

Bardeen-Cooper-Schrieffer state representation and pairing detection in the fermionic Tonks-Girardeau gas

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We demonstrate that the ground state of the fermionic Tonks-Girardeau gas is a number-conserving Bardeen-Cooper-Schrieffer (BCS) state. This offers a compelling new perspective on describing the ground-state wave function of the fermionic Tonks-Girardeau gas, enabling a complete description in second quantization. The proposed wave function is constructed based on the occupation numbers and natural orbitals of the one-body density matrix and is valid under any external potential. Specifically, we provide the exact coefficients that allow one to describe the fermionic Tonks-Girardeau gas as a number-conserving BCS state. Furthermore, the proposed wave function's form in second quantization allows us to derive the necessary expectation values to detect pairing in the fermionic Tonks-Girardeau gas. Through this approach, we demonstrate and show how to detect that the fermionic Tonks-Girardeau gas not only exhibits nontrivial quantum correlations but also is a paired state.

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Introduction. One-dimensional fermionic and bosonic systems are of particular interest due to their pronounced quantum effects and mathematical tractability, which aid in the development and validation of theoretical models [1]. Ultracold gases serve as a platform for studying one-dimensional quantum many-body systems with short-range interactions [2]. Such one-dimensional systems have already been experimentally realized using confinement in quasi-one-dimensional (highly elongated) harmonic traps [3–8]. At the same time, the interaction between particles can be precisely controlled and tuned [9].

A characteristic feature of one-dimensional systems is the mapping between fermionic and bosonic systems. This mapping allows one to describe strongly correlated bosonic (fermionic) systems as the symmetrized (antisymmetrized) version of simpler fermionic (bosonic) systems. This fermion-boson duality emerges from the Girardeau mapping, for which the bosonic and fermionic systems share identical diagonal properties [10–12]. A prominent example is the Tonks-Girardeau gas, describing one-dimensional, strongly repulsive, hard-core bosons mapped onto a system of ideal fermions under the same external potential [10–17]. The generalization of this mapping to arbitrary interaction strengths, by Cheon and Shigehara [11], allows for mapping the ground state of strongly attracting p -wave fermions to the ground state of noninteracting bosons. This system is referred to as the fermionic Tonks-Girardeau (FTG) gas. Its ground-state wave

function (WF) is

$$\psi_F(x_1, \dots, x_N) = \psi_B(x_1, \dots, x_N) \prod_{j < k}^N \text{sgn}(x_j - x_k) \quad (1)$$

and is related to that of an ideal Bose gas, $\psi_B(x_1, \dots, x_N) = \prod_{i=1}^N \phi(x_i)$, where $\phi(x)$ is the single-particle ground state [18–25].

In a recent work, we computed the one-body density matrix (OBDM) $\rho_1(x, x') = \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x') \rangle$ for the FTG gas and diagonalized it [26]. We presented analytical expressions for the eigenvalues of the OBDM, $\lambda_k^{(N)}$, i.e., the occupation numbers. We showed that the occupation numbers are always doubly degenerate, and we presented expressions for the corresponding degenerate natural orbitals, $\chi_{k+}(x)$ and $\chi_{k-}(x)$.

With these expressions, we showed that the total density of the FTG gas, which is equivalent to the density of N noninteracting bosons,

$$n(x) = N|\phi(x)|^2, \quad (2)$$

can be explained by the formation of $N/2$ pairs of fermions in the orbitals $\chi_{k+}(x)$ and $\chi_{k-}(x)$ with a density of $2|\phi(x)|^2$ per pair in the even case. In the odd case, we reasoned that $(N - 1)/2$ pairs are formed, with the remaining fermion occupying the single-particle ground state [26].

In this Letter, we show that the FTG gas is a number-conserving BCS state, implying that the FTG gas can be fully described and characterized by the same type of states appearing in the BCS theory of superconductivity [27]. This provides a new way to express the ground state WF of the FTG gas, as given by Eq. (1), explicitly containing the observed pairing in Ref. [26]. We provide a detailed proof that the proposed WF is equivalent to the pair-product one (1). Remarkably, we derive the proposed ground-state WF solely from information

obtained from the OBDM, that is, its occupation numbers, natural orbitals, and our understanding of the pairing mechanism. This is noteworthy because in most systems recovering the WF from OBDM information alone is neither straightforward nor possible. The proposed WF is more suitable for working within a second quantization framework than the pair product form (1). It enables the calculation of important expectation values involving creation and annihilation operators, paving the way for new research directions on the FTG gas that were not possible with the WF (1). Lastly, we provide the expectation values of a set of measurable operators, which show that the FTG gas is a paired state within the framework for pairing derived in Ref. [28].

