of the last lemma above. Because the first-order moments of the state (446) are zero, the CM of the mixture is the mixture of CMs and we have

$$\gamma_{xx} = \frac{1}{3} \begin{pmatrix} \cosh(2r) + \frac{1}{2} & \frac{1}{2} \sinh(2r) & \frac{1}{2} \sinh(2r) \\ \frac{1}{2} \sinh(2r) & \cosh(2r) + \frac{1}{2} & \frac{1}{2} \sinh(2r) \\ \frac{1}{2} \sinh(2r) & \frac{1}{2} \sinh(2r) & \cosh(2r) + \frac{1}{2} \end{pmatrix},$$

$$\gamma_{pp} = \frac{1}{3} \begin{pmatrix} \cosh(2r) + \frac{1}{2} & -\frac{1}{2} \sinh(2r) & -\frac{1}{2} \sinh(2r) \\ -\frac{1}{2} \sinh(2r) & \cosh(2r) + \frac{1}{2} & -\frac{1}{2} \sinh(2r) \\ -\frac{1}{2} \sinh(2r) & -\frac{1}{2} \sinh(2r) & \cosh(2r) + \frac{1}{2} \end{pmatrix}.$$
(447)

This CM is of the form given by Eq. (392) with

$$\alpha = \frac{1}{3}(\cosh(2r) + \frac{1}{2}), \quad \beta = \frac{1}{6} \sinh(2r). \quad (448)$$

This parametric curve is also shown in Fig. 15.

From the proof of the first part of the last lemma above, it follows that the parametric curve described by these covariance matrices forms a part of the boundary of the biseparable states for $|r| \leq r^*$. Note that this parametric curve has the following explicit equation:

$$\alpha = \frac{1}{3} \sqrt{36\beta^2 + 1} + \frac{1}{6}. \quad (449)$$

This curve and the curve $\alpha = R_1(\beta)$ are very close near the origin $\beta = 0$ (Fig. 16). One would need to enlarge Fig. 15 at least 1000 $\times$ to see the difference, so for most, if not all, practical purposes this difference is irrelevant, but nevertheless these are different curves. For larger β , the curve [Eq. (449)] diverges from the biseparability boundary and intersects the boundaries of all the regions. The intersection with the 1|2|3-

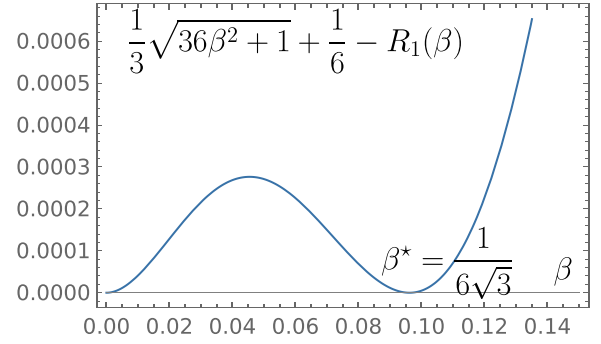


FIG. 16. The difference between the curve (449) and the curve $\alpha = R_1(\beta)$.

separability region happens at $\zeta = r_0$, where

$$r_0 = \frac{1}{2} \ln \left(\frac{13 + \sqrt{193}}{2} \right) = 1.2993 \dots \quad (450)$$

This result guarantees that the state (446) with $\zeta = r < r_0$ is not 1|2|3-separable, but for a larger value of squeezing we cannot say that it is separable. The Gaussian state with the CM given by Eq. (447) is 1|2|3-separable if $r > r_0$, but for the non-Gaussian state (446) with the same CM, this test is inconclusive.

The intersection with 1|23-separability regions happens at $\zeta = r'_0$, where

$$r'_0 = \frac{1}{2} \ln \left(\frac{7 + 2\sqrt{31} + 2\sqrt{41 + 7\sqrt{31}}}{3} \right) = 1.2427 \dots$$

For $\zeta = r < r'_0$, the state (446) is guaranteed to be 1|23-entangled (and due to symmetry also 2|13- and 3|12-entangled), but for $r \geq r'_0$ we cannot say anything. This number was also obtained in Refs. [24,42].

