

Entanglement dynamics and Page curves in random permutation circuits

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The characterization of ensembles of many-qubit random states and their realization via quantum circuits are crucial tasks in quantum-information theory. In this work, we study the ensembles generated by quantum circuits that randomly permute the computational basis, thus acting classically on the corresponding states. We focus on the averaged entanglement and present two main results. First, we derive generically tight upper bounds on the entanglement that can be generated by applying permutation circuits to arbitrary initial states. We show that the late-time “entanglement Page curves” are bounded in terms of the initial state participation entropies and their overlap with the “maximally antilocalized” state. Generally speaking, this result states that the quantum correlations generated by classical circuits are bounded in terms of some quantum property of the initial state (namely, the degree to which it can be written as a superposition of classical states). Second, comparing the averaged Rényi-2 entropies generated by (1) an infinitely deep random circuit of two-qubit gates and (2) global random permutations, we show that the two quantities are different for finite N but the corresponding Page curves coincide in the thermodynamic limit. We also discuss how these conclusions are modified by additional random phases or considering circuits of k -local gates with $k \geq 3$. Our results are exact and highlight the implications of classical features on entanglement generation in many-body systems.

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Introduction. Random ensembles of quantum operators are fundamental tools in quantum-information theory [1,2], now being routinely employed in several experimental settings such as randomized measurement protocols [3–6] or in the context of quantum advantage demonstrations [7–10]. At the same time, models evolving through discrete random operations provide a powerful computational framework to tackle nonperturbative problems in many-body physics [11,12].

The point of contact is given by the recent identification of quantum circuits—many-body systems in discrete spacetime—as useful toy models for generic many-body dynamics [13–15]. Indeed, their minimal structure provides a concrete handle to study hard topical questions pertaining to, e.g., entanglement growth [15–21], thermalization processes [21–24], and information scrambling [25–29]. Circuit models are also much easier to implement on current quantum devices than continuous-time dynamics [30], therefore providing ideal benchmarking platforms for current prototypes [31–33]. In analogy to random matrix theory, the introduction of randomness simplifies the analytical treatment of circuit models while retaining the basic features of the dynamics.

Random circuit models are defined by a few ingredients: the number k of qubits coupled by the local gates, the ge-

ometric connectivity of the circuit, and the set of unitaries from which the gates are sampled [11]. In this setting, one is typically interested in the averaged dynamics of quantum correlations, their asymptotic values, and the circuit depths required to reach a stationary output distribution. These features strongly depend on the precise model under consideration: Significant theoretical work has recently allowed us to characterize several families of circuits, with systematic results obtained in the case of Haar-random gates [34–39], random Cliffords [40–42], or random matchgate circuits [43–47].

In this work, we consider *random permutation* circuits, made of gates reshuffling the local computational basis, reproducing the action of reversible classical automata [48–52]. As recently pointed out in Ref. [53], these circuits are appealing because, by acting classically on a given basis of states, they allow one to address the role played by “quantumness” in quantum-circuit dynamics. Here, we focus on entanglement, a hallmark of quantum correlations, and present two main results, which appear to be unique features of permutation dynamics. First, we derive upper bounds on the entanglement that can be generated by permutation circuits acting on arbitrary initial states and show that they are generically saturated. This result states that the quantum correlations generated by classical circuits are bounded in terms of some quantum property of the initial state (namely, the degree to which it can be written as a superposition of classical states). Second, starting from simple N -qubit states, we compare the averaged Rényi-2 entropies generated by (1) an infinitely deep random circuit of two-qubit gates and (2) global random permutations. We show that the two quantities differ for finite N , but their Page curves [54,55] coincide in the thermodynamic limit (see Fig. 1 for a visual summary). This result highlights an unexpected locality

