

Decidabilities of local unitary equivalence for entanglement witnesses and states

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(Received 10 April 2025; accepted 29 August 2025; published 16 September 2025)

The problem of determining whether two states are equivalent by local unitary (LU) operations is important for quantum information processing. In this paper, we propose an alternative perspective to study this problem by comparing the decidabilities of LU equivalence (also known as LU decidabilities for short) between entanglement witnesses and states. We introduce a relation between sets of Hermitian operators in terms of the LU decidability. Then, we compare the LU decidability for the set of entanglement witnesses to the LU decidabilities for several sets of states. By comparison, we establish a hierarchy of these sets in terms of LU decidabilities. Moreover, we realize that the simultaneous LU (SLU) equivalence between tuples of mutually orthogonal projectors is crucial to LU equivalent operators. We reveal by examples that for two tuples of projectors, the partial SLU equivalence cannot ensure the overall SLU equivalence. Generally, we present a necessary and sufficient condition such that two tuples of states are SLU equivalent.

DOI: [10.1103/bzfp-wxht](https://doi.org/10.1103/bzfp-wxht)

I. INTRODUCTION

At the heart of quantum mechanics lies the concept of entanglement, a phenomenon that distinguishes quantum systems from classical ones, and serves as a key resource for various information processing protocols. An efficient way to characterize entanglement is to classify entangled states under several equivalence relations relying on different kinds of local operations such as local unitary (LU) operations [1], local operations and classical communication (LOCC) [2], and stochastic LOCC [3]. The fundamental one is LU equivalence, which refers to the equivalence relation between two multipartite states if they can be transformed into each other by applying LU operations. This equivalence plays an essential role in understanding the structures and properties of entanglement, as it defines an equivalence class of states that share identical entanglement characteristics under local operations. In addition, since the process of LU transformation is reversible, the LU classification of states is thus crucial for determining the resource interconvertibility among different entangled states [4], and has significant implications for quantum computing [5], communication [6], and metrology [7].

For bipartite pure states, the Schmidt decomposition provides an effective technique to classify them under LU equivalence. Although it lacks a generalization of the Schmidt

decomposition for multipartite scenarios, the complete classification for multipartite pure states under LU equivalence has been progressively developed [8–10]. Despite the breakthroughs on the LU classification of pure states, the problem of characterizing LU orbits of mixed states remains challenging. A possible way to attack this problem is to discover algebraic invariants [11] under LU transformations that serve as necessary conditions for determining whether two states are LU equivalent. Studying along this way has been of great interest in recent years [12–18]. However, many open questions remain, especially regarding the completeness of invariant sets and the practical implementation of classification algorithms for higher-dimensional systems.

Indeed, the classification under LU equivalence is not confined to states. The study has been extended to several other objects in entanglement theory, e.g., the nonlocal unitary operations [19,20] and the entanglement witnesses (EWs) [21,22]. The characterizations of nonlocal unitary operations and EWs under LU equivalence are also of physical significance [23–25]. From an operational perspective, to implement a complicated nonlocal unitary gate or EW, an efficient approach is to first implement a simpler one which is LU equivalent to the target, assisted by a sequence of LU operations [26,27]. An entangled state with NPT (non-positive partial transpose) property can be linked to an EW by virtue of the partial transposition. This connection motivates us to jointly investigate the LU equivalence of EWs and that of states. Since EWs and states are both expressed by Hermitian operators that have spectral decompositions, some EWs and entangled states can be mathematically converted by adjusting eigenvalues. Thus, we propose an alternative perspective on the LU equivalence of states by connecting with the LU equivalence of EWs.

In this paper, we introduce the concept of LU decidable sets where any two bipartite operators can be decided as LU

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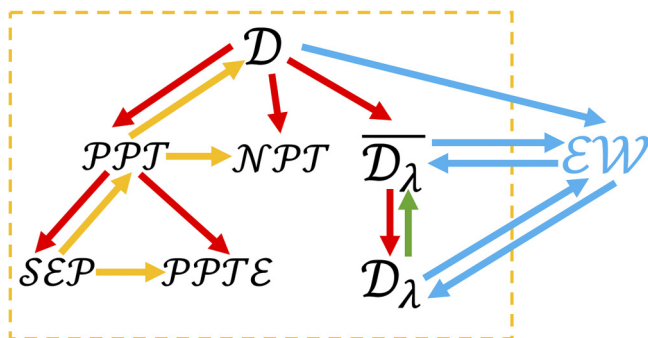


FIG. 1. The relations in terms of the LU decidability among several sets. Denote by $\mathcal{D}$, $\mathcal{NPT}$, $\mathcal{PPT}$, $\mathcal{PPTE}$, $\mathcal{SEP}$, and $\mathcal{EW}$ the sets of all states, NPT states, PPT states, PPT entangled states, separable states, and EWs, respectively, supported on the same bipartite Hilbert space. We refer to $\mathcal{D}_\lambda$ and $\overline{\mathcal{D}}_\lambda$, respectively, as the set of bipartite states satisfying that at least one eigenspace contains no product vector, and the set of bipartite states satisfying that at least one eigenspace is not spanned by product vectors. The arrow “ $\rightarrow$ ” represents the notation “ $\succeq$ ” indicating the relation in terms of the LU decidability. The arrows in red are concluded from the inclusion relation between the two sets. The arrows in yellow are derived by Theorem 1, the arrows in blue are related to Theorem 2, and the arrow in green is derived by Proposition 1. The arrows in the dashed box reveal the relations among the sets of states, and the arrows in blue establish the connection between $\mathcal{EW}$ and those sets of states.

equivalent or LU inequivalent. By comparing the LU decidabilities between two sets, we further introduce a relation in terms of LU decidabilities. For two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of bipartite operators, we use $\mathcal{S}_1 \preceq \mathcal{S}_2$ to indicate that the LU decidability of $\mathcal{S}_1$ is not stronger than that of $\mathcal{S}_2$, which alternatively means that $\mathcal{S}_1$ is LU decidable if $\mathcal{S}_2$ is LU decidable. By virtue of this relation, we connect the LU decidability of the set of EWs with those of several sets of states. Based on the comparison results, we establish a hierarchy in terms of LU decidabilities for various sets under consideration. To visualize this hierarchy, we illustrate it in Fig. 1. In Theorem 1, we build the relations among the set of EWs and the sets of states divided by the PPT (positive-partial-transpose) criterion [28]. In order to more accurately evaluate the LU decidability of the set of EWs, we figure out a set of states each of which satisfies that there is an eigenspace containing no product vector, and show by Theorem 2 that the LU decidability of such set is equal to that of the set of EWs. Furthermore, we extend our study to a larger set of states each of which satisfies that there exists some eigenspace not spanned by product vectors. We conclude in Proposition 1 that the LU decidability of this larger set is equal to that of the abovementioned set. By analyzing the spectral decompositions of LU equivalent operators, we realize that the simultaneous LU (SLU) equivalence between tuples of mutually orthogonal projectors plays an essential role. In light of this, we consider the conditions ensuring the SLU equivalence between such tuples. A necessary condition requires that any parts of such two tuples have to be SLU equivalent, which we call the partial SLU equivalence. By Example 1, we indicate that the partial SLU equivalence cannot ensure the SLU equivalence between the whole tuples of mutually orthogonal projectors. In a more

general context, we propose a necessary and sufficient condition in Theorem 3 such that two tuples of states are SLU equivalent.

The remainder of this paper is organized as follows: In Sec. II, we clarify the notations and definitions, and present some known results as necessary preliminaries. In Sec. III, we establish connections of the LU decidabilities among the set of EWs and several sets of states, which also reveals a hierarchy of these sets in terms of the LU decidability. In Sec. IV, we investigate the SLU equivalence between tuples of projectors, which is quite essential to the problem of determining whether two arbitrary states are LU equivalent or not. Finally, the concluding remarks are given in Sec. V.

II. PRELIMINARIES

In this section, we perform mathematical preparations to make our discussions clear and concise. We first clarify some common notations. The LU equivalence is the core concept in this paper, which implies mutual transformations of bipartite operators by LU operations. Given two bipartite Hermitian operators M and N , we use $M \sim N$ to represent that they are LU equivalent, specifically defined by $M = (U \otimes V)N(U \otimes V)^\dagger$ for some LU operator $U \otimes V$. Conversely, we may use $M \not\sim N$ to represent that they are LU inequivalent, which means there is no LU operation such that M and N can be transformed to each other. One can analogously formulate the LU equivalence of bipartite subspaces by $X = (U \otimes V)Y$ for two subspaces X and Y of $\mathbb{C}^m \otimes \mathbb{C}^n$. We also use $X \sim Y$ and $X \not\sim Y$ to represent that such two subspaces are LU equivalent and LU inequivalent, respectively.

For any Hermitian operator H , its spectral decomposition reads as $H = \sum_j \lambda_j P_j$, where λ_j 's are distinct eigenvalues and P_j 's are orthogonal projectors with $P_j P_k = \delta_{jk} P_j$. Unless specified otherwise, the eigenvalues in the spectral decomposition are sorted by descending order. We shall use the shorthand notation λ_j eigenspace to represent the eigenspace associated with the j th eigenvalue λ_j . Unless stated otherwise, the states addressed in this paper are assumed as non-normalized; i.e., the traces of states are just positive. Denote by $\mathcal{D}$ the set of all bipartite states. Based on the spectral decomposition of a bipartite state, we introduce some subsets of $\mathcal{D}$ that are closely related to our main results as follows. These subsets are denoted by $\mathcal{D}_{\lambda_j}$, $\mathcal{D}_\lambda$, and $\overline{\mathcal{D}_\lambda}$, respectively. A bipartite state $\rho \in \mathcal{D}_{\lambda_j}$ satisfies that the eigenspace of ρ associated with the j th largest eigenvalue contains no product vector. Then $\mathcal{D}_\lambda$ is a union of all $\mathcal{D}_{\lambda_j}$ for all distinct eigenvalues λ_j 's. Specifically, a bipartite state $\rho \in \mathcal{D}_\lambda$ satisfies that there exists some eigenvalue λ of ρ , such that the corresponding eigenspace contains no product vector. A subspace contains no product vector is also known as the completely entangled subspace in the field of entanglement theory. The completely entangled subspace by definition is a subspace not spanned by product vectors, while the reverse is not true. Thus, we may further expand $\mathcal{D}_\lambda$ to $\overline{\mathcal{D}_\lambda}$. Specifically, a bipartite state $\rho \in \overline{\mathcal{D}_\lambda}$ satisfies that there exists some eigenvalue λ of ρ , such that the corresponding eigenspace is not spanned by product vectors. We propose a state in $\overline{\mathcal{D}_\lambda}$ but not in $\mathcal{D}_\lambda$ as follows to clarify the distinction between $\mathcal{D}_\lambda$ and $\overline{\mathcal{D}_\lambda}$. Let ρ be

a state with the spectral decomposition as

$$\rho = \frac{1}{3}(|\phi_+\rangle\langle\phi_+| + |01\rangle\langle 01|) + \frac{1}{6}(|\phi_-\rangle\langle\phi_-| + |10\rangle\langle 10|), \quad (1)$$

where $|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. By observation, both eigenspaces corresponding to eigenvalues $\frac{1}{3}$ and $\frac{1}{6}$ contain a product vector but are not spanned by product vectors. Hence, we determine by definition that the two-qubit state ρ given by Eq. (1) belongs to $\overline{\mathcal{D}_\lambda}$ but not to $\mathcal{D}_\lambda$. It implies that $\mathcal{D}_\lambda$ is a proper subset of $\overline{\mathcal{D}_\lambda}$, i.e., $\mathcal{D}_\lambda \subsetneq \overline{\mathcal{D}_\lambda}$.