A novel expression for the FTG gas ground state. For clarity, we initially present the simpler case of $N = 2$. From Ref. [26], we know that, when dealing only with two fermions, these two fermions form a pair occupying the orbitals $\chi_{k+}(x)$ and $\chi_{k-}(x)$ for a given k with probability $\lambda_k^{(2)} = 8/[\pi(2k-1)]^2$. Based on this, we postulate that the ground state of the two-particle FTG gas can be written as

$$\psi_T(x_1, x_2) = \sum_{k=1} \sqrt{\lambda_k^{(2)}} \varphi_k(x_1, x_2), \quad (3)$$

where $\varphi_k(x_1, x_2)$ is the Slater determinant formed by the states $\chi_{k+}(x)$ and $\chi_{k-}(x)$. We denote our proposed WF as ψ_T to distinguish it from the well-known FTG gas ground-state ψ_F given by Eq. (1). The proposed WF is normalized since

$$\iint |\psi_T(x_1, x_2)|^2 dx_1 dx_2 = \sum_{k=1} \lambda_k^{(2)} = N/2 = 1. \quad (4)$$

The proof that $\psi_T = \psi_F$ is straightforward and is included here. For larger numbers of particles, we provide the proof in the Supplemental Material [29]. We start by noting that

$$\varphi_k(x_1, x_2) = \phi(x_1)\phi(x_2)\sqrt{2} \sin[(2k-1)\pi(y_1 - y_2)], \quad (5)$$

where $y_i = F(x_i) = \int_{-\infty}^{x_i} |\phi(z)|^2 dz$. Then,

$$\begin{aligned} \frac{\psi_T(x_1, x_2)}{\phi(x_1)\phi(x_2)} &= \sum_{k=1} \sqrt{\lambda_k^{(2)}} \sqrt{2} \sin[(2k-1)\pi(y_1 - y_2)] \\ &= \frac{4}{\pi} \sum_{k=1} \frac{\sin[(2k-1)\pi(y_1 - y_2)]}{2k-1} \\ &= \text{sgn}(y_1 - y_2) = \text{sgn}[F(x_1) - F(x_2)], \end{aligned} \quad (6)$$

for $F(x_1) - F(x_2) \in (-1, 1)$. Since $F(x)$ is a monotonically increasing function, $\text{sgn}[F(x_1) - F(x_2)] = \text{sgn}(x_1 - x_2)$. Therefore,

$$\psi_T(x_1, x_2) = \phi(x_1)\phi(x_2)\text{sgn}(x_1 - x_2) = \psi_F(x_1, x_2), \quad (7)$$

as we wanted to prove. Remarkably, Eq. (3) can be interpreted as the Slater decomposition of the FTG gas. The Slater decomposition can be understood as analogous to the Schmidt decomposition for fermionic, thus indistinguishable, systems [30,31]. Since the Slater decomposition obtained contains more than one nonzero coefficient $\lambda_k^{(2)}$, the FTG gas has a Slater rank greater than 1. This implies that the FTG gas exhibits nontrivial quantum correlations according to Refs. [30,31].

The proposed WF for the $N = 3$ case follows from reasoning similar to that for the $N = 2$ case. As shown in Ref. [26], only one pair is formed in the states $\chi_{k+}(x)$ and $\chi_{k-}(x)$ while the third fermion occupies the single-particle ground state. Based on this, we postulate

$$\psi_T(x_1, x_2, x_3) = \sum_{k=1} \sqrt{\lambda_k^{(3)}} \varphi_{0,k}(x_1, x_2, x_3), \quad (8)$$

where $\varphi_{0,k}(x_1, x_2, x_3)$ is the Slater determinant formed by the states $\chi_{k+}(x)$, $\chi_{k-}(x)$, and $\phi(x)$. Again, the WF is properly normalized [26]:

$$\sum_{k=1} \lambda_k^{(3)} = \frac{(N-1)}{2} = 1. \quad (9)$$

Note that the nondegenerate eigenvalue $\lambda_0^{(3)} = 1$ is not included in the summation. In the Supplemental Material, we provide a proof that $\psi_T(x_1, x_2, x_3) = \psi_F(x_1, x_2, x_3)$ [29].

