

Modeling of beam envelope instabilities in high-intensity circular accelerators

Yao-shuo Yuan ^{*}

*Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China;
Spallation Neutron Source Science Center, Dongguan 523803, China;
and University of Chinese Academy of Sciences, Beijing 100049, China*



(Received 27 May 2025; accepted 12 October 2025; published 1 December 2025)

With the growing demand for the high-intensity hadron beams in various applications in recent years, coherent beam instabilities driven by space charge are of critical concern for those high-intensity hadron accelerator projects, where the plasma dynamics of beams is pronounced and requires careful consideration. In this paper, an analytical model is developed to characterize coherent beam instabilities in circular machines. Based on the proposed model, a coherent mode, namely the vertical coupled mode, is identified. It is demonstrated that the tilting instability is more enhanced in circular machines because of the increased anisotropy from dispersion. Moreover, the well-known envelope instability in linear accelerator can be generalized and classified into “a family of envelope instabilities” for high-intensity circular accelerators. The proposed analytical model has been compared to particle-in-cell simulations and shows good agreement. The chart of the envelope instability family can be adopted for optimization of the working point to achieve higher beam intensity.

DOI: [10.1103/c1dn-z4n9](https://doi.org/10.1103/c1dn-z4n9)

I. INTRODUCTION

In recent years, several projects of high-intensity hadron (proton or heavy ion) circular accelerators have been built, under construction, or newly planned worldwide for achieving various scientific goals [1], such as high-intensity heavy ion rings for radioactive beam facilities [2,3], the proton drivers for spallation neutron sources [4,5], the neutrino factories [6], the muon collider [7], and so on. For such large-scale accelerator complexes, space-charge effects pose a key challenge to increasing beam intensities and optimizing the efficiency of accelerator operations. Significant research efforts have been dedicated to space-charge-driven incoherent resonances, which are caused by the external periodic focusing and/or enhanced by the magnetic errors and nonlinearities of accelerator components, for example, in Refs. [8–25].

For the proposed or currently under-construction high-intensity hadron accelerators that will operate with significantly enhanced beam intensities in the future, such intense beams exhibit dynamical behavior characteristic of non-neutral plasmas (see, for example, Refs. [26–29]). Therefore, besides the aforementioned well-known incoherent properties, another essential aspect of space charge in determining the beam intensity limitation is its coherent phenomena of the density oscillations in the beam, caused by the beam’s intrinsic response to the strong beam anisotropy or the external

periodic focusing and characterized by the eigenmodes of space-charge oscillations [30,31]. The space-charge eigenmodes define the perturbed space-charge potential, playing a pivotal role in high-intensity beam dynamics. Under certain conditions, the eigenmodes become unstable, causing coherent beam instabilities, such as the emittance exchange (equipartitioning) and the well-known envelope instability [32–36], which have become standard criteria for the design and operation of high-intensity linear accelerators (linacs) [28,37]. The well-known 90° envelope instability is typically initiated when the phase advance per periodic cell exceeds 90° and is depressed to a value slightly below 90° due to space charge. It has been experimentally studied in high-intensity linacs [38], where the conclusion was drawn that the envelope instability is dominated by the fourth-order structural resonance. However, further more detailed studies have been conducted to explore and interpret the “competitive” relationship between the 90° envelope instability and the fourth-order resonance [39].

The frequency of the eigenmodes can be theoretically obtained based on the linearized Vlasov-Poisson approach [40]. Among these eigenmodes, the most dominant modes are the lowest second-order modes [41]. A theoretical approach to investigate the stability of the second-order modes is based on the perturbation of the Cherenin’s equation set [42], which shows that the second-order odd (skew) modes cause the tilting instability, characterized with a linear resonant emittance exchange [36], while the second-order even modes can cause the parametric sum envelope instabilities [43]. A set of linearized rms moments to characterize the coherent properties is established in Refs. [44,45]. Besides the second-order modes, another important contribution of space charge is the first-order modes (dipole modes), where the space charge plays a crucial role in beam collective instabilities together with wake fields [46].

^{*}Contact author: ysyuan@ihep.ac.cn

The coherent properties of space charge, which have primarily been studied in the context of high-intensity linacs, will become increasingly prominent in future circular machines due to the unprecedented demand for high-intensity hadron beams. Compared to high-intensity linacs, the situation for high-intensity circular accelerators becomes even more complicated because of the coupled effect of space charge and dispersion. In the past few decades, progress has been made in understanding of the combined effect of the space charge and dispersion. Marco Venturini and Martin Reiser (V-R) developed a system of envelope equations with a generalized invariant emittance in the presence of both dispersion and space charge [47,48]. S. Y. Lee and H. Okamoto (L-O) proposed a Hamiltonian expression with a modified dispersion formula including higher-order space-charge potential terms [49]. Studies on the combined effect of space charge and dispersion were also performed with the smooth approximation approach [50] and modified particle-core model [18,51,52]. A perturbation analysis on the generalized envelope equations with dispersion is conducted to identify the beam coherent instability induced by space charge and dispersion, with a corresponding unstable dispersion mode identified. The experimental measurement of the second-order coherent modes in circular accelerators has been proposed using the quadrupolar pickup signals [53]. The stability of intense beams transported in circular machines requires ongoing and systematic investigation.

In this article, we attempt to explore a framework of beam coherent instabilities, under the close interconnection of space charge and dispersion, including all the possible coherent second-order mode of the beams in high-intensity circular accelerators. Specifically, a new coherent mode, characterized by the horizontal-to-vertical coupling, referred to as the vertical coupled mode is identified. The mechanism of emittance exchange driven by tilting instability is discussed in detail.

The paper is structured as follows: After the introduction in Sec. I, in Sec. II we perform canonical transformation on the Hamiltonian with dispersion and linear coupling. In Sec. III, the analytical model of beam second-order moments under the coupling effect of space charge and dispersion is developed based on the transformed Hamiltonian and the generalized Cherenin's equation, with the vertical coupled mode identified and discussed. The tilting instabilities in circular accelerators are analyzed in Sec. V. Section VI presents an overview of the envelope instability family for high-intensity circular accelerators. Finally, conclusions are given in Sec. VII.

