

Two-detector reconstruction of multiphoton states in linear optical networks

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We propose a method for partial state reconstruction of multiphoton states in multimode (N -photon M -mode) linear optical networks (LONs) employing only two bucket photon-number-resolving detectors. The reconstructed Heisenberg-Weyl-reduced density matrix captures quantum coherence and symmetry with respect to Heisenberg-Weyl operators. Employing deterministic quantum computing with one qubit (DQC1) circuits, we reduce the detector requirement from M to 2, while the requirement on measurement configurations is retained $2M^3 - 2M$. To ensure physicality, maximum likelihood estimation is incorporated into the DQC1 reconstruction process, with numerical simulations demonstrating the efficiency of our approach and its robustness against interferometer noises. This method offers a resource-efficient solution for state characterization in large-scale LONs to advance photonic quantum technologies.

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Introduction. Photonic quantum computing (PQC) is a promising platform for universal quantum computation, offering advantages such as long decoherence times, room-temperature operation, and seamless integration with quantum networks. In particular, linear optical implementations of PQC [1,2] provide a scalable platform for various computational paradigms, including measurement-based one-way quantum computation [3–8], gate teleportation [9,10], KLM protocol [11,12], and boson sampling [13–17].

In linear optical quantum computing (LOQC), high-quality quantum state preparation, particularly the generation of entangled states, is crucial for enabling high-fidelity quantum processing in LOQC. Various approaches for specific entanglement generation and detection have been theoretically proposed and experimentally validated [18–25]. In qubit or qudit systems, the quality of state preparation is typically evaluated through quantum state tomography [26–28]. For the evaluation of an N -photon state preparation in an M -mode linear optical network (LON), one can employ the full state tomography with an M' -mode reconfigurable LON interferometer followed by M' photon-number-resolving (PNR) detectors, as outlined by Banchi *et al.* [29] [see Fig. 1(a)]. It requires $\mathcal{O}((M+N)^N)$ LON configurations with M PNR detectors, or one LON configuration with at least approximately $\mathcal{O}(M^2/N)$ PNR detectors for $N \ll M$ [30]. The latter scheme reduces the number of configurations, which improve the efficiency in time, but demands additional M^2 modes and

PNR detectors, significantly increasing the experimental cost. Such a trade-off between the efficiency in measurement time and cost set up the minimum experimental requirement on the full state tomography in multiphoton multimode LON systems. In particular, as LON systems continuously scale up [31], the full state tomography is prohibitively expensive in terms of both time and resources, presenting a major challenge for experimental implementations.

To evaluate state preparation with limited experimental resources, we propose a method to reconstruct a reduced $M \times M$ density matrix, derived from the full density matrix, using only two PNR detectors. While this reduced matrix coarsens the full information of the multiphoton state, it retains the symmetry with respect to Heisenberg-Weyl (HW) operators [32–35]. We refer to this reduced representation as the HW-reduced density matrix of a multiphoton state. Such an HW-reduced density matrix encodes information about multiphoton indistinguishability, which can be quantified through the expectation value of a multimode permutation operator, one of the HW operators. Since multiphoton indistinguishability is a key requirement for achieving quantum advantage in boson sampling [36,37], the HW-reduced density matrix can serve as a useful tool for characterizing the quality of multiphoton light sources used in photonic quantum computing.

The proposed reconstruction of the HW-reduced density matrix with two PNR detectors extends the state tomography techniques developed for qudit systems [38,39], leveraging the principles of deterministic quantum computing with one qubit (DQC1) [40]. Our method requires solely two bucket PNR detectors and $2M^3 - 2M$ measurement configurations. To ensure the physicality of the reconstructed HW-reduced density matrix, we incorporate maximum likelihood estimation (MLE) [41–43] into our approach. Numerical simulations shows that the proposed reconstruction method effectively

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extracts partial yet meaningful information about multiphoton states employing even noisy LON interferometers.

