Dynamical large deviations, self-similarity, and criticality

Vittoria Sposini,* Francesco Arceri,† Enzo Orlandini,‡ Flavio Seno,§ and Fulvio Baldovin‖
Dipartimento di Fisica e Astronomia 'G. Galilei' (DFA), Sezione INFN, Università di Padova, Via Marzolo 8, 35131 Padova (PD), Italy

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Large-deviation theory describes the statistics of rare fluctuations, while self-similarity captures scale-invariant behavior. Both are central pillars of statistical physics, yet the extent of their overlap is not well understood. Here, we ask whether a stochastic process can simultaneously satisfy both. We show that, under a precise set of conditions—exact self-similarity governed by one Hurst exponent and a scaled cumulant-generating function that is analytic around the origin—compatibility holds only if the scaling distribution is Gaussian. We illustrate this result with a broad class of subordinated stochastic processes, including continuous-time random walks, diffusing-diffusivity models, and polymers in the grand-canonical ensemble. In non-Gaussian cases, a characteristic timescale separates regular large-deviation and self-similar regimes. At criticality, the divergence of the correlation time and the emergence of a singular cumulant generating function make a regular large-deviation description inappropriate and lead to a strong enhancement of extreme dynamical fluctuations.

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Large-deviation (LD) theory is a cornerstone of both equilibrium and nonequilibrium statistical mechanics, providing a unified toolbox for quantifying rare fluctuations and their thermodynamic structure [1–4]. At its core, LD theory replaces exact probabilities with asymptotic exponential estimates and uses Laplace/saddle-point techniques to extract the information encoded in the rate function [2,5–7]. In many settings, the central part of the probability density function (PDF) becomes asymptotically Gaussian (a manifestation of the central limit regime), while the rate function governs the tails responsible for large deviations [1,3,8]. A second pillar of statistical mechanics is the theory of critical phenomena, understood through anomalous scaling emerging at nontrivial self-similar (SS) fixed points of the renormalization group [9–13]. At criticality, the divergence of the correlation length leads to non-Gaussian (anomalous) scale-invariant PDFs associated with the fixed-point theory [12,13]—in contrast to the generic Gaussian bulk of LD theory. This juxtaposition raises a natural question: Is the LD toolbox applicable to anomalous self-similarity, or does critical scaling fall outside its domain [1,2]?

Here, we address this question by focusing on a specific setting: We ask whether *exact monofractal* SS and a regular dynamical LD structure can simultaneously describe a stochastic observable. Throughout, "exact monofractal" refers to a scaling law that is governed by a single Hurst exponent in the asymptotic regime under study; approximate, preasymptotic, multifractal, or crossover scaling laws therefore lie outside the hypotheses considered here. We first establish a compatibility criterion directly from the cumulant scalings implied by SS and by a regular scaled cumulant generating function. Within this framework, we show that exact monofractal SS and regular large-deviation (RLD) scalings can coexist only if the statistics are Gaussian. We then use *subordinated* stochastic processes [14,15] as a broad and analytically tractable test bed. Our analysis covers continuous-time random walks (CTRWs) with both finite and infinite mean waiting times [16–21], diffusing-diffusivity models [22–24], the diffusion of polymers undergoing conformational changes, and polymers in the grand-canonical ensemble [25,26]. In all non-Gaussian cases, a timescale separates RLD regimes from SS ones. While LD behavior is often associated with long times, for CTRWs we find an RLD scaling as an intermediate regime, followed by an asymptotic SS regime [16,18,19,21]. In contrast, for a critical system a long-lasting SS emerges due to the diverging correlation length [9,10,12,13], so RLD descriptions underestimate the likelihood of extreme dynamical fluctuations. Explicit derivations are reported in the Supplemental Material [27].

Let us start by recalling the notion of self-similarity. To keep things simple, we consider a one-dimensional, centered stochastic process X_t ($t \geqslant 0$) initially located in $X_0 = 0$. X_t is SS if, for any $a > 0$, X_{at} and $a^H X_t$ share the same distribution [28,29]; setting $a = t_0/t$ with $t_0 > 0$ some reference time, in terms of the PDF, this means

$$p_{X_t}(x,t) = (t_0/t)^H \, g((t_0/t)^H x), \tag{1}$$

*Contact author: vittoria.sposini@unipd.it
†Contact author: francesco.arceri@pd.infn.it
‡Contact author: enzo.orlandini@unipd.it
§Contact author: flavio.seno@unipd.it
‖Contact author: fulvio.baldovin@unipd.it

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for some scaling function $g(x) \equiv p_{X_{t_0}}(x, t_0)$ and scaling (or Hurst) exponent $H > 0$. Equation (1) assumes a *monofractal* self-similar scaling, namely, a single Hurst exponent H governing all moments. Multifractal generalizations, in which different orders of moments may scale with different effective exponents, lie outside the scope of the present work.

