

Quantum geometric tensor for the group SU(1,1)

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The quantum geometric tensor sets the local relation between variations of external parameters and distances between states in Hilbert space. Its real and imaginary parts are, respectively, the quantum Fisher information matrix, which determines the ultimate precision bound in parameter estimation, and the Berry curvature. We study the geometry of SU(1,1) Hamiltonian eigenstates and propose two dynamical methods for extracting the components of the SU(1,1) quantum geometric tensor. A time-separated quench method extracts both the real and imaginary parts of the quantum geometric tensor from transition probabilities following two sudden parameter changes separated by free evolution, while a weak resonant modulation method relates quasiparticle production to the same geometric tensor. These results provide an operational method for measuring the local geometry of single-mode squeezed states and, in particular, for extracting the quantum Fisher information matrix directly from experimental observables.

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Introduction. The evolution of a quantum mechanical system traces a curve through projective Hilbert space, whereas the external parameters entering the Hamiltonian provide only a coordinate description of this motion. The Euclidean geometry of parameter space has no direct physical significance; the relevant notion of distance is the separation between neighboring rays in projective Hilbert space. This relation is made precise by the quantum geometric tensor (QGT), which maps infinitesimal displacements in parameter space onto the intrinsic geometry of the Hilbert space [1–3].

The geometry of the Hilbert space is described by the QGT, $\chi_{\mu\nu}$, which decomposes as $\chi_{\mu\nu} = \frac{1}{4}\mathcal{F}_{\mu\nu} - i\Omega_{\mu\nu}/2$ [4,5]. The real component, $\mathcal{F}_{\mu\nu}$, is the quantum Fisher information matrix, which quantifies the distinguishability of nearby quantum states induced by infinitesimal changes in the control parameters [6–9]. Through the quantum Cramér-Rao bound, it sets the ultimate precision limit for unbiased parameter estimation, making it a central quantity in quantum metrology and sensing [10–13]. The imaginary component, $\Omega_{\mu\nu}$, is the Berry curvature, which may be viewed as an effective magnetic field in parameter space that determines the Berry phase accumulated per unit parameter-space area [3].

The state space geometry is not merely formal; it produces measurable effects on the dynamics and response of a system. A canonical example is the Berry phase acquired under adiabatic transport [14]. Another important example is

the quantum Hall effect in two-dimensional systems, where the quantized Hall conductivity is determined by the integral of the Berry curvature over the Brillouin zone [15]. More generally, quantum geometry provides the broader theoretical basis for topological phases in many-body systems [16–18]. These observable effects have motivated several methods for directly measuring QGT components from the effects they generate [19]. Under weak periodic modulation of a control parameter, the integrated transition rate between eigenstates is proportional to the QGT components [20]. When the modulation is driven near resonance between two states, the QGT enters the generalized Rabi frequency that can be inferred from coherent Rabi oscillations [21–23]. In contrast, a single-parameter quench method probes only the real part of the QGT, by measuring the resulting transition probability after a sudden change of parameters [24,25]. Access to the imaginary part generally requires a phase-sensitive method, such as the dynamical quantum Hall effect in which the Berry curvature is extracted from the generalized force it generates in parameter space [26,27].

The QGT extraction methods discussed above have largely been developed in finite-dimensional systems, while the corresponding geometry of infinite-dimensional SU(1,1)-type systems remains less explored, despite early studies of the SU(1,1) Berry phase in parametric amplifiers and polarization optics already revealing the geometric structure of these systems [28–31]. In quantum optics, the SU(1,1) group appears prominently in the description of second-order nonlinear materials [32], atomic ion traps [33], and SU(1,1) interferometry [34]. Related SU(1,1)-type Bogoliubov transformations arise in Bogoliubov–de Gennes descriptions of collective excitations in Bose-Einstein condensates [35] and Lorentz transformations in quantum field theory [36], where recent work has begun to examine the quantum geometry of Bo-

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goliubov quasiparticles [37]. Direct methods for extracting the full QGT in SU(1,1) systems, however, remain comparatively underdeveloped.