HW-reduced density matrix. A single-photon M -mode LON system encodes a qudit system with a dimension of M . Its quantum state can be fully reconstructed by evaluating the HW operators $\hat{\Lambda}_{k,l} := \hat{X}^k \hat{Z}^l$ [32–35], where the HW operator is constructed using the mode shift operator $\hat{X}$ and the phase shift operator $\hat{Z}$ [44]. The mode shift operator cyclically shifts a photon to the next neighboring mode, while the phase shift operator introduces a mode-dependent phase to each output mode. These operations are defined as $\hat{X} \hat{a}_m^\dagger \hat{X}^\dagger = \hat{a}_{(m+1) \pmod M}^\dagger$ and $\hat{Z} \hat{a}_m^\dagger \hat{Z}^\dagger = \omega^m \hat{a}_m^\dagger$, where $\omega := \exp(i2\pi/M)$ is the M th root of the unity. As shown in Refs. [38,39,45], the expectation values of the HW operators $\hat{\Lambda}_{k,l}$ can be used to reconstruct the full density matrix of a single-photon M -mode state via

$$\rho_{1ph} = \frac{1}{M} \sum_{k,l} \langle \hat{\Lambda}_{k,l} \rangle L_{k,l}^\dagger, \quad (1)$$

where $L_{k,l}$ is the matrix representation of $\hat{\Lambda}_{k,l}$ in the single-photon M -mode system. Explicitly, the matrix element $L_{k,l}(m, m') := \langle \dots 01_m 0 \dots | \hat{\Lambda}_{k,l} | \dots 01_{m'} 0 \dots \rangle$ is the transition coefficient of mapping the single-photon state from the m' th mode to the m th mode.

In an N -photon M -mode LON system, namely, a $d_{M,N}$ -dimensional Hilbert space spanned the Fock state basis,

$$\{|\mathbf{n}\rangle = |n_0, \dots, n_{M-1}\rangle : |\mathbf{n}| = N, \# \text{mode}(\mathbf{n}) = M\}, \quad (2)$$

the representation of an HW operator is diagonal with respect to HW-irreducible subspaces [23]. The expectation value $\langle \hat{\Lambda}_{k,l} \rangle$ of a multiphoton state $\hat{\rho}$ is therefore invariant under the decoherence with respect to HW-irreducible subspaces $\langle \hat{\Lambda}_{k,l} \rangle_\rho = \text{tr}((\sum_{\mathbb{E}} \hat{\Pi}_{\mathbb{E}} \hat{\rho} \hat{\Pi}_{\mathbb{E}}) \hat{\Lambda}_{k,l})$, where $\hat{\Pi}_{\mathbb{E}}$ is the projection onto the $\mathbb{E}$ -spanned HW-irreducible subspaces, and $\mathbb{E}_n := \{\hat{X}^m |\mathbf{n}\rangle\}_m$ is generated by the mode shift operator $\hat{X}$ applied to the Fock state $|\mathbf{n}\rangle$. The expectation value of an HW operator is a quantity that contains coarsened information associated with an $M \times M$ density matrix ρ_{HW} reduced from the full $d_{M,N} \times d_{M,N}$ density matrix ρ via a mixture of the density matrix $\rho_{\text{HW}}^{(\mathbb{E})}$ within each HW subspace,

$$\rho_{\text{HW}} := \sum_{\mathbb{E}} p_{\mathbb{E}} \rho_{\text{HW}}^{(\mathbb{E})}, \quad (3)$$

where the entries of $\rho_{\text{HW}}^{(\mathbb{E})}$ are obtained from $\rho_{\text{HW}}^{(\mathbb{E})}[m, m'] = \langle \hat{X}^m \mathbf{n} | \hat{\rho} | \hat{X}^{m'} \mathbf{n} \rangle_{d_{\mathbb{E}}} / (M p_{\mathbb{E}})$, $d_{\mathbb{E}} = |\mathbb{E}|$ is the dimension of the $\mathbb{E}$ -spanned HW-irreducible subspace, and $p_{\mathbb{E}} = \text{tr}(\hat{\Pi}_{\mathbb{E}} \hat{\rho} \hat{\Pi}_{\mathbb{E}})$ is the weight of the quantum state in the subspace (see Definition S1 in the Supplemental Material [46] for the detailed construction of an HW-reduced density matrix).

For a mode number M and a photon number N that fulfill $\text{gcd}(N, M) = 1$, the quantum system within each $\mathbb{E}$ -spanned HW subspace is a well-defined M -dimensional qudit system. Analogous to the representation of a single-photon state in Eq. (1), the density matrix $\rho_{\text{HW}}^{(\mathbb{E})}$ within an HW-irreducible subspace $\mathbb{H}_{\mathbb{E}}$ can be decomposed as a sum of the HW density matrices $L_{k,l}^\dagger$. As a result, the HW-reduced density matrix of a multiphoton state can be reconstructed via evaluating the HW operators according to the following theorem.