The next step is to obtain the WF for $N = 4$ fermions. In this case, there are two different pairs at two different k 's, k_1 and k_2 . Based on that, we propose

$$\psi_T(x_1, \dots, x_4) = \sum_{k_1 < k_2} \sqrt{p_4(k_1, k_2)} \varphi_{k_1, k_2}(x_1, \dots, x_4), \quad (10)$$

where $\varphi_{k_1, k_2}(x_1, \dots, x_4)$ is the Slater determinant containing the four states involved in the two pairs k_1 and k_2 . We denote by $p_4(k_1, k_2)$ the coefficients that represent the probability of having pairs k_1 and k_2 occupied. Initially, one might consider $p_4(k_1, k_2) = \lambda_{k_1}^{(4)} \lambda_{k_2}^{(4)}$, but these coefficients imply improper normalization. This is because we are treating the probabilities of filling each pair as independent when they are not. The probability of filling the second pair k_2 depends on which pair, k_1 , is filled since, to fulfill the Pauli exclusion principle, we cannot fill again k_1 . Therefore, to obtain the proper coefficients $p_4(k_1, k_2)$, we should account for the dependencies between the probabilities of filling each pair.

Following the same reasoning regarding the formation of pairs, we find that the WF in the case of N fermions (here N is even) is

$$\begin{aligned} \psi_T(x_1, \dots, x_N) &= \sum_{k_1 < \dots < k_{N_p}} \sqrt{p_N(k_1, \dots, k_{N_p})} \varphi_{k_1, \dots, k_{N_p}}(x_1, \dots, x_N), \end{aligned} \quad (11)$$

where $N_p = N/2$ denotes the number of pairs and $\varphi_{k_1, \dots, k_{N_p}}(x_1, \dots, x_N)$ is the Slater determinant containing the N states $\chi_{k_1+}(x)$, $\chi_{k_1-}(x)$, ..., $\chi_{k_{N_p}+}(x)$, $\chi_{k_{N_p}-}(x)$. The expression for an odd number of fermions is similar, but it includes an unpaired fermion in the single-particle ground state,

$$\begin{aligned} \psi_T(x_1, \dots, x_N) &= \sum_{k_1 < \dots < k_{N_p}} \sqrt{p_N(k_1, \dots, k_{N_p})} \varphi_{0, k_1, \dots, k_{N_p}}(x_1, \dots, x_N), \end{aligned} \quad (12)$$

where now $N_p = (N-1)/2$ and $\varphi_{0, k_1, \dots, k_{N_p}}(x_1, \dots, x_N)$ is the Slater determinant containing the N states $\chi_{k_1+}(x)$, $\chi_{k_1-}(x)$, ..., $\chi_{k_{N_p}+}(x)$, $\chi_{k_{N_p}-}(x)$, and importantly, $\phi(x)$.

Inspired by the fact that the probability of filling a pair k cannot be treated independently from the other filled pairs, we

postulate the following recursive formula for $p_N(k_1, \dots, k_{N_p})$,

$$p_N(k_1, \dots, k_{N_p}) = \lambda_{k_1}^{(N)} \frac{p_{N-2}(k_2, \dots, k_{N_p})}{1 - \sum_{(l < \dots) \neq k_1} p_{N-2}(k_1, l, \dots)}. \quad (13)$$

This formula can be understood as a Bayes conditioned probability $P(A, B) = P(A)P(B|A)$. Where in our case A is the probability of filling the pair k_1 , here $P(B|A)$ is the probability of filling the $N_p - 1$ remaining pairs conditioned on k_1 being already filled. Importantly, we can check that, if p_{N-2} is properly normalized (which we know for $N = 2$ and $N = 3$), then p_N is also properly normalized. The short demonstration of proper normalization starts with

$$\sum_{k_1 < \dots < k_{N_p}} p_N(k_1, \dots, k_{N_p}) = \frac{1}{N_p!} \sum_{k_1 \neq \dots \neq k_{N_p}} p_N(k_1, \dots, k_{N_p}), \quad (14)$$

which equals

$$\frac{(N_p - 1)!}{N_p!} \sum_{k_1} \lambda_{k_1}^{(N)} \sum_{(k_2 < \dots < k_{N_p}) \neq k_1} \frac{p_{N-2}(k_2, \dots, k_{N_p})}{1 - \sum_{(l < \dots) \neq k_1} p_{N-2}(k_1, l, \dots)}. \quad (15)$$