For any bipartite Hermitian operator M , we may refer to M^{Γ_i} as the partial transpose of M with respect to the i th subsystem for $i = 1, 2$. Since the two partial transposes of M are the same up to a global transposition, i.e., $M^{\Gamma_1} = (M^{\Gamma_2})^T$, we usually consider M^{Γ_1} and simply denote it as M^Γ . According to the PPT criterion, $\mathcal{D}$ is divided into several common subsets. Here, we use the calligraphic terms as $\mathcal{NPT}$, $\mathcal{PPT}$, $\mathcal{PPT\mathcal{E}}$, $\mathcal{SEP}$, and $\mathcal{EW}$ to represent the sets of NPT states, PPT states, PPT entangled states, separable states, and EWs, respectively, where the operators in these sets are all supported on the same bipartite Hilbert space. We also use $\mathcal{R}(\rho)$ to represent the range of positive semidefinite ρ .

Second, based on LU equivalence, we introduce two kinds of relations between sets of bipartite Hermitian operators as follows. Definition 1 establishes a hierarchy in terms of the LU decidability. Definition 2 characterizes whether a set of bipartite Hermitian operators can be correspondingly transformed to another set by a common LU operator.

Definition 1. (i) For a given set $\mathcal{S}$ of bipartite Hermitian operators, if the LU equivalence or LU inequivalence between any two operators in $\mathcal{S}$ is known, the set $\mathcal{S}$ is called LU decidable.

(ii) For two given sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of bipartite Hermitian operators, if $\mathcal{S}_1$ is LU decidable when $\mathcal{S}_2$ is LU decidable, then the LU decidability of $\mathcal{S}_1$ is not stronger than that of $\mathcal{S}_2$, denoted by $\mathcal{S}_1 \preceq \mathcal{S}_2$.

(iii) We call the LU decidabilities of $\mathcal{S}_1$ and $\mathcal{S}_2$ equal, denoted by $\mathcal{S}_1 \stackrel{d}{=} \mathcal{S}_2$, if both two relations $\mathcal{S}_1 \preceq \mathcal{S}_2$ and $\mathcal{S}_2 \preceq \mathcal{S}_1$ hold simultaneously.

By definition one can verify that the relation “ $\preceq$ ” admits the transitivity. That is, $\mathcal{S}_1 \preceq \mathcal{S}_3$ holds if both relations $\mathcal{S}_1 \preceq \mathcal{S}_2$ and $\mathcal{S}_2 \preceq \mathcal{S}_3$ hold. One can also check that the relation “ $\stackrel{d}{=}$ ” is an equivalence relation.

Moreover, the relation of LU decidability is linked to the inclusion relation. We give some remarks on this link as follows. For two sets with an inclusion relation, e.g., $\mathcal{S}_1 \subseteq \mathcal{S}_2$, their relation of LU decidability naturally admits as $\mathcal{S}_1 \preceq \mathcal{S}_2$. For such two sets, if it is further known that $\mathcal{S}_1 \stackrel{d}{=} \mathcal{S}_2$, one can verify that for any set $\mathcal{S}_3$ with $\mathcal{S}_1 \subseteq \mathcal{S}_3 \subseteq \mathcal{S}_2$, the equivalence relation $\mathcal{S}_1 \stackrel{d}{=} \mathcal{S}_2 \stackrel{d}{=} \mathcal{S}_3$ holds. Therefore, it establishes a hierarchy of sets of bipartite Hermitian operators in terms of the LU decidability.

It is necessary to simultaneously consider the LU equivalence of multiple pairs of bipartite Hermitian operators, in order to determine the relation of LU decidability for two given sets. Therefore, the following introduced simultaneous LU equivalence is essential to study the hierarchy in terms of the LU decidability.

Definition 2. Suppose that $\mathcal{S}_1 = (M_1, \dots, M_k)$ and $\mathcal{S}_2 = (N_1, \dots, N_k)$ are two k -tuples of bipartite Hermitian operators supported on $\mathbb{C}^m \otimes \mathbb{C}^n$. We call that $\mathcal{S}_1$ is SLU equivalent to $\mathcal{S}_2$, denoted by $\mathcal{S}_1 \sim_s \mathcal{S}_2$, if there exists a common LU operator $U \otimes V$ such that $(U \otimes V)M_i(U \otimes V)^\dagger = N_i$ for each $1 \leq i \leq k$.

The definition above can be directly extended to the context of bipartite subspaces. We here do not repeat the definition of SLU equivalence for bipartite subspaces.

Characterizing PPT entangled states is a difficult task. There is an effective way to construct PPT entangled states based on the unextendible product basis (UPB) [29,30] with the formal definition below.

Definition 3. Unextendible product basis is a set of orthonormal product vectors such that no product vector is orthogonal to all of them.

By definition, the orthogonal complement subspace of the subspace spanned by a UPB contains no product vector, i.e., a completely entangled subspace. It implies that the range of the state given by

$$\rho := \frac{1}{d-k} \mathbb{I} - \sum_{j=1}^k |a_j, b_j\rangle\langle a_j, b_j| \quad (2)$$

contains no product vector, and thus ρ is a PPT entangled state, where $\{|a_j, b_j\rangle\}_{j=1}^k$ forms a UPB.

Finally, we present some known lemmas which are necessary to derive our main results. Lemma 1 indicates that the entanglement of a state will vanish completely by adding large enough noise. Lemma 2 proposes a necessary and sufficient condition in terms of SLU equivalence given by Definition 2, such that two bipartite Hermitian operators are LU equivalent.

Lemma 1. Given a multipartite state ρ , there is a large enough coefficient $x > 0$ such that $\rho + x\mathbb{I}$ is a separable state, where $\mathbb{I}$ is supported on the same space as ρ .

Lemma 2. Let H and K be two bipartite Hermitian operators. Their spectral decompositions read as $H = \sum_{j=1}^m \lambda_j M_j$ and $K = \sum_{j=1}^n \mu_j N_j$, where λ_j 's and μ_j 's are nonzero eigenvalues of H and K , respectively, in descending order. Then H is LU equivalent to K , i.e., $H \sim K$, if and only if $m = n$, and for each j , $\lambda_j = \mu_j$ and $M_j \sim N_j$ by a common LU operator.

It is worthy to note that the SLU equivalence of the two m -tuples $(M_j)_{j=1}^m$ and $(N_j)_{j=1}^m$ is indispensable for Lemma 2. In other words, for two given pairs of bipartite projectors (A_1, B_1) and (A_2, B_2) and two real coefficients x and y , the two operators formulated by $\alpha_i := xA_i + yB_i$ for $i = 1, 2$ may not be LU equivalent, even if $A_1 \sim A_2$ and $B_1 \sim B_2$. We explain it by example. Let $A_1 = A_2 = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ for $\theta \in (0, \frac{\pi}{2})$. Then, we have $A_1 \sim A_2$. Let $B_1 = |\Psi^+\rangle\langle\Psi^+|$ and $B_2 = |\Psi^-\rangle\langle\Psi^-|$, where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Then, we have that B_1 and B_2 are LU equivalent by the LU operator as $\mathbb{I}_2 \otimes \text{diag}(-1, 1)$. However, one can verify that the eigenvalues of α_1 are not the same as those of α_2 . Thus, α_1 and α_2 are not LU equivalent. It is also known by Lemma 2 that (A_1, B_1) is not SLU equivalent to (A_2, B_2) .

The following lemma characterizes the unitary matrices that maintain diagonal matrices invariant under the unitary similarity.

Lemma 3. Let $\Lambda = \oplus_{j=1}^n \lambda_j \mathbb{I}_{d_j}$ be a diagonal matrix. The unitary matrix U such that $U \Lambda U^\dagger = \Lambda$ has to be a direct sum of unitary matrices, namely, $U = \oplus_{j=1}^n U_j$, where U_j is supported on $\mathbb{C}^{d_j}$.

III. RELATIONS AMONG THE SET OF EWS AND SETS OF STATES IN TERMS OF LU DECIDABILITIES

EWs provide a powerful tool to detect entanglement based on negative expectation values, namely, $\text{tr}(W\rho) < 0$ only for some entangled states. Due to the inherent connection between LU operations and entanglement characteristics, it inspires us to jointly consider the LU orbits of EWs and the LU orbits of states. According to the relation in terms of the LU decidability introduced by Definition 1, we shall focus on such relations among sets of bipartite states and the set of EWs. Our discussion in this section not only establishes a connection of the two problems on the LU decidabilities, respectively, for states and EWs, but also reveals a hierarchy of the sets of states and that of EWs. To better understand these relations, one may refer to Fig. 1, which visualizes the main results of this section.

In Theorem 1, we compare the LU decidabilities among the set of EWs and some common sets of bipartite states, namely, $\mathcal{NPT}$, $\mathcal{PPT}$, $\mathcal{PPT}\mathcal{E}$, and $\mathcal{SEP}$. In Theorem 2, we introduce a subset $\mathcal{D}_\lambda \subsetneq \mathcal{D}$ and show that the LU decidability of $\mathcal{D}_\lambda$ is equal to that of $\mathcal{EW}$. Such two main results in this section indicate that the LU decidabilities for such two sets including different types of Hermitian operators could be related. It sheds light on the problem of verifying the LU equivalence or inequivalence between two arbitrary states, assisted by introducing other types of Hermitian operators, e.g., EWs. Then, we extend our study to a larger set of states, denoted by $\overline{\mathcal{D}_\lambda}$, which includes the set $\mathcal{D}_\lambda$. We show by Proposition 1 that the two LU decidabilities of $\mathcal{D}_\lambda$ and $\overline{\mathcal{D}_\lambda}$ are equal. It is valuable to establish the equivalence relation between the LU decidabilities of the two sets that admit an inclusive relation, as it may reduce the difficulty of analyzing the LU decidability of the larger set.

The PPT criterion is a celebrated tool to identify entangled states, which divides $\mathcal{D}$ into two nonintersected parts, namely, $\mathcal{NPT}$ and $\mathcal{PPT}$, and tells that each state of $\mathcal{NPT}$ is entangled. The set $\mathcal{PPT}$ can be further divided to two disjoint subsets $\mathcal{PPT}\mathcal{E}$ and $\mathcal{SEP}$, where $\mathcal{PPT}\mathcal{E}$ is nonempty when the supporting space has a global dimension greater than 6. Here, we first compare the LU decidabilities for several common sets of bipartite states and the set of EWs.

Lemma 4. (i) One can decide whether two arbitrary states in $\mathcal{NPT}$ are LU equivalent or not, if the LU equivalence or LU inequivalence between two arbitrary EWs is decidable.

(ii) One can decide whether two arbitrary EWs are LU equivalent or not, if the LU equivalence or LU inequivalence between two arbitrary states in $\mathcal{D}$ is decidable.

(iii) One can decide whether two arbitrary states are LU equivalent or not, if and only if the LU equivalence or LU inequivalence between two arbitrary states in $\mathcal{SEP}$ is decidable.

Proof. (i) The assertion follows from the fact that the partial transpose of an NPT state is an EW. Suppose that ρ_{AB} and σ_{AB} are two NPT states. It follows that ρ_{AB}^Γ and σ_{AB}^Γ are

both EWs, where ρ_{AB}^Γ means the partial transpose of ρ_{AB} with respect to the first subsystem A , similarly for σ_{AB}^Γ . One can verify that the LU equivalence between two bipartite operators M and N is the same as the LU equivalence between M^Γ and N^Γ . Thus, we conclude that $\rho_{AB} \sim \sigma_{AB}$ is equivalent to $\rho_{AB}^\Gamma \sim \sigma_{AB}^\Gamma$. Due to the assumption, the LU equivalence or LU inequivalence between ρ_{AB}^Γ and σ_{AB}^Γ is decidable. Thus, the LU equivalence or LU inequivalence between ρ_{AB} and σ_{AB} is also decidable. This assertion implies that $\mathcal{NPT} \leq \mathcal{EW}$.