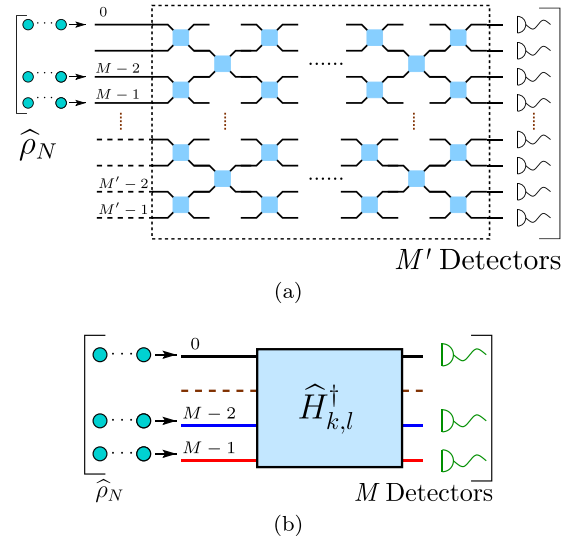


FIG. 1. (a) A general schematic for full state tomography of an N -photon M -mode system in an M' -mode ($M' \geq M$) reconfigurable LON followed by M' PNR detectors. Blue squares represent tunable Mach-Zehnder interferometers. (b) Direct evaluation of an HW operator $\hat{\Lambda}_{k,l}$ through generalized Hadamard transformation $\hat{H}_{k,l}$ followed by M PNR detectors.

Theorem 1 (HW-reduced state reconstruction). For an N -photon M -mode state with $\text{gcd}(N, M) = 1$, its HW-reduced density matrix of an N -photon M -mode quantum state can be reconstructed via

$$\rho_{\text{HW}} = \frac{1}{M} \sum_{k,l=0}^{M-1} \langle \hat{\Lambda}_{k,l} \rangle_\rho L_{k,l}^\dagger, \quad (4)$$

where $L_{k,l}(m, m') := \langle \dots 01_m 0 \dots | \hat{\Lambda}_{k,l} | \dots 01_{m'} 0 \dots \rangle$ is the matrix representation of the HW operator $\hat{\Lambda}_{k,l}$ in the single-photon subspace.

Proof. This theorem is a special case of Theorem S4 in the Supplemental Material [46]. ■

The measurement configuration for the reconstruction of an HW-reduced state via direct HW-operator evaluation is illustrated in Fig. 1(b). It employs the generalized Hadamard transforms $\hat{H}_{k,l}$ [32–35] to map the phase shift operator $\hat{Z}$ into the HW operator $\hat{\Lambda}_{k,l} = \hat{H}_{k,l} \hat{Z} \hat{H}_{k,l}^\dagger$. The expectation value of an HW operator is explicitly obtained via $\langle \hat{\Lambda}_{k,l} \rangle = \text{Tr}(\hat{Z}(\hat{H}_{k,l}^\dagger \hat{\rho} \hat{H}_{k,l}))$. It requires $M^2 - 1$ measurement configurations and M PNR detectors.

DQC1 reconstruction using two PNR detectors. Introducing a time delay to each output mode in the direct reconstruction scheme shown in Fig. 1(b), one can then employ time demultiplexing to reduce the number of PNR detectors to one. Although this approach is feasible for systems with a small number of modes, the required delay-line length grows rapidly $\mathcal{O}(M^2)$ as with the mode number M , which makes it impractical for large-mode systems. To reduce the number of PNR detectors in large-mode systems, we adopt the DQC1 method in our state tomography, which lowers the requirement to only two PNR detectors.

The DQC1 method [40] has been adapted for state reconstruction in qudit systems [38,39]. In DQC1-based state

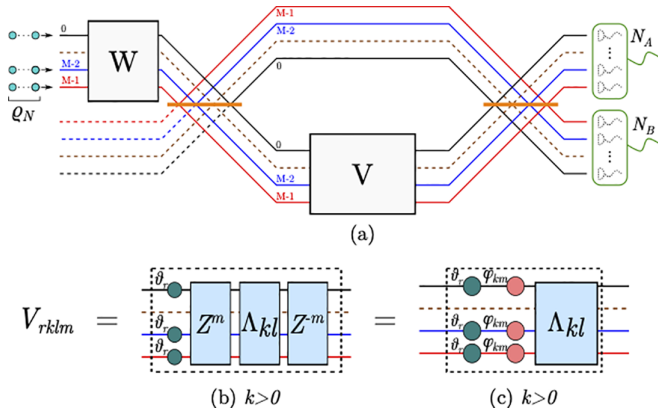


FIG. 2. DQC1 measurement. (a) Measurement configuration of DQC1 in multiphoton LONs for a unitary $\hat{U} = \hat{W}^\dagger \hat{V} \hat{W}$. (b) Construction of the erasure channel for evaluating HW operators $\hat{\Lambda}_{k,l}$ with $k \geq 1$. (c) Implementation of the erasure channel.