II. COUPLED HAMILTONIAN TRANSFORMATION

In the following, we assume that x and y are the transverse degrees in horizontal and vertical directions, respectively, and s is the longitudinal coordinate; the dispersion effect acts on x and there is no rf focusing or acceleration process in the longitudinal direction.

Let us start with a coasting beam transported in a periodic focusing channel with bending magnets and skew quadrupoles. For simplicity, here we ignore any imperfection of the periodic channel and any nonlinear effect due to external focusing, as well as all the chromatic terms. The Hamiltonian of such a

coupled transport system can be expressed as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\kappa_{x0}(s)}{2}x^2 + \frac{\kappa_{y0}(s)}{2}y^2 + q_{xy0}(s)xy - \frac{x}{\rho(s)}\delta + \frac{m^2c^4}{E_0^2}\delta^2 - V_{sc}(x, y, s), \quad (1)$$

where $\kappa_{x0} = (1/B\rho)(\partial B_y/\partial x)$ and $\kappa_{y0} = -(1/B\rho)(\partial B_x/\partial y)$ are the external focusing gradients, $q_{xy0} = (1/B\rho)(\partial B_x/\partial x)$ is the external linear coupling between x and y , $\rho(s)$ is the radius of curvature, and $B\rho$ is the magnetic rigidity. $\delta = (p - p_0)/p_0$ is the fractional momentum deviation from the design momentum p_0 , with E_0 being the corresponding energy. For a round beam with its center located at the pipe center, the space-charge potential V_{sc} can be expanded as [49]

$$V_{sc} = \frac{1}{2}V_{sc,xx}x^2 + V_{sc,xy}xy + \frac{1}{2}V_{sc,yy}y^2 + o(x^2, y^2), \quad (2)$$

with the space-charge coefficients $V_{sc,xx} = \partial^2 V_{sc}/\partial x^2$, $V_{sc,yy} = \partial^2 V_{sc}/\partial y^2$, and $V_{sc,xy} = \partial^2 V_{sc}/\partial x\partial y$. The symbol “ $o(\cdot)$ ” denotes the negligible higher-order terms. Following canonical transformation introduced in the V-R and L-O frameworks, we introduce $\bar{x} = x - D_x\delta$, $\bar{p}_x = p_x - D'_x\delta$, $\bar{y} = y$, and $\bar{p}_y = p_y$ as the betatron coordinates affected by space charge, in which $D_x\delta$ and $D'_x\delta$ are the off-momentum coordinates (here, the prime symbol denotes the derivative with respect to s). With the “new” coordinates $\bar{x}$ and $\bar{y}$, the Hamiltonian in Eq. (1) can be transformed to

$$\bar{H} = \frac{1}{2}(\bar{p}_x^2 + \bar{p}_y^2) + \frac{\kappa_x(s)}{2}\bar{x}^2 + \frac{\kappa_y(s)}{2}\bar{y}^2 + q_{xy}\bar{x}\bar{y} + q_{xy}D_x\delta\bar{y} + \frac{m^2c^4}{E_0^2}\delta^2, \quad (3)$$

in which we define the external focusing gradients $\kappa_{x,y}$ and the linear coupling q_{xy} depressed by space charge, respectively, as

$$\begin{aligned} \kappa_x &= \kappa_{x0} - V_{sc,xx}, & \kappa_y &= \kappa_{y0} - V_{sc,yy}, \\ q_{xy} &= q_{xy0} - V_{sc,xy}. \end{aligned} \quad (4)$$

The coefficients D_x and D'_x can be identified as the space-charge-modified dispersion function and its derivative,

$$D_x'' + \kappa_x(s)D_x = \frac{1}{\rho(s)}. \quad (5)$$

Next, in order to decouple the $\delta - \bar{y}$ coupling term in Eq. (3), we further introduce a second coordinate transformation, given by

$$\begin{aligned} \hat{x} &= \bar{x} = x - D_x\delta, \\ \hat{y} &= \bar{y} - D_y\delta = y - D_y\delta, \end{aligned} \quad (6)$$

with the coefficients D_y and D'_y defined as the *coupled vertical dispersion* and its derivative, satisfying

$$D_y'' + \kappa_y(s)D_y = q_{xy}D_x. \quad (7)$$

Since it is assumed that there are no vertical bending magnets in the system, the vertical dispersion D_y and its derivative D'_y are completely determined by the horizontal dispersion and the $x - y$ coupling. The Hamiltonian after the second

transformation reads

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{\kappa_x(s)}{2}\hat{x}^2 + \frac{\kappa_y(s)}{2}\hat{y}^2 + q_{xy}(s)\hat{x}\hat{y} + \frac{m^2 c^4}{E_0^2}\delta^2. \quad (8)$$

By introducing the (horizontal) space-charge-modified dispersion function of Eq. (5) and the vertical coupling dispersion of Eq. (7), in the new Hamiltonian expression of Eq. (8) the dispersion term $x\delta/\rho$ is eliminated.

III. GENERALIZED MODEL OF SECOND-ORDER MODES

The equation of motion for a single particle in a transporting channel with bending magnets and skew coupling magnets using the transverse coordinates $\hat{x}$ and $\hat{y}$ in Eq. (8) can be written as

$$\mathbf{v}' = M\mathbf{v} + \boldsymbol{\omega}, \quad (9)$$

where $\mathbf{v}$ is a 6×1 column vector with six elements: the transformed coordinates $\hat{x}(s)$, $\hat{x}'(s)$, $\hat{y}(s)$, $\hat{y}'(s)$ defined in Eq. (6), and the horizontal dispersion $D_x(s)$ and its derivative $D'_x(s)$ defined, respectively, in Eq. (5). M is a 6×6 transfer matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\kappa_x & 0 & -q_{xy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -q_{xy} & 0 & -\kappa_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\kappa_x & 0 \end{pmatrix}, \quad (10)$$

in which $\kappa_x = \kappa_{x0} - \Delta\kappa_x$, $\kappa_y = \kappa_{y0} - \Delta\kappa_y$, and $q_{xy} = q_{xy,0} - \Delta q_{xy}$ defined in Eq. (4), with $\Delta\kappa_x$, $\Delta\kappa_y$, and Δq_{xy} the space-charge-induced shifts. The 6×1 column vector $\boldsymbol{\omega}$ takes the form

$$\boldsymbol{\omega} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1/\rho)^T. \quad (11)$$