In this Letter, we investigate the quantum geometry of SU(1,1) systems and formulate dynamical protocols for extracting the associated SU(1,1) QGT components. We first introduce a time-separated sudden parameter quench method capable of extracting the real and imaginary components of the QGT. We then develop a second method based on weak, resonant parametric modulation, showing that the static geometry of the vacuum is encoded in the number of quasiparticles generated. Together, these protocols provide operational methods for probing the geometry of squeezed states and for directly extracting the quantum Fisher information matrix from experimentally accessible observables.

SU(1,1) Hamiltonian and states. The group SU(1,1) is characterized by the Lie algebra $\mathfrak{su}(1,1)$, with generators $K_{\pm}$ and K_3 satisfying the commutation relations

$$[K_-, K_+] = 2K_3, \quad [K_3, K_+] = K_+, \quad [K_-, K_3] = K_- \quad (1)$$

An SU(1,1) Hamiltonian may be written as a linear combination of these generators,

$$\hat{H} = 2\omega(\hat{K}_3 - \frac{1}{4}) + \xi\hat{K}_- + \xi^*\hat{K}_+ \quad (2)$$

The simplest realization of the $\mathfrak{su}(1,1)$ algebra is in terms of single-mode boson operators $\hat{a}$ and $\hat{a}^\dagger$, satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ with vacuum state $|0\rangle$ defined by $\hat{a}|0\rangle = 0$ [38]. In this bosonic representation,

$$\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2, \quad \hat{K}_- = \frac{1}{2}\hat{a}^2, \quad \hat{K}_3 = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \quad (3)$$

for which the Hamiltonian in Eq. (2) becomes

$$\hat{H}(\lambda) = \omega\hat{a}^\dagger\hat{a} + \frac{1}{2}(\xi\hat{a}^2 + \xi^*(\hat{a}^\dagger)^2), \quad (4)$$

with the mode frequency ω and complex squeezing amplitude $\xi = \kappa e^{i\theta}$. The operator $\hat{K}_3$ measures the mode occupation up to a constant shift, while $\hat{K}_{\pm}$ the photon pair creation and annihilation responsible for squeezing [39]. We focus on the stable regime $\omega > \kappa$. In this case, the Hamiltonian is diagonalized by the Bogoliubov transformation $\hat{a} = \cosh(r)\hat{b} - e^{-i\theta}\sinh(r)\hat{b}^\dagger$, $r = \frac{1}{2}\text{arctanh}(\frac{\kappa}{\omega})$, which brings $\hat{H}(\lambda)$ to the form

$$\hat{H}(\lambda) = \varepsilon\hat{b}^\dagger\hat{b} + \frac{\varepsilon - \omega}{2}, \quad (5)$$

with $\varepsilon = \sqrt{\omega^2 - \kappa^2}$ (see Sec. I of the Supplemental Material [40]). The eigenenergies are $E_n = \varepsilon n + \frac{\varepsilon - \omega}{2}$ and the corresponding eigenstates are Bogoliubov number states defined by $\hat{b}|0_b\rangle = 0$. The Bogoliubov vacuum, written in the Fock basis, is the squeezed vacuum:

$$|0_b(r, \theta)\rangle = (1 - |\zeta|^2)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \zeta^n |2n\rangle, \quad (6)$$

where $\zeta = -\tanh(r)e^{-i\theta}$. The corresponding Bogoliubov number states are obtained by applying the Bogoliubov creation operator, $|n_b\rangle = (b^\dagger)^n / \sqrt{n_b!} |0_b\rangle$. Although the spectrum is simple, the eigenvectors depend nontrivially on the parameters $\lambda^\mu = (\kappa, \theta)$ at fixed ω . In what follows, we characterize the local response of the eigenstates, and in particular

the ground state, to variations in κ and θ , thereby determining the SU(1,1) geometry of the state space.

SU(1,1) QGT evaluation. Let $|n_b(\lambda)\rangle$ denote an eigenstate of $\hat{H}(\lambda)$ with eigenvalue $E_n(\lambda)$. The QGT associated with the ray $|n_b(\lambda)\rangle$ is defined as [4]

$$\chi_{\mu\nu} = \langle \partial_\mu n_b | (1 - |n_b\rangle\langle n_b|) | \partial_\nu n_b \rangle, \quad (7)$$