reconstruction, an auxiliary qubit is introduced to control HW operators on the target qudit system, allowing all HW operators to be efficiently evaluated through qubit measurements of the auxiliary qubit. Extending DQC1 to N -photon M -mode LON systems, one can evaluate all HW operators $\langle \hat{\Lambda}_{k,l} \rangle$ with solely two PNR detectors.

The DQC1 measurement setup for an M -mode LON system is illustrated in Fig. 2(a). An M -mode LON system is extended to $2M$ modes by adding M auxiliary modes via two M -mode beam splitters that interfere the upper and lower arms. It constructs a dual- M -rail encoding system. For the evaluation of a target unitary $\hat{U} = \hat{W}^\dagger \hat{V} \hat{W}$, one first inputs an N -photon testing state ρ_N to the upper M modes. After the M -mode transform $\hat{W}$, the state is split by an M -mode beam splitter into upper and lower arms followed by an M -mode transform $\hat{V}$ applied in the lower arm. Combining the two arms by a second beam splitter, at the output, two bucket PNR detectors measure the total photon number N_A and N_B of the two arms. In practice, a superconducting-nanowire-single-photon-detector array [47] could be a good candidate for such a bucket detector.

The dual- M -rail qubit system in this N -photon M -mode DQC1 configuration is defined by two projectors, $\hat{\Pi}_\pm$, associated with the parity of the photon number N_B . The corresponding Pauli-Z observable $\hat{\zeta}(U)$ for the unitary $\hat{U}$ is given by

$$\hat{\zeta}(U) = \hat{\Pi}_+ - \hat{\Pi}_-, \quad \text{with} \quad \hat{\Pi}_\pm := \sum_{(-1)^{N_B} = \pm 1} \hat{\Pi}_{N_A} \otimes \hat{\Pi}_{N_B}, \quad (5)$$

where $\hat{\Pi}_{N_{A,B}} = \sum_{|n_{A,B}|=N_{A,B}} |n_{A,B}\rangle \langle n_{A,B}|$ is the projector onto the total photon numbers $N_{A,B}$ on the output arms A and B, respectively. The DQC1 Z-observable $\hat{\zeta}(U)$ is evaluated from the experimental data by

$$\langle \hat{\zeta}(U) \rangle_{\text{DQC1}} = \sum (-1)^{N_B} p(N_B | N_A + N_B = N). \quad (6)$$

One can show that the expectation value of the DQC1 Z-observable evaluates the observable $(\hat{U} + \hat{U}^\dagger)/2$ (see Lemma

S5 in the Supplemental Material [46]),

$$\langle \hat{\zeta}(U) \rangle_{\text{DQC1}} = \left\langle \frac{1}{2} (\hat{U} + \hat{U}^\dagger) \right\rangle_{\rho_N} = \frac{1}{2N} \langle \hat{U} + \hat{U}^\dagger \rangle_{\rho_N}. \quad (7)$$

For single-photon systems $N = 1$, which is a well-defined qudit system, the expectation value is linear with respect to the sum of operators, $\langle \hat{U} + \hat{U}^\dagger \rangle = \langle \hat{U} \rangle + \langle \hat{U}^\dagger \rangle$. The expectation value of the DQC1 Z-observable is therefore equal to the real part of the expectation value of the target unitary operator $\langle \hat{U} \rangle_\rho$, expressed as $\langle \hat{\zeta}(U) \rangle_{\text{DQC1}} = \Re \langle \hat{U} \rangle_\rho$. Employing the following simple configuration,

$$\hat{W} = \hat{1} \quad \text{and} \quad \hat{V}_{rkl} = (-i)^r \hat{\Lambda}_{k,l}, \quad (8)$$

one can extract both the real ($r = 0$) and imaginary ($r = 1$) parts of the expectation value of an HW operator $\langle \hat{\Lambda}_{k,l} \rangle_\rho$. This enables the full reconstruction of the state $\hat{\rho}_{N=1}$ employing solely two PNR detectors via Eq. (4).