Substituting the motion of single particle in Eq. (9) into the Chernin's equation [42], the motion of the beam second-order moments with the inclusion of dispersion can be generalized to

$$\boldsymbol{\Sigma}' = M\boldsymbol{\Sigma} + (M\boldsymbol{\Sigma})^T + \boldsymbol{\omega}\mathbf{v}^T + \mathbf{v}\boldsymbol{\omega}^T. \quad (12)$$

Here, $\boldsymbol{\Sigma}$ represents the 6×6 matrix of the second-order moments, with elements Σ_{ij} defined by $\Sigma_{ij} \equiv \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$, where the averages $\langle \rangle$ are taken over the phase-space variables and the subscripts i, j run from 1 to 6 representing $\hat{x}(s)$, $\hat{x}'(s)$, $\hat{y}(s)$, $\hat{y}'(s)$, $D_x(s)$ and $D'_x(s)$. Clearly, the elements Σ_{13} , Σ_{14} , Σ_{23} , and Σ_{24} denote the coupling moments between x and y . The elements Σ_{15} , Σ_{16} , Σ_{25} , and Σ_{26} represent the dispersion coupling in x while the elements Σ_{35} , Σ_{36} , Σ_{45} , and Σ_{46} are the vertical coupling dispersion in y .

In fact, due to the irrelevance between the longitudinal quantity δ and the transverse quantity $\hat{x}$ with $\langle \hat{x}\delta \rangle = \langle \hat{x}'\delta \rangle = 0$, one can safely assume that $\Sigma_{15} = \Sigma_{16} = \Sigma_{25} = \Sigma_{26} = 0$. Furthermore, as will be proved in Sec. IV, the coupling effect in the y direction can be described via the coupled vertical dispersion function D_y and its derivative D'_y in Eq. (7), which contains the complete dynamical information of the four moments Σ_{35} , Σ_{36} , Σ_{45} , and Σ_{46} . With the above two

considerations, the equation set of motion for the second moments can be expressed explicitly as

$$\begin{aligned} \Sigma'_{11} &= 2\Sigma_{12}, \\ \Sigma'_{12} &= \Sigma_{22} - \kappa_x \Sigma_{11} + q_{xy} \Sigma_{13}, \\ \Sigma'_{13} &= \Sigma_{23} + \Sigma_{14}, \\ \Sigma'_{14} &= \Sigma_{24} - \kappa_y \Sigma_{13} + q_{xy} \Sigma_{11}, \\ \Sigma'_{22} &= -2\kappa_x \Sigma_{12} + 2q_{xy} \Sigma_{23}, \\ \Sigma'_{23} &= \Sigma_{24} - \kappa_x \Sigma_{13} + q_{xy} \Sigma_{33}, \\ \Sigma'_{24} &= -\kappa_x \Sigma_{14} - \kappa_y \Sigma_{23} + q_{xy} (\Sigma_{12} + \Sigma_{34}), \\ \Sigma'_{33} &= 2\Sigma_{34}, \\ \Sigma'_{34} &= \Sigma_{44} + q_{xy} \Sigma_{13} - \kappa_y \Sigma_{33}, \\ \Sigma'_{44} &= 2q_{xy} \Sigma_{14} - 2\kappa_y \Sigma_{34}, \\ D'_x &= D_{xs}, \\ D'_{xs} &= -\kappa_x D_x + 1/\rho + q_{xy} D_y, \\ D'_y &= D_{ys}, \\ D'_{ys} &= -\kappa_y D_y + q_{xy} D_x, \end{aligned} \quad (13)$$

which can be further written as a concise form

$$\tilde{\boldsymbol{\Sigma}}' = G(\tilde{\boldsymbol{\Sigma}}(s), s). \quad (14)$$

Here, the vector matrix $\tilde{\boldsymbol{\Sigma}}$ is a “simplified” form of $\boldsymbol{\Sigma}$, consisting of 14 elements as listed in Eq. (13).

For a transport system in the absence of external linear coupling ($q_{xy,0} = 0$), the moments Σ_{22} and Σ_{44} can be replaced by the generalized rms emittance with dispersion defined by $\varepsilon_{dx} = \sqrt{\Sigma_{11}\Sigma_{22} - \Sigma_{12}^2}$ and $\varepsilon_{dy} = \sqrt{\Sigma_{33}\Sigma_{44} - \Sigma_{34}^2}$ [11]. In this case, the 14 equations of motion in Eq. (13) can be further simplified to the following 12 equations:

$$\begin{aligned} \Sigma'_{11} &= 2\Sigma_{12}, \\ \Sigma'_{12} &= \frac{\varepsilon_{dx}^2 + \Sigma_{12}^2}{\Sigma_{11}} - \kappa_x \Sigma_{11}, \\ \Sigma'_{13} &= \Sigma_{23} + \Sigma_{14}, \\ \Sigma'_{14} &= \Sigma_{24} - \kappa_y \Sigma_{13} + \Delta q_{xy} \Sigma_{11}, \\ \Sigma'_{23} &= \Sigma_{24} - \kappa_x \Sigma_{13} + \Delta q_{xy} \Sigma_{33}, \\ \Sigma'_{24} &= -\kappa_x \Sigma_{14} - \kappa_y \Sigma_{23} + \Delta q_{xy} (\Sigma_{12} + \Sigma_{34}), \\ \Sigma'_{33} &= 2\Sigma_{34}, \\ \Sigma'_{34} &= \frac{\varepsilon_{dy}^2 + \Sigma_{34}^2}{\Sigma_{33}} - \kappa_y \Sigma_{33} + \Delta q_{xy} \Sigma_{13}, \\ D'_x &= D_{xs}, \\ D'_{xs} &= -\kappa_x D_x + 1/\rho + \Delta q_{xy} D_y, \\ D'_y &= D_{ys}, \\ D'_{ys} &= -\kappa_y D_y + \Delta q_{xy} D_x. \end{aligned} \quad (15)$$

Here, the coupling between x and y is purely from space charge Δq_{xy} . For a matched case with zero coupling moments: $\Sigma_{13,0} = \Sigma_{14,0} = \Sigma_{23,0} = \Sigma_{24,0} = 0$ (here we use the subscript “0” to denote the matched case), and Eq. (15) becomes the

well-known envelope equation

$$\begin{aligned}\Sigma'_{11,0} &= 2\Sigma_{12,0}, \\ \Sigma'_{12,0} &= \frac{\varepsilon_{dx}^2 + \Sigma_{12,0}^2}{\Sigma_{11,0}} - \kappa_x \Sigma_{11,0}, \\ \Sigma'_{33,0} &= 2\Sigma_{34,0}, \\ \Sigma'_{34,0} &= \frac{\varepsilon_{dy}^2 + \Sigma_{34,0}^2}{\Sigma_{33,0}} - \kappa_y \Sigma_{33,0},\end{aligned}\quad (16)$$

with the dispersion function of Eq. (5).