The second summation $\sum_{(k_2 < \dots < k_{N_p}) \neq k_1}$ equals unity. Finally,

$$\sum_{k_1 < \dots < k_{N_p}} p_N(k_1, \dots, k_{N_p}) = \frac{1}{N_p} \sum_{k_1} \lambda_{k_1}^{(N)} = 1, \quad (16)$$

as we wanted to prove. Both Eq. (13) and the proof of proper normalization are valid for the even and odd cases. From the recursive formula Eq. (13), we can derive an explicit formula for $p_N(k_1, \dots, k_{N_p})$. First,

$$\sum_{(l < \dots) \neq k_1} p_{N-2}(k_1, l, \dots) = \lambda_{k_1}^{(N-2)}, \quad (17)$$

which can be easily derived using Eq. (13) since the summation does not involve k_1 . As a consequence, Eq. (13) is simplified to

$$p_N(k_1, \dots, k_{N_p}) = \lambda_{k_1}^{(N)} \frac{p_{N-2}(k_2, \dots, k_{N_p})}{1 - \lambda_{k_1}^{(N-2)}}. \quad (18)$$

Then, it follows from expressing $\lambda_{k_1}^{(N)}$ as a function of $\lambda_{k_1}^{(N-2)}$, using the recursive formula for the occupation numbers derived in Ref. [26], that

$$p_N(k_1, \dots, k_{N_p}) = \frac{N(N-1)}{(k_1 - 1/2)^2 \pi^2} p_{N-2}(k_2, \dots, k_{N_p}), \quad (19)$$

for the even case, and

$$p_N(k_1, \dots, k_{N_p}) = \frac{N(N-1)}{k_1^2 \pi^2} p_{N-2}(k_2, \dots, k_{N_p}), \quad (20)$$

for the odd case. Finally, by starting from $p_2(k) = \lambda_k^{(2)}$ and $p_3(k) = \lambda_k^{(3)}$, we arrive at the final explicit formulas,

$$p_N(k_1, \dots, k_{N_p}) = \begin{cases} \frac{N!}{\pi^N} \prod_{i=1}^{N_p} \frac{1}{(k_i - 1/2)^2}, & \text{if } N \text{ is even,} \\ \frac{N!}{\pi^{(N-1)}} \prod_{i=1}^{N_p} \frac{1}{k_i^2}, & \text{if } N \text{ is odd.} \end{cases} \quad (21)$$

This form of p_N fulfills normalization, as already shown. Additionally, it is invariant under the chosen order of k 's. While these two properties are necessary for a suitable form of p_N , they do not guarantee that our guess is the correct one. In the Supplemental Material, we provide a detailed proof that the proposed WF ψ_T , Eqs. (11) and (12), along with the proposed form of p_N , Eq. (21), satisfies $\psi_T = \psi_F$ [29]. In addition to the mathematical proof, we performed several numerical checks on the equivalency of the two WFs.

Our proposed WF, ψ_T , is mathematically equivalent to the original WF of the FTG gas, ψ_F (1). This equivalence demonstrates that ψ_T , which is constructed solely based on the formation of pairs without including higher-order correlation terms such as three-body correlations, fully describes the FTG gas. The ability to describe the FTG gas state entirely through pair formation is particularly noteworthy, as previous studies have shown that three-body properties in spinless p -wave fermions are non-negligible [32–34].

The FTG gas as a BCS state. In this section, we demonstrate that the WF of the FTG gas in second quantization form can be expressed as a number-conserving BCS state. The paired states appearing in the BCS theory of superconductivity [27] can be described as a superposition of number-conserving BCS states $|\psi_{\text{BCS}}^{(N)}\rangle$,

$$|\psi_{\text{BCS}}\rangle = \sum_N \beta_N |\psi_{\text{BCS}}^{(N)}\rangle. \quad (22)$$

The specific form of $|\psi_{\text{BCS}}^{(N)}\rangle$ is [27]

$$|\psi_{\text{BCS}}^{(N)}\rangle = C_N \left(\sum_k \alpha_k P_k^\dagger \right)^{N_p} |0\rangle, \quad (23)$$

where C_N is a normalization constant, $P_k^\dagger = a_k^\dagger a_{-k}^\dagger$, and $|0\rangle$ is the vacuum state. Here, $a_k^\dagger$ is the creation operator that creates a fermion in the single-particle orbital labeled by k . In our system, this means $a_k^\dagger$ creates a fermion in the orbital $\chi_{k+}(x)$, while $a_{-k}^\dagger$ creates one in $\chi_{k-}(x)$.