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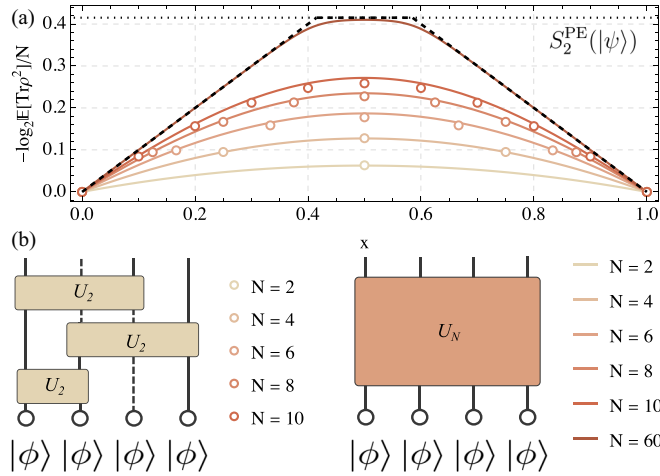


FIG. 1. (a) Averaged Rényi-2 entropies generated by a deep random circuit of two-qubit gates (open markers) and global random permutations (solid lines) acting on $|\psi\rangle = |\phi(\theta)\rangle^{\otimes N}$, with $|\phi(\theta)\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$ and $\theta = \pi/8$. The black dashed line shows our rigorous bound [Eq. (6)] and the black dotted one, the second participation entropy $S_2^{\text{PE}}(|\psi\rangle)$. (b) Schematic depiction of the two random circuits considered.

constraint on the set of permutation operators, similar, in spirit, to the one recently established on the set of symmetric unitaries [56–58].

Random permutation circuits. We consider a system of N qubits, corresponding to the Hilbert space $\mathcal{H} = \otimes_{j=1}^N \mathcal{H}_j$, with $\mathcal{H}_j \simeq \mathbb{C}^2$. We denote by σ_j^α the Pauli matrices, with $\alpha = 0, 1, 2, 3$ ($\sigma^0 = \mathbf{1}$ being the identity) and by $\{|0\rangle, |1\rangle\}$ the local computational basis. The permutation operators of interest here are those implementing a reshuffling of the states of the computational basis. Namely, given a computational basis state of k qubits, a permutation operator acts as

$$U^{(\pi)}|s\rangle = |\pi(s)\rangle. \quad (1)$$

Here, $\pi \in S_{2^k}$ is a permutation of 2^k elements, $|s\rangle = |s_1\rangle \otimes |s_2\rangle \cdots \otimes |s_k\rangle$, and $\{s_1, s_2, \dots, s_k\}$ are the coefficients of the binary representation of $s \in \mathbb{Z}_{2^k-1}$.

We consider two random-operator ensembles. The first, $\mathcal{E}_{\text{GP}}$, is the set of random global permutations and is defined by sampling from a uniform distribution the set of $(2^N)!$ permutation operators acting on N qubits. The second, $\mathcal{E}_{\text{PC}}(D)$, corresponds to the set of random permutation circuits of depth D . While most of our results will be independent of the circuit connectivity, we focus on nonlocal circuits made of two-qubit gates, which are defined by choosing, at each discrete step, a random pair of qubits i, j and applying a random permutation operator $U_{i,j}^{(\pi)}$ to them. The number of discrete steps is the circuit depth D . Following Ref. [59], we in fact consider a continuous-time version of this model: We associate an interval Δt with each discrete step and apply each random gate $U_{i,j}^{(\pi)}$ with probability $p = N\lambda\Delta t$, where $\lambda > 0$. In this way, expectation values of observables computed at time t display a well-defined limit for $\Delta t \rightarrow 0$, which allows us to ease some of the analytic computations.

Given an initial state $|\psi_0\rangle$, we denote by $|\psi_t\rangle$ the state evolved by the circuit up to time t , and consider the averaged

bipartite Rényi entropies:

$$\mathbb{E}[S_\alpha(\rho_A(t))] = \mathbb{E}[(1 - \alpha)^{-1} \log_2 \text{Tr}(\rho_A^\alpha(t))], \quad (2)$$

where $\rho_A(t) = \text{Tr}_{\bar{A}}[|\psi_t\rangle\langle\psi_t|]$, $\bar{A}$ is the complement of the region A , while $\mathbb{E}[\dots]$ denotes the ensemble average. The von Neumann entropy is obtained in the limit $\alpha \rightarrow 1$.