IV. VERTICAL COUPLED MODE

It is known that the second-order modes consist of the even modes (also known as the envelope breathing and quadrupolar modes), the odd modes (often termed “tilting” modes), and the recently identified dispersion mode in circular machines. In the following, we will show that in the presence of the coupled effects of space charge and dispersion, a new second-order mode—referred to as the *vertical coupled mode*—is identified based on the coupled model in Eq. (14).

To this end, we assume a slightly mismatched case, where the matrix of the second moments $\tilde{\Sigma}(s) = \tilde{\Sigma}_0(s) + \tilde{\Sigma}_p(s)$ (here the subscript “0” represents the matched moments and “p” denotes a small perturbation on the matched solution). After Taylor expanding and keeping only the first-order terms, the equation of motion for the moments in Eq. (14) can be written as

$$\tilde{\Sigma}' = (\tilde{\Sigma}_0 + \tilde{\Sigma}_p)' = \mathbf{G}(\tilde{\Sigma}_0) + \mathbf{J}(\tilde{\Sigma}_p), \quad (17)$$

which consists of two parts: the motion for the matched moments

$$\tilde{\Sigma}'_0 = \mathbf{G}(\tilde{\Sigma}_0) \quad (18)$$

and the motion for the perturbations

$$\tilde{\Sigma}'_p = \mathbf{J}(\tilde{\Sigma}_p), \quad (19)$$

where $\mathbf{J}$ represents the Jacobian matrix of $\mathbf{G}$ with elements $J_{k,l} = \partial G_k(\Sigma_l)/\partial \Sigma_l$ [here k, l run from 1 to 16 along the possible combinations of the second-order moments Σ_{ij} in Eq. (15)].

The coupled model described by Eqs. (18) and (19) can be employed to calculate the frequencies of all the possible second-order eigenmodes in synchrotrons and thus analyze the beam stability: With a given lattice and beam current, the matched second moments can be solved via integrating Eq. (18) over a periodic focusing length, and the frequencies of the second-order modes can be calculated from the Jacobian matrix $\mathbf{J}$ in Eq. (19).

For the sake of simplicity, we approximate the lattice by a constant focusing transport system with dispersion (abbreviated as CFD in the following [44]). For CFDs, the differential equation set of the perturbations in Eq. (19) is simplified to

$$\xi'' = \mathbf{C}\xi, \quad (20)$$

where ξ is a 6×1 column matrix, given by

$$\xi = (\Sigma_{11}^p \quad \Sigma_{33}^p \quad d_x \quad d_y \quad \Sigma_{13}^p \quad \Sigma_{24}^p)^T, \quad (21)$$

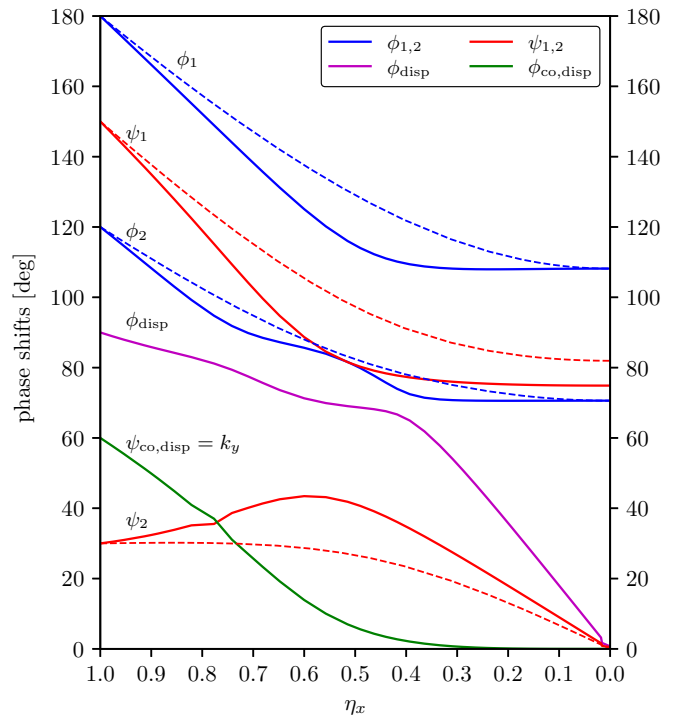


FIG. 1. Phase shifts of the second-order modes versus horizontal tune depression in a CFD with $k_{0,x} = 90^\circ$ and $k_{0,y} = 60^\circ$ with (solid line) and without (dashed line) dispersion: the two even modes $\phi_{1,2}$ (blue), the two odd modes $\psi_{1,2}$ (red), the dispersion mode ϕ_{disp} (purple), and the coupling dispersion mode $\psi_{\text{co,disp}}$ (green).

and $\mathbf{C}$ is the transfer matrix of the perturbations for constant focusing, where its elements are the functions composed with the matched beam second moments, the generalized rms emittance, the external focusing, and the space-charge strength. Clearly, six eigenmodes can be decoupled from the matrix $\mathbf{C}$. An explicit expression of $\mathbf{C}$ can be found in the Appendix.