The CMs (447) intersect the {1|23, 2|13}-separability regions at the squeezing parameter r''_0 , where

$$r''_0 = \frac{1}{2} \ln(\tilde{r}) = 1.0128 \dots,$$

$$3\tilde{r}^4 - 19\tilde{r}^3 - 25\tilde{r}^2 - 25\tilde{r} - 6 = 0.$$

The number $\tilde{r}$ is the fourth root of its minimal polynomial, and one can give an explicit expression for it in terms of radicals. As before, for the squeezing parameter $\zeta = r < r''_0$, the state (446) is definitely not {1|23, 2|13}-separable, but for $r \geq r''_0$, we cannot say anything.

The plot of different entanglement measures for CMs of the form (447) is shown in Fig. 17. The genuine entanglement measure is always nonpositive, which means that the state is not genuine entangled. Since our state is biseparable by construction, this is expectable. For $\zeta = r = 0$, the CM becomes the CM of the product of vacuum states, and this measure must be zero. This measure is exactly zero up to where the squeezing parameter becomes r^* , for larger values the measure becomes strictly negative, which means that the corresponding CMs are no longer on the boundary of the biseparability region. The points $(\alpha^*, \pm\beta^*)$, corresponding to

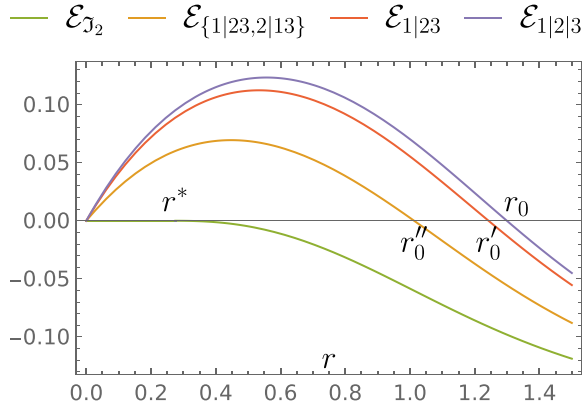


FIG. 17. Genuine entanglement measure of CM (447).

$\pm r^*$ read as

$$\begin{aligned} (\alpha^*, \pm\beta^*) &= \left(\frac{1}{3} \left[\cosh(2r^*) + \frac{1}{2} \right], \pm \frac{1}{6} \sinh(2r^*) \right) \\ &= \left(\frac{4 + \sqrt{3}}{6\sqrt{3}}, \pm \frac{1}{6\sqrt{3}} \right). \end{aligned} \quad (451)$$

These points are simple nontrivial solutions of Eq. (420). Also note the inequality

$$\mathcal{E}_{J_2} \leq \mathcal{E}_{\{1|23,2|13\}} \leq \mathcal{E}_{1|23} \leq \mathcal{E}_{1|2|3}, \quad (452)$$

which fully agrees with the inequalities given by Eq. (128).

We said before that an analytical construction of the optimal solution for $\{1|23, 2|13\}$ -separability is quite large and has not presented it explicitly. Now, we show that the CM (447) with the parameters given by Eq. (451) is $\{1|23, 2|13\}$ -entangled. If we do not aim at the optimal violation of the separability condition, any pair of positive semidefinite matrices X and P , not necessarily trace normalized, that violate the condition will suffice. We compute the exact analytical solution, which is expressed by algebraic numbers of third order, rationalize it and get a much simpler witness,

$$X = \begin{pmatrix} \frac{87}{368} & -\frac{87}{368} & 0 \\ -\frac{87}{368} & \frac{87}{368} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{54}{215} & \frac{54}{215} & \frac{23}{291} \\ \frac{54}{215} & \frac{54}{215} & \frac{23}{291} \\ \frac{23}{291} & \frac{23}{291} & \frac{5}{201} \end{pmatrix}. \quad (453)$$

These matrices are positive semidefinite, and for the primal objective value of the $\{1|23, 2|13\}$ -entanglement optimization problem, we have

$$\begin{aligned} \text{Tr}(X\gamma_{xx} + P\gamma_{pp}) - \text{str}(X[1], P[1]) - \text{str}(X[2, 3], P[2, 3]) \\ = \frac{771322369 + 696435289\sqrt{3}}{4627807920} - 9\sqrt{\frac{29}{9890}} = -0.0600 \dots \end{aligned}$$