Entanglement bounds for permutation circuits. Consider a bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, so that any state can be written as $|\psi_0\rangle = \sum_{n,m} c_{n,m} |n\rangle_A |m\rangle_B$, where $|n\rangle_A$ and $|m\rangle_B$ denote basis states for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We begin by deriving upper bounds on the Rényi- α entropies generated by applying arbitrary permutation circuits to $|\psi_0\rangle$.

First, we introduce the inverse participation ratio

$$I_\alpha(|\psi\rangle) = \sum_s |\langle s|\psi\rangle|^{2\alpha}, \quad (3)$$

where $\{|s\rangle\}$ is the computational basis of $\mathcal{H}$. $I_\alpha(|\psi\rangle)$ is closely related to the so-called participation entropy (PE) $S_\alpha^{\text{PE}}(|\psi\rangle) = (1 - \alpha)^{-1} \log_2 I_\alpha(|\psi\rangle)$ [60–64]. We also introduce the overlap

$$z(|\psi\rangle) = |\langle a_N|\psi\rangle|, \quad (4)$$

where we denote by $|a_l\rangle = |+\rangle^{\otimes l} = (|0\rangle + |1\rangle)^{\otimes l}/2^{l/2}$ the “maximally antilocalized” state of l qubits. The state $|a_N\rangle$ can be easily seen to maximize the PE. Crucially, $z(|\psi\rangle)$ and $I_\alpha(|\psi\rangle)$ are invariant under any operator that permutes the computational basis (therefore are time independent), and the idea is to bound $S_\alpha(\rho_A(t))$ in terms of these quantities. Such a bound is also consistent with the numerical observations of Ref. [50].

To proceed, we consider the initial reduced density matrix $\rho_A(0) = \sum_{n,m,p} c_{n,m} c_{p,m}^* |n\rangle_A \langle p|$ and the dephasing channel $\mathcal{D}_l(\cdot) = \sum_s \langle s| \cdot |s\rangle |s\rangle \langle s|$, where $\{|s\rangle\}$ denotes the computational basis of $\otimes_{j=1}^l \mathcal{H}_j$. Since the dephasing channel is unital, $\mathcal{D}_l(\mathbf{1}) = \mathbf{1}$, the data-processing inequality for the sandwiched Rényi divergences [65] implies $S_\alpha(\rho_A) \leq S_\alpha(\mathcal{D}_{|A|}(\rho_A))$. Next, we note $S_\alpha(\mathcal{D}_{|A|}(\rho_A)) = (1 - \alpha)^{-1} \log_2 \sum_n p_n^\alpha$, where $p_n = \sum_m |c_{n,m}|^2$. Finally, using that the (classical) Rényi entropies of marginal probability distributions are bounded by the Rényi entropy of the joint distribution [66], we obtain

$$S_\alpha(\rho_A(t)) \leq S_\alpha^{\text{PE}}(|\psi_0\rangle). \quad (5)$$

Equation (5) provides a bound that is typically not tight. For instance, when initializing the system in the state $|\psi_0\rangle = |a_N\rangle$, the Rényi entropies remain identically zero, while $S_\alpha^{\text{PE}}(|\psi_0\rangle)$ scales linearly in N . This remark suggests improving the bound by taking into account the overlap $z(|\psi_0\rangle)$. To this end, we introduce the dephasing channel in the x basis, $\bar{\mathcal{D}}_l(\cdot) = \sum_s \langle v_s| \cdot |v_s\rangle |v_s\rangle \langle v_s|$, where $|v_s\rangle = (\sigma_1^z)^{s_1} \cdots (\sigma_l^z)^{s_l} |a_l\rangle$ and $s_1, \dots, s_l$ are the digits in the binary representation of s . Using again the data-processing inequality ($\bar{\mathcal{D}}_l$ is also unital), we obtain $S_\alpha(\rho_A) \leq S_\alpha(\bar{\mathcal{D}}_{|A|}(\rho_A))$, and combining with $\sum_s \langle v_s|\rho_A|v_s\rangle^\alpha \geq \langle a_{|A|}|\rho_A|a_{|A|}\rangle^\alpha \geq z^{2\alpha}$, we arrive at $S_\alpha(\rho_A(t)) \leq (1 - \alpha)^{-1} \log_2 z^{2\alpha}$. Therefore, putting everything together, we obtain

$$S_\alpha(\rho_A(t)) \leq \min[|A|, S_\alpha^{\text{PE}}(|\psi_0\rangle), (1 - \alpha)^{-1} \log_2 z^{2\alpha}], \quad (6)$$

where we also took into account that the Rényi entropy cannot exceed $|A|$, the number of qubits in A (assuming $|A| \leq N/2$). Equation (6) is the first main result of our work.