As a first application of the generalized coupled model to calculate the frequency of the modes and verify the newly identified vertical coupling dispersion mode, we use an example CFD with $k_{0,x} = 90^\circ$ and $k_{0,y} = 60^\circ$ to calculate the frequencies of the six eigenmodes. We adopt the notation $k_{0,x,y}$ and $k_{x,y}$ to represent the phase advance per focusing cell in x and y without and with space charge, respectively. In the example, the ratio of the generalized emittance $\varepsilon_{dx}/\varepsilon_{dy}$ is set to 1/2 and beam rms momentum spread $\sigma_p = 0.002$.

The phase shifts of the six modes with varying beam current in the unit of the horizontal tune depression $\eta_x \equiv k_x/k_{0,x}$ are numerically solved by Eq. (20) and shown in Fig. 1, with the comparison of the four modes (two even modes and two odd modes) in the absence of dispersion (shown as dashed line in Fig. 1). It can be seen that for both cases with and without dispersion, two even modes $\phi_{1,2}$ share the same space-charge limits $5/8 \times 2k_{0,x,y}$, which is in agreement with the generalized envelope theory with dispersion [53]. In comparison, the dispersion mode ϕ_{disp} and slow odd mode ψ_2 tend to zero in the space-charge limit. The fast odd modes ψ_1 with and without dispersion have different space-charge limits. We attribute this to the fact that the dispersion breaks the symmetry between x and y [44].

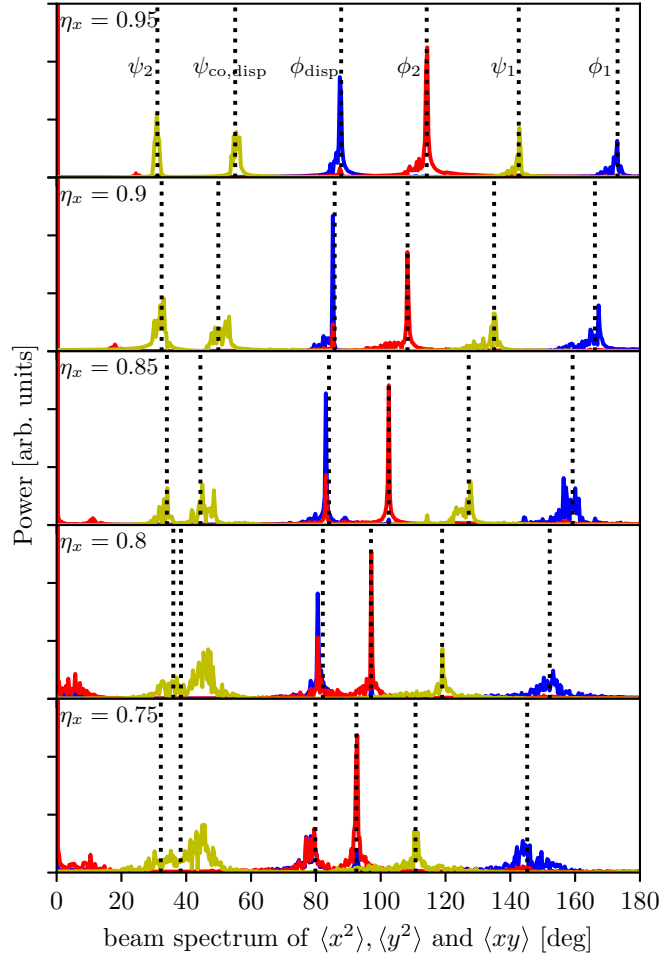


FIG. 2. Beam spectrum of $\langle x^2 \rangle$, $\langle y^2 \rangle$, and $\langle xy \rangle$ in a CFD with $k_{0,x} = 90.0^\circ$ and $k_{0,y} = 60.0^\circ$ from PIC simulation compared with the analytical results (dotted lines). The mode signals marked as blue, red, and yellow are obtained from the moments $\langle x^2 \rangle$, $\langle y^2 \rangle$, and $\langle xy \rangle$, respectively.

The *vertical coupled mode* $\psi_{\text{co,disp}}$ can be identified as green curve in Fig. 1. Since neither vertical bending magnets nor skew magnets exist in the CFD example, this mode is purely coupled from *horizontal* dispersion and space charge. Moreover, the phase shift of the vertical coupled dispersion mode is equal to the space-charge depressed phase advance in y , $\psi_{\text{co,disp}} = k_y$. We attribute this to the fact that the mode behaves like a single particle, which is analogous to the horizontal dispersion modes [54].

Particle-in-cell (PIC) simulations utilizing the particle tracking code ORBIT [55] are conducted to validate the analytical results of the second-order modes from the generalized coupled model. In the simulation, 50 000 particles are tracked over 1000 focusing periods in the example CFD with $k_{0,x} = 90^\circ$ and $k_{0,y} = 60^\circ$. The initial particle distribution is chosen as waterbag and rms-matched to the space charge and the CFD structure.

The beam spectrum of the second moments $\langle x^2 \rangle$, $\langle y^2 \rangle$, and $\langle xy \rangle$ with various beam intensities is Fourier analyzed, as displayed in Fig. 2. The modes are shifted with varying beam current. For strong beam intensity ($\eta_x < 0.85$), the

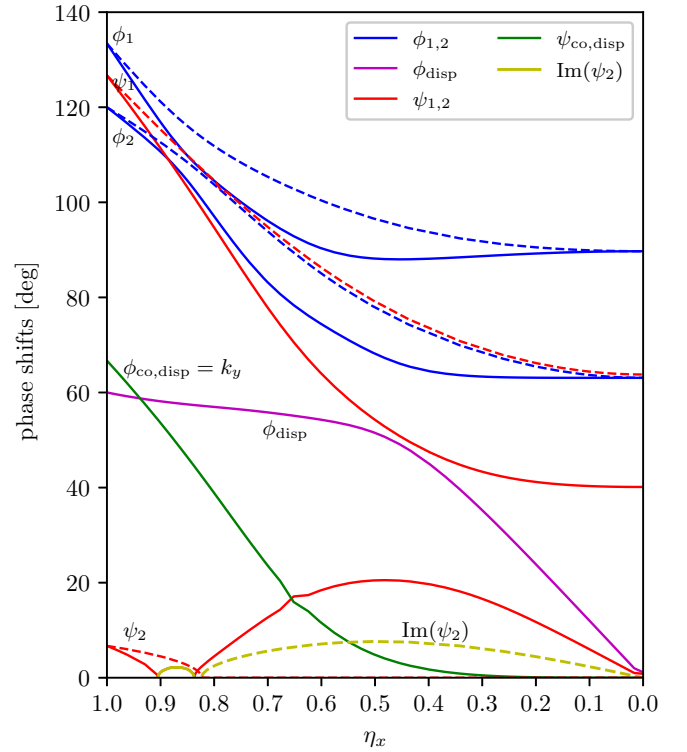


FIG. 3. Phase shifts of the second-order modes vs horizontal tune depression in a CFD of $k_{0,x} = 60^\circ$ and $k_{0,y} = 66.7^\circ$ with $\varepsilon_{dx}/\varepsilon_{dy} = 3$ (solid lines), compared to the case without dispersion (dashed lines).