where $\partial_\mu = \frac{\partial}{\partial \lambda^\mu}$. Differentiating the eigenvalue equation $\hat{H}(\lambda)|n_b(\lambda)\rangle = E_n|n_b(\lambda)\rangle$ with respect to μ and projecting onto a state $|m_b\rangle \neq |n_b\rangle$, one finds $\langle m_b | \partial_\mu n_b \rangle (E_n - E_m) = \langle m_b | \partial_\mu \hat{H} | n_b \rangle$ (see Sec. II of the Supplemental Material [40]). Substituting this relation into Eq. (7) gives the spectral representation:

$$\chi_{\mu\nu} = \sum_{m \neq n} \frac{\langle n_b | \partial_\mu \hat{H} | m_b \rangle \langle m_b | \partial_\nu \hat{H} | n_b \rangle}{(E_n - E_m)^2}. \quad (8)$$

Expressed in the Bogoliubov basis, the parametric derivative of the Hamiltonian take the form

$$\begin{aligned} \partial_\kappa \hat{H} &= \frac{\cosh(2r)}{2} (e^{i\theta} \hat{b}^2 + e^{-i\theta} (\hat{b}^\dagger)^2) - \frac{\sinh(2r)}{2} (2\hat{b}^\dagger \hat{b} + 1), \\ \partial_\theta \hat{H} &= \frac{i\kappa}{2} (e^{i\theta} \hat{b}^2 - e^{-i\theta} (\hat{b}^\dagger)^2). \end{aligned} \quad (9)$$

Up to an additive scalar term, these operators may be written in the general form

$$\partial_\mu \hat{H} = c_\mu \hat{b}^2 + c_\mu^* (\hat{b}^\dagger)^2 + d_\mu \hat{b}^\dagger \hat{b}, \quad (10)$$

where $\mu \in \{\kappa, \theta\}$ denotes the experimental control parameters. The complex two-photon drive coefficients c_μ and the particle-number shifts coefficients d_μ are

$$c_\kappa = \frac{\omega}{2\varepsilon} e^{i\theta}, \quad d_\kappa = -\frac{\kappa}{\varepsilon}, \quad c_\theta = \frac{i\kappa}{2} e^{i\theta}, \quad d_\theta = 0. \quad (11)$$

The operators $\hat{b}^2$ and $(\hat{b}^\dagger)^2$ in Eq. (10) change the number of Bogoliubov mode excitations by two, and when evaluated in Eq. (7) for the ground state $|0_b\rangle$, only the state $|2_b\rangle$ contributes and one obtains

$$\chi_{\mu\nu} = \sum_{m \neq 0} \frac{(\sqrt{2})^2 c_\mu c_\nu^* \langle 2_b | m_b \rangle \langle m_b | 2_b \rangle}{(E_m - E_0)^2} = \frac{c_\mu c_\nu^*}{2\varepsilon^2}, \quad (12)$$

where $E_2 - E_0 = 2\varepsilon$. The QGT is thus

$$\chi_{\mu\nu} = \frac{1}{8\varepsilon^4} \begin{pmatrix} \omega^2 & -i\varepsilon\kappa\omega \\ i\varepsilon\kappa\omega & \kappa^2\varepsilon^2 \end{pmatrix}, \quad (13)$$

for the parameters $\mu, \nu \in \{\kappa, \theta\}$ (see Sec. III of the Supplemental Material [40]). We now proceed to methods for extracting the QGT components experimentally.

Extraction of the quantum Fisher information matrix. The first method is inspired by the work of Tan *et al.* [24], who extracted the real part of the QGT by measuring the transition probability from the ground state following a sudden quench of the system parameters along coordinate directions,

$$P(\lambda + d\lambda) = 1 - |\langle 0_b(\lambda) | 0_b(\lambda + d\lambda) \rangle|^2. \quad (14)$$

Expanding the quenched state $|0_b(\lambda + d\lambda)\rangle$ to second order in the basis of the unquenched Hamiltonian (see Sec. IV of the

Supplemental Material [40]) one finds

$$P(\lambda + d\lambda) = \frac{1}{4} \sum_{\mu\nu} d\lambda^\mu d\lambda^\nu \mathcal{F}_{\mu\nu} \\ = \frac{\omega^2}{8\varepsilon^4} d\kappa^2 + \frac{\kappa^2}{8\varepsilon^2} d\theta^2 + \mathcal{O}(d\lambda^3). \quad (15)$$

The real components of the QGT $\mathcal{F}_{\kappa\kappa}$ and $\mathcal{F}_{\theta\theta}$ can therefore be measured by preparing the system in the ground state $|0_b(\kappa, \theta)\rangle$ of the Hamiltonian in Eq. (4), sudden quenching the parameters by $d\kappa$ and $d\theta$, and measuring the excitation probability in the Bogoliubov mode basis.