A few remarks are in order. First, we stress that the intermediate result [Eq. (5)] already appeared in Ref. [67], although its derivation there was not fully formalized (cf. also Ref. [68]). Second, we note that the quantities appearing on the right-hand side of Eq. (6) can be efficiently measured in families of low-entangled, short-correlated states. For instance, both the overlap and the PE may be obtained by applying the method of Ref. [69] (see also the recent proposal [70] for measuring the von Neumann PE). Third, we note that $(1 - \alpha)^{-1} \log_2 z^{2\alpha} \rightarrow \infty$ for $\alpha \rightarrow 1$, thus trivializing the z dependence of the upper bound. Still, using the Fannes inequality [71], we are able to derive the following alternative bound for $\alpha = 1$ (see the Supplemental Material [72]):

$$S_1(t) \leq \min[|A|, S_1^{\text{PE}}(|\psi\rangle), |A|\sqrt{1 - z^2} + O(1)]. \quad (7)$$

Finally, an important question is whether the bound is tight, as in principle there could be other missing dynamical invariants. In the rest of this Letter, we provide analytical and numerical evidence that, for *typical* permutation operators, the bound in Eq. (6) for the Rényi-2 entropy is indeed saturated at the leading order in N .

To this end, we now proceed to compute the averaged bipartite entanglement in the global permutation $[\mathcal{E}_{\text{GP}}]$ and the permutation-circuit $[\mathcal{E}_{\text{PC}}(t)]$ ensembles, respectively. We are interested in two aspects: estimating the time needed for averages over $\mathcal{E}_{\text{PC}}(t)$ to approach stationary values, and understanding whether the entanglement averaged over $\mathcal{E}_{\text{PC}}(\infty)$ and $\mathcal{E}_{\text{GP}}$ coincide. These problems are well understood in the Haar random case [35–39]: Sufficiently connected Haar random circuits over N qubits approach the ensemble of global Haar random operators at infinite depths, while for two-local circuits, the entanglement approaches its asymptotic value at the *scrambling time* $t_s \sim O(\log N)$ [13,73,74]. In the case at hand, however, the situation is much less clear as two-qubit permutation gates belong to the Clifford group [75,76] and circuits generated by Clifford gates are well known to display special features (including being efficiently simulable when initialized in so-called stabilizer states [1]).

Effective dynamics in replica space. As pointed out in Refs. [51,52], several features of permutation circuits can be simulated by sampling the basis states appearing in the decomposition of the input state and computing the corresponding classical evolution [51]. The computational cost to simulate the expectation value of an observable is then determined by the statistical variance of the sampled values. While this method is provably efficient for observables with bounded operator norm, entanglement-related quantities require a number of samples growing exponentially in N , thus offering no real advantage over a brute-force state-vector simulation of the quantum dynamics [72]. Therefore, studying entanglement dynamics under permutation circuits is nontrivial.

Here, we present an analytic approach to compute the purity $\mathbb{E}[2^{S_2(t)}]$ [77], which makes use of several recently developed tools for random quantum circuits [11]. We start by exploiting the Choi-Jamiołkowski isomorphism [1] to map the operator $\rho_A(t) \otimes \rho_A(t)$ to the state

$$|\rho_A(t) \otimes \rho_A(t)\rangle = \mathbf{1} \otimes \rho_A(t) \otimes \mathbf{1} \otimes \rho_A(t) |\mathcal{I}\rangle, \quad (8)$$