analytical predictions of $\psi_{\text{co,disp}}$ derived from Eq. (20) show a slight discrepancy with the simulation results, primarily due to the increased nonlinear effects. Nevertheless, for $\eta_x > 0.85$, which is typical for high-intensity circular machines, the beam spectra are in excellent agreement with the analytical calculations in Fig. 1.

V. TILTING INSTABILITIES IN CIRCULAR ACCELERATORS

It is known that the tilting instability of the beams in linacs is driven by an unstable (imaginary) second-order odd mode with an energy anisotropy caused by space charge, characterized by a linear emittance exchange between two degrees of freedom. It occurs in a nonaxisymmetric focusing transport system with stronger external focusing in one degree but stronger in another degree driven by space charge, for instance, $k_{0,y} > k_{0,x}$ and $k_y < k_x$, and vice versa [40,43,45]. In high-intensity circular accelerators, we will show that the dispersion effects can bring additional energy anisotropy in x , and consequently modify the stop band of the tilting instability.

To illustrate this, the second-order modes in an example CFD with $k_{0,x} = 60^\circ$, $k_{0,y} = 66.7^\circ$ with varying beam current are numerically calculated and plotted in Fig. 3, compared to the case without dispersion. It can be seen that with beam current increasing, the lower branch of the second-order mode ψ_2 becomes an imaginary number, indicating the occurrence of the tilting instability. By closer observation of Fig. 3, in the absence of dispersion, the tilting instability exists when

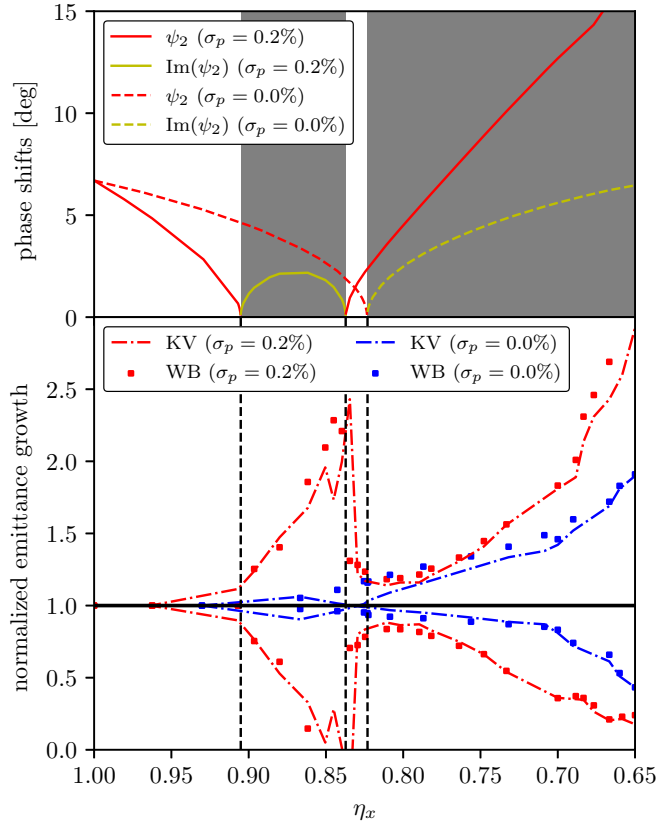


FIG. 4. Enlarged plot of Fig. 3 of the unstable second-order odd modes with and without dispersion (shaded areas), compared with the simulation results of the vertical emittance growth (KV distribution: chain lines; and waterbag distribution: dotted lines).

$\eta_x < 0.82$, where the lower odd mode becomes an imaginary number $\text{Im}(\psi_2)$, as shown in the dashed yellow line in Fig. 3. In comparison, in the presence of dispersion, the stop band of the tilting instability is modified and consequently becomes much narrower, indicated by the imaginary ψ_2 in the range of $0.91 > \eta_x > 0.84$, shown as the solid yellow line in Fig. 3. We attribute the shifted stop band with dispersion to the break of the asymmetry in x and y caused by dispersion.

PIC simulations are conducted within the stop band of the tilting instability, and the corresponding results are presented in Fig. 4. In the simulation, two types of particle distribution, KV and waterbag, are adopted.

For the linac case (in the absence of dispersion, $\sigma_p = 0.0\%$), it can be seen from Fig. 4 that the emittance growth with an initial KV distribution (blue chain line) is in good agreement with the analytical results (blue solid line). In comparison, the range of the emittance growth for the waterbag distribution (blue dotted line) is beyond the stop band of the tilting instability, since the emittance growth is also contributed from particle resonance driven by the high-order space-charge potentials.

For the circular accelerator case (in the presence of dispersion), the emittance growth is mainly in the range of $0.93 > \eta_x > 0.85$, which is in good agreement with the stopband of the analytical result ($0.91 > \eta_x > 0.84$). One can observe that the emittance growth for the two different distributions

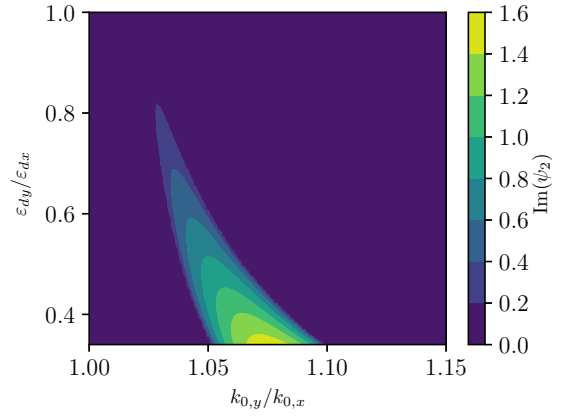


FIG. 5. Scanning of the growth rate of the unstable second-order mode with varying $k_{0,y}$ and emittance ratio for constant $k_{0,x} = 60.0^\circ$ and $\varepsilon_{dx} = 30.0$ mm mrad.

is nearly identical. There exists slight emittance growth outside of $\eta_x > 0.91$. In fact, as the beam starts to propagate through the CFD, the initial KV (uniform) beam evolves into a nonuniform distribution due to dispersion coupling, resulting in enhanced nonlinear space-charge effect and exhibiting behavior similar to that of the waterbag distribution.