Extraction of the QGT by time-separated quench. The single-quench method isolates only the real part of the quantum geometric tensor and is therefore insufficient to extract the Berry curvature $\Omega_{\mu\nu}$. To recover the full tensor, we introduce a time-separated quench method. In this scheme, the system is first prepared in a slightly displaced ground state, then quenched to the Hamiltonian at a fixed parameter point and allowed to evolve freely for a time τ , and finally projected onto a second displaced ground state. The accumulated dynamical phase rotates the imaginary part of the QGT into a measurable contribution to the transition probability. This method is reminiscent of Ramsey's method of separated oscillatory fields [39,41], in which the excited-state transition probability oscillates as a function of the time interval between two applied fields. In the present protocol, the analogous quantity is the transition probability between the two ground states, which oscillates with the time separation between successive parameter quenches. More precisely, let λ_0 be a fixed parameter point and let $\delta\lambda$ be a small parameter displacement. Prepare the ground state $|0_b(\lambda')\rangle$, with $\lambda' = \lambda_0 + \delta\lambda$. The Hamiltonian is then suddenly quenched to λ_0 and the state is allowed to evolve under the Hamiltonian $\hat{H}(\lambda_0)$ for a time period τ , and the final state is projected onto a second displaced ground state $|0_b(\lambda'')\rangle$ obtained by quenching parameters to $\lambda'' = \lambda_0 - \Delta\lambda$. The resulting in the transition probability is

$$P(\tau) = 1 - |\langle 0_b(\lambda'') | e^{-i\hat{H}(\lambda_0)\tau} | 0_b(\lambda') \rangle|^2. \quad (16)$$

Expanding the initial and final ground states to second order about a fixed parameter point λ_0 (see Sec. V of the Supplemental Material [40]), one obtains

$$P(\tau) = \sum_{\mu\nu} \left(\frac{1}{4} \delta\lambda^\mu \delta\lambda^\nu \mathcal{F}_{\mu\nu} + \frac{1}{4} \Delta\lambda^\mu \Delta\lambda^\nu \mathcal{F}_{\mu\nu} \right. \\ + 2\Delta\lambda^\mu \delta\lambda^\nu \sum_{m \neq 0} \cos(\omega_m \tau) \text{Re}(K_{\mu\nu}^m) \\ + 2\Delta\lambda^\mu \delta\lambda^\nu \sum_{m \neq 0} \sin(\omega_m \tau) \text{Im}(K_{\mu\nu}^m) \Big) \\ + \mathcal{O}(\delta^3, \delta^2\Delta, \Delta^2\delta, \Delta^3), \quad (17)$$

where $\omega_m = E_m(\lambda_0) - E_0(\lambda_0)$ and $K_{\mu\nu}^m = \langle \partial_\mu 0_b(\lambda_0) | m_b(\lambda_0) \rangle \langle m_b(\lambda_0) | \partial_\nu 0_b(\lambda_0) \rangle$. In the special case $\tau = 0$, one recovers the single-quench method for extracting the real part of the QGT:

$$P(0) = \frac{1}{4} \sum_{\mu\nu} \Lambda^\mu \Lambda^\nu \mathcal{F}_{\mu\nu} + \mathcal{O}(\delta\lambda^3, \Delta\lambda^3), \quad (18)$$

with $\Lambda^\mu = \delta\lambda^\mu + \Delta\lambda^\mu$ by setting δ or Δ to zero. We now show how the time-separated quench method can be used to extract the Berry curvature. If the coupling operator, $\partial_\mu \hat{H}$, selects a single excited state coupled to the ground state, as in Eq. (10), then $K_{\mu\nu}^m = \frac{c_\mu c_\nu}{2\varepsilon^2} \delta_{m2}$ so that the sums in Eq. (17) collapse to a single term with $\omega_2 = 2\varepsilon$. Choosing the evolution time $\tau = \frac{\pi}{4\varepsilon}$ and the transition probability becomes