For the $\{2|13\}$ -partition, we get the same negative value, which proves that this state is $\{1|23, 2|13\}$ -entangled. Not surprisingly, for the partition $3|12$ we have

$$\begin{aligned} \text{Tr}(X\gamma_{xx} + P\gamma_{pp}) - \text{str}(X[3], P[3]) - \text{str}(X[1, 2], P[1, 2]) \\ = \frac{771322369 + 696435289\sqrt{3}}{4627807920} = 0.4273 \dots > 0, \end{aligned}$$

so the matrices X and P certify only the $\{1|23, 2|13\}$ -entanglement and not the genuine entanglement (which would contradict the fact that our state is biseparable by construction). This gives an explicit example of a three-mode biseparable state which is not separable with respect to any other form of separability (except biseparability).

For most parameters α and β , the exact solution cannot be expressed in terms of radicals and one has to work with algebraic numbers in terms of their minimal polynomials. Consider a very simple case with easily memorable parameters $\alpha = 3/2$ and $\beta = 2/3$. The CM reads as

$$\gamma_{xx} = \frac{1}{6} \begin{pmatrix} 9 & 4 & 4 \\ 4 & 9 & 4 \\ 4 & 4 & 9 \end{pmatrix}, \quad \gamma_{pp} = \frac{1}{6} \begin{pmatrix} 9 & -4 & -4 \\ -4 & 9 & -4 \\ -4 & -4 & 9 \end{pmatrix}. \quad (454)$$

The corresponding point is shown in Fig. 15 and lies between the green (biseparability) and yellow ($\{1|23, 2|13\}$ -separability) lines. We show that this state is biseparable, but cannot be represented as a combination of one or two separable states with respect to a fixed partition.

The analytical solution of the KKT equations cannot be obtained even if we set the above concrete values of α and β , but the numerical solution produces $f^* = -2.0$ in Eq. (423). Adding this equation to the KKT system of equation allows one to obtain an analytical expression for both primal [Eq. (421)] and dual [Eq. (423)] solutions. Here, we need just the dual solution to demonstrate that the state with the CM given by Eq. (454) is biseparable. The other matrix elements of the dual solution are algebraic numbers with the minimal polynomials given by

$$\begin{aligned} 7808a^7 - 54064a^6 - 100448a^5 + 159960a^4 + 25112a^3 - 3475a^2 - 122a - 12 &= 0, \\ 562176b^7 + 2838528b^6 - 118416b^5 - 4411968b^4 - 587704b^3 + 6640b^2 - 3913b - 24 &= 0, \\ 281088c^7 - 1883520c^6 + 2786016c^5 - 509001c^4 - 638456c^3 - 93400c^2 - 4576c - 144 &= 0, \\ 3072d^7 + 7808d^6 + 55600d^5 - 100448d^4 - 159960d^3 + 25112d^2 + 3379d - 122 &= 0, \\ 48e^7 - 160e^6 - 24e^5 + 1472e^4 - 2109e^3 - 3270e^2 + 5644e - 512 &= 0. \end{aligned} \quad (455)$$

In *Mathematica* nomenclature, the concrete roots have the following numbers:

$$a^* : 4, \quad b^* : 5, \quad c^* : 3, \quad d^* : 4, \quad e^* : 5. \quad (456)$$

Numerically, the dual solution reads as

$$\begin{aligned} \gamma_{xx}^{1*} &= \gamma_{xx}^{2*} = \gamma_{xx}^{3*} = (1.1689 \dots), \\ \gamma_{xx}^{23*} &= \gamma_{xx}^{13*} = \gamma_{xx}^{12*} = \begin{pmatrix} 1.1981 \dots & 1.1373 \dots \\ 1.1373 \dots & 1.1981 \dots \end{pmatrix}, \\ \gamma_{pp}^{1*} &= \gamma_{pp}^{2*} = \gamma_{pp}^{3*} = (0.2138 \dots), \\ \gamma_{pp}^{23*} &= \gamma_{pp}^{13*} = \gamma_{pp}^{12*} = \begin{pmatrix} 2.1070 \dots & -2 \\ -2 & 2.1070 \dots \end{pmatrix}. \end{aligned} \quad (457)$$