(ii) One may transform the problem of determining whether two EWs are LU equivalent to the problem for two states, by the following relation between EWs and states: for an arbitrary EW W supported on the bipartite Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$, there exists a large enough coefficient x such that

$$\rho_x := W + x\mathbb{I}_{mn} \quad (3)$$

is positive semidefinite, thus a non-normalized state. Let W_1 and W_2 be two arbitrary EWs supported on $\mathbb{C}^m \otimes \mathbb{C}^n$. Using the technique given by Eq. (3), one can find two large enough coefficients x_1 and x_2 such that $\rho_{x_1} := W_1 + x_1\mathbb{I}_{mn}$ and $\rho_{x_2} := W_2 + x_2\mathbb{I}_{mn}$ are two states. Denote by x the maximum of $\{x_1, x_2\}$. Let $\rho_1 := W_1 + x\mathbb{I}_{mn}$ and $\rho_2 := W_2 + x\mathbb{I}_{mn}$. It follows that ρ_1 and ρ_2 are both states. Then one can verify that the LU equivalence of W_1 and W_2 is simultaneous with that of ρ_1 and ρ_2 . Thus, if one can decide whether two states are LU equivalent or not, then whether two EWs are LU equivalent or not is also decidable by the transformation as Eq. (3). This assertion implies that $\mathcal{EW} \leq \mathcal{D}$.

(iii) The “only if” part follows directly by the relation $\mathcal{SEP} \subset \mathcal{D}$, which implies that $\mathcal{SEP} \leq \mathcal{D}$. Next, we prove the “if” part which means that $\mathcal{SEP} \geq \mathcal{D}$. Let ρ and σ be two arbitrary states supported on the same bipartite Hilbert space. By Lemma 1, we conclude that there exist two large enough coefficients x_1 and x_2 such that $\rho + x_1\mathbb{I}$ and $\sigma + x_2\mathbb{I}$ are both separable. Denote $x = \max\{x_1, x_2\}$. Thus, we have that $\rho_x \equiv \rho + x\mathbb{I}$ and $\sigma_x \equiv \sigma + x\mathbb{I}$ are both separable. Similar to assertion (i), we know that the LU equivalence of ρ and σ is simultaneous with that of ρ_x and σ_x . Due to the assumption, the LU equivalence or inequivalence of ρ_x and σ_x is decidable, and thus the LU equivalence or inequivalence of ρ and σ is also decidable, i.e., $\mathcal{SEP} \geq \mathcal{D}$. To sum up, we obtain that $\mathcal{SEP} \stackrel{d}{=} \mathcal{D}$ by definition.

This completes the proof. $\blacksquare$

Combining Lemma 1 and the inclusion relations among the sets of bipartite states, we may summarize the relations in terms of LU decidabilities as follows.

Theorem 1. The decidabilities of LU equivalence for the set of EWs and several common sets of bipartite states admit the following chain of relations:

$$\mathcal{NPT} \leq \mathcal{EW} \leq \mathcal{D} \stackrel{d}{=} \mathcal{SEP} \stackrel{d}{=} \mathcal{PPT} \geq \mathcal{PPT}\mathcal{E}. \quad (4)$$

Proof. The chain of relations is a combination of the four parts as below:

(1) The relation $\mathcal{NPT} \leq \mathcal{EW}$ follows by Lemma 1(i).

(2) The relation $\mathcal{EW} \leq \mathcal{D}$ follows by Lemma 1(ii).

(3) The equivalence relation $\mathcal{D} \stackrel{d}{=} \mathcal{SEP} \stackrel{d}{=} \mathcal{PPT}$ follows by Lemma 1(iii) and the inclusion relation of such three sets, namely, $\mathcal{SEP} \subset \mathcal{PPT} \subset \mathcal{D}$.

(4) The relation $PPT \geq PPT\mathcal{E}$ follows directly by $PPT\mathcal{E} \subset PPT$.

This completes the proof. ■

To prove Lemma 1, it is essential to observe the fact that for any two bipartite Hermitian operators ρ and σ supported on $\mathbb{C}^m \otimes \mathbb{C}^n$ and any real number x , the LU equivalence between ρ and σ is simultaneous with that between $\rho + x\mathbb{I}_{mn}$ and $\sigma + x\mathbb{I}_{mn}$. In the following lemma, we reveal the relation between $\mathcal{D}_{\lambda_1}$ and the set $\mathcal{EW}$ in terms of the LU decidability, by virtue of the above observation. Here, $\mathcal{D}_{\lambda_1}$ is the set given by $\mathcal{D}_{\lambda_j}$ when the j th largest eigenvalue λ_j is specified as the largest eigenvalue λ_1 . One may refer to the second paragraph of Sec. II for the definition of $\mathcal{D}_{\lambda_j}$.

Lemma 5. Denote by $\mathcal{D}_{\lambda_1}$ a subset of $\mathcal{D}$. A state $\rho \in \mathcal{D}_{\lambda_1}$ satisfies that the eigenspace of ρ associated with the largest eigenvalue λ_1 contains no product vector. Then the LU decidabilities of $\mathcal{D}_{\lambda_1}$ and $\mathcal{EW}$ admit the relation as $\mathcal{D}_{\lambda_1} \leq \mathcal{EW}$.

Recall that a state in $\mathcal{D}_{\lambda_j}$ satisfies that the λ_j eigenspace contains no product vector, and the set $\mathcal{D}_{\lambda_j}$ is a situation of $\mathcal{D}_{\lambda_1}$ by specifying λ_j as the largest eigenvalue λ_1 . In light of this, Lemma 5 is covered by Lemma 6, which applies to all sets $\mathcal{D}_{\lambda_j}$. Despite this, we propose another idea of proving Lemma 5 in Appendix A, to help understand such two results better.

Lemma 6. For each λ_j , the LU decidabilities of the corresponding set $\mathcal{D}_{\lambda_j}$ and $\mathcal{EW}$ admit $\mathcal{D}_{\lambda_j} \leq \mathcal{EW}$.

Proof. For any two states ρ_1 and ρ_2 , one can decide that the two are not LU equivalent by Lemma 2, if the eigenvalues of ρ_1 are not exactly the same as those of ρ_2 . We thus consider whether there exists the LU equivalence between $\rho_1, \rho_2 \in \mathcal{D}_{\lambda_j}$ with identical eigenvalues. In this case, let the spectral decompositions of ρ_1 and ρ_2 , respectively, be

$$\begin{aligned}\rho_1 &= \lambda_j P_j + \sum_{i \neq j} \lambda_i P_i, \\ \rho_2 &= \lambda_j Q_j + \sum_{i \neq j} \lambda_i Q_i,\end{aligned}\quad (5)$$

where P_i 's (including P_j) and Q_i 's (including Q_j) are two families of orthogonal projectors. By the definition of $\mathcal{D}_{\lambda_j}$, we obtain that the ranges of P_j and Q_j , respectively, contain no product vector. One can verify that for any number $x > 0$, the two non-normalized states $\rho_1 + x\mathbb{I}$ and $\rho_2 + x\mathbb{I}$ are still in $\mathcal{D}_{\lambda_j}$, and they are of full rank. Recall that the LU equivalence between ρ_1 and ρ_2 is simultaneous with that between $\rho_1 + x\mathbb{I}$ and $\rho_2 + x\mathbb{I}$. In light of this, it suffices to consider the states of full rank in $\mathcal{D}_{\lambda_j}$, and thus each eigenvalue written in Eq. (5) can be assumed as positive.

Based on Eq. (5), we construct two Hermitian operators by

$$\begin{aligned}W_1 &:= -\mu P_j + \sum_{i \neq j} \lambda_i P_i, \\ W_2 &:= -\mu Q_j + \sum_{i \neq j} \lambda_i Q_i,\end{aligned}\quad (6)$$

where $\mu > 0$. We claim that there exists a small enough μ such that W_1 and W_2 are both non-normalized EWs. By definition an EW W is a Hermitian operator that is not positive semidefinite and satisfies $\langle a, b|W|a, b \rangle \geq 0$ for any product vector $|a, b\rangle$. First, from Eq. (6), both W_1 and W_2 are not positive semidefinite. Second, to ensure W_1 and W_2 are EWs,

the following two inequalities should hold for any product vector $|a, b\rangle$ with norm 1:

$$\langle a, b| \left(\sum_{i \neq j} \lambda_i P_i \right) |a, b \rangle - \mu \langle a, b| P_j |a, b \rangle \geq 0, \quad (7)$$

$$\langle a, b| \left(\sum_{i \neq j} \lambda_i Q_i \right) |a, b \rangle - \mu \langle a, b| Q_j |a, b \rangle \geq 0. \quad (8)$$

For convenience, we note that

$$\begin{aligned}p_{\max} &:= \max_{\| |a, b\rangle \| = 1} \langle a, b| P_j |a, b \rangle, \\ q_{\max} &:= \max_{\| |a, b\rangle \| = 1} \langle a, b| Q_j |a, b \rangle.\end{aligned}\quad (9)$$

Since $P_j \neq 0$, there always exists a product vector $|a, b\rangle$ such that $\langle a, b| P_j |a, b \rangle > 0$, and similarly for Q_j . We then consider that both $p_{\max}$ and $q_{\max}$ are positive. We also note that

$$\begin{aligned}p_{\min} &:= \min_{\| |a, b\rangle \| = 1} \langle a, b| \left(\sum_{i \neq j} \lambda_i P_i \right) |a, b \rangle, \\ q_{\min} &:= \min_{\| |a, b\rangle \| = 1} \langle a, b| \left(\sum_{i \neq j} \lambda_i Q_i \right) |a, b \rangle.\end{aligned}\quad (10)$$

In fact the following map:

$$|x\rangle \mapsto \langle x| \hat{P} |x\rangle, \quad (11)$$

where $\hat{P}$ is positive semidefinite and subnormalized (i.e., $\text{tr}(\hat{P}) \leq 1$), is a continuous map, because it maps $|x\rangle$ to the square of the norm of $\sqrt{\hat{P}}|x\rangle$ that is bounded by the square of $\|\sqrt{\hat{P}}\| \cdot \| |x\rangle \|$. Then, by the knowledge from real analysis [31], the two minimums $p_{\min}$ and $q_{\min}$ given by Eq. (10) can be attained. That is, there exist some $|a_1, b_1\rangle$ and $|a_2, b_2\rangle$ such that $p_{\min} = \langle a_1, b_1| \left(\sum_{i \neq j} \lambda_i P_i \right) |a_1, b_1\rangle$ and $q_{\min} = \langle a_2, b_2| \left(\sum_{i \neq j} \lambda_i Q_i \right) |a_2, b_2\rangle$.

We next show by contradiction that for any product vector $|a, b\rangle$, $\langle a, b| \left(\sum_{i \neq j} \lambda_i P_i \right) |a, b \rangle > 0$. It further implies that $p_{\min} > 0$. One can similarly show $q_{\min} > 0$. Assume that if there is some $|a, b\rangle$ such that $\langle a, b| \left(\sum_{i \neq j} \lambda_i P_i \right) |a, b \rangle = 0$. It follows that $\langle a, b| P_i |a, b \rangle = 0$ for each $i \neq j$, as each eigenvalue λ_i is positive and P_i 's are orthogonal projectors. It means that $|a, b\rangle$ lies in the orthogonal complement space of $\oplus_{i \neq j} \mathcal{R}_i$, where $\mathcal{R}_i$ denotes the range of P_i for each i . Recall that ρ_1 and ρ_2 are both of full rank. By Eq. (5), we conclude that the orthogonal complement space of $\oplus_{i \neq j} \mathcal{R}_i$ is indeed the range of P_j , namely, $\mathcal{R}_j$. Thus, $|a, b\rangle$ lies in $\mathcal{R}_j$. It contradicts that $\mathcal{R}_j$ contains no product vector by the definition of $\mathcal{D}_{\lambda_j}$. Therefore, we conclude that both $p_{\min} > 0$ and $q_{\min} > 0$.

Let

$$0 < \mu \leq \min \left\{ \frac{p_{\min}}{p_{\max}}, \frac{q_{\min}}{q_{\max}} \right\}. \quad (12)$$

One can verify that both Eqs. (7) and (8) hold if μ satisfies Eq. (12). By the definition of EW, we obtain that W_1 and W_2 formulated by Eq. (6) are both EWs, when μ is given by Eq. (12).

Finally, we show that the LU equivalence between ρ_1 and ρ_2 is decidable, if that between W_1 and W_2 given by Eq. (6) is decidable. By Lemma 2, we conclude that $W_1 \sim W_2$ if and only if $P_i \sim Q_i$ by the same LU operator for each i .