In multiphoton systems, photon bunching disrupts the linearity of the expectation value $\langle \hat{U} + \hat{U}^\dagger \rangle \neq \langle \hat{U} \rangle + \langle \hat{U}^\dagger \rangle$. This nonlinearity prevents the DQC1 evaluation of $\hat{\Lambda}_{k,l}$ employing the simple configuration in Eq. (8). The nonlinearity arises from photon bunching, which induces transitions between different HW subspaces for the operator $\hat{\Lambda}_{k,l} + \hat{\Lambda}_{k,l}^\dagger$. It leads to nonzero off-diagonal elements in its matrix representation with respect to HW subspaces. On the contrary, the matrices of the individual operators $\hat{\Lambda}_{k,l}$ and $\hat{\Lambda}_{k,l}^\dagger$ are both diagonal with respect to HW subspaces.

To restore the linearity, we introduce an erasing channel $\mathcal{E}_k(\cdot)$ to cancel out the unwanted transitions among HW subspaces [see Fig. 2(b)],

$$\mathcal{E}_k(\hat{U}) = \frac{2^{(1-\delta_M^N)}}{M} \sum_m \cos\left(\frac{2\pi}{M} kmN\right) \hat{Z}^{-m} \hat{U} \hat{Z}^m. \quad (9)$$

This erasing channel works for $N \leq M/\gcd(2k, M)$ and $k \geq 1$, ensuring that $\langle \mathcal{E}_k(\hat{\Lambda}_{k,l} + \hat{\Lambda}_{k,l}^\dagger) \rangle_\rho = \langle \hat{\Lambda}_{k,l} \rangle_\rho + \langle \hat{\Lambda}_{k,l}^\dagger \rangle_\rho$ (see Lemmas S6 and S7 in the Supplemental Material [46]). The DQC1 evaluation of the erasing channel can be derived from Eq. (7), which leads to the following theorem for DQC1 evaluation of the operator $\hat{U}_{k,l}$ that are equivalent to HW operators through an M -mode unitary $\hat{W}$, $\hat{U}_{k,l} = \hat{W}^\dagger \hat{\Lambda}_{k,l} \hat{W}$.

Theorem 2 (DQC1 evaluation with two detectors). For an N -photon M -mode state $\hat{\rho}$, where $N \leq M/\gcd(2k, M)$, the expectation value of an operator $\hat{U}_{k,l} = \hat{W}^\dagger \hat{\Lambda}_{k,l} \hat{W}$ with $k \geq 1$ can be evaluated by

$$\begin{aligned} \Re(e^{i\theta N} \langle \hat{U}_{k,l} \rangle_\rho) &= \frac{2^{N-\delta_0^{2kN}}}{M} \sum_{m=0}^{M-1} \cos\left(\frac{2\pi}{M} kmN\right) \langle \hat{\zeta}(\hat{W}^\dagger \hat{V}_{\theta,klm} \hat{W}) \rangle_{\text{DQC1}}, \end{aligned} \quad (10)$$

where $\delta_0^{2kN} = 1$ for $2kN \pmod{M} = 0$,

$$\hat{V}_{\theta,klm} = e^{i(\theta + \varphi_{km})\hat{N}} \hat{\Lambda}_{k,l}, \quad \text{with} \quad \varphi_{km} = km(2\pi/M), \quad (11)$$

and $\hat{N} = \sum_m \hat{a}_m^\dagger \hat{a}_m$ counts the total photon number.

Proof. See the Supplemental Material [46]. ■

The construction of DQC1 evaluation of HW-equivalent operators for multiphoton states is shown in Figs. 2(a) and 2(c). With this construction, for a photon number smaller

