

Continuous matrix-product operators for quantum fields

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In this work, we introduce an ansatz for continuous matrix-product operators for quantum field theory. We show that (1) they admit a closed-form expression in terms of finite number of matrix-valued functions without reference to any lattice parameter; (2) they are obtained as a suitable continuum limit of matrix-product operators; and (3) they preserve the entanglement area law directly in the continuum, and in particular they map a continuous matrix-product state (cMPS) to another cMPS. As an application, we use this ansatz to construct several families of continuous matrix-product unitaries beyond quantum cellular automata.

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Introduction. Tensor networks have been very useful in providing a compact description of quantum many-body physics with local interactions. Many analytical insights into tensor network theory were also made possible by its compatibility with tools from quantum information theory [1]. Tensor networks can be used to describe physically relevant states known as matrix-product states (MPS) [2,3], and also to describe operators known as matrix-product operators (MPOs) [4,5]. The latter are particularly useful to describe mixed states [4–9], symmetries [10–13], and approximate short-time evolutions [14,15].

Tensor networks are designed to capture the notion of area-law entanglement in quantum many-body systems [16–19]. This feature allows low-energy states of local Hamiltonians to be well approximated by MPS [2,20–22]. Similarly, the fact that quantum cellular automata (QCA) [23–25] (unitaries with exact light cone) can only create little entanglement is captured by its equivalent representation as translationally invariant matrix-product unitaries (MPUs) [26,27]. However, formulating a good notion of area-law entanglement is non-trivial in the continuum, i.e., quantum fields. This is relevant whenever one wishes to preserve some properties that only exist in the continuum (e.g., symmetries) or when the system of interest is naturally defined as a quantum field.

The first attempt toward porting tensor network techniques directly to the continuum was the construction of continuous matrix-product states (cMPS) [28–30]. They have been successful as a variational ansatz for strongly interacting (non)relativistic quantum systems [31–35] and as a continuum limit of MPS; they capture the entanglement area law directly

in the continuum. Furthermore, cMPS can be viewed as a real-space renormalization procedure applied to one-dimensional MPS run backward, i.e., performing a fine-graining procedure rather than coarse graining by blocking [36,37].

In this work, we provide an ansatz for *continuous matrix-product operator* (cMPO) for quantum fields [38]. We formulate cMPO as a suitable continuum limit of discrete MPO without reference to any underlying lattice spacing. We show that it admits a closed-form expression in terms of a path-ordered exponential similar to traced Wilson line operator in non-Abelian gauge theory. By construction, it preserves area-law entanglement natively in the continuum and maps a cMPS to another cMPS. As an application of this ansatz, we are able to construct continuous matrix-product unitaries (cMPUs) that are a continuum limit of MPUs beyond QCA [39].

Review of cMPS. Let us first review the basic features of cMPS. Our setting is a nonrelativistic bosonic field defined on a one-dimensional interval $I \subset \mathbb{R}$ of length ℓ . In the second quantization formalism, a bosonic field operator $\psi(x)$ satisfies the canonical commutation relations $[\psi(x), \psi^\dagger(y)] = \delta(x - y)\mathbb{1}$ and a vacuum state $|\Omega\rangle$ with respect to ψ is defined to be the state for which $\psi(x)|\Omega\rangle = 0$ for all x . The Hilbert space of the field is a Fock space $\mathfrak{F}(\mathcal{H}) := \bigoplus_{N=0}^{\infty} \mathcal{H}^{\odot N}$, where $\mathcal{H} \cong L^2(I)$ is the one-particle sector Hilbert space and $\mathcal{H}^{\odot N}$ is the symmetrized subspace of N identical particles. By construction, any arbitrary state in $\mathfrak{F}(\mathcal{H})$ must have finite total particle number.

Definition 1 [28,29]. A bosonic cMPS with bond dimensions D is the state $|\psi[B, Q, L]\rangle \in \mathfrak{F}(\mathcal{H})$ given by

$$|\psi[B, Q, L]\rangle := \text{Tr}_D(B\mathcal{P}e^{\int_I dx Q(x) \otimes \mathbb{1} + L(x) \otimes \psi^\dagger(x)}) |\Omega\rangle, \quad (1)$$

where $\mathcal{P}$ denotes path ordering, $|\Omega\rangle \in \mathfrak{F}(\mathcal{H})$ is the Fock vacuum state annihilated by $\psi(x)$, $Q(x)$ and $L(x)$ are one-parameter family of matrices [40] in $M_D(\mathbb{C})$, $B \in M_D(\mathbb{C})$ is the boundary matrix, and Tr_D is the trace over the auxiliary space.