The stopband of tilting instability in circular machines shifts toward the lower beam intensity regime, since the dispersion brings additional anisotropy. As a result, compared to the linac case, the susceptibility of the beam to tilting instability is markedly increased in circular machines and becomes an important potential contributor to beam loss. The scan of the growth rate of the tilting instability with a fixed beam current and varying phase advance and emittance is shown in Fig. 5, which can be used in the design and operation of a high-intensity circular accelerator.

VI. FAMILY OF ENVELOPE INSTABILITIES

Our analysis in the preceding sections focused on transport systems of constant focusing with dispersion. For modern high-intensity accelerators with alternating gradient structures, beams can be exposed to quality degradation caused by resonantly acting space-charge forces as collective structural instabilities. The most dominant structural instabilities are the resonance of the second-order (eigen)modes, known as envelope instabilities. The instabilities driven by eigenmodes higher than second order play relatively minor role and can be Landau damped [56]. In fact, the nonlinearity of space charge in non-KV beams, such as waterbag and Gaussian distributions, contributes to incoherent tune spread, within which the frequencies of higher-order modes are located and induce Landau damping. In comparison, the envelope instabilities cannot be Landau damped, since the envelope modes reside outside the frequency range of the tune spread [41].

Compared to linacs, envelope instabilities in circular machines have several unique characteristics; to illustrate this, the phase shifts of second modes in an example of alternating focusing structure with dispersion (AGD) with $k_{0,x} = 140^\circ$, $k_{0,y} = 60^\circ$ with varying beam current are numerically calculated and plotted in Fig. 6.

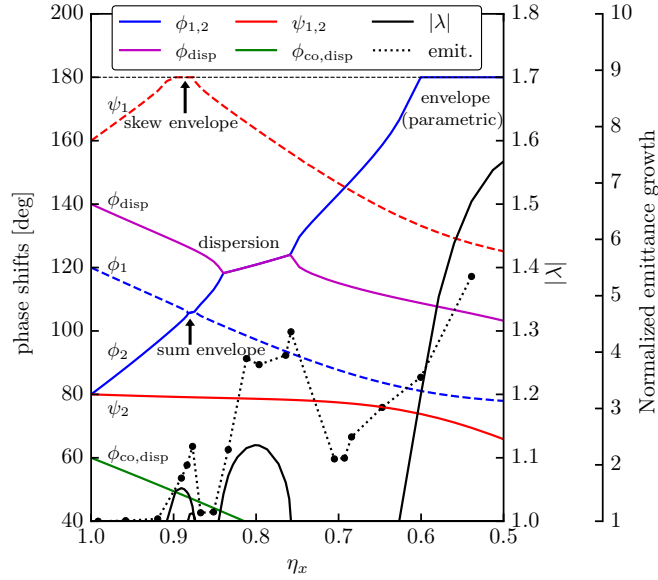


FIG. 6. Phase shifts of the second-order modes vs horizontal tune depression in an AGD with $k_{0,x} = 140^\circ$ and $k_{0,y} = 60^\circ$, compared with the simulation results of total normalized emittance growth in x and y (dotted black lines). The analytical results of the growth rate are also shown (solid black lines).

With beam current increasing, the beam experiences four types of envelope instabilities, indicated by the corresponding growth rates: the skew envelope instability induced by one of the skew modes resonant with the lattice $\psi_{1,2} = 180^\circ$; the sum envelope instability induced by the resonance between the two even modes $\phi_1 + \phi_2 = 360^\circ$; the dispersion-induced instability, driven by one of the even modes, resonates with the dispersion mode $\phi_{1,2} + \phi_{\text{disp}} = 360^\circ$; and the 90° envelope (parametric) instability induced by one of the even modes resonant with the lattice $\phi_1 = 180^\circ$. It can be seen that the range of the growth rates calculated from the coupled envelope model are in good agreement with the normalized emittance growth from PIC simulation results. It is worth pointing out that around $\eta_x = 0.7$, we have $|\lambda| = 0$, indicating that there is no envelope instabilities. However, emittance growth in simulations can be observed in this region. We attribute the observed emittance growth in the simulation around this region to the contribution from the nonlinear resonances, since the space charge when $\eta_x = 0.7$ is rather strong.

In general, for high-intensity circular accelerators, “a unified family of envelope instabilities” can be obtained based on the generalized envelope mode. In Fig. 7, a tune diagram is scanned by solving Eq. (17). The scan is performed in the range $60^\circ \leq k_{0,x,y} \leq 140^\circ$ of a circular accelerator with FODO lattice, for a fixed beam intensity. As illustrated in Fig. 7, six types of envelope instabilities are identified and discussed in detail below.

(1) *90° envelope instabilities*. It is the well-known traditional envelope instability, which is a critical consideration in the design of linacs. It is characterized via $\phi_{1,2} = 180^\circ$ for $k_{x,0} > 90^\circ$, shown as the two horizontal and vertical bands around $k_{0,x,y} = 100^\circ$ in Fig. 7 (also named as parametric resonance in the literature).