Since ρ_1 has the same projector as W_1 and ρ_2 has the same projector as W_2 , then the LU equivalence between ρ_1 and ρ_2 is simultaneous with that between W_1 and W_2 . If $\mathcal{EW}$ is LU decidable, which means that the LU equivalence between W_1 and W_2 given by Eq. (6) is decidable, thus the LU equivalence between ρ_1 and ρ_2 is decidable. That is, $\mathcal{D}_{\lambda_j} \leq \mathcal{EW}$.

This completes the proof. $\blacksquare$

Since the relation “ $\leq$ ” admits the transitivity, by combining Lemma 6 with Theorem 1, we obtain another chain of relations as

$$\mathcal{D}_{\lambda_j} \leq \mathcal{EW} \leq \mathcal{D} \stackrel{d}{=} \mathcal{SEP} \stackrel{d}{=} \mathcal{PPT} \succeq \mathcal{PPT} \mathcal{E}. \quad (13)$$

Recall that the set $\mathcal{D}_\lambda$ is a union of all $\mathcal{D}_{\lambda_j}$ where these sets $\mathcal{D}_{\lambda_j}$ may intersect. For each $\mathcal{D}_{\lambda_j}$, we obtain $\mathcal{D}_{\lambda_j} \leq \mathcal{D}_\lambda$, and $\mathcal{D}_\lambda$ is LU decidable if and only if each $\mathcal{D}_{\lambda_j}$ is LU decidable. There is a corollary derived from Lemma 6 that $\mathcal{D}_\lambda \leq \mathcal{EW}$. Suppose that ρ lies in $\mathcal{D}_\lambda$, and thus by definition ρ lies in some specific $\mathcal{D}_{\lambda_j}$. For another state $\sigma \in \mathcal{D}_\lambda$, we conclude that σ is not LU equivalent to ρ , if σ is outside $\mathcal{D}_{\lambda_j}$. Further, if ρ and σ both lie in $\mathcal{D}_{\lambda_j}$, by Lemma 6, the LU equivalence between ρ and σ is decidable when the set $\mathcal{EW}$ is LU decidable. It implies that $\mathcal{D}_\lambda \leq \mathcal{EW}$.

We can also conclude that two states are not LU equivalent if one lies in $\mathcal{D}_\lambda$ and the other lies outside $\mathcal{D}_\lambda$. If ρ lies in $\mathcal{D}_\lambda$ and σ lies outside $\mathcal{D}_\lambda$, it implies that each eigenspace of σ contains at least one product vector, then for any local unitary operator $U \otimes V$, each eigenspace of $(U \otimes V)\sigma(U \otimes V)^\dagger$ also contains at least one product vector. It follows that $(U \otimes V)\sigma(U \otimes V)^\dagger$ cannot be the same as ρ , and thus the two states ρ and σ are LU inequivalent.

We further show the inverse part of the LU decidability relation between $\mathcal{D}_\lambda$ and $\mathcal{EW}$, namely, $\mathcal{EW} \leq \mathcal{D}_\lambda$.

Lemma 7. Denote by $\mathcal{D}_\lambda$ the set of states each of which satisfies that at least one eigenspace contains no product vector. Then, the LU decidabilities of $\mathcal{D}_\lambda$ and $\mathcal{EW}$ admit $\mathcal{EW} \leq \mathcal{D}_\lambda$.

Proof. For any two given EWs, $W_1, W_2 \in \mathcal{EW}$, their spectral decompositions read as

$$W_1 = \sum_{j=1}^k \lambda_j P_j, \quad W_2 = \sum_{j=1}^k \lambda_j Q_j, \quad (14)$$

where λ_j 's are eigenvalues in descending order. Here, we may assume that W_1 and W_2 share the identical eigenvalues. Otherwise, one can decide that W_1 and W_2 cannot be LU equivalent by Lemma 2. Since W_1 and W_2 are EWs, there is at least one negative eigenvalue for W_1 and W_2 , respectively. By the definition of EWs, for each negative eigenvalue λ_x , the corresponding eigenspace is a completely entangled subspace, i.e., contains no product vector. Otherwise, if there is a product vector $|a, b\rangle$ in λ_x eigenspace of W_1 for some $\lambda_x < 0$, then one can verify that $\langle a, b | W_1 | a, b \rangle < 0$ which contradicts that W_1 is an EW. Based on Eq. (14), we propose two states as follows:

$$\rho_1 = \sum_{j=1}^k \mu_j P_j, \quad \rho_2 = \sum_{j=1}^k \mu_j Q_j, \quad (15)$$

where each μ_j is positive. According to the analysis above, we know that ρ_1 and ρ_2 both have one eigenspace including no

product vector, and thus both ρ_1 and ρ_2 are in $\mathcal{D}_\lambda$. Therefore, if $\mathcal{D}_\lambda$ is LU decidable, the LU equivalence between ρ_1 and ρ_2 is clear, and correspondingly the LU equivalence between W_1 and W_2 given by Eq. (14) can be determined. In other words, we obtain $\mathcal{EW} \leq \mathcal{D}_\lambda$. This completes the proof. $\blacksquare$

The idea of connecting the LU equivalence of bipartite states with that of entanglement witnesses may present different intuition. For example, the authors in Ref. [22] discussed the set of EWs that are LU equivalent to some real EW whose matrix expression is over $\mathbb{R}$, the field of real numbers, and analyzed the detection power of such EWs. It is known from Ref. [22] that the states of real density matrices must be detected by a real EW that could be easier to construct than a complex EW whose matrix expression is over $\mathbb{C}$. Then, given a real EW W_r and a known complex EW $W_c \sim W_r$ whose spectral decompositions read as $W_r = \sum_{j=1}^k \lambda_j P_j$ and $W_c = \sum_{j=1}^k \lambda_j Q_j$, one can identify multiple pairs of LU equivalent states by virtue of the technique in the proof of Lemma 7. That is, for any fixed eigenvalues $\mu_1, \dots, \mu_k > 0$ with $\sum_{j=1}^k \mu_j = 1$, the two states $\rho_1 = \sum_{j=1}^k \mu_j P_j$ and $\rho_2 = \sum_{j=1}^k \mu_j Q_j$ are identified to be LU equivalent. However, by only observing the density matrices of ρ_1 and ρ_2 without the assistance of the two EWs W_r and W_c , it is generally difficult to determine the LU equivalence between ρ_1 and ρ_2 .

By combining Lemmas 6 and 7 above, we establish straightforwardly an equivalence relation in terms of the LU decidability between two sets of different types of bipartite operators, as the following theorem. Theorem 2 shows that the set of EWs $\mathcal{EW}$ is LU decidable if and only if a subset of bipartite states, namely, $\mathcal{D}_\lambda$, is LU decidable.

Theorem 2. The LU decidability of $\mathcal{EW}$ is equal to that of $\mathcal{D}_\lambda$, namely, $\mathcal{EW} \stackrel{d}{=} \mathcal{D}_\lambda$.

Based on Theorem 1 and Theorem 2, we establish the relations in terms of LU decidabilities among the set of EWs $\mathcal{EW}$ and several sets of bipartite states. These known relations present a hierarchy depicted in Fig. 1.

The set $\mathcal{D}_\lambda$ intersects with $\mathcal{NPT}$, $\mathcal{PPT} \mathcal{E}$, and $\mathcal{SEP}$, respectively. The following remark roughly characterizes the states in $\mathcal{D}_\lambda$ by examples.

Remark The set $\mathcal{D}_\lambda$ contains $\mathcal{NPT}$ states, $\mathcal{PPT}$ entangled states, and separable states. We construct examples to show the existence of such three types of states in $\mathcal{D}_\lambda$. First, we propose an $\mathcal{NPT}$ state in $\mathcal{D}_\lambda$ as follows:

$$\rho_1 := \frac{1}{10}(|01\rangle\langle 01| + |10\rangle\langle 10| + 2|\phi_+\rangle\langle \phi_+| + 6|\phi_-\rangle\langle \phi_-|), \quad (16)$$

where $|\phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. One can verify that ρ_1 given by Eq. (16) belongs to $\mathcal{D}_\lambda$ as Eq. (16) is the spectral decomposition of ρ_1 . According to Eq. (16), the eigenspaces, respectively, corresponding to the eigenvalues $1/5$ and $3/5$ contain no product vector. Further, by calculation, the four eigenvalues of ρ_1^Γ are $0.4, 0.4, 0.3$, and -0.1 , and thus ρ_1 is an $\mathcal{NPT}$ state.

Second, we propose a $\mathcal{PPT}$ entangled state in $\mathcal{D}_\lambda$. Recall the method of constructing $\mathcal{PPT}$ entangled states by virtue of UPBs. Denote by $\mathcal{B} = \{|a_i, b_i\rangle\}_{i=1}^l$ a UPB. Then, we may

construct a PPT entangled state as

$$\sigma = \frac{1}{n-l} \left(\mathbb{I}_n - \sum_{i=1}^l |a_i, b_i\rangle\langle a_i, b_i| \right). \quad (17)$$

By Definition 3 for the UPB, we know that the range of σ is a completely entangled subspace, and thus σ is entangled. It follows that there exists a small enough $\epsilon > 0$ such that $\sigma + \epsilon \mathbb{I}$ is still PPT entangled and has full rank. Thus, we obtain a PPT entangled state $\rho_2 := \sigma + \epsilon \mathbb{I} \in \mathcal{D}_\lambda$.

Finally, we construct a separable state in $\mathcal{D}_\lambda$. The following is a two-qubit separable state of rank three:

$$\rho'_3 := \frac{1}{6}(|00\rangle\langle 00| + |11\rangle\langle 11| + 4|+\rangle\langle +|), \quad (18)$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. One may check that $|01\rangle - |10\rangle$ is orthogonal to the range of ρ'_3 . Thus, we conclude that $\rho_3 := \rho'_3 + \epsilon \mathbb{I}$ for some positive ϵ has an eigenspace spanned by $|01\rangle - |10\rangle$ only. It means that ρ_3 is a separable state and has an eigenspace including no product vector. Hence, ρ_3 belongs to $\mathcal{D}_\lambda$. $\square$

We then extend our study to a larger set including $\mathcal{D}_\lambda$, denoted by $\overline{\mathcal{D}_\lambda}$. The state in $\overline{\mathcal{D}_\lambda}$ satisfies that there exists some eigenspace not spanned by product vectors. A subspace including no product vector is sufficiently a space that is not spanned by product vectors. Hence, $\mathcal{D}_\lambda$ is a subset of $\overline{\mathcal{D}_\lambda}$. More specifically, by the example given in Eq. (1), we know that $\mathcal{D}_\lambda$ is a proper subset of $\overline{\mathcal{D}_\lambda}$, i.e., $\mathcal{D}_\lambda \subsetneq \overline{\mathcal{D}_\lambda}$. Due to this inclusive relation, we naturally obtain that $\mathcal{D}_\lambda \leq \overline{\mathcal{D}_\lambda}$ in terms of the LU decidability. Despite $\mathcal{D}_\lambda \subsetneq \overline{\mathcal{D}_\lambda}$, by Proposition 1, we further show that the LU decidability of $\overline{\mathcal{D}_\lambda}$ is equal to that of $\mathcal{D}_\lambda$, namely, $\overline{\mathcal{D}_\lambda} \stackrel{d}{=} \mathcal{D}_\lambda$. It is instructive to reduce the difficulty of verifying the LU equivalence between states in $\overline{\mathcal{D}_\lambda}$.

Proposition 1. Denote by $\overline{\mathcal{D}_\lambda}$ the set of states whose element ρ satisfies that there exists some eigenspace of ρ that are not spanned by product vectors. Then $\overline{\mathcal{D}_\lambda}$ is LU decidable if and only if $\mathcal{D}_\lambda$ is LU decidable. That is, $\overline{\mathcal{D}_\lambda} \stackrel{d}{=} \mathcal{D}_\lambda$.