In general, B is allowed to depend on ℓ but not x . We say that a cMPS is bulk uniform if $Q(x), L(x)$ are constant

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matrices independent of $x \in I$, and uniform if $B = \mathbb{1}$. For concreteness, we use $I = [-\ell/2, +\ell/2]$. In principle, one can generalize this to multiple fields [29], but for clarity, we restrict to one species of bosons.

A useful representation of the cMPS as an element of the Fock space is obtained by writing explicitly the sum over j -particle sectors, namely,

$$|\psi[B, Q, L]\rangle = \sum_{j=0}^{\infty} \int D^j x \text{Tr}(B V_-^1 L_1 V_1^2 \cdots L_j V_j^+) \psi^\dagger(x_1) \cdots \psi^\dagger(x_j) |\Omega\rangle, \quad (2)$$

where we use a shorthand for the path-ordered measure $\int D^j x \equiv \int_I \cdots \int_I dx_1 \cdots dx_j \Theta(x_1 - x_2) \cdots \Theta(x_{j-1} - x_j)$, and the wavefunction coefficients are given in terms of B, Q, L :

$$V_i^{i+1} = \mathcal{P} \exp \int_{x_i}^{x_{i+1}} dx Q(x), \quad L_i \equiv L(x_i). \quad (3)$$

We write $V_-^1 \equiv V_{-\ell/2}^{x_1}$ and $V_j^+ \equiv V_{x_j}^{+\ell/2}$ when the lower and upper limits of the integrals are at the boundaries, respectively. For convenience, we refer to states of the form $\int D^j x f(x_1, \dots, x_j) \psi^\dagger(x_1) \cdots \psi^\dagger(x_j) |\Omega\rangle$ as j -particle Fock states.

The cMPS can be constructed as a continuum limit of discrete MPS, by setting the local dimension $d \rightarrow \infty$ and identifying the tensors at site x to be $A_x^0 \approx \mathbb{1} + \epsilon Q(x)$, $A_x^1 \approx \sqrt{\epsilon} L(x)$ at leading order in lattice spacing ϵ , and all $A_x^{n \geq 2}$ are fixed by physical consistency. For physically relevant situations such as computation of energy expectation values or correlation functions, in principle one may also need to impose regularity conditions for $Q(x), L(x)$ [29].

Definition of cMPO. A theory of continuous matrix-product operator should provide us with a sufficiently large class of operators with the following desirable properties: (1) it is parametrized by some finite number of matrix-valued functions in $M_D(\mathbb{C})$; (2) it is obtained as a continuum limit of some MPO with constant bond dimension D ; and (3) Given two cMPO O_1, O_2 with bond dimensions D_1, D_2 , respectively, the product $O_1 O_2$ is another cMPO with bond dimension $D \leq D_1 D_2$. If O is also unitary, we would also like to have (4) a systematic way of verifying unitarity in terms of the local tensors. These considerations suggest the following definition of cMPO.

Definition 2 (cMPO). Let $Q(x), L(x), R(x), T(x) \in M_D(\mathbb{C})$ be matrix-valued functions on I and $B \in M_D(\mathbb{C})$ the boundary matrix that can depend on system size $\ell \in (0, \infty]$. A cMPO with bond dimension D is an operator O acting on $\mathfrak{F}(H)$ defined as

$$O \equiv O[B, Q, L, R, T] = \text{Tr}_D(B \mathcal{P} e^{\int dx \mathfrak{L}_x[\cdot]}) (|\Omega\rangle \langle \Omega|), \quad (4)$$

where $[\cdot]$ takes operators of the field theory as input,

$$\mathfrak{L}_x := Q(x) \otimes \text{Id} + L(x) \otimes l_x + R(x) \otimes r_x + T(x) \otimes \text{Ad}_x, \quad (5)$$

and the supermaps acting on the field are defined as

$$l_x[\cdot] := \psi^\dagger(x)[\cdot], \quad r_x[\cdot] := [\cdot]\psi(x), \\ \text{Id}[\cdot] := [\cdot], \quad \text{Ad}_x[\cdot] := \psi^\dagger(x)[\cdot]\psi(x), \quad (6)$$

where “Ad_x” refers to adjoint action by $\psi(x)$. We remark that O is defined to act specifically on the Fock space $\mathfrak{F}(H)$ where the Fock vacuum $|\Omega\rangle$ lives for any $\ell \in (0, \infty]$, as this allows us to work with the thermodynamic limit. Once we fix $|\Omega\rangle$, all bulk-uniform cMPS in the thermodynamic limit that are not equivalent to the Fock vacuum are not in the domain of O .