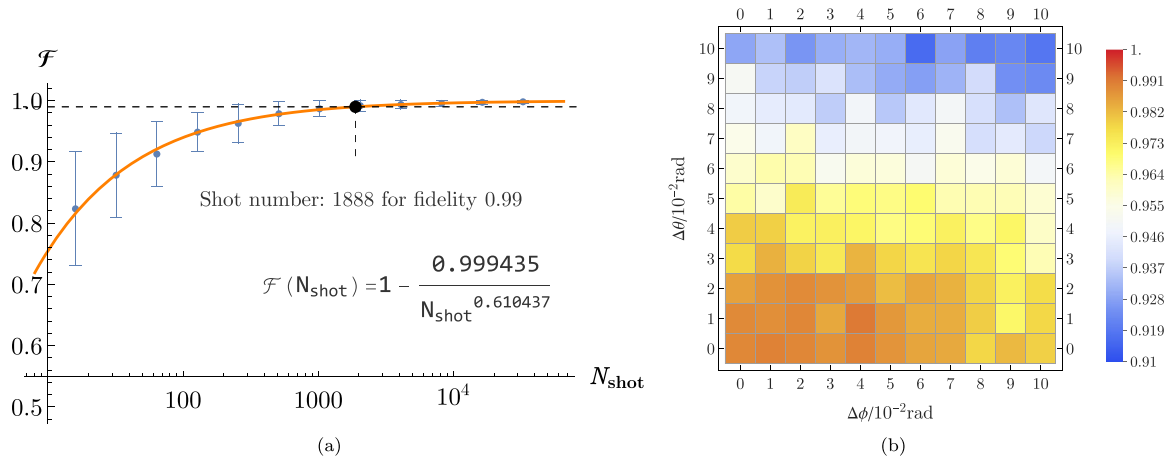


FIG. 3. DQC1 reconstruction using MLE for two-photon three-mode states. (a) Fidelity of the DQC1 reconstruction of random two-photon three-mode pure states. (b) Fidelity of noisy DQC1 reconstruction with 2048 shots each DQC1 measurement for random two-photon three-mode pure states.

than the nontrivial minimum divisor $D_{\min}$ of M , namely, $N < D_{\min}(M)$, one can employ Theorems 1 and 2 to reconstruct the HW-reduced density matrix via evaluating the real and imaginary parts of all the HW operators $\langle \hat{\Lambda}_{k,l} \rangle$ with the following configurations. An example of the applicable scenarios is $N = 1, 2, 3, 4$ and $M = 25$.

Corollary 1 (DQC1 HW-reduced reconstruction). In an N -photon M -mode LON system with $N < D_{\min}(M)$, one can reconstruct the HW-reduced density matrix of a quantum state with two PNR detectors:

$$\rho_{HW} = \frac{1}{M} \sum_{k,l=0}^{M-1} \langle \hat{\Lambda}_{k,l} \rangle \rho L_{k,Nl}^\dagger, \quad (12)$$

via the DQC1 evaluation of HW operators employing the following configurations:

$$\begin{aligned} \langle \hat{\Lambda}_{k \geq 1, l} \rangle : \hat{W} &= \hat{\mathbb{1}}, \quad \hat{V}_{r,k,l,m} = e^{i(\theta_r + \varphi_{km})\hat{N}} \hat{\Lambda}_{k,l}, \\ \langle \hat{\Lambda}_{0, l \geq 1} \rangle : \hat{W} &= \hat{H}_{1,0}, \quad \hat{V}_{r,l,0,m} = e^{i(\theta_r + \varphi_{lm})\hat{N}} \hat{\Lambda}_{l,0}. \end{aligned} \quad (13)$$

By choosing the phase $\theta_r = -r\pi/(2N)$, one can obtain the real ($r = 0$) and imaginary ($r = 1$) parts of $\langle \hat{\Lambda}_{k,l} \rangle$. For $k = l = 0$, we simply take $\langle \hat{\Lambda}_{0,0} \rangle = 1$. Such a DQC1 reconstruction of HW-reduced density matrix has a measurement-configuration complexity of $2M^3 - 2M$.

Although the method can only reconstruct the HW-reduced density matrix, it can be extended by incorporating the approach in Ref. [48], which allows ancillary coherent states as inputs. By introducing coherent states into the lower arms of the DQC1 configuration, one can access the quantum coherence between two HW-irreducible subspaces and reconstruct the corresponding off-diagonal elements.

DQC1 reconstruction with MLE. In experiments, limited shot numbers N_{shot} lead to unphysical HW-reduced density matrix after a straightforward state reconstruction employing Corollary 1. To address this problem, maximum likelihood estimation [41–43] can be employed to restore the physicality of the reconstructed density matrix $\rho_{HW}^{(\text{MLE})}$. The quality of DQC1 reconstruction of an input state $\rho_{HW}^{(\text{in})}$ is evaluated using the state fidelity $F_{N_{\text{shot}}}(\rho_{HW}^{(\text{in})}, \rho_{HW}^{(\text{MLE})})$ [49].