Proof. First, we conclude that $\mathcal{D}_\lambda \leq \overline{\mathcal{D}_\lambda}$ due to $\mathcal{D}_\lambda \subset \overline{\mathcal{D}_\lambda}$. For $\overline{\mathcal{D}_\lambda} \stackrel{d}{=} \mathcal{D}_\lambda$ to hold, we next have to show $\overline{\mathcal{D}_\lambda} \leq \mathcal{D}_\lambda$. It means to show that for any two states in $\overline{\mathcal{D}_\lambda}$, whether they are LU equivalent or not is decidable, if the set $\mathcal{D}_\lambda$ is LU decidable. Suppose that ρ and σ are two arbitrary states in $\overline{\mathcal{D}_\lambda}$. It suffices to consider that ρ and σ have identical eigenvalues, otherwise the two states are LU inequivalent. We may also assume that ρ and σ both have full rank by means of adding the same white noise, namely, $\rho + x\mathbb{I}$ and $\sigma + x\mathbb{I}$.

Let the spectral decompositions of ρ and σ be

$$\begin{aligned} \rho &= \sum_{j=1}^l \lambda_j P_j, \\ \sigma &= \sum_{j=1}^l \lambda_j Q_j, \end{aligned} \quad (19)$$

where λ_j 's are all positive, and for each j , the projectors P_j and Q_j have the same rank. By the definition of $\overline{\mathcal{D}_\lambda}$, it means that for each of ρ and σ , there exists some eigenspace not spanned by product vectors. For the case when $\mathcal{R}(P_j)$ is not spanned by product vectors while $\mathcal{R}(Q_j)$ is spanned by prod-

uct vectors, one can verify that $P_j \approx Q_j$, and thus conclude that $\rho \approx \sigma$ by Lemma 2. Then, it suffices to consider that for some j , both $\mathcal{R}(P_j)$ and $\mathcal{R}(Q_j)$ are not spanned by product vectors. Without loss of generality, we may assume that both $\mathcal{R}(P_1)$ and $\mathcal{R}(Q_1)$ are not spanned by product vectors. We may decompose P_1 and Q_1 as below:

$$\begin{aligned} P_1 &= P_{11} + P_{12}, \\ Q_1 &= Q_{11} + Q_{12}, \end{aligned} \quad (20)$$

where P_{11} and P_{12} are orthogonal projectors, Q_{11} and Q_{12} are orthogonal projectors, $\mathcal{R}(P_{11})$ is spanned by product vectors and $\mathcal{R}(P_{12})$ contains no product vector, and $\mathcal{R}(Q_{11})$ is spanned by product vectors and $\mathcal{R}(Q_{12})$ contains no product vector. In other words, all product vectors in $\mathcal{R}(P_1)$ are included in $\mathcal{R}(P_{12})$, the same for $\mathcal{R}(Q_1)$. Substituting the decompositions into Eq. (19), we obtain that

$$\begin{aligned} \rho &= \lambda_1 P_{11} + \lambda_1 P_{12} + \sum_{j=2}^l \lambda_j P_j, \\ \sigma &= \lambda_1 Q_{11} + \lambda_1 Q_{12} + \sum_{j=2}^l \lambda_j Q_j. \end{aligned} \quad (21)$$

We claim that $\rho \sim \sigma$ if and only if $(P_{11}, P_{12}, P_2, \dots, P_l)$ is SLU equivalent to $(Q_{11}, Q_{12}, Q_2, \dots, Q_l)$ for the following reason: The “if” part can be verified directly. Then, we show the “only if” part. With the condition $\rho \sim \sigma$, we obtain that $(P_1, P_2, \dots, P_l)$ is SLU equivalent to $(Q_1, Q_2, \dots, Q_l)$. It follows that there exists an LU operator $U \otimes V$ such that

$$(U \otimes V)(P_{11} + P_{12})(U \otimes V)^\dagger = Q_{11} + Q_{12}, \quad (22)$$

$$(U \otimes V)P_j(U \otimes V)^\dagger = Q_j, \quad j > 1. \quad (23)$$

One can verify that the range of $(U \otimes V)P_{11}(U \otimes V)^\dagger$ is spanned by product vectors, and the range of $(U \otimes V)P_{12}(U \otimes V)^\dagger$ contains no product vector. By Eq. (22), we conclude that the range of $(U \otimes V)P_{11}(U \otimes V)^\dagger$ is the same as $\mathcal{R}(Q_{11})$, and the range of $(U \otimes V)P_{12}(U \otimes V)^\dagger$ is the same as $\mathcal{R}(Q_{12})$. It is known that two projectors are the same if they have the same range. Thus, we derive that $(U \otimes V)P_{11}(U \otimes V)^\dagger = Q_{11}$ and $(U \otimes V)P_{12}(U \otimes V)^\dagger = Q_{12}$. Combining this conclusion with Eq. (23), we obtain that $(P_{11}, P_{12}, P_2, \dots, P_l)$ is SLU equivalent to $(Q_{11}, Q_{12}, Q_2, \dots, Q_l)$, and thus the “only if” part holds.

By adjusting the eigenvalues of ρ and σ , we may construct two states in $\mathcal{D}_\lambda$ as

$$\begin{aligned} \rho' &= \mu_{11} P_{11} + \mu_{12} P_{12} + \sum_{j=2}^l \mu_j P_j, \\ \sigma' &= \mu_{11} Q_{11} + \mu_{12} Q_{12} + \sum_{j=2}^l \mu_j Q_j, \end{aligned} \quad (24)$$

where the coefficients $\mu_{11}, \mu_{12}, \mu_2, \dots, \mu_l$ are all positive. Because P_{11} is orthogonal to P_{12} and Q_{11} is orthogonal to Q_{12} , the decompositions given by Eq. (24) are spectral decompositions of ρ' and σ' . It is known that ρ' and σ' are included in $\mathcal{D}_\lambda$, as $\mathcal{R}(P_{12})$ and $\mathcal{R}(Q_{12})$ both contain no product vector. By Lemma 2, we conclude that the LU equivalence between

ρ and σ is simultaneous with that between ρ' and σ' . The LU equivalence between ρ' and σ' can be decided when $\mathcal{D}_\lambda$ is LU decidable. Therefore, we have shown that whether ρ and σ are LU equivalent or not is decidable if $\mathcal{D}_\lambda$ is LU decidable, i.e., $\overline{\mathcal{D}_\lambda} \preceq \mathcal{D}_\lambda$.

To sum up, the relation $\overline{\mathcal{D}_\lambda} \stackrel{d}{=} \mathcal{D}_\lambda$ holds. This completes the proof. ■

Now, all the relations in terms of the LU decidability depicted in Fig. 1 are verified by the results in this section. According to Fig. 1, the two chains of relations, denoted by $SEP \stackrel{d}{=} PPT \stackrel{d}{=} \mathcal{D}$ and $\overline{\mathcal{D}_\lambda} \stackrel{d}{=} \mathcal{D}_\lambda \stackrel{d}{=} \mathcal{E}W$, are worth noting. The former implies that the problem of determining whether two arbitrary states are LU equivalent or not can be restricted to a proper subset SEP , which is a kind of simplification of the problem. The latter implies that for some specific states, e.g., the states in $\mathcal{D}_\lambda$, it contributes to identifying the LU equivalence among them by introducing some EWs whose LU equivalences are known. Thus, it is meaningful to characterize the LU orbits of EWs [22] which are closely connected to identifying LU equivalent states by virtue of the techniques developed in this section.

Finally, we illustrate how to generalize the results focused on bipartite systems to multipartite systems. A straightforward way is to take the bipartition of a multipartite system into consideration, analogous to the generalization of PPT criterion into multipartite systems. This is useful to identify multipartite LU inequivalent states. That is, two multipartite states are not multipartite LU equivalent, if they are not bipartite LU equivalent in some bipartition of the system. Unfortunately, as we all know, the characterization of multipartite entanglement is much more complicated than bipartite entanglement. Even if two multipartite states are LU equivalent in all bipartitions of the system, it does not guarantee they are genuine multipartite LU equivalent, i.e., equivalent by a multipartite LU operator. An intuitive example is the three-qubit GHZ state and W state, namely, $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. One can verify that $|\text{GHZ}\rangle$ and $|W\rangle$ are LU equivalent in all bipartitions of the three-qubit system, but there is no tripartite LU operator $U \otimes V \otimes W$ such that $|W\rangle = (U \otimes V \otimes W)|\text{GHZ}\rangle$ [8].

Another way of extending the results on the relations of LU decidability to the case of multipartite systems is to correspondingly consider the set of multipartite EWs and the multipartite forms of the two sets $\mathcal{D}_\lambda$ and $\overline{\mathcal{D}_\lambda}$. Specifically, based on the analysis of the bipartite case, the essence of the bipartite states in $\mathcal{D}_\lambda$ is the existence of an eigenspace that contains no bipartite product vector, and similarly the essence of the bipartite states in $\overline{\mathcal{D}_\lambda}$ is the existence of an eigenspace that is not spanned by bipartite product vectors. The bipartite product vector $|a_1, a_2\rangle$ can be naturally generalized to multipartite product vectors $|a_1, a_2, \dots, a_m\rangle$. Thus, we may define the m -partite EW W as the nonpositive semidefinite operator satisfying

$$\langle a_1, a_2, \dots, a_m | W | a_1, a_2, \dots, a_m \rangle > 0$$

for any m -partite product vector $|a_1, a_2, \dots, a_m\rangle$, and denote by $\mathcal{E}W^m$ the set of m -partite EWs. Moreover, we may generalize $\mathcal{D}_\lambda$ and $\overline{\mathcal{D}_\lambda}$ to the multipartite forms, defined as follows. Denote by $\mathcal{D}_\lambda^m$ the set of states each of which satisfies that

there is at least one eigenspace including no product vector, and by $\overline{\mathcal{D}_\lambda^m}$ the set of states each of which satisfies that there is at least one eigenspace not spanned by product vectors, where the superscript m represents the number of subsystems. Based on the definitions above and by virtue of the techniques developed from the bipartite case, one can analogously derive a similar conclusion on the relations among $\mathcal{E}W^m$, $\mathcal{D}_\lambda$, and $\overline{\mathcal{D}_\lambda}$, namely, $\overline{\mathcal{D}_\lambda^m} \stackrel{d}{=} \mathcal{D}_\lambda^m \stackrel{d}{=} \mathcal{E}W^m$. It indicates that some results focused on the bipartite system also apply to multipartite systems.

IV. SLU EQUIVALENCE BETWEEN TUPLES OF BIPARTITE PROJECTORS

In this section, we consider how to determine whether two tuples of projectors are SLU equivalent or not, where the SLU equivalence between tuples is defined by Definition 2. From Lemma 2, we realize that the SLU equivalence between tuples of mutually orthogonal projectors is essential for LU equivalent Hermitian operators, by analyzing their spectral decompositions. By Example 1, we show that for two tuples each of which contains more than two projectors, the partial SLU equivalence between parts of such two tuples is necessary but not sufficient to ensure the overall SLU equivalence between the two entire tuples. To reconcile this gap, we propose a necessary and sufficient condition, in Theorem 3, on bipartite states, such that two tuples of states are SLU equivalent.