Let us show that Definition 2 satisfies our requirements (1)–(3). By construction, (1) is satisfied. To show that it fulfills (2), we first express a discrete MPO in a “Fock-like representation” (see Sec. I of Supplemental Material [41]). We first define the shorthand for the “free propagator” $W_{x_i}^{x_j} = A_{x_i}^{00} A_{x_i+1}^{00} \cdots A_{x_j}^{00}$ and a collection of *ladder maps* at site x :

$$f_x^{ij}[\cdot] := (J_x^+)^i [\cdot] (J_x^-)^j, \quad (7)$$

where $f_x^{00}[\cdot] \equiv \text{Id}[\cdot]$ is the identity map and $J^\pm$'s are D -dimensional matrix representations of the ladder operators for the $\mathfrak{su}(2)$ algebra. Then, we can write any MPO as (see Sec. I of Supplemental Material [41])

$$O_N[B, \{A_x^{ij}\}] = \sum_{m=0}^N \sum_{x_1 < \cdots < x_m} \text{Tr}(B W_{-}^{x_1-1} A_{x_1}^{i_1 j_1} \cdots A_{x_m}^{i_m j_m} W_{x_m+1}^+) \\ \times f_{x_1}^{i_1 j_1} \cdots f_{x_m}^{i_m j_m} [|\Omega_N\rangle \langle \Omega_N|] + \mathcal{S}_{N,d}, \quad (8)$$

where for fixed m the summation contains exactly m ladder operators $J_{x_k}^\pm$ and $i_k, j_k = 0, 1$. The remainder $\mathcal{S}_{N,d}$ contains all terms that involve $(J_{x_k}^\pm)^{n \geq 2}$ on either side of $|\Omega_N\rangle \langle \Omega_N|$. Now the strategy is to follow the cMPS construction by rescaling the matrices and operators as

$$A_x^{00} \approx \mathbb{1}_D + \epsilon Q(x) \quad f_x^{00}[\cdot] = \text{Id}[\cdot], \\ A_x^{10} \approx \sqrt{\epsilon} L(x) \quad f_x^{10}[\cdot] \approx \sqrt{\epsilon} (\psi^\dagger(x))[\cdot], \\ A_x^{01} \approx \sqrt{\epsilon} R(x) \quad f_x^{01}[\cdot] \approx \sqrt{\epsilon} [\cdot] (\psi(x)), \\ A_x^{11} \approx T(x) \quad f_x^{11}[\cdot] \approx \epsilon \psi^\dagger(x)[\cdot] \psi(x), \quad (9)$$

and all other matrices A_x^{ij} are fixed by consistency. We then take the limit $N \rightarrow \infty, \epsilon \rightarrow 0$ while keeping ℓ constant, which gives Eq. (2). It is important that A_x^{11} is $O(1)$ and not necessarily close to $\mathbb{1}_D$ like A_x^{00} .

To show that O satisfies (3), we note that given two cMPOs O_i with bond dimension D_i ($i = 1, 2$) that has the form (2), the product $O = O_1 O_2$ also takes the same form, with bond dimension $D \leq D_1 D_2$, and the local tensors are given by

$$B = B_1 \otimes B_2, \quad T = T_1 \otimes T_2, \\ Q = Q_1 \otimes \mathbb{1}_{D_2} + \mathbb{1}_{D_1} \otimes Q_2 + R_1 \otimes L_2, \\ L = L_1 \otimes \mathbb{1}_{D_2} + T_1 \otimes L_2, \quad R = \mathbb{1}_{D_1} \otimes R_2 + R_1 \otimes T_2. \quad (10)$$

This can be proven from the Dyson series expansion using the product relations between l_x, r_x , and Ad_x (see Sec. II of Supplemental Material [41]). Consequently, $O_1 O_2$ indeed takes the form of Eq. (2) with bond dimension $D \leq D_1 D_2$ and (3) is satisfied.