It is possible to check that these are physical CMs and $a^* d^* = 1/4$, so the single-mode states are squeezed vacuum states. In addition, the two-mode states satisfy the equality

$$\gamma_{xx}^{23*} \gamma_{pp}^{23*} = \frac{1}{4} \Omega_2, \quad (458)$$

and are thus pure. We have that

$$\begin{aligned} \gamma_{xx} &\succcurlyeq \frac{1}{3} (\gamma_{xx}^{1*} \oplus \gamma_{xx}^{23*} + \gamma_{xx}^{2*} \oplus \gamma_{xx}^{13*} + \gamma_{xx}^{3*} \oplus \gamma_{xx}^{12*}), \\ \gamma_{pp} &\succcurlyeq \frac{1}{3} (\gamma_{pp}^{1*} \oplus \gamma_{pp}^{23*} + \gamma_{pp}^{2*} \oplus \gamma_{pp}^{13*} + \gamma_{pp}^{3*} \oplus \gamma_{pp}^{12*}), \end{aligned} \quad (459)$$

since the eigenvalues of the differences of the left-hand side and the right-hand side are positive and given by

$$\begin{aligned} (0.8867 \dots, 0.0240 \dots, 0.0240 \dots), \\ (0.0240 \dots, 0.0240 \dots, 0.0240 \dots), \end{aligned} \quad (460)$$

respectively. Note that $\lambda_{\min}^* = 0.0240 \dots$. We emphasize that these statements can be verified exactly by computing the minimal polynomial of the quantities in question.

X. OPEN PROBLEMS

In all the problems studied analytically above in the KKT condition (123), which we reproduce here as

$$\begin{pmatrix} \gamma_{xx}^{(I)*} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp}^{(I)*} \end{pmatrix} \begin{pmatrix} X^*[I] & -Z^{(I)*} \\ -Z^{(I)*T} & P^*[I] \end{pmatrix} = 0, \quad (461)$$

both terms have the same rank, equal to $k = |I|$, the size of the matrices $\gamma_{xx}^{(I)}$, $\gamma_{pp}^{(I)}$, $X[I]$, and $P[I]$. Since rank of the first term is always $\geq k$, the statement that both terms have rank k has a simple interpretation. The optimal $\gamma_{xx}^{(I)*}$ and $\gamma_{pp}^{(I)*}$ are such that the last k column vectors are linearly dependent on the first k ,

$$\begin{pmatrix} * & \frac{1}{2}E \\ \frac{1}{2}E & * \end{pmatrix}. \quad (462)$$

Under the assumption that $\text{rank}(X^{(I)*}) = \text{rank}(P^{(I)*}) = k$, the optimal $Z^{(I)*}$ is such that it makes the last k column vectors linearly dependent on the first k ,

$$\begin{pmatrix} X^{(I)*} & * \\ * & P^{(I)*} \end{pmatrix}. \quad (463)$$

This assumption turns out to be valid in all the analytical cases considered above and for all I participating in the optimal solution. The full optimal matrices X^* and P^* are not necessarily of full rank. In fact, they might even be of different rank.

Numerical experiments show that the two terms in the product (461) do not always have the same rank, but the following less strong relation holds:

$$\text{rank} \begin{pmatrix} \gamma_{xx}^{(I)*} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp}^{(I)*} \end{pmatrix} + \text{rank} \begin{pmatrix} X^*[I] & -Z^{(I)*} \\ -Z^{(I)*T} & P^*[I] \end{pmatrix} = 2k. \quad (464)$$

If A and B are matrices of the same size $2k$ and $AB = 0$, then a weaker inequality is always satisfied,

$$\text{rank}(A) + \text{rank}(B) \leq 2k. \quad (465)$$

Our observation is the stronger statement that this inequality is always equality, and in some cases an even stronger relation,

$$\text{rank}(A) = \text{rank}(B) = k, \quad (466)$$

is valid. In the full case, the similar numerical observation states that

$$\text{rank} \mathcal{P}(\gamma^{(I)*}) + \text{rank}(M^*[I] + iK^{(I)*}) = 2k. \quad (467)$$

So, the first open problem is to prove this conjecture.