We simulated DQC1 reconstruction for two-photon three-mode states. To test the efficiency with respect to measurement shot numbers, for a given shot number N_{shot} , we randomly select 100 input pure states $|\psi_i\rangle$, implement the DQC1 reconstruction of $\rho_{HW,i}$, and calculate the fidelities $F_{N_{\text{shot}},i} = \langle \psi_i | \rho_{HW,i}^{(\text{MLE})} | \psi_i \rangle$. The average and standard deviation of $\{F_{N_{\text{shot}},i}\}_i$ for different shot numbers N_{shot} are evaluated and plotted in Fig. 3(a). One can observe that the average reconstruction fidelity converges to 1 and achieves a value of 0.99 with 1888 shots. The simulated result for two-photon five-mode, three-photon five-mode, and two-photon seven-mode are provided in the Supplemental Material [46].

To test the robustness of our method against the noises of interferometers in LONs, we adopt the noise model in Ref. [50] and numerically evaluate the average reconstruction fidelities using noisy Mach-Zehnder (MZ) interferometers in the Clements construction [51] of DQC1 LON circuits. The two parameters $(\theta \pm \Delta\theta, \phi \pm \Delta\phi)$ of each MZ interferometer determine the reflectivity and the phase shift of each MZ interferometers, respectively. They are sampled under the Gaussian distribution with the standard deviations $\Delta\theta$ and $\Delta\phi$. The DQC1 reconstruction fidelity with 2048 shots per measurement configuration using noisy MZ interferometers is plotted in Fig. 3(b). One can observe that the noises in reflectivity angles θ affect the fidelity of the DQC1 reconstruction of HW-reduced states more than the noises in the phase-shift angles ϕ . Under a noise up to $(\Delta\theta, \Delta\phi) = (0.1 \text{ rad}, 0.1 \text{ rad})$, one can achieve an average reconstruction fidelity greater than 0.9.

Conclusion. We have developed a method for partial state reconstruction of multiphoton states in multimode (N -photon M -mode) LON systems using only two bucket PNR detectors. The reconstructed density matrix is an $M \times M$ HW-reduced density matrix, which is projected from a multiphoton state onto HW subspaces [Eq. (3)]. It contains the symmetry properties with respect to HW operators $\hat{\Lambda}_{k,l}$. The HW-reduced density matrix is reconstructed via evaluating HW operators (Theorem 1), originally requiring M PNR detectors and $\mathcal{O}(M^2)$ measurement configurations. Employing DQC1 circuits, we reduce the detector requirement to two. Due to

nonlinearity introduced by photon bunching, additional M DQC1 measurement configurations are needed for each HW operator $\hat{\Lambda}_{k,l}$ (Theorem 2). It allows us to reconstruct the HW-reduced density matrix with only $2M^3 - 2M$ measurement configurations (Corollary 1). Maximum likelihood estimation is incorporated into the two-detector DQC1 reconstruction to ensure the physicality of the reconstructed HW-reduced density matrix. Numerical simulations of MLE reconstruction have demonstrated the robustness and efficiency of our method.

The two-detector DQC1 reconstruction method is applicable when the photon number N is smaller than the smallest nontrivial divisor of the mode number M . Although the two PNR detectors could, in principle, be replaced by a single parity-check measurement of N_B on the lower arms, such a configuration is not photon-loss-aware. To detect photon loss for fixed-photon-number inputs, we employ two PNR detectors to resolve both N_A and N_B , allowing photon-loss events to be identified and removed via postselection on outcomes within the targeted photon-number subspace. This postselection enables accurate reconstruction of the HW-reduced density matrix for fixed-photon-number states. However, it is not resilient to photon loss when the input is generated

from continuous-variable sources, which may include higher-photon-number components. It is therefore essential to extend the method to a more photon-loss-resilient scheme in future work.

As LON systems for photonic quantum computing continue to scale up [31], direct characterization becomes increasingly resource-intensive due to the growing number of required detectors. Our method addresses this challenge by providing a two-detector solution for state reconstruction in large-mode systems. Beyond state tomography, this approach is versatile. Whenever HW operators need to be evaluated, one can employ this method for other applications, such as multiphoton indistinguishability evaluation [52,53], quantum state fidelity estimation [45,54], and entanglement detection [22–25]. It is a practical and resource-efficient tool for advancing photonic quantum technologies.

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Data availability. The data supporting this study's findings are available within the article.

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