where $|\mathcal{I}\rangle = |I^+\rangle^{\otimes N} \in \mathcal{H}^{\otimes 4}$ and $|I^+\rangle = \sum_{a,b=0}^1 |aabb\rangle$. The purity of $\rho_A(t)$ can then be written as [72]

$$\text{Tr}_A \mathbb{E}[\rho_A^2(t)] = \langle G_{|A|, N-|A|} | \rho_A(t) \otimes \rho_A(t) \rangle, \quad (9)$$

where we have introduced the boundary state $|G_{n_-, n_+}\rangle = |I^-\rangle^{\otimes n_-} \otimes |I^+\rangle^{\otimes n_+}$, with $|I^+\rangle$ defined above and $|I^-\rangle = \sum_{a,b=0}^1 |abba\rangle$. This formalism is useful because it allows one to reformulate the computation of the time-dependent purity in terms of the dynamics of the averaged vector $\mathbb{E}[|\rho(t) \otimes \rho(t)\rangle]$. Exploiting now the fact that the randomness is uncorrelated in space and time, a standard derivation shows [72]

$$\frac{d}{dt} \mathbb{E}[|\rho(t) \otimes \rho(t)\rangle] = -\mathcal{L} \mathbb{E}[|\rho(t) \otimes \rho(t)\rangle], \quad (10)$$

where

$$\mathcal{L} = \frac{2\lambda}{N-1} \sum_{(j,k)} (1 - \mathcal{U}_{j,k}). \quad (11)$$

Here, the sum is over sites that are nearest neighbors according to the circuit connectivity; we set $\mathcal{U}_{j,k} = \mathbb{E}[U_{j,k}^* \otimes U_{j,k} \otimes U_{j,k}^* \otimes U_{j,k}]$ and $(\cdot)^*$ denotes complex conjugation. By explicit computation [53], the average $\mathcal{U}_{j,k}$ is seen to be a nonunitary operator acting on the tensor product of local subspaces of dimension $d_{\text{eff}} = 15$. Equation (10) describes a deterministic evolution in a *replica* Hilbert space of local dimension d_{eff} . Although it does not require sampling over disorder realizations, its solution is still challenging due to the exponential growth of the Hilbert-space dimension. In the following, we simplify the problem relying on two key properties.

First, for the nonlocal circuits studied in our work, the sum in Eq. (11) runs over arbitrary pairs of sites so that $\mathcal{L}$ is invariant under site permutation. This introduces a significant simplification [59,78]: When starting from permutation-symmetric initial conditions, the dynamics takes place in the symmetric subspace, whose dimension is $D_{\text{symm}} = \binom{N+d_{\text{eff}}-1}{d_{\text{eff}}-1}$. Unfortunately, despite growing only polynomially in N , it does so with a large exponent, $D_{\text{symm}} \sim N^{14}$, thus making the direct numerical solution of Eq. (11) unfeasible for large N . To make progress, we exploit the fact that two-qubit permutation gates belong to the Clifford group. This property allows us to further simplify the problem and obtain analytic results for certain families of initial states, using a method that is reminiscent of the twirling technique of Ref. [79].

Exact dynamics from typical initial states. We consider random product states $|\psi\rangle = \bigotimes_j |\phi_j\rangle$, where $|\phi_j\rangle = v_j |0\rangle$ with v_j Haar random unitaries. Denoting by $\mathcal{P}_N$ the set of Pauli strings without overall phases, the invariance of the Haar measure under left multiplication implies that $(P^* \otimes P)^{\otimes 2}$ leaves $\mathbb{E}_{\{\phi_j\}}[\bigotimes_j (|\phi_j\rangle^* \langle \phi_j|)^{\otimes 2}]$ invariant for any P in $\mathcal{P}_N$. Therefore, also

$$\frac{1}{4^N} \sum_{P \in \mathcal{P}_N} (P^* \otimes P)^{\otimes 2} = \left[\frac{1}{4} \sum_{\alpha=0}^3 \sigma_j^{\alpha*} \otimes \sigma_j^\alpha \otimes \sigma_j^{\alpha*} \otimes \sigma_j^\alpha \right]^{\otimes N} \quad (12)$$