Definition 2 can be used to show that a cMPO preserves the entanglement area law natively in the continuum; that is, a cMPO O_1 with bond dimension D_1 maps a cMPS $|\psi[B_2, Q_2, L_2]\rangle$ with bond dimension D_2 to a cMPS $|\psi[B, Q, L]\rangle$ with bond dimension $D \leq D_1 D_2$. This follows

from the fact that, according to Definition 1, we can write the cMPS as $|\psi[B_2, Q_2, L_2]\rangle = O_2 |\Omega\rangle$ for some operator O_2 acting on the vacuum. It can be shown that O_2 is a cMPO according to Definition 2 (see Sec. III of Supplemental Material [41]).

There is some gauge freedom in the definition of cMPO if we view the path-ordered exponential

$$W_x^y := \mathcal{P}e^{\int_x^y dz \mathcal{L}_z[-]} \quad (11)$$

as a kind of Wilson line found in non-Abelian gauge theory, generated by matrix-valued “gauge field” $\mathcal{L}_x$. A sufficient condition for two sets of matrices to generate the same cMPO is that $\text{Tr}[\widetilde{B}W^+] = \text{Tr}[BW^+]$, i.e., that the “traced Wilson line” is invariant under the local gauge transformation [42]

$$\begin{aligned} \widetilde{W}_x^y &= g(y)W_x^y g(x)^{-1}, \quad g(y) \in GL_D(\mathbb{C}), \\ \widetilde{B} &= g(x_-)Bg(x_+)^{-1}, \end{aligned} \quad (12)$$

where $x_{\pm}$ are the end points, conventionally taken to be $x_{\pm} = \pm\ell/2$ for a box of length ℓ . To see how the tensors transform under the local gauge transformation, we use the fact that the operator $\mathcal{L}_x$ is formally a generator of W_x^y satisfying the first-order differential equation

$$\frac{dW}{dy} = \mathcal{L}_y W, \quad (13)$$

which gives the following transformations for the tensors:

$$\begin{aligned} Q(x) &= g(x)^{-1} \widetilde{Q}(x) g(x) + g(x)^{-1} \frac{dg}{dx}, \\ T(x) &= g(x)^{-1} \widetilde{T}(x) g(x), \\ L(x) &= g(x)^{-1} \widetilde{L}(x) g(x), \quad R(x) = g(x)^{-1} \widetilde{R}(x) g(x). \end{aligned} \quad (14)$$

Notably, $Q(x)$ transforms like the vector potential in non-Abelian gauge theory.

Application: cMPU. As a nontrivial application of the cMPO ansatz, we show that we can construct several interesting families of cMPUs (unitary cMPOs), where unitarity is known to impose a strong constraint on the structure of MPOs. These families are a natural continuum limit of nonuniform MPUs beyond the QCA family [39]. In discrete settings, uniform MPUs are known to be equivalent to translationally invariant QCAs [26], and by allowing nontrivial boundary we can have MPUs beyond QCAs [39].

The first example is the displacement operator

$$D(\alpha) := e^{\int dx \alpha(x) \psi^\dagger(x) - \alpha^*(x) \psi(x)}, \quad (15)$$

a $D = 1$ cMPU with $B = 1$, $T = 1$, $Q = -\frac{1}{2}|\alpha(x)|^2$, $L = \alpha(x)$, and $R = -\alpha^*(x)$. This follows by noting that the supermaps $l_x, r_x, \text{Ad}_x, \text{Id}$ are pairwise commuting, and

$$\mathcal{P}e^{\int dx \text{Ad}_x[-]}[|\Omega\rangle\langle\Omega|] = \sum_{j=0}^{\infty} \int D^j x \text{Ad}_x^j[\Omega] = \mathbb{1}. \quad (16)$$

We write $\text{Ad}_x^j(\Omega) := \psi^\dagger(x_1) \cdots \psi^\dagger(x_j) |\Omega\rangle\langle\Omega| \psi(x_j) \cdots \psi(x_1)$ for convenience. This also shows that $\mathbb{1}$ is a $D = 1$ cMPU with tensors $B = T = 1$, $Q = L = R = 0$, so we know that the set of cMPUs is nonempty.

Similar to the discrete MPU setting, in general, we cannot check the unitarity of a cMPO by looking at the local tensors

alone. In Refs. [26,39], a characterization for MPUs is given by formulating unitarity as a condition over auxiliary space. Below, we provide an analogous condition.