$$P\left(\frac{\pi}{4\varepsilon}\right) = \sum_{\mu\nu} \left(\frac{1}{4} \delta\lambda^\mu \delta\lambda^\nu \mathcal{F}_{\mu\nu} + \frac{1}{4} \Delta\lambda^\mu \Delta\lambda^\nu \mathcal{F}_{\mu\nu} \right. \\ \left. - \Delta\lambda^\mu \delta\lambda^\nu \Omega_{\mu\nu} \right) + \mathcal{O}(\delta^3, \delta^2\Delta, \Delta^2\delta, \Delta^3). \quad (19)$$

The appearance of the Berry curvature in the transition probability allows it to be measured by a suitable choice of quench direction. For example, taking the first quench along κ , with $\delta\lambda^\mu = \delta\kappa$, and the second along θ , with $\Delta\lambda^\nu = \Delta\theta$, gives

$$P_-\left(\frac{\pi}{4\varepsilon}\right) \approx \frac{1}{4} \delta\kappa^2 \mathcal{F}_{\kappa\kappa} + \frac{1}{4} \Delta\theta^2 \mathcal{F}_{\theta\theta} - \Delta\theta \delta\kappa \Omega_{\kappa\theta}. \quad (20)$$

Because the Berry curvature is antisymmetric, $\Omega_{\kappa\theta} = -\Omega_{\theta\kappa}$, exchanging the order of the two quenches changes the sign of the curvature. Reversing the quench order, one then obtains

$$P_+\left(\frac{\pi}{4\varepsilon}\right) \approx \frac{1}{4} \delta\kappa^2 \mathcal{F}_{\kappa\kappa} + \frac{1}{4} \Delta\theta^2 \mathcal{F}_{\theta\theta} + \Delta\theta \delta\kappa \Omega_{\kappa\theta}. \quad (21)$$

Taking the difference of these two transition probabilities yields the Berry curvature

$$\Omega_{\kappa\theta} \approx \frac{P_+\left(\frac{\pi}{4\varepsilon}\right) - P_-\left(\frac{\pi}{4\varepsilon}\right)}{2\delta\kappa\Delta\theta}. \quad (22)$$

The parameter-space trajectories associated with the separated-quench method are shown schematically in Fig. 1. Experimentally, the transition probability $P(\tau)$ may be written in a form adapted to SU(1,1) state readout:

$$|\langle 0_b(\lambda'') | e^{-i\hat{H}(\lambda_0)\tau} | 0_b(\lambda') \rangle|^2 \\ = |\langle 0 | S^\dagger(\lambda'') S(\lambda_0) e^{-i\varepsilon\tau \hat{a}^\dagger \hat{a}} S^\dagger(\lambda_0) S(\lambda') | 0 \rangle|^2 \\ = \langle 0 | S^\dagger(\lambda'') \rho(\tau) S(\lambda'') | 0 \rangle, \quad (23)$$

where the squeezed vacuum state resulting from the composition of squeezing operations is

$$\rho(\tau) = T |0_a\rangle \langle 0_a| T^\dagger, \quad (24)$$

with composite squeezing operator

$$T = S(\lambda_0) e^{-i\varepsilon\tau \hat{a}^\dagger \hat{a}} S^\dagger(\lambda_0) S(\lambda') \quad (25)$$

and squeezing operator

$$S(\lambda) = e^{(\frac{1}{2} r e^{-i\theta} \hat{a}^2 - \frac{1}{2} r e^{i\theta} (\hat{a}^\dagger)^2)}. \quad (26)$$

Thus, the time-separated quench probability is the vacuum projection probability of the transformed state $S^\dagger(\lambda'') \rho(\tau) S(\lambda'')$. Experimentally, this quantity may be measured by passing the squeezed vacuum state $\rho(\tau)$ to a degenerate parametric amplifier, or quadratic operation relevant to the system, with complex pump parameters fixed at λ'' , and then measuring the Fock vacuum probability. This is a single-mode analog of the SU(1,1) state reconstruction scheme of Agarwal and Banerji [42].