Another problem is to find the maximal values of the measures $\mathcal{E}_{\mathcal{J}_2}^+$ and $\mathcal{T}_{\mathcal{J}_2}^+$. For the latter, we have $\mathcal{T}_{\mathcal{J}_2}^+(\gamma) < 1$ for all γ and the symmetric states defined by Eq. (92) numerically show that

$$\sup_{\lambda, \mu} \mathcal{T}_{\mathcal{J}_2}^+(\gamma^S(\lambda, \mu)) = 1. \quad (468)$$

An analytical proof of this equality would demonstrate that

$$\sup_{\gamma} \mathcal{T}_{\mathcal{J}_2}^+(\gamma) = 1. \quad (469)$$

For the same n -mode symmetric states, we have, numerically,

$$\sup_{\lambda, \mu} \mathcal{E}_{\mathcal{J}_2}^+(\gamma^S(\lambda, \mu)) = \frac{1}{n}. \quad (470)$$

Our conjecture is then that the right-hand side of this relation is also the maximal value of the $\mathcal{E}^+$ measure,

$$\sup_{\gamma} \mathcal{E}_{\mathcal{J}_2}^+(\gamma) = \frac{1}{n}. \quad (471)$$

Verifying the validity of the Eqs. (469) and (471) is our second open problem. A difficulty in maximizing these quantities is that they are convex. Standard convex optimization refers to minimizing convex functions or maximizing concave ones, but here we need to maximize convex functions.

The last open problem, we formulate in this work is finding the genuine entangled states with as many positive partial transpositions (PTs) as possible and with $\mathcal{E}_{\mathcal{J}_2}^+$ (or $\mathcal{T}_{\mathcal{J}_2}^+$) as large as possible. The proof given in Ref. [41] states that $1 + n$ bipartitions of an $(n + 1)$ -mode state are equivalent to $1 + n$ separability, so we should exclude such bipartitions from the consideration. The state with CM given by Eq. (180) has all three $2 + 2$ PTs positive (and all its $1 + 3$ PTs nonpositive), but this state is $12|34$ -entangled. This is a very simple form of entanglement, but we are interested in a much stronger notion of genuine entanglement. One way to look for such states is to randomly generate a positive semidefinite 8×8 matrix M with

trace 1 and then solve the following optimization problem:

$$\min_{\gamma} \left[\text{Tr}(M\gamma) - \min_{I \in \mathcal{I}_2} \sum_{I \in \mathcal{I}} \text{str}(M[I]) \right], \quad (472)$$

where the minimization is over all physical CMs γ with all three 2 + 2 PTs positive and $\mathcal{I}_2$ is the set of all seven bipartitions of four modes. So, given M , we optimize over γ . The optimal solution γ^* might or might not be genuine entangled. Testing a few billion of random matrices, we found a matrix M for which the optimal solution of the problem (472) satisfies the inequality

$$\mathcal{E}_{\mathcal{I}_2}^+(\gamma^*) > 0.01. \quad (473)$$

The measure $\mathcal{E}_{\mathcal{I}_2}^+$ has been obtained by solving the problem (116) where, given γ , we optimize over M . The optimal matrix M^* corresponding to γ^* (and which is typically different from the initial M) can be used as a new input to the problem (472) and produce a new state γ^{**} with slightly larger $\mathcal{E}_{\mathcal{I}_2}^+$. This state can be used as an input to the problem (116) and produce yet another matrix M^{**} , which can be used as an input to Eq. (472) and so on. This process quickly leads to unwieldy numerical results, but $\mathcal{E}_{\mathcal{I}_2}^+(\gamma^{**\dots})$ seems to converge. Our last open problem is to find

$$\sup_{\gamma} \mathcal{E}_{\mathcal{I}_2}^+(\gamma), \quad (474)$$

where the supremum is taken over the physical CMs with all PTs, except the ones of the form 1 + n , positive. In the case of four-mode states, we just found that this supremum is larger than 0.01.