To further emphasize the key role of SLU equivalence, we also consider the SLU equivalence between two tuples of subspaces from the perspective of studying ranges of projectors. Considering two bipartite states $\rho \sim \sigma$, it follows directly that their ranges admit the relation $\mathcal{R}(\rho) = (U \otimes V)\mathcal{R}(\sigma)$ for some local unitary $U \otimes V$. In other words, if there exists some $|\alpha\rangle \in \mathcal{R}(\rho)$ such that $(U \otimes V)|\alpha\rangle \notin \mathcal{R}(\sigma)$ for any local unitary $U \otimes V$, one can decide $\rho \not\sim \sigma$. Conversely, the relation $\mathcal{R}(\rho) = (U \otimes V)\mathcal{R}(\sigma)$ is not sufficient to ensure $\rho \sim \sigma$. We propose a pair of states as below:

$$\begin{aligned} \rho &:= \frac{1}{2}|00\rangle\langle 00| + \frac{1}{4}|01\rangle\langle 01| + \frac{1}{8}(|10\rangle\langle 10| + |11\rangle\langle 11|), \\ \sigma &:= \frac{1}{2}|\phi_+\rangle\langle \phi_+| + \frac{1}{4}|\phi_-\rangle\langle \phi_-| + \frac{1}{8}(|01\rangle\langle 01| + |10\rangle\langle 10|), \end{aligned} \quad (25)$$

where $|\phi_+\rangle, |\phi_-\rangle$ are a pair of Bell states. Since ρ and σ both have full rank, their ranges are both $\mathbb{C}^2 \otimes \mathbb{C}^2$, and thus the relation $\mathcal{R}(\rho) = (U \otimes V)\mathcal{R}(\sigma)$ is naturally satisfied. However, the eigenvector $|00\rangle$ of ρ is not LU equivalent to the eigenvector $|\phi_+\rangle$ of σ , corresponding to the same eigenvalue $\frac{1}{2}$. By Lemma 2, we conclude that ρ and σ are LU inequivalent. It motivates us to further consider how to determine the LU equivalence between two bipartite states by their eigenspaces. A criterion of determining $\rho \sim \sigma$ is presented as below:

Lemma 8. Suppose that two bipartite states ρ and σ have spectral decompositions as $\rho = \sum_{j=1}^r \lambda_j P_j$ and $\sigma = \sum_{j=1}^r \lambda_j Q_j$. Then $\rho \sim \sigma$, if and only if for each j , $\mathcal{R}(P_j) = (U \otimes V)\mathcal{R}(Q_j)$ by a common LU operator $U \otimes V$.

We shall show the detailed proof of Lemma 8 in Appendix B. Lemma 8 is a counterpart of Lemma 2 from the perspective of ranges of positive semidefinite operators. One may refer to the pair of states (α_1 and α_2) proposed below

Lemma 2 to fully understand the essential role played by a common LU operator.

In light of the importance of SLU equivalence, we next deeply investigate the conditions on the SLU equivalence relation between two tuples of mutually orthogonal projectors. By virtue of the Schmidt decomposition, such conditions are effective to check the LU equivalence between two arbitrary bipartite Hermitian operators. For two binary tuples of orthogonal projectors (P_1, P_2) and (Q_1, Q_2) , the conditions both $P_1 \sim P_2$ and $Q_1 \sim Q_2$ are generally not sufficient for the SLU equivalence relation between the two binary tuples (P_1, P_2) and (Q_1, Q_2) . One may refer to the counterexample formulated in the paragraph below Lemma 2.

We further consider the cases where the tuples contain more than two mutually orthogonal projectors, by focusing on the tuples of three projectors. The cases where the tuples contain more than three mutually orthogonal projectors can be similarly investigated. Suppose that (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are two tuples of mutually orthogonal projectors, namely, $\text{tr}(P_i P_j) = \text{tr}(Q_i Q_j) = \delta_{ij}$. In order to make $(P_1, P_2, P_3) \sim_s (Q_1, Q_2, Q_3)$, we shall assign necessary preconditions that $(P_1, P_2) \sim_s (Q_1, Q_2)$, $(P_1, P_3) \sim_s (Q_1, Q_3)$, and $(P_2, P_3) \sim_s (Q_2, Q_3)$. Otherwise, if one of such three SLU equivalence relations is violated, one can conclude that (P_1, P_2, P_3) is not SLU equivalent to (Q_1, Q_2, Q_3) . Then, we continue to study whether these preconditions are sufficient to ensure $(P_1, P_2, P_3) \sim_s (Q_1, Q_2, Q_3)$.

By the following lemma, we may simply consider the two tuples with $P_1 = Q_1$ and $P_2 = Q_2$. Then, the number of projectors to be processed is reduced from six to four.

Lemma 9. Suppose that (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are two tuples of mutually orthogonal projectors, which satisfy that $(P_1, P_2) \sim_s (Q_1, Q_2)$, $(P_1, P_3) \sim_s (Q_1, Q_3)$, and $(P_2, P_3) \sim_s (Q_2, Q_3)$. By simultaneously applying any LU operations to Q_i 's, namely, $\tilde{Q}_i := (U \otimes V)Q_i(U \otimes V)^\dagger$ for any LU operator $(U \otimes V)$, then

(i) the SLU equivalences between (P_i, P_j) and $(\tilde{Q}_i, \tilde{Q}_j)$, for any $1 \leq i < j \leq 3$, are all maintained;

(ii) (P_1, P_2, P_3) is SLU equivalent to (Q_1, Q_2, Q_3) , if and only if (P_1, P_2, P_3) is SLU equivalent to $(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$.

We shall prove Lemma 9 in Appendix B. Recall one of the preconditions $(P_1, P_2) \sim_s (Q_1, Q_2)$, which means there exists an LU operator $U \otimes V$, such that $P_1 = (U \otimes V)Q_1(U \otimes V)^\dagger$ and $P_2 = (U \otimes V)Q_2(U \otimes V)^\dagger$. Then, let $\tilde{Q}_1 := (U \otimes V)Q_1(U \otimes V)^\dagger = P_1$, $\tilde{Q}_2 := (U \otimes V)Q_2(U \otimes V)^\dagger = P_2$, and $\tilde{Q}_3 := (U \otimes V)Q_3(U \otimes V)^\dagger$. By Lemma 9(i), we conclude that $(P_i, P_j) \sim_s (\tilde{Q}_i, \tilde{Q}_j)$, for any $1 \leq i < j \leq 3$, are all maintained when $(P_1, P_2) \sim_s (Q_1, Q_2)$, $(P_1, P_3) \sim_s (Q_1, Q_3)$, and $(P_2, P_3) \sim_s (Q_2, Q_3)$. According to Lemma 9(ii), we consequently obtain that $(P_1, P_2, P_3) \sim_s (Q_1, Q_2, Q_3)$ if and only if $(P_1, P_2, P_3) \sim_s (\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$, where $\tilde{Q}_1 = P_1$ and $\tilde{Q}_2 = P_2$. Then we may equivalently examine whether there exists the SLU equivalence between (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) with $P_1 = Q_1$ and $P_2 = Q_2$, under the preconditions.

Here, we claim that the preconditions mentioned above are not sufficient to ensure $(P_1, P_2, P_3) \sim_s (Q_1, Q_2, Q_3)$, and show our claim by proposing examples as follows.

Example 1. By constructing specific examples, we conclude that the two tuples of mutually orthogonal projectors,

(P_1, P_2, P_3) and (Q_1, Q_2, Q_3) , may not be SLU equivalent, even if $(P_i, P_j) \sim_s (Q_i, Q_j)$ holds for any $1 \leq i < j \leq 3$. According to the analysis above, we may assume that $P_1 = Q_1$ and $P_2 = Q_2$. Then, the four distinct projectors supported on $\mathbb{C}^4 \otimes \mathbb{C}^4$ are formulated as below:

$$\begin{aligned} P_1 &= Q_1 = |0\rangle\langle 0| \otimes \text{diag}(1, 1, 0, 0), \\ P_2 &= Q_2 = |1\rangle\langle 1| \otimes \text{diag}(1, 0, 1, 0), \\ P_3 &= |2\rangle\langle 2| \otimes \text{diag}(1, 0, 0, 1), \\ Q_3 &= |2\rangle\langle 2| \otimes \text{diag}(0, 1, 1, 0). \end{aligned} \quad (26)$$

First, due to $P_1 = Q_1$ and $P_2 = Q_2$, the relation $(P_1, P_2) \sim_s (Q_1, Q_2)$ naturally holds by the LU matrix $\mathbb{I} \otimes \mathbb{I}$. Second, one can verify that (P_1, P_3) is SLU equivalent to (Q_1, Q_3) by the LU matrix as

$$\mathbb{I} \otimes \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right). \quad (27)$$

Third, one can verify that (P_2, P_3) is SLU equivalent to (Q_2, Q_3) by the LU matrix as

$$\mathbb{I} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (28)$$

So the two tuples of projectors satisfy the preconditions. ■

In the following proof, we show that the two tuples given in Example 1 cannot be SLU equivalent.

Proof. We show by contradiction that (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) formulated by Eq. (26) are not SLU equivalent. Assume that $(P_1, P_2, P_3) \sim_s (Q_1, Q_2, Q_3)$, and denote by $U_1 \otimes U_2$ an arbitrary LU matrix such that $P_j = (U_1 \otimes U_2)Q_j(U_1 \otimes U_2)^\dagger$ for each j . By virtue of Eq. (26) and observing the second subsystems of P_1, P_2 , we obtain that

$$\begin{aligned} U_2(\text{diag}(1, 1, 0, 0))U_2^\dagger &= \text{diag}(1, 1, 0, 0), \\ U_2(\text{diag}(1, 0, 1, 0))U_2^\dagger &= \text{diag}(1, 0, 1, 0). \end{aligned} \quad (29)$$

It means that the unitary matrix U_2 leaves both $\text{diag}(1, 1, 0, 0)$ and $\text{diag}(1, 0, 1, 0)$ invariant under the unitary similarity. By Lemma 3, we conclude that U_2 must be diagonal, namely, $\text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})$. Nevertheless, such diagonal U_2 cannot make that $U_2(\text{diag}(1, 0, 0, 1))U_2^\dagger = \text{diag}(0, 1, 1, 0)$ holds. It means that P_3 cannot be locally transformed to Q_3 by $U_1 \otimes U_2$, which contradicts the assumption. Therefore, the formulated two tuples cannot be SLU equivalent. This completes the proof. ■

Example 1 indicates the partial SLU equivalence cannot ensure the overall SLU equivalence. It implies that the problem of determining the SLU equivalence between two arbitrary tuples of mutually orthogonal projectors could become more and more difficult as the number of projectors included in a tuple increases. A projector can be regarded as a non-normalized state. Thus, we may derive a more general conclusion that for two tuples of states, the partial SLU equivalence cannot ensure the overall SLU equivalence. Specifically, for two tuples of states denoted by (ρ_1, ρ_2, ρ_3) and $(\sigma_1, \sigma_2, \sigma_3)$, they may not be SLU equivalent, even if $(\rho_i, \rho_j) \sim_s (\sigma_i, \sigma_j)$ holds for any $1 \leq i < j \leq 3$.

Along with this broader perspective by extending projectors to states, we study the SLU equivalence between tuples of states, and propose a necessary and sufficient condition for the SLU equivalence in Theorem 3. This necessary and sufficient condition is formulated in terms of the twirling operation which has been widely used in the resource theory of asymmetry [32–34]. It suffices to assume that $\rho_1 = \sigma_1$ and $\rho_2 = \sigma_2$ for the two tuples (ρ_1, ρ_2, ρ_3) and $(\sigma_1, \sigma_2, \sigma_3)$ as the similar idea given below Lemma 9.

Theorem 3. Suppose that $\rho_1 = \sigma_1, \rho_2 = \sigma_2, \rho_3, \sigma_3$ are normalized states that satisfy that $(\rho_i, \rho_j) \sim_s (\sigma_i, \sigma_j)$ for any $1 \leq i < j \leq 3$. Then, the tuple (ρ_1, ρ_2, ρ_3) is SLU equivalent to $(\sigma_1, \sigma_2, \sigma_3)$, if and only if there exist two intersected groups $\mathcal{U}_1$ and $\mathcal{U}_2$ of LU operators such that

(i) $\rho_1 = \int_{\mathcal{U}_1} U \alpha U^\dagger dU$ for some state α and $\rho_2 = \int_{\mathcal{U}_2} U \beta U^\dagger dU$ for some state β , with the two integrals over the Haar measure;

(ii) $\rho_3 = (U \otimes V) \sigma_3 (U \otimes V)^\dagger$ for some $U \otimes V \in \mathcal{U}_1 \cap \mathcal{U}_2$.