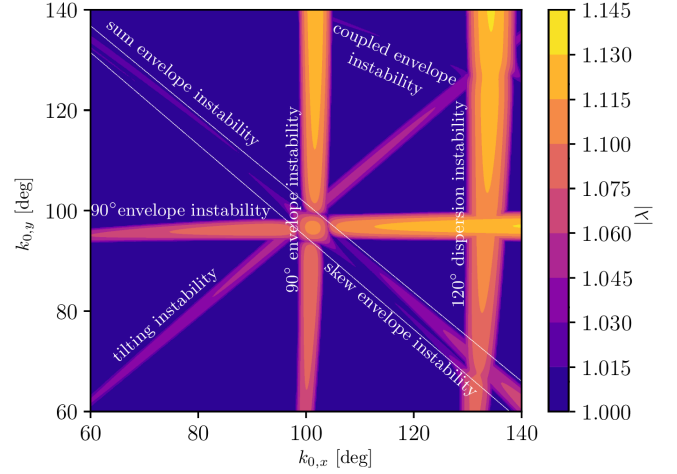


FIG. 7. Picture of full classification of envelope instabilities in circular accelerators. Tune diagram with a fixed beam current and a constant asymmetric emittance.

(2) *Coupled envelope instability*. It is characterized via $\phi_1 = \phi_2$, shown as the short line in the top right-hand corner in Fig. 7. In linac, it is mixed with the parametric resonance. However, in circular accelerators, it can independently exist as a distinct phenomenon.

(3) *Sum envelope instability*. It occurs when the sum of ϕ_1 and ϕ_2 resonates with the lattice ($\phi_1 + \phi_2 = 360^\circ$), shown as the main diagonal in Fig. 7. The sum envelope instability is not confused with the coupled envelope instability. The former exists in a lattice with a large split $k_{0,x}$ and $k_{0,y}$, while the latter exists for close $k_{0,x}$ and $k_{0,y}$.

(4) *120° dispersion instability*. It is driven by the resonance between the dispersion mode and the envelope mode with $\phi_{\text{disp}} + \phi_{1,2} = 360^\circ$ for $k_{x,0} > 120^\circ$, shown as the vertical band at around $k_{0,x} = 130^\circ$ in Fig. 7. In this context, it represents a distinct form of sum envelope instability.

(5) *Skew envelope instability*. It is triggered by one of the skew mode resonances with the lattice $\psi_{1,2} = 180^\circ$, shown as the border band of the main diagonal in Fig. 7.

(6) *Dispersion-enhanced tilting instability*. It arises when $k_{0,x} \approx k_{0,y}$ and the beam current exceeds a critical threshold, causing an imaginary skew mode (ψ_2). It is induced by the anisotropy between x and y , shown as the back diagonal in Fig. 7. Compared to the case in linac, the susceptibility of beams to tilting instabilities in circular accelerators is enhanced.

The dispersion-enhanced tilting instability and the coupled envelope instability are not existing in Fig. 6. This is understandable because split tunes are employed in Fig. 6 ($k_{0,x} = 140^\circ$, $k_{0,y} = 60^\circ$), which is located far away from the criterion $k_{0,x} \approx k_{0,y}$ of the two envelope instabilities.

VII. SUMMARY

In this work, coherent beam instabilities in high-intensity circular accelerators are examined. In particular, a new vertical coupled mode is identified. It is found that the tilting instability becomes of practical concern for design and operation of high-intensity circular machines. The well-known

90° envelope instability is generalized to a family of envelope instability, providing a framework to describe the key coherent beam dynamics, including the integrated effects of space charge, dispersion, and linear coupling. It is shown that coupled-envelope instability is separated from the conventional envelope instability near 90°. It is expected that the model of the envelope instability family provides a clear information of the stop bands of the second-order space-charge-driven instabilities, which is essential for the working point optimization in high-intensity synchrotrons in the future.

ACKNOWLEDGMENTS

The author expresses his gratitude to Professor Giuliano Franchetti (GSI) for his valuable suggestions. This work was jointly supported by the National Natural Science

Foundation of China (Grant No. 12475155), the Guangdong Basic and Applied Basic Research Foundation (Project No. 2024A1515012658), and the International Partnership Program of Chinese Academy of Sciences (Project No. 013GJHZ2023026FN).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: PERTURBATION TRANSFER MATRIX IN CONSTANT FOCUSING WITH DISPERSION

For transport systems of constant focusing with dispersion, the matrix $\mathbf{C}$ can be obtained from Eq. (19), taking the form

$$\mathbf{C} = \begin{pmatrix} 2\frac{\partial G_2}{\partial \Sigma_{11}} & 2\Sigma_{11}\frac{\partial q_{xx}}{\partial \Sigma_{33}} & 2\Sigma_{11}\frac{\partial q_{xx}}{\partial D_x} & 0 & 0 & 0 \\ 2\Sigma_{33}\frac{q_{yy}}{\partial \Sigma_{11}} & 2\frac{\partial G_8}{\partial \Sigma_{33}} & 2\Sigma_{33}\frac{\partial q_{yy}}{\partial D_x} & 0 & 0 & 0 \\ D_x\frac{\partial q_{xx}}{\partial \Sigma_{11}} & D_x\frac{\partial q_{xx}}{\partial \Sigma_{33}} & \frac{G_{10}}{D_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_y & D_x\frac{\partial q_{xy}}{\partial \Sigma_{13}} & 0 \\ 0 & 0 & 0 & 0 & (\Sigma_{11}\frac{\partial q_{xy}}{\partial \Sigma_{13}} + \Sigma_{33}\frac{\partial q_{xy}}{\partial \Sigma_{13}} - k_x - k_y) & 2 \\ 0 & 0 & 0 & 0 & (-k_x\Sigma_{11}\frac{\partial q_{xy}}{\partial \Sigma_{13}} - k_y\Sigma_{33}\frac{\partial q_{xy}}{\partial \Sigma_{13}} + 2k_xk_y) & -(k_x + k_y) \end{pmatrix}, \quad (\text{A1})$$

in which

$$\frac{\partial G_2}{\partial \Sigma_{11}} = -\frac{\varepsilon_{dx}^2 + \Sigma_{12}^2}{\Sigma_{11}^2} - k_x + \Sigma_{11}\frac{\partial q_{xx}}{\partial \Sigma_{11}}\frac{\partial G_8}{\partial \Sigma_{33}} = -\frac{\varepsilon_y^2 + \Sigma_{34}^2}{\Sigma_{33}^2} - k_y + \Sigma_{33}\frac{\partial q_{yy}}{\partial \Sigma_{33}}. \quad (\text{A2})$$

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