XI. CONCLUSION

In this work, we developed several approaches to determine the true physical CM from uncertain experimentally measured nonphysical data and test the corresponding quantum state for entanglement. All these approaches are based on solving convex optimization problems. Compared with existing theoretical treatments, our current work is applicable to continuous-variable multimode states of a larger size, consistent with the growing scale of optical quantum computing experiments based on Gaussian cluster states [45–48], though with the aim to go beyond the scope of these experiments and their limited focus on full inseparability. For other relevant recent works on Gaussian states, we refer to Refs. [49,50]. We presented two ways to determine the most likely physical CM given a measured nonphysical one. The first way is to compute the closest physical CM, and the other is to compute the most probable CM under assumption of the Gaussian distribution of the measured CM elements. These two procedures can be routinely applied to 200-mode states on a high-end desktop PC.

Having recovered the physical CM, the next step is to test it for entanglement. These tests are expressed in additive and multiplicative forms, for exactly known CMs and for CMs with uncertainty. Using these tests, we analytically establish the entanglement of one simple class of symmetric CMs of pure states for arbitrary number of parts. These states are used for benchmarking of our numerical procedures.

Though the representation of the symplectic trace as a minimum of some quantity has already been known, neither its representation as a maximum of another quantity, nor an explicit expression for it has been known before (to our knowledge). These two results play an important role in the entanglement tests we developed.

Our tests are completely general in the sense that they can test for any combination of arbitrary partitions of the modes of the state. For example, in the case of $n = 5$ we can test not only for entanglement of concrete partitions like 12|345 or 1|24|35, but also for arbitrary combinations of partitions like 12|345, 1|24|35, 15|2|3|4. We mostly work with genuine multipartite entanglement, which involves all possible bipartitions, including their convex combinations, and thus requires a more refined and less easily scalable approach in terms of entanglement witnesses than just full inseparability. All optimization problems are specialized for the practically relevant reduced case, where xp part of CM is zero. We show that the specialized problems perform much faster and take much less RAM than the problems in the full case, so we can quickly solve the typical case without losing the ability to solve the general case when we really need to.

Testing the multipartite states with 16 parts (depending on whether it is the full or reduced case) can be easily done on a high-end desktop PC. We use three different solvers and show that for all of them, the time to find a solution may vary greatly depending on the state (with the same number of parts). It is a feature of all iterative algorithms where the number of iterations (and thus the time to converge to a solution) depends on some kind of condition on the input data. We observe that different solvers work best for different kinds of problems, so there is no one that is superior to others in all cases. In addition, our figures show that more than one metric is needed to estimate the quality of the produced solution. We are also able to test larger states, up to $n = 18$, but the accuracy of the results is not always satisfactory. We show that the time needed to test for genuine multipartite entanglement grows exponentially with n , while the RAM grows superexponentially. As the software for solving convex optimization problems will further advance, it will be possible to solve larger problems on the given hardware (though this advancement will not eliminate the exponential complexity of the entanglement tests).

Besides purely numerical tests, in some cases the optimization problems we presented here admit an analytical solution. We show how to analytically construct the optimal solution of two kinds of the entanglement test optimization problems for a few well-known examples, also including a family of non-Gaussian states, and for one representative, interesting parametrical family of states. In addition, for this parametric family of states we show the regions of parameters corresponding to various kinds of separability properties and demonstrate how these regions expand from the strongest property, the full separability to the weakest property, the biseparability. All our examples are either new or they are known from the literature, but were only numerically treated before. We are confident that our “modern” treatment in this work will find useful applications in Gaussian quantum optical technology and information theory, and beyond.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: PROOF OF THEOREM 1

Introducing a variable s defined by Eq. (33), the objective function to minimize is just $f_0(\mathbf{x}) = s$, where the variables of the problem $\mathbf{x} = (s, \gamma)$ are s and the elements of γ . There is one semidefinite condition $\mathcal{P}(\gamma) \succcurlyeq 0$, which in real form reads as

$$-\begin{pmatrix} \gamma & \frac{1}{2}\Omega \\ -\frac{1}{2}\Omega & \gamma \end{pmatrix} \preccurlyeq 0, \quad (\text{A1})$$