Proof. Due to $\rho_1 = \sigma_1, \rho_2 = \sigma_2$, by definition one can verify that (ρ_1, ρ_2, ρ_3) is SLU equivalent to $(\sigma_1, \sigma_2, \sigma_3)$ if and only if there exists a common LU operator $U \otimes V$ which simultaneously commutes with ρ_1 and ρ_2 , and satisfies $\rho_3 = (U \otimes V) \sigma_3 (U \otimes V)^\dagger$. Next, we show that the combination of conditions (i) and (ii) is equivalent to this necessary and sufficient condition.

First we show that the LU operators commuting with a bipartite operator form a group under the matrix multiplication. Let ρ be a bipartite operator. Then the identity operator $\mathbb{I} \otimes \mathbb{I}$ must commute with ρ . Moreover, assume that ρ commutes with two LU operator $U_1 \otimes V_1$ and $U_2 \otimes V_2$, i.e., $\rho(U_1 \otimes V_1) = (U_1 \otimes V_1)\rho$ and $\rho(U_2 \otimes V_2) = (U_2 \otimes V_2)\rho$. One can directly verify that ρ also commutes with the LU operator $(U_1 U_2) \otimes (V_1 V_2)$ as

$$\begin{aligned} \rho[(U_1 U_2) \otimes (V_1 V_2)] &= \rho(U_1 \otimes V_1)(U_2 \otimes V_2) \\ &= (U_1 \otimes V_1)\rho(U_2 \otimes V_2) \\ &= (U_1 \otimes V_1)(U_2 \otimes V_2)\rho \\ &= [(U_1 U_2) \otimes (V_1 V_2)]\rho. \end{aligned} \quad (30)$$

Hence, the LU operators commuting with a bipartite operator form a multiplication group.

Second, for a given LU group $\mathcal{U}_g$, the bipartite states commuting with $\mathcal{U}_g$ is denoted by $\text{inv}(\mathcal{U}_g)$,

$$\text{inv}(\mathcal{U}_g) \equiv \{\rho \in \mathcal{D}(\mathcal{H}^{AB}) | \forall X \in \mathcal{U}_g : X \rho X^\dagger = \rho\}. \quad (31)$$

It is known that the set $\text{inv}(\mathcal{U})$ can be characterized by the useful twirling operation [33], namely,

$$\mathcal{T}[\rho] \equiv \int_{\mathcal{U}} U \rho U^\dagger dU, \quad (32)$$

for some unitary group $\mathcal{U}$. The twirling operation averages over the action of the unitary group $\mathcal{U}$ with the Haar measure. If the unitary group $\mathcal{U}$ is finite, one simply replaces the integral with a sum as

$$\mathcal{T}[\rho] \equiv \frac{1}{|\mathcal{U}|} \sum_{U \in \mathcal{U}} U \rho U^\dagger. \quad (33)$$

By virtue of the twirling operation, we may characterize $\text{inv}(\mathcal{U}_g)$ as follows:

$$\text{inv}(\mathcal{U}_g) = \{\rho \in \mathcal{D}(\mathcal{H}^{AB}) | \mathcal{G}[\rho] = \rho\}, \quad (34)$$

where $\mathcal{G}[\rho]$ is given by

$$\mathcal{G}[\rho] = \int_{\mathcal{U}_g} U \rho U^\dagger dU. \quad (35)$$

Since $\mathcal{G}$ is idempotent, namely, $\mathcal{G}^2 \equiv \mathcal{G} \circ \mathcal{G} = \mathcal{G}$, the set $\text{inv}(\mathcal{U}_g)$ is indeed the image of $\mathcal{G}$, i.e.,

$$\text{inv}(\mathcal{U}_g) = \{\rho | \rho = \mathcal{G}[\sigma], \forall \sigma \in \mathcal{D}(\mathcal{H}^{AB})\}. \quad (36)$$

Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are two intersected groups of LU operators. Based on the analysis above, condition (i) ensures that ρ_1 commutes with $\mathcal{U}_1$ and ρ_2 commutes with $\mathcal{U}_2$. Since $\mathcal{U}_1 \cap \mathcal{U}_2$ is nonempty, it follows that the elements in $\mathcal{U}_1 \cap \mathcal{U}_2$ commute with ρ_1 and ρ_2 simultaneously. Additionally, combining with condition (ii), we conclude that there exists a common LU operator which simultaneously commutes with ρ_1 and ρ_2 , and makes $\rho_3 \sim \sigma_3$. Therefore, the addition of conditions (i) and (ii) is necessary and sufficient to $(\rho_1, \rho_2, \rho_3) \sim_s (\sigma_1, \sigma_2, \sigma_3)$ with $\rho_1 = \sigma_1, \rho_2 = \sigma_2$.

This completes the proof. $\blacksquare$

By viewing projectors as special states, one may similarly derive a necessary and sufficient condition for the SLU equivalence between tuples of projectors. We may take the tuples of projectors as an example to illustrate the condition proposed in Theorem 3. We construct the following four distinct projectors:

$$\begin{aligned} P_1 &= Q_1 = |00\rangle\langle 00|, \\ P_2 &= Q_2 = |11\rangle\langle 11|, \\ P_3 &= \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|), \\ Q_3 &= \frac{1}{2}(-|01\rangle + |10\rangle)(-\langle 01| + \langle 10|). \end{aligned} \quad (37)$$

Then, (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are two tuples of mutually orthogonal projectors. One can verify that

$$\mathcal{U}_1 = \{\text{diag}(e^{i\theta_1}, e^{i\theta_2}) \otimes \text{diag}(e^{i\theta_3}, e^{i\theta_4}) | \forall \theta_j \in [0, 2\pi)\} \quad (38)$$

is a group of LU matrices which leaves $P_1 = |00\rangle\langle 00|$ invariant. It follows by calculation that $P_1 = \int_{\mathcal{U}_1} U P_1 U^\dagger dU$. Then, we present the other group $\mathcal{U}_2$ of LU matrices, whose elements are formulated by

$$\begin{aligned} X_1 &= \mathbb{I}_2 \otimes \mathbb{I}_2, & X_2 &= \mathbb{I}_2 \otimes \text{diag}(1, -1), \\ X_3 &= \mathbb{I}_2 \otimes \text{diag}(1, i), & X_4 &= \mathbb{I}_2 \otimes \text{diag}(1, -i). \end{aligned} \quad (39)$$

It follows by observation that $\mathcal{U}_2$ is a subgroup of $\mathcal{U}_1$, and thus the intersection is indeed $\mathcal{U}_2$. Since $\mathcal{U}_2$ is a group of finite elements, the twirling of $P_2 = |11\rangle\langle 11|$ over the group $\mathcal{U}_2$ is calculated by

$$\mathcal{T}[P_2] = \frac{1}{4} \sum_{j=1}^4 X_j P_2 X_j^\dagger = P_2. \quad (40)$$

In light of this, for the two tuples (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) , there exist two intersected groups satisfying condition (i) in Theorem 3. Further, one can verify that the LU matrix $X_2 \in \mathcal{U}_2$ makes the equality $P_3 = X_2 Q_3 X_2^\dagger$ holds,

and thus the two groups also satisfy condition (ii) in Theorem 3. Then by applying the LU matrix X_2 to Q_j 's, we simultaneously obtain P_j 's. Thus, the two tuples (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) are SLU equivalent by definition.

V. CONCLUDING REMARKS

In this paper, we introduced a relation in terms of the LU decidabilities between different sets of bipartite Hermitian operators, and connected the LU decidabilities for the set of EWs and several sets of states. This idea may shed light on identifying LU equivalent states.

By virtue of the relation " $\leq$ " defined by Definition 1, we compared the LU decidability of the set of EWs with that of several sets of states. Based on the comparisons, we established a hierarchy in terms of LU decidabilities for these sets. The hierarchy has been illustrated by Fig. 1. By Theorem 1, we have shown the relations of LU decidability among the sets of states classified by the PPT criterion, and the set of EWs. Then, we proposed a set of states satisfying that at least one eigenspace contains no product vector, denoted by $\mathcal{D}_\lambda$. We indicated in Theorem 2 that the two LU decidabilities for $\mathcal{D}_\lambda$ and the set of EWs are equal. We further extended our study to a larger set including $\mathcal{D}_\lambda$, each of whose states satisfies that there is one eigenspace not spanned by product vectors. We refer to such a set as $\overline{\mathcal{D}_\lambda}$, and have shown that the LU decidability of $\overline{\mathcal{D}_\lambda}$ is equal to that of $\mathcal{D}_\lambda$ by Proposition 1. By analyzing the spectral decomposition, we realized that the SLU equivalence is crucial to LU equivalent operators. In light of this, we studied the SLU equivalence between tuples of mutually orthogonal projectors. By Example 1, we revealed that for two tuples of mutually orthogonal projectors, the partial SLU equivalence cannot ensure the overall SLU equivalence. For this reason, we provided a necessary and sufficient condition in Theorem 3, such that two tuples of bipartite states are SLU equivalent.

There are some interesting problems remaining for future work. One direction is to complement the relations in terms of the LU decidability for the sets appearing in Fig. 1. It is useful for assessing the difficulty of determining the LU equivalence among states in some given set, compared with the LU decidability of the set of EWs. Another direction is to further characterize the SLU equivalent tuples of mutually orthogonal projectors, in order to propose more efficient criteria of determining whether two states are LU equivalent or not.

ACKNOWLEDGMENTS

The authors appreciate the anonymous reviewers for helping improve the quality of this paper. This work is funded by the NNSF of China (Grants No. 12401597 and No. 12471427), the Basic Research Program of Jiangsu (Grant No. BK20241603), and the Wuxi Science and Technology Development Fund Project (Grant No. K20231008).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ANOTHER PROOF OF LEMMA 5

Although Lemma 5 is covered by Lemma 6 which applies to all $\mathcal{D}_{\lambda_j}$, we here show another idea of proving Lemma 5. This idea relies on the special property of the eigenspace corresponding to the largest eigenvalue.

Proof of Lemma 2. For any two given states $\rho_1, \rho_2 \in \mathcal{D}_{\lambda_1}$, their spectral decompositions read as $\rho_1 = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ and $\rho_2 = \sum_j \mu_j |\phi_j\rangle\langle\phi_j|$, where λ_j 's and μ_j 's are all positive, and sorted by descending order, respectively. By Lemma 2, one can decide ρ_1 and ρ_2 are not LU equivalent, if there exists some j such that $\lambda_j \neq \mu_j$. We thus consider the case $\lambda_j = \mu_j$ for each j . Based on the formulas of ρ_1 and ρ_2 , we construct two operators by

$$W_1 := \mu \mathbb{I} - \rho_1, \quad W_2 := \mu \mathbb{I} - \rho_2, \quad (\text{A1})$$

where μ is defined as

$$\mu := \max_{\| |a, b\rangle \| = 1} \{ \langle a, b | \rho_1 | a, b \rangle, \langle a, b | \rho_2 | a, b \rangle \}. \quad (\text{A2})$$

It follows from $\rho_1, \rho_2 \in \mathcal{D}_{\lambda_1}$ that the λ_1 eigenspaces of ρ_1 and ρ_2 , respectively, contain no product vector. Due to this fact, one can verify that μ is strictly smaller than λ_1 , the largest eigenvalue for both ρ_1 and ρ_2 . Hence, W_1 and W_2 given by Eq. (A1) are nonpositive semidefinite. We further obtain that for any product vector $|a, b\rangle$ with norm 1,

$$\begin{aligned} \langle a, b | W_1 | a, b \rangle &= \mu - \langle a, b | \rho_1 | a, b \rangle \geq 0, \\ \langle a, b | W_2 | a, b \rangle &= \mu - \langle a, b | \rho_2 | a, b \rangle \geq 0, \end{aligned} \quad (\text{A3})$$

where the two inequalities follow directly by Eq. (A2). Thus, we conclude that both W_1 and W_2 are EWs by definition. According to Eq. (A1), the LU equivalence or inequivalence between ρ_1 and ρ_2 is decidable, if one can decide whether W_1 and W_2 are LU equivalent or not. Due to $W_1, W_2 \in \mathcal{EW}$, we finally derive that $\mathcal{D}_{\lambda_1} \leq \mathcal{EW}$.