Lemma 1. Let O be a cMPO with bond dimension D and let

$$\begin{aligned} O^\dagger O &= \text{Tr}_D(B_+ \mathcal{P}e^{\int dx \mathcal{L}_+(x)[-]})[\Omega], \\ O O^\dagger &= \text{Tr}_D(B_- \mathcal{P}e^{\int dx \mathcal{L}_-(x)[-]})[\Omega] \end{aligned}$$

be two cMPOs with bond dimension $\leq D^2$, where

$$\mathcal{L}_{\pm} := Q_{\pm} \otimes \text{Id} + L_{\pm} \otimes l_x + R_{\pm} \otimes r_x + T_{\pm} \otimes \text{Ad}_x.$$

Let $V_{x,\pm}^y$ be the free propagator generated by $Q_{\pm}$ and

$$K_{j,\pm}^L := L_{\pm}(x_j), \quad K_{j,\pm}^R := R_{\pm}(x_j), \quad K_{j,\pm}^A := T_{\pm}(x_j).$$

Then, O is unitary if and only if for all $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &\text{Tr}(B_{\pm} V_{-, \pm}^1 K_{1, \pm}^{\alpha_1} V_{1, \pm}^2 K_{2, \pm}^{\alpha_2} \cdots K_{j, \pm}^{\alpha_j} V_{j, \pm}^+) \\ &= \begin{cases} 1, & \alpha_k = A \quad \forall 1 \leq k \leq j, \\ 0, & \exists 1 \leq k \leq j : \alpha_k \neq A \end{cases} \end{aligned} \quad (17)$$

for any $x_1, x_2, \dots, x_j \in I$.

This follows because if O is unitary, then the matrix-product coefficients of both $O^\dagger O$ and $O O^\dagger$ must produce the same coefficients as the cMPU for $\mathbb{1}$ in Eq. (16), so it cannot contain any L, R matrices and the coefficients containing only $K_{\pm}^A$ must be all equal to 1. Lemma 1 is simpler to use for bulk-uniform matrices. In this case, we can rewrite the matrices in the “interaction picture” with respect to Q , so that $\mathcal{K}_{\pm}^{\alpha}(x) := V_{x, \pm}^x K_{\pm}^{\alpha} V_{x, \pm}^{x-}$ and $\mathcal{B}_{\pm}(\ell) := V_{x, \pm}^{x+} B_{\pm}$, in which case the unitarity condition becomes

$$\text{Tr}(\mathcal{B}_{\pm}(\ell) \mathcal{K}_{\pm}^{\alpha_1}(x_1) \cdots \mathcal{K}_{\pm}^{\alpha_j}(x_j)) = \begin{cases} 1, & \alpha_k = A \quad \forall k, \\ 0, & \exists k : \alpha_k \neq A. \end{cases} \quad (18)$$

Below, we consider mainly three different families of cMPUs (see Sec. IV of Supplemental Material [41] for derivations and other cMPU families). The first is the *phase cMPUs* based on phase MPUs [39], which has $L = R = 0$. For example, the following tensors

$$\begin{aligned} Q(x) &= i \text{diag}(q_1(x), \dots, q_D(x)), \quad T(x) \in U(1)^D \rtimes S_D, \\ B &= V_{x+}^{x-} |k\rangle\langle +|, \quad k = 0, 1, 2, \dots, D-1 \end{aligned}$$

give rise to a phase cMPU. Here, $U(1)^D \equiv \bigoplus_{j=1}^D U(1)$, S_D is the symmetric group of D elements, V_{x+}^{x-} is defined in Eq. (3), and q_j, t_j are real-valued functions. More concretely, $T(x)$ is a generalized permutation matrix $T(x) = e^{i \text{diag}(t_1(x), \dots, t_D(x))} P$, with $P \in S_D$. Here, we use rounded bracket notation $|k\rangle\langle +|$ for vectors in the bond space and $|+\rangle = (1, 1, 1, \dots, 1)$. To get a better sense of these objects, consider the special case $D = 2$ with $B \propto |0\rangle\langle +|$, $Q = i \frac{\omega}{2} Z$, $T = X$. The resulting cMPU has an explicit form involving a string operator [43], namely (see Sec. IV of Supplemental Material for the derivation [41]),

$$U_{\theta} = e^{-i\omega K}, \quad K = \int_x^{\ell/2} dx x (-1)^{\Pi(x)} n(x), \quad (19)$$

where $\Pi(x) := \int_x^{\ell/2} dz n(z)$ and $n(x) := \psi^\dagger(x) \psi(x)$.