and $2n(2n+1)$ linear conditions

$$\pm\gamma \mp \gamma^\circ - s\sigma \leq 0. \quad (\text{A2})$$

Thus, the dual problem has three symmetric matrix variables $(\lambda_1, \lambda_2, \lambda_3) = (Z, U^+, U^-)$, a $4n \times 4n$ matrix $Z \succcurlyeq 0$ and two $2n \times 2n$ matrices $U^\pm \succcurlyeq 0$. The Lagrangian of this problem reads as follows:

$$\begin{aligned} L(s, \gamma, Z, U^+, U^-) = & s - \text{Tr} \left[Z \begin{pmatrix} \gamma & \frac{1}{2}\Omega \\ -\frac{1}{2}\Omega & \gamma \end{pmatrix} \right] \\ & + \text{Tr}[U^+(\gamma - \gamma^\circ - s\sigma)] \\ & - \text{Tr}[U^-(\gamma - \gamma^\circ + s\sigma)]. \end{aligned}$$

The objective of the dual problem is obtained from the Lagrangian by minimizing it over the primal variables

$$g(Z, U^+, U^-) = \inf_{s, \gamma} L(s, \gamma, Z, U^+, U^-), \quad (\text{A3})$$

where the infimum is taken over all real s and all symmetric matrices γ . Let us decompose Z as

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{pmatrix}, \quad (\text{A4})$$

then the Lagrangian can be written as follows:

$$\begin{aligned} L = & \frac{1}{2} \text{Tr}[(Z_2 - Z_2^T)\Omega] - \text{Tr}[(U^+ - U^-)\gamma^\circ] \\ & + (1 - \text{Tr}[(U^+ + U^-)\sigma])s \\ & - \text{Tr}[(Z_1 + Z_3 - U^+ + U^-)\gamma]. \end{aligned} \quad (\text{A5})$$

A linear function is bounded from below if and only if it is identically equal to zero, so the dual objective is

$$g(Z, U^+, U^-) = \frac{1}{2} \text{Tr}[(Z_2 - Z_2^T)\Omega] - \text{Tr}[(U^+ - U^-)\gamma^\circ], \quad (\text{A6})$$

provided that the dual variables satisfy the conditions

$$\begin{aligned} \text{Tr}[(U^+ + U^-)\sigma] &= 1 \\ U^+ - U^- &= Z_1 + Z_3, \end{aligned} \quad (\text{A7})$$

and $g(Z, U^+, U^-) = -\infty$ otherwise. Using the second condition, we can write the first, finite case as

$$g(Z, U^+, U^-) = \frac{1}{2} \text{Tr}[(Z_2 - Z_2^T)\Omega] - \text{Tr}[(Z_1 + Z_3)\gamma^\circ]. \quad (\text{A8})$$

We now introduce the complex Hermitian matrix

$$\Lambda = \Lambda(Z) = Z_1 + Z_3 + i(Z_2 - Z_2^T). \quad (\text{A9})$$

It is easy to verify that

$$\text{Tr}[\Lambda \mathcal{P}(\gamma^\circ)] = \text{Tr}[(Z_1 + Z_3)\gamma^\circ] - \frac{1}{2} \text{Tr}[(Z_2 - Z_2^T)\Omega], \quad (\text{A10})$$

so we derive that in the finite case the dual objective reads as

$$g(Z, U^+, U^-) = \text{Tr}[\Lambda(Z) \mathcal{P}(\gamma^\circ)]. \quad (\text{A11})$$

The matrix $\Lambda(Z)$ is positive semidefinite since

$$\begin{aligned} \widetilde{\Lambda(Z)} &= \begin{pmatrix} Z_1 + Z_3 & Z_2 - Z_2^T \\ -Z_2 + Z_2^T & Z_1 + Z_3 \end{pmatrix} \\ &= Z + \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}^T Z \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \succcurlyeq 0. \end{aligned} \quad (\text{A12})$$