This completes the proof. $\blacksquare$

APPENDIX B: PROOFS OF RESULTS IN SEC. IV

In this Appendix, we show two results of Sec. IV in detail. First, we give the proof of Lemma 8. As preliminaries, we present the following lemma which allows the unitary freedom in the ensemble for density matrices.

Lemma B1. A density matrix can be decomposed as $\rho = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_i |\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$ if and only if

$$|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle, \quad (\text{B1})$$

where u_{ij} is the (i, j) element of a unitary matrix.

With Lemma B1, we are ready to prove Lemma 8.

Proof of Lemma 8. For two bipartite states ρ and σ with spectral decompositions as $\rho = \sum_{j=1}^r \lambda_j P_j$ and $\sigma = \sum_{j=1}^r \lambda_j Q_j$, it follows from Lemma 2 that $\rho \sim \sigma$ if and only if for each j , P_j and Q_j are SLU equivalent, i.e., LU equivalent by a common LU operator. We claim that for two given projectors P and Q , $P \sim Q$ if and only if their ranges $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ are LU equivalent, i.e., $U \otimes V$ such that $(U \otimes V)\mathcal{R}(P) = \mathcal{R}(Q)$ for some LU operator $U \otimes V$. The "only if" part of this claim can be verified directly. We

next show the “if” part of this claim. Let $\{|a_1\rangle, \dots, |a_r\rangle\}$ be an orthonormal basis of $\mathcal{R}(P)$. According to the condition $(U \otimes V)\mathcal{R}(P) = \mathcal{R}(Q)$, it follows that $(U \otimes V)|a_j\rangle \in \mathcal{R}(Q)$ for each j . Denote that $|b_j\rangle = (U \otimes V)|a_j\rangle$ for each j , and thus $\mathcal{R}(Q) = \text{span}\{|b_1\rangle, \dots, |b_r\rangle\}$. Applying Lemma B1, we know that Q has the following decomposition:

$$Q = \sum_{j=1}^r \left(\sum_{k=1}^r u_{jk} |b_k\rangle \right) \left(\sum_{l=1}^r u_{jl}^* \langle b_l| \right), \quad (\text{B2})$$

where u_{jk} is the (j, k) element of a unitary matrix. It follows by simplifying Eq. (B2) that Q has the spectral decomposition as $Q = \sum_{j=1}^r |b_j\rangle \langle b_j|$. Due to $|b_j\rangle = (U \otimes V)|a_j\rangle$ for each j , we conclude that $P \sim Q$. By the claim above, we conclude that $\rho \sim \sigma$ if and only if for each j , the ranges of P_j and Q_j are SLU equivalent, i.e., $\mathcal{R}(P_j) = (U \otimes V)\mathcal{R}(Q_j)$ by a common LU operator $U \otimes V$ for each j . This completes the proof. ■

Second, we show Lemma 5 in detail as below.

Proof of Lemma 5. According to the preconditions, we may assume that

$$\begin{aligned} P_1 &= (U_1 \otimes V_1)Q_1(U_1 \otimes V_1)^\dagger, \\ P_2 &= (U_1 \otimes V_1)Q_2(U_1 \otimes V_1)^\dagger, \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} P_1 &= (U_2 \otimes V_2)Q_1(U_2 \otimes V_2)^\dagger, \\ P_3 &= (U_2 \otimes V_2)Q_3(U_2 \otimes V_2)^\dagger, \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} P_2 &= (U_3 \otimes V_3)Q_2(U_3 \otimes V_3)^\dagger, \\ P_3 &= (U_3 \otimes V_3)Q_3(U_3 \otimes V_3)^\dagger, \end{aligned} \quad (\text{B5})$$

with local unitary $U_j \otimes V_j$, $j = 1, 2, 3$. By simultaneously applying any LU operation to Q_i 's, it follows that

$$\begin{aligned} \tilde{Q}_1 &= (U \otimes V)Q_1(U \otimes V)^\dagger, \\ \tilde{Q}_2 &= (U \otimes V)Q_2(U \otimes V)^\dagger, \\ \tilde{Q}_3 &= (U \otimes V)Q_3(U \otimes V)^\dagger. \end{aligned} \quad (\text{B6})$$

Next, we show the two assertions as follows.

(i) By Eq. (B3), we have

$$\begin{aligned} P_1 &= (U_1 U^\dagger \otimes V_1 V^\dagger) \tilde{Q}_1 (U U_1^\dagger \otimes V V_1^\dagger), \\ P_2 &= (U_1 U^\dagger \otimes V_1 V^\dagger) \tilde{Q}_2 (U U_1^\dagger \otimes V V_1^\dagger). \end{aligned} \quad (\text{B7})$$

By Eq. (B7), we conclude that (P_1, P_2) is SLU equivalent to $(\tilde{Q}_1, \tilde{Q}_2)$, namely, $(P_1, P_2) \sim_s (\tilde{Q}_1, \tilde{Q}_2)$. Analogously, one can verify that $(P_1, P_3) \sim_s (\tilde{Q}_1, \tilde{Q}_3)$ by virtue of Eqs. (B4) and (B6), and $(P_2, P_3) \sim_s (\tilde{Q}_2, \tilde{Q}_3)$ by virtue of Eqs. (B5) and (B6). Then, assertion (i) holds.

(ii) We show the “if” part, and the “only if” part can be proven similarly, due to the dual relation. Assume that $(P_1, P_2, P_3) \sim_s (\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$. It means there exists a common LU operator $W \otimes X$ such that $P_j = (W \otimes X) \tilde{Q}_j (W \otimes X)^\dagger$ for each j . By Eq. (B6), we obtain

$$\begin{aligned} P_1 &= (WU \otimes XV)Q_1(WU \otimes XV)^\dagger, \\ P_2 &= (WU \otimes XV)Q_2(WU \otimes XV)^\dagger, \\ P_3 &= (WU \otimes XV)Q_3(WU \otimes XV)^\dagger. \end{aligned} \quad (\text{B8})$$

It follows that $(P_1, P_2, P_3) \sim_s (Q_1, Q_2, Q_3)$. Then, assertion (ii) holds.

This completes the proof. ■

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- [1] B. Kraus, Local unitary equivalence and entanglement of multipartite pure states, *Phys. Rev. A* **82**, 032121 (2010).
 - [2] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, *Phys. Rev. A* **62**, 062314 (2000).
 - [3] L. Chen and M. Hayashi, Multicopy and stochastic transformation of multipartite pure states, *Phys. Rev. A* **83**, 022331 (2011).
 - [4] F. G. S. L. Brandão and G. Gour, Reversible framework for quantum resource theories, *Phys. Rev. Lett.* **115**, 070503 (2015).
 - [5] I. Marvian, Restrictions on realizable unitary operations imposed by symmetry and locality, *Nat. Phys.* **18**, 283 (2022).
 - [6] L. Piroli, G. Styliaris, and J. I. Cirac, Quantum circuits assisted by local operations and classical communication: Transformations and phases of matter, *Phys. Rev. Lett.* **127**, 220503 (2021).
 - [7] M. Oszmaniec, R. Augusiak, C. Gogolin, J. Kołodyński, A. Acín, and M. Lewenstein, Random bosonic states for robust quantum metrology, *Phys. Rev. X* **6**, 041044 (2016).
 - [8] A. Acín, A. Andrianov, L. Costa, E. Jane, J. I. Latorre, and R. Tarrach, Generalized Schmidt decomposition and classification of three-quantum-bit states, *Phys. Rev. Lett.* **85**, 1560 (2000).
 - [9] B. Kraus, Local unitary equivalence of multipartite pure states, *Phys. Rev. Lett.* **104**, 020504 (2010).
 - [10] B. Liu, J.-L. Li, X. Li, and C.-F. Qiao, Local unitary classification of arbitrary dimensional multipartite pure states, *Phys. Rev. Lett.* **108**, 050501 (2012).
 - [11] Y. Makhlin, Nonlocal properties of two-qubit gates and mixed states, and the optimization of quantum computations, *Quantum Inf. Proc.* **1**, 243 (2002).
 - [12] T.-G. Zhang, M.-J. Zhao, M. Li, S.-M. Fei, and X. Li-Jost, Criterion of local unitary equivalence for multipartite states, *Phys. Rev. A* **88**, 042304 (2013).
 - [13] M. Li, T. Zhang, S.-M. Fei, X. Li-Jost, and N. Jing, Local unitary equivalence of multiqubit mixed quantum states, *Phys. Rev. A* **89**, 062325 (2014).
 - [14] A. M. Martins, Necessary and sufficient conditions for local unitary equivalence of multiqubit states, *Phys. Rev. A* **91**, 042308 (2015).
 - [15] N. Jing, S.-M. Fei, M. Li, X. Li-Jost, and T. Zhang, Local unitary invariants of generic multiqubit states, *Phys. Rev. A* **92**, 022306 (2015).
 - [16] B.-Z. Sun, S.-M. Fei, and Z.-X. Wang, On local unitary equivalence of two and three-qubit states, *Sci. Rep.* **7**, 4869 (2017).
 - [17] Q. Zhou, Y.-Z. Zhen, X.-Y. Xu, S. Zhao, W.-L. Yang, S.-M. Fei, L. Li, N.-L. Liu, and K. Chen, Local unitary equivalence of arbitrary-dimensional multipartite quantum states, *Phys. Rev. A* **109**, 022427 (2024).
 - [18] L. Zhang, B. Xie, and Y. Tao, Local unitary equivalence and entanglement by Bargmann invariants, *arXiv:2412.17237*.

- [19] L. Chen and L. Yu, Decomposition of bipartite and multipartite unitary gates into the product of controlled unitary gates, *Phys. Rev. A* **91**, 032308 (2015).
- [20] Y. Shen, L. Chen, and L. Yu, Classification of Schmidt-rank-two multipartite unitary gates by singular number, *J. Phys. A: Math. Theor.* **55**, 465302 (2022).
- [21] D. Chruściński and G. Sarbicki, Entanglement witnesses: Construction, analysis and classification, *J. Phys. A: Math. Theor.* **47**, 483001 (2014).
- [22] Y. Shen, L. Chen, and Z. Bian, Detection power of real entanglement witnesses under local unitary equivalence, *Phys. Rev. A* **112**, 022415 (2025).
- [23] L. Chen and L. Yu, Entangling and assisted entangling power of bipartite unitary operations, *Phys. Rev. A* **94**, 022307 (2016).
- [24] X. Qiu, Z. Song, and L. Chen, Multipartite entangling power by von Neumann entropy, *Phys. Rev. A* **111**, 022407 (2025).
- [25] Y. Liang, J. Yan, D. Si, and L. Chen, Inertia of partial transpose of positive semidefinite matrices, *J. Phys. A: Math. Theor.* **57**, 125203 (2024).
- [26] L. Yu, R. B. Griffiths, and S. M. Cohen, Efficient implementation of bipartite nonlocal unitary gates using prior entanglement and classical communication, *Phys. Rev. A* **81**, 062315 (2010).
- [27] D. Amaro and M. Müller, Design and experimental performance of local entanglement witness operators, *Phys. Rev. A* **101**, 012317 (2020).
- [28] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, *Phys. Lett. A* **232**, 333 (1997).
- [29] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible product bases and bound entanglement, *Phys. Rev. Lett.* **82**, 5385 (1999).
- [30] C. Cheng and L. Chen, The construction of multiqubit unextendible product bases, [arXiv:2402.11785](https://arxiv.org/abs/2402.11785) [quant-ph].
- [31] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed. (Wiley, USA, 1999).
- [32] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Reference frames, superselection rules, and quantum information, *Rev. Mod. Phys.* **79**, 555 (2007).
- [33] G. Gour and R. W. Spekkens, The resource theory of quantum reference frames: Manipulations and monotones, *New J. Phys.* **10**, 033023 (2008).
- [34] G. Gour, I. Marvian, and R. W. Spekkens, Measuring the quality of a quantum reference frame: The relative entropy of frame-ness, *Phys. Rev. A* **80**, 012307 (2009).