The second family of cMPU can be obtained by modifying the phase unitary in order to go beyond diagonal unitaries. We

use the fact that any qubit MPU whose vectorization admits a nontrivial compression as a locally maximally entangleable state [44] is locally unitary equivalent to a phase unitary [39]. Since the phase cMPU U_θ is based on qubit MPUs, we can construct an infinite family of cMPUs that are not diagonal by concatenating U_θ with displacement operator that takes the role of local unitaries (see Sec. IV of Supplemental Material [41]).

Last but not least, the third example we consider is based on Example 14 of Ref. [39], where a class of MPU called “unitary action over product subspace” is described by a $D = 5$ MPU

$$U = \mathbb{1}^{\otimes N} + \sum_{i,j=0}^1 (V_{ij} - \delta_{ij}) |i\rangle^{\otimes N} \langle j|^{\otimes N}, \quad (20)$$

where U acts as $V \in U(2)$ over the two-dimensional subspace $\text{span}\{|0\rangle^{\otimes N}, |1\rangle^{\otimes N}\}$. This was constructed as an example where bulk-uniform MPU with nontrivial boundary does not in general admit a block-diagonal decomposition for its local tensors [39]. This particular example does *not* generalize to the continuum, but we can modify this slightly to give a similar cMPU, namely,

$$U = \mathbb{1} + \sum_{i,j=1}^2 (V_{ij} - \delta_{ij}) |\psi_i\rangle \langle \psi_j|, \quad (21)$$

where $|\psi_i\rangle \equiv |\psi_i[B_i, Q_i, L_i]\rangle$ are pairwise orthonormal cMPS with bond dimension $D = D_i$. The idea is to use the cMPO ansatz (2) to show that the operator

$$P_j = |\psi_i[B_i, Q_i, L_i]\rangle \langle \psi_j[B_j, Q_j, L_j]| \quad (22)$$

is a cMPO with bond dimension $D_i D_j$, so the cMPU has bond dimension $D = 1 + \sum_{i,j=1}^2 D_i D_j$. An analog to the above MPU in Eq. (20) can be constructed by choosing $|\psi_1\rangle = |\Omega\rangle$ and $|\psi_2\rangle$ to be normalized one-particle Fock states $|1_f\rangle := \int dx f(x) \psi^\dagger(x) |\Omega\rangle$, which is a cMPS with bond dimension $D = 2$ with $B_2 \propto \sigma^-$, $Q_2 = 0$, and $L_2 = f(x) \sigma^+$ (see Sec. IV of Supplemental Material [41]).

Discussion and outlook. In this work, we have introduced the family of cMPOs for bosonic quantum fields. They admit a

closed-form expression in terms of path-ordered exponential of finitely many matrix-valued functions, they are obtained as a suitable continuum limit of discrete MPOs and by construction they preserve entanglement area law directly in the continuum, and they map cMPS to cMPS. As an application, we constructed a continuum limit of MPUs beyond QCA studied in Ref. [39].

Several open questions arise from these considerations. First, the cMPO ansatz (2) provides a natural starting point for studying systematically the continuum limit of MPDOs [4–9], i.e., continuous matrix-product density operators (cMPDOs): indeed, convex combinations of cMPS projectors form a subclass of cMPDOs. Second, since the cMPOs are in general non-Gaussian, they allow explorations into non-Gaussian operations in $(1+1)$ -dimensional quantum field theories. Third, ideally one would like to obtain the generalization to fermionic setting natively as a continuum limit of fermionic tensor networks [45,46], and also generalization to higher dimensions so that it preserves area-law entanglement of continuous tensor networks [47]. Last but not least, it would be interesting to see whether it is possible to formulate a notion of cMPO symmetry using the cMPO ansatz.

We mention in passing that the cMPOs defined in this work seem to exclude some physically interesting operators in the context of quantum field theory, such as translation or unitaries generated by field derivatives. In principle, the ansatz can be generalized to cover such operators, although it is no longer automatic that the resulting cMPO arises from some continuum limit of discrete MPO. We provide some heuristic examples in Sec. V of Supplemental Material [41] and we leave the general theory of such “generalized cMPOs” for future work.

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- [41] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/d61y-cxx3> for details, which includes Refs. [44,49].
- [42] In practice, one may also need to impose additional smoothness conditions on g (such as k -differentiability) to ensure that the cMPO action on a cMPS does not worsen the well-behavedness of the initial cMPS [29].
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