leaves invariant the averaged state. Denoting now by Π_j the quantity in brackets on the right-hand side, we note that Π_j is a projector onto a four-dimensional subspace of $\mathcal{H}_j^{\otimes 4}$. Therefore, the averaged state is left invariant by the projector $(\bigotimes_j \Pi_j)$. Moreover, we also have $(\bigotimes_j \Pi_j) \mathcal{L} = \mathcal{L} (\bigotimes_j \Pi_j)$,

which follows from the fact that the two-qubit gates are Clifford operators [72]. Putting all together and setting $\tilde{\mathcal{L}} \equiv (\otimes_j \Pi_j) \mathcal{L} (\otimes_j \Pi_j)$, we obtain that the solution to Eq. (10) after averaging over the random initial states can be written as

$$|\rho(t) \otimes \rho(t)\rangle = e^{-\tilde{\mathcal{L}}t} \mathbb{E}_{\phi_j} [\otimes_j \Pi_j (|\phi_j\rangle^* |\phi_j\rangle)^{\otimes 2}]. \quad (13)$$

That is, the dynamics takes place in the site-permutation symmetric subspace of $\tilde{\mathcal{H}} = \otimes_j \tilde{\mathcal{H}}_j$ with $\dim(\tilde{\mathcal{H}}_j) = 4$. The corresponding dimension is now $\binom{N+3}{3} \sim N^3$, which allows one to perform efficient numerical computations.

In fact, Eq. (13) allows us to go further and derive a system of differential equations encoding the purity dynamics. Specifically, we set $G_{p,q,r,s} = \langle G_{p,q,r,s} | \rho_A(t) \otimes \rho_A(t) \rangle$, where $|G_{p,q,r,s}\rangle = |I^-\rangle^{\otimes p} \otimes |I^+\rangle^{\otimes q} \otimes |I^0\rangle^{\otimes r} \otimes |I^x\rangle^{\otimes s}$ with $|I^\pm\rangle$ are defined above, while $|I^0\rangle = \sum_{a=0}^1 |aaaa\rangle$, $|I^x\rangle = \sum_{a,b=0}^1 |abab\rangle$. Using Eq. (13), we find [72]

$$\frac{dG_{p,q,r,s}(t)}{dt} = \sum_{\{\ell_k\}_{k=1}^4} c[\{\ell_k\}] G_{p+\ell_1, q+\ell_2, r+\ell_3, s+\ell_4}(t). \quad (14)$$

The coefficients $c[\{\ell_k\}]$ are reported explicitly in Ref. [72], together with an expression for the initial condition $G_{p,q,r,s}(0)$ corresponding to random initial product states. Equation (14) is the second main result of our work.

Entanglement dynamics and Page curves. Before illustrating the solution to Eq. (14), we complete our analysis by discussing the averaged bipartite purity over the global ensemble $\mathcal{E}_{\text{GP}}$. This computation can be performed via a similar replica-space approach, but it is now much simpler due to the absence of spatial structures and time dependence. In this case, we can perform the computation for arbitrary initial product states, obtaining an explicit, although unwieldy, formula [72]. For instance, the averaged purity for random initial product states reads

$$\begin{aligned} & \mathbb{E}_{\text{GP}}[\text{Tr} \rho_A^2] \\ &= \frac{y(\xi^{1-x} + \xi^x)q - [(\xi - \xi^x)(\xi^{1-x} - \xi) + 2y]p}{y(\xi - 3)(\xi - 2)(\xi - 1)}, \end{aligned} \quad (15)$$

where $p = 7 + (-4 + \xi)\xi$, $q = 8 + (-5 + \xi)\xi$, $\xi = 2^N$, and $y = 3^N$. As another example, Fig. 1 reports the evaluation of the analytic formula for an initial state

$$|\psi_0\rangle = |\phi(\theta)\rangle^{\otimes N}, \quad |\phi(\theta)\rangle = \cos \theta |0\rangle + \sin(\theta) |1\rangle. \quad (16)$$