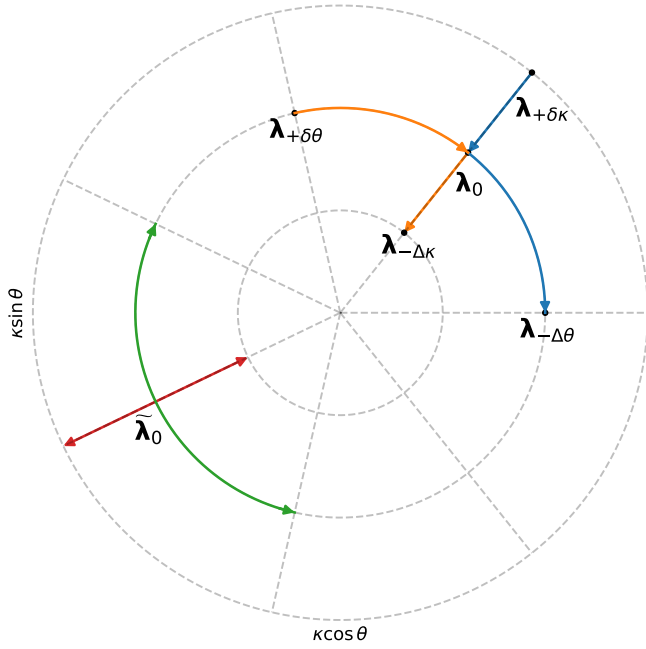


FIG. 1. Parameter space (κ, θ) illustrating two dynamical protocols for extracting the off-diagonal components of the quantum geometric tensor. The time-separated-quench protocol is shown on the right by two ordered paths through the operating point $\lambda_0 = (\kappa_0, \theta_0)$: $(\lambda_{+\delta\kappa} \rightarrow \lambda_0 \rightarrow \lambda_{-\Delta\theta})$ (blue) and $(\lambda_{+\delta\theta} \rightarrow \lambda_0 \rightarrow \lambda_{-\Delta\kappa})$ (orange), where $\lambda_{\pm\kappa} = (\kappa_0 \pm \kappa, \theta_0)$, $\lambda_{\pm\theta} = (\kappa_0, \theta_0 \pm \theta)$. The periodic-modulation protocol is shown on the left, centered at $\tilde{\lambda}_0 = (\tilde{\kappa}_0, \tilde{\theta}_0)$. The red and green arrows denote radial and angular modulation, respectively, with $\kappa(t) = \tilde{\kappa}_0 + A^\kappa \cos(\omega_d t + \varphi_\kappa)$, $\theta(t) = \tilde{\theta}_0 + A^\theta \cos(\omega_d t + \varphi_\theta)$. Dashed circles and rays indicate constant- κ and constant- θ coordinate lines, respectively.

Extraction of the QGT by periodic modulation. A second method for extracting the QGT components is based on periodic modulation of control parameters. Such methods have been used to extract the SU(2) QGT [20–22]. Consider a weak periodic modulation of the parameters $\lambda^\mu = (\kappa, \theta)$ about a fixed parameter point λ_0^μ , of the form $\lambda^\mu(t) = \lambda_0^\mu + A^\mu \cos(\omega_d t + \varphi_\mu)$, with amplitudes $A^\mu \ll 1$, parametric drive frequency ω_d , and phases φ_μ . The parametrically driven Hamiltonian can be expanded in A^μ as

$$\hat{H}(\lambda^\mu(t)) \approx \hat{H}(\lambda_0^\mu) + \sum_\mu A^\mu \cos(\omega_d t + \varphi_\mu) \partial_\mu \hat{H}(\lambda_0^\mu), \quad (27)$$

where the parametric derivatives $\partial_\mu \hat{H}(\lambda_0^\mu)$ are given by Eq. (9). Passing to the interaction picture and invoking the rotating-wave approximation for drive frequencies near $\omega_d \approx 2\varepsilon$, we neglect terms oscillating far from the parametric resonance and retain only the slowly varying resonant contributions (see Sec. VI of the Supplemental Material [40]). The resulting time-independent effective Hamiltonian is

$$\hat{H}_{\text{eff}} = \frac{1}{2} \sum_\mu A^\mu [c_\mu e^{i\varphi_\mu} \hat{b}^2 + c_\mu^* e^{-i\varphi_\mu} (\hat{b}^\dagger)^2], \quad (28)$$

where the coefficients c_μ are given in Eq. (11). The corresponding time evolution operator under this effective