In the state $|\psi_{\text{BCS}}^{(N)}\rangle$, all N_p pairs occupy the same two-particle state given by $\sum_k \alpha_k P_k^\dagger$. The specific case where the single-particle orbitals are spin-momentum states, i.e., $k = (\vec{k}, \uparrow)$ and $-k = (-\vec{k}, \downarrow)$ corresponds to the standard BCS theory of superconductivity. Equation (23) can be alternatively written as [28]

$$|\psi_{\text{BCS}}^{(N)}\rangle = C_N N_p! \sum_{k_1 < \dots < k_{N_p}} \alpha_{k_1} \dots \alpha_{k_{N_p}} P_{k_1}^\dagger \dots P_{k_{N_p}}^\dagger |0\rangle. \quad (24)$$

Since our new formulation of the ground state WF is a superposition of Slater determinants, it is particularly convenient to express it in second quantization. For the even case, we have

$$|\psi_T\rangle = \sum_{k_1 < \dots < k_{N_p}} \sqrt{p_N(k_1, \dots, k_{N_p})} P_{k_1}^\dagger \dots P_{k_{N_p}}^\dagger |0\rangle, \quad (25)$$

where $P_k^\dagger = a_{k+}^\dagger a_{k-}^\dagger$. Since p_N (21) factorizes in all the k_i 's, we can rewrite our state as

$$|\psi_T\rangle = \sum_{k_1 < \dots < k_{N_p}} \alpha_{k_1} \dots \alpha_{k_{N_p}} P_{k_1}^\dagger \dots P_{k_{N_p}}^\dagger |0\rangle, \quad (26)$$

with

$$\alpha_{k_i} = \frac{N!^{1/N}}{\pi} \frac{1}{(k_i - 1/2)}. \quad (27)$$

Thus, we have clearly expressed the ground-state WF of the even FTG gas as a number-conserving BCS state and provided the explicit expressions for the coefficients α_{k_i} . The normalization constant C_N is found to be $C_N = 1/N_p!$. To express the odd-case WF in second quantization, we need to add the creation operator for a single particle in the single-particle ground state,

$$|\psi_T\rangle = a_0^\dagger \sum_{k_1 < \dots < k_{N_p}} \sqrt{p_N(k_1, \dots, k_{N_p})} P_{k_1}^\dagger \dots P_{k_{N_p}}^\dagger |0\rangle. \quad (28)$$

Again, we express p_N (21) as a product of α_{k_i} 's, getting to

$$|\psi_T\rangle = a_0^\dagger \sum_{k_1 < \dots < k_{N_p}} \alpha_{k_1} \dots \alpha_{k_{N_p}} P_{k_1}^\dagger \dots P_{k_{N_p}}^\dagger |0\rangle, \quad (29)$$

where now

$$\alpha_{k_i} = \frac{N!^{1/(N-1)}}{\pi} \frac{1}{k_i}. \quad (30)$$

Thus, we have expressed the ground-state WF of the odd FTG gas as a number-conserving BCS state, with the addition of a particle in the single-particle ground state. Therefore, we have seen that, starting from the general form of a BCS state, we can fully characterize and describe the FTG ground-state WF. The demonstration is based on the novel form of the ground-state WF and the fact that p_N factorizes in the k_i 's.

Detecting pairing in the FTG gas. The pairing phenomenon can be treated as a quantum correlation and can be formally defined within quantum information theory [28]. A fermionic state is said to be paired if there exists a set of operators containing at most two creation and two annihilation operators such that their expectation values cannot be reproduced by any separable state [28]. Following this definition, it can be formally proved that any number-conserving BCS state is a paired state [28]. The proof is based on the set of operators

$$\vec{O}_3 = \begin{pmatrix} n_k + n_{-k} + n_l + n_{-l} \\ n_k n_{-k} + n_l n_{-l} \\ a_k^\dagger a_{-k}^\dagger a_{-l} a_l + \text{H.c.} \end{pmatrix}. \quad (31)$$

The expectation values of $\vec{O}_3$ obtained with a number-conserving BCS state cannot be reproduced by any separable state [28]. Specifically, the expectation value of

$$H_1 = \frac{1}{2}(n_k + n_{-k} + n_l + n_{-l}) - (n_k n_{-k} + n_l n_{-l}) - (a_k^\dagger a_{-k}^\dagger a_{-l} a_l + \text{H.c.}), \quad (32)$$

is greater than or equal to 0, $\langle H_1 \rangle \geq 0$, for any separable state. For a number-conserving BCS state, $\langle H_1 \rangle < 0$ [28].