As we vary the subsystem size $|A| = xN$ in the limit $N \rightarrow \infty$, $x \in (0, 1)$, the logarithm of the averaged purity yields the Rényi-2 entropy Page curve corresponding to the ensemble $\mathcal{E}_{\text{GP}}$ [80]. We have derived the infinite- N Page curve for an arbitrary product state of the form (16): Setting $\bar{s}_2(x) = \lim_{N \rightarrow \infty} (-\log(\mathbb{E}_{\text{GP}}[\text{Tr} \rho_A^2]))/N$ and choosing $x \leq 1/2$, we find [72]

$$\bar{s}_2(x) = \min\{x, s_2^{\text{PE}}(\theta), -2 \log_2 \zeta^2\}, \quad (17)$$

where $s_2^{\text{PE}} = \lim_{N \rightarrow \infty} S_2^{\text{PE}}(|\psi\rangle)/N$ and $\zeta^2 = |\langle \phi(\theta) | + \rangle|^2$. Note that since $-2 \log_2 \zeta^2 = \lim_{N \rightarrow \infty} [-\log_2(z^2)/N]$ with $z = |\langle \psi | a_N \rangle|$, Eq. (17) shows that the bipartite entanglement saturates the bound in Eq. (6) as $N \rightarrow \infty$.

We finally discuss the entanglement dynamics of the permutation circuit, as described by Eq. (14). Figure 2 shows

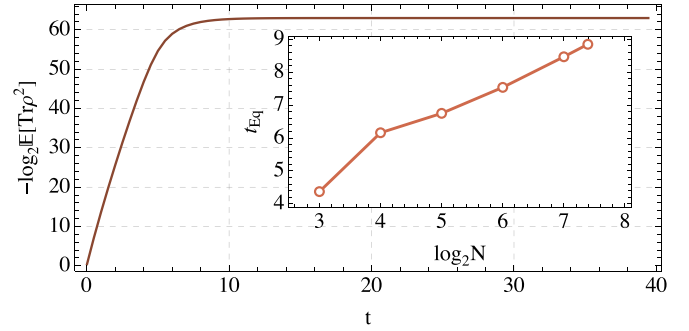


FIG. 2. Rényi-2 entropy dynamics for random permutation circuits and random initial product states. The system size is $N = 128$. Inset: the equilibration time, as a function of N .

the time evolution from random initial states. Based on a scaling analysis performed at increasing system sizes, we see that the time needed for the entanglement to equilibrate to its maximum value scales as $t_e \sim O(\log N)$, similarly to the scrambling time in Haar random circuits. Next, we compare the averaged purity computed by solving Eq. (14) up to large times and the one corresponding to global random permutations [cf. Eq. (15)]. Our results are shown in Fig. 3. The steady-state solution of Eq. (14) is obtained by solving the system $dG_{p,q,r,s}(t)/dt = 0$, and the exact expression of the averaged purity is reported in Ref. [72]. Comparing the latter with Eq. (15) shows that the finite- N results are different, whereas the Page curves coincide in the thermodynamic limit. One can repeat the stationary-state analysis also for arbitrary initial states, which are not invariant under Eq. (12). Interestingly, in this case the difference between the two Page curves does not necessarily vanish in the $N \rightarrow \infty$ limit [72].

We stress that, although our gates belong to the Clifford group, here we are applying the circuit to arbitrary (random) initial states, yielding nonstabilizer states for which entanglement dynamics is typically hard to simulate [81,82]. Therefore, it is not *a priori* obvious that the entanglement Page curves corresponding to $\mathcal{E}_{\text{GP}}$ and $\mathcal{E}_{\text{PC}}(\infty)$ should differ for finite N , and only coincide in the thermodynamic limit. A natural question, however, is what happens if we break the Clifford property. This can be easily achieved by increasing the support k of our local permutation gates or introducing phases.