Hamiltonian is a squeezing operation

$$e^{-i\hat{H}_{\text{eff}}t} = e^{(\frac{1}{2}z^* \hat{b}^2 - \frac{1}{2}z(\hat{b}^\dagger)^2)}, \quad (29)$$

with time-dependent complex squeezing parameter $z = it \sum_\mu A^\mu c_\mu^* e^{-i\varphi_\mu}$. Under this evolution, the average number of quasiparticles excited from the Bogoliubov vacuum is

$$\langle \hat{n}_b(t) \rangle = \sinh^2(|z|). \quad (30)$$

In the weak squeezing limit $|z| \ll 1$, generated by the weak drive field, this reduces to

$$\langle \hat{n}_b(t) \rangle = |z|^2 + \mathcal{O}(|z|^4). \quad (31)$$

Using Eq. (12), one may relate the QGT to the squeezing amplitude as

$$|z|^2 = 2\varepsilon^2 t^2 \sum_{\mu\nu} A^\mu A^\nu e^{i(\varphi_\mu - \varphi_\nu)} \chi_{\mu\nu}. \quad (32)$$

Note, although the definition of $\chi_{\mu\nu}$ involves only the state $|2_b\rangle$, the state generated by the effective Hamiltonian is an infinite linear combination of even number of Fock states. Evaluating the above expression for $\mu = \kappa$ and $\nu = \theta$, the quasiparticle number becomes

$$\langle \hat{n}_b(t) \rangle \approx 2\varepsilon^2 t^2 \left(\frac{1}{4} (A^\kappa)^2 \mathcal{F}_{\kappa\kappa} + \frac{1}{4} (A^\theta)^2 \mathcal{F}_{\theta\theta} + \frac{1}{2} A^\kappa A^\theta \mathcal{F}_{\kappa\theta} \cos(\phi) + A^\kappa A^\theta \Omega_{\kappa\theta} \sin(\phi) \right), \quad (33)$$

where $\phi = \varphi_\kappa - \varphi_\theta$. Thus, by a suitable choice of phase difference between the parametric drives, and parametric drive frequency $\omega_d = 2\varepsilon$, the QGT components may be extracted from measurements of the average quasiparticle number. The diagonal components of the QGT are isolated by modulating each control parameter independently:

$$\mathcal{F}_{\kappa\kappa} \approx \frac{2\langle \hat{n}_b(t) \rangle}{\varepsilon^2 t^2 (A^\kappa)^2}, \quad \mathcal{F}_{\theta\theta} \approx \frac{2\langle \hat{n}_b(t) \rangle}{\varepsilon^2 t^2 (A^\theta)^2}, \quad (34)$$

with the former obtained for $A^\theta = 0$ and the latter for $A^\kappa = 0$. The off-diagonal component may be extracted by taking the difference between measurement at $\phi = 0$ and $\phi = \pi$,

$$\mathcal{F}_{\kappa\theta} \approx \frac{\langle \hat{n}_b(t) \rangle_0 - \langle \hat{n}_b(t) \rangle_\pi}{2\varepsilon^2 t^2 A^\kappa A^\theta}. \quad (35)$$

Likewise, using phase differences $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$, the Berry curvature may be extracted from

$$\Omega_{\kappa\theta} \approx \frac{\langle \hat{n}_b(t) \rangle_{\frac{\pi}{2}} - \langle \hat{n}_b(t) \rangle_{\frac{3\pi}{2}}}{4\varepsilon^2 t^2 A^\kappa A^\theta}. \quad (36)$$

The parameter-space paths associated with the periodic modulation method are shown in Fig. 1. As in the SU(1,1) state reconstruction method discussed above, the number of quasiparticles may be measured by unsqueezing the final state and measuring the Fock state populations.

Conclusion. In summary, we have studied the quantum geometric tensor for the $SU(1,1)$ systems and formulated two dynamical methods for extracting its components. A time-separated-quench sequence isolates both the metric and the Berry curvature, while weak resonant modulation connects the same geometric structure to quasiparticle generation. These results provide an operational interpretation of the $SU(1,1)$ geometry and establishes practical methods to measure the quantum

Fisher information matrix and Berry curvature in $SU(1,1)$ systems.

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Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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