On the other hand, given a complex Hermitian positive semidefinite matrix:

$$\Lambda = M + iK, \quad (\text{A13})$$

we can construct a real positive semidefinite matrix

$$Z = \frac{1}{2} \tilde{\Lambda} = \frac{1}{2} \begin{pmatrix} M & K \\ -K & M \end{pmatrix}, \quad (\text{A14})$$

such that

$$\Lambda(Z) = \Lambda. \quad (\text{A15})$$

It means that when the variable Z runs over all symmetric positive semidefinite $4n \times 4n$ matrices, the corresponding $\Lambda(Z)$ runs over all Hermitian positive semidefinite $2n \times 2n$ matrices and instead of Z , we can use Λ . The dual problem now read as follows: Maximize

$$g(\Lambda, U^+, U^-) = \text{Tr}[\Lambda \mathcal{P}(\gamma^\circ)], \quad (\text{A16})$$

over all $\Lambda \succcurlyeq 0$ and $U^\pm \succcurlyeq 0$, provided that

$$\begin{aligned} \text{Tr}[(U^+ + U^-)\sigma] &= 1 \\ U^+ - U^- &= \text{Re}(\Lambda). \end{aligned} \quad (\text{A17})$$

It is exactly the problem given in the main text.

In the reduced case the variables are $\mathbf{x} = (s, \gamma_{xx}, \gamma_{pp})$. The semidefinite constrain of the primal problem is

$$-\begin{pmatrix} \gamma_{xx} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \preccurlyeq 0, \quad (\text{A18})$$

and the $2n(n+1)$ linear constraints are

$$\begin{aligned} \pm\gamma_{xx} \mp \gamma_{xx}^\circ - s\sigma_{xx} &\leq 0, \\ \pm\gamma_{pp} \mp \gamma_{pp}^\circ - s\sigma_{pp} &\leq 0, \end{aligned} \quad (\text{A19})$$

so in this case there are five dual variables, a real positive semidefinite $2n \times 2n$ matrix $Z \succcurlyeq 0$, and four $n \times n$ matrices

$U^\pm, V^\pm \geq 0$ with non-negative elements. The Lagrangian reads as

$$L(s, \gamma_{xx}, \gamma_{pp}, \Lambda, U^\pm, V^\pm) = s - \text{Tr} \left[Z \begin{pmatrix} \gamma_{xx} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \right] + \text{Tr}[U^+(\gamma_{xx} - \gamma_{xx}^\circ - s\sigma_{xx}) - U^-(\gamma_{xx} - \gamma_{xx}^\circ + s\sigma_{xx})] \\ + \text{Tr}[V^-(\gamma_{pp} - \gamma_{pp}^\circ + s\sigma_{pp}) - V^+(\gamma_{pp} - \gamma_{pp}^\circ - s\sigma_{pp})].$$

The dual objective function is

$$g(\Lambda, U^\pm, V^\pm) = \min_{s, \gamma_{xx}, \gamma_{pp}} L(s, \gamma_{xx}, \gamma_{pp}, \Lambda, U^\pm, V^\pm),$$

where the minimization is over all real s and all symmetric γ_{xx} and γ_{pp} . For g to be bounded the coefficient of s must be zero, so we have

$$\text{Tr}[(U^+ + U^-)\sigma_{xx} + (V^+ + V^-)\sigma_{pp}] = 1. \quad (\text{A20})$$

Using the same decomposition for Λ as Eq. (A4) for Z above, we derive

$$\text{Tr} \left[\begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2^\top & \Lambda_3 \end{pmatrix} \begin{pmatrix} \gamma_{xx} & \frac{1}{2}E \\ \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \right] = \text{Tr}[\Lambda_1 \gamma_{xx} + \Lambda_3 \gamma_{pp} + \Lambda_2].$$

Equating the coefficients of γ_{xx} and γ_{pp} to zero, we obtain

$$\Lambda_{xx} \equiv \Lambda_1 = U^+ - U^-$$

$$\Lambda_{pp} \equiv \Lambda_3 = V^+ - V^-. \quad (\text{A21})$$

The dual objective is now readily seen to be given by

$$g(\Lambda, U^\pm, V^\pm) = -\text{Tr}[(U^+ - U^-)\gamma_{xx}^\circ + (V^+ - V^-)\gamma_{pp}^\circ] \\ - \text{Tr}(\Lambda_2) = -\text{Tr}[\Lambda \mathcal{P}(\gamma_{xx}^\circ, \gamma_{pp}^\circ)].$$

The KKT conditions follow from the constraints (A18) and (A19) of the primal problem.

The other theorems of the main text are derived in a similar way, by writing the Lagrangian and simplifying it with matrix manipulations.

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