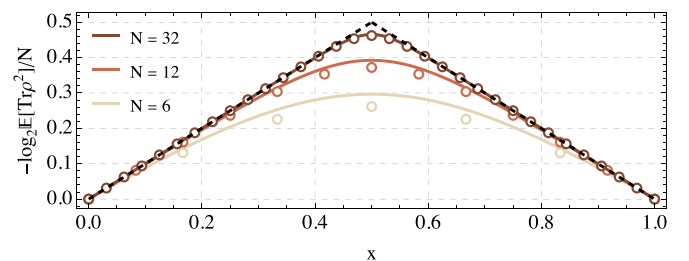


FIG. 3. Comparison between the Page curves obtained by averaging over the late-time circuit ensemble, $\mathcal{E}_{\text{PC}}(\infty)$ (open markers), and the global-permutation ensemble, $\mathcal{E}_{\text{GP}}$ (solid lines). The system is initialized in a random product state. The upper bound is shown in black dashed lines.

k-local permutation and random-phase gates. Permutation gates are typically not Clifford for $k \geq 3$. In fact, it is known that circuits made of random three-qubit gates form permutation t designs for all values of t [83–85]. That is, for sufficiently connected and deep circuits, all the moments of the permutation-circuit ensemble match those of the ensemble of global permutations taken from the *alternating group*, which is formed by the even permutations. At the same time, the moments of the alternating and full permutation groups coincide up to order $t = 2^N - 2$ [84]. Then, since the full entanglement spectrum of the reduced density matrix is uniquely determined by its Rényi- α entropies with integer $\alpha \leq 2^{N/2}$, the entanglement Page curves corresponding to the infinite-time circuit ensemble and $\mathcal{E}_{\text{GP}}$ coincide for all N .

Going beyond permutations, one can consider more general unitaries that preserve the computational basis, by allowing for additional phases setting $U_{ij}^{(\pi)}|s_1, s_2\rangle = e^{i\varphi_{s_1, s_2}}|\pi(s)_1, \pi(s)_2\rangle$. Circuits made of such gates were studied in Refs. [51,52], where they were referred to as automaton circuits. We can define automaton-circuit ensembles that are analogous to $\mathcal{E}_{\text{PC}}(t)$ and $\mathcal{E}_{\text{GP}}$. The first one is obtained by considering a circuit of random two-qubit gates, where the permutations and phases are randomly drawn from the uniform measure. As before, one can then take a continuous-time limit, defining the ensemble $\mathcal{E}_{\text{PPC}}(t)$. The second ensemble, $\mathcal{E}_{\text{GPP}}$, is constructed by applying global random unitaries acting as $U^{(\pi)}|s\rangle = e^{i\varphi_{s_1, \dots, s_N}}|\pi(s)\rangle$, with random π and φ .

In this setting, it can be checked that the maximally antilocalized state $|a_N\rangle$ is not conserved by the circuit, so that the bipartite entanglement can only be bounded by the PE as in Eq. (5). Similarly to the case of permutations, one can ask whether the bound is tight and the entanglement entropies averaged over $\mathcal{E}_{\text{GPP}}$ and $\mathcal{E}_{\text{PPC}}(t)$ coincide. These questions can be addressed following the steps described above for the random permutations. It turns out that the random phases

introduce great simplifications, allowing us to arrive at explicit results for very general initial states, including highly entangled states [72]. Our analysis shows that the entanglement entropies averaged over $\mathcal{E}_{\text{PPC}}(\infty)$ and $\mathcal{E}_{\text{GPP}}$ coincide for arbitrary N , and that, similarly to the case of random permutations, the bound in Eq. (5) is saturated in the thermodynamic limit.

Outlook. Our work raises several questions. First, it is important to understand whether the bound obtained for the von Neumann entropy [Eq. (7)] is also typically tight. Because of the different functional dependence on z compared to Eq. (6), answering this question in the affirmative would reveal an interesting phenomenology: For initial states close to $|a_N\rangle$, the Rényi Page curves corresponding to $\mathcal{E}_{\text{GP}}$ would be all vanishing, while the von Neumann Page curve would not. Moreover, further investigation should be devoted to understanding which types of initial states, beyond random ones, saturate the bound (6) in the thermodynamic limit. Next, a natural direction would be to consider adding minimal ingredients to the permutation circuit, exploring how the late-time output distribution is modified. For instance, by adding measurements, a natural question is to characterize the so-called projected ensemble [86,87] arising from permutation-circuit dynamics.

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Data availability. The data that support the findings of this study are openly available at the following [88].

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