The set of operators $\vec{O}_3$ does not include any creation or annihilation operator acting on the ground state $k = 0$.

Therefore, the proof can be immediately extended to the even-number FTG gas and also to the odd-number FTG gas, since the addition of a particle in the single-particle ground state does not alter the expectation values of $\vec{O}_3$. Therefore, using the newly proposed shape of the WF, we conclude that the FTG gas forms a paired state.

Making use of our proposed ground-state WF, we are also able to compute the values of $\vec{O}_3$ for the FTG gas as

$$\langle \vec{O}_3^{\text{FTG}} \rangle = \begin{pmatrix} 2(\lambda_k^{(N)} + \lambda_l^{(N)}) \\ \lambda_k^{(N)} + \lambda_l^{(N)} \\ 2 \sum_{(k_2 < \dots < k_{N_p}) \neq l, k} \sqrt{p(l, k_2, \dots) p(k, k_2, \dots)} \end{pmatrix}. \quad (33)$$

These values imply

$$\langle H_1 \rangle = -2 \sum_{(k_2 < \dots < k_{N_p}) \neq l, k} \sqrt{p(l, k_2, \dots) p(k, k_2, \dots)} < 0. \quad (34)$$

The detailed calculations of these expectation values are provided in the Supplemental Material [29]. The calculations are relatively straightforward due to the suitable form in second quantization of the proposed WF. The specific values of $\langle \vec{O}_3^{\text{FTG}} \rangle$ are of interest because they are the main quantities to measure in order to detect pairing in a fermionic system. Therefore, if $\langle \vec{O}_3^{\text{FTG}} \rangle$ were to be measured, one could observe the existence of pairing in the FTG gas and determine that the prepared system is indeed a FTG gas.

Conclusions. In this Letter, we present a new expression for the ground-state wave function of the fermionic Tonks-Girardeau gas, as given by Eqs. (11) and (12) for even and odd numbers of fermions, respectively. This wave function is based exclusively on the occupation numbers and natural orbitals obtained from the one-body density matrix. This is remarkable because, generally, it is not possible to recover the wave function of a quantum many-body system solely from the one-body density matrix. The proposed ground state can be used to describe the ground state of the fermionic Tonks-Girardeau gas under any external potential, due to the universality of the coefficients p_N , which is a consequence of the universality of the occupation numbers $\lambda_k^{(N)}$ [23]. By expressing the newly formulated wave function in the second quantization, as given by Eqs. (25) and (28), we demonstrate that the proposed wave function is a specific case of a number-conserving BCS state and provide the explicit form of the α coefficients in Eqs. (27) and (30). Finally, in Eq. (33), we explicitly derive the expectation values of the operators to be measured in order to experimentally detect pairing in the fermionic Tonks-Girardeau gas.

We expect our results to have an impact on understanding one-dimensional quantum systems, in particular the fermionic Tonks-Girardeau gas, and to be useful for the study of quantum phenomena and quantum correlations occurring in p -wave fermions. The system under study is particularly appealing as it is not a Luttinger liquid, which instead describes compressible one-dimensional gases. Therefore, it is of great interest to understand how quantum correlations form beyond the Luttinger liquid paradigm. All previous studies on the fermionic Tonks-Girardeau gas were based on pair-product form (1) [18–25]. Since our proposed wave function is well-suited for working within a second quantization framework,

we anticipate that our new formulation of the ground state will lead to novel investigations on the fermionic Tonks-Girardeau gas that were not possible with the previous wave function. For instance, we anticipate that the proposed wave function can be used to predict and study whether pairing is also present in the excitations of the system. On the other hand, the proposed wave function is well suited for calculating relevant quantities in the study of entanglement between fermionic orbitals, as has already been done with other systems [35–37]. Furthermore, we expect this Letter to be of interest to experimentalists, as we present the values of a set of observables for the fermionic Tonks-Girardeau gas that can be measured in order to detect pairing in fermionic systems.

Beyond the specific relevance of this Letter for the study of the fermionic Tonks-Girardeau gas, we anticipate our findings to have a broader impact. Our results present a novel example of a system that can be exactly described using the same quantum states that describe a BCS superconductor [27], which have raised a lot of interest from a wide range of fields in

physics. Thus, we expect our results to be of significance to the study of BCS superconductivity.

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