Central to our discussion is the cumulant-generating function (CGF):

$$K_{X_t}(k,t) \equiv \ln \mathbb{E}[\mathrm{e}^{k X_t}] \equiv \sum_{m=1}^{\infty} \kappa_{X_t}^{(m)}(t)\, \frac{k^m}{m!}, \tag{2}$$

where $\kappa_{X_t}^{(m)}(t)$ is the cumulant of order m of X_t at time t, and we have assumed a regular Taylor expansion around $k = 0$. When it exists, the CGF plays a role similar to that of a free energy in statistical mechanics [1]. It is interesting to note the behavior of PDFs and CGFs under a rescaling $\overline{X}_{t,\theta} \equiv (t_0/t)^\theta X_t$ of the stochastic process: We have $p_{\overline{X}_{t,\theta}}(x,t) = (t/t_0)^\theta\, p_{X_t}((t/t_0)^\theta x, t)$ and $K_{\overline{X}_{t,\theta}}(k,t) = K_{X_t}((t_0/t)^\theta k, t)$. A rescaling with $\theta = H$ yields thus the time-independent relevant information about an SS process, $p_{\overline{X}_{t,H}}(x,t) = p_{X_{t_0}}(x,t_0) = g(x)$, $K_{\overline{X}_{t,H}}(k,t) = K_{X_{t_0}}(k,t_0) \equiv \ln \mathbb{E}[\mathrm{e}^{k X_{t_0}}]$, implying for the mth cumulant of X_t the time scaling

$$\kappa_{X_t}^{(m)}(t) = (t/t_0)^{Hm}\, \kappa_{X_{t_0}}^{(m)} \quad \text{(SS regime)}. \tag{3}$$

Thus, for an SS process, each cumulant has a different scaling in time, defined by its order m.

On the other hand, $\overline{X}_{t,\theta}$ is said to satisfy an LD principle [1] if its PDF behaves as [30]

$$p_{\overline{X}_{t,\theta}}(x,t) \asymp \mathrm{e}^{-(t/t_0)^\theta I(x)}. \tag{4}$$

Here, "$\asymp$" stands for "the dominant part as $t \to \infty$", $(t/t_0)^\theta$ is the *speed* ($\theta > 0$), and $I(x)$ the *rate function*. In the present work, we use Eq. (4) in the context of *dynamical* large deviations in time for one-time marginals of a stochastic process. This setting, while different from equilibrium large deviations with respect to system size, can be understood in terms of time-integrated observables, similarly to what is done in nonequilibrium many-body systems [1]. Indeed, within our framework, the end-point displacement is itself a variable accumulated over time: For subordinated diffusion, it is a stochastic time integral, while for a CTRW it is a random sum of all jumps completed up to a certain time. Such an additive or integrated representation is linked to the conventional dynamical-LD viewpoint, but does not by itself guarantee a temporal LD principle. The latter must still be established from the long-time scaling and correlation structure of the process; confinement, long-range correlations, or aging can alter or prevent the RLD asymptotics considered here. LD theory typically focuses on a $\theta = 1$ rescaling; to account for a novel result [31], we are instead considering general values for θ, taking into account the possibility of an anomalous speed [32,33] for the variable $\overline{X}_{t,\theta}$, which occurs when $\theta \neq 1$ [34]. In the regular setting in which the rate function is twice continuously differentiable and the corresponding scaled cumulant-generating function exists and admits a regular Taylor expansion around the origin, a Laplace calculation extracts the time-independent information in terms of the *scaled* CGF

$$\lambda(k) \equiv \lim_{t\to\infty} (t_0/t)^\theta\, \ln \mathbb{E}[\mathrm{e}^{(t/t_0)^\theta\, k\, \overline{X}_{t,\theta}}] \equiv \sum_{m=1}^{\infty} \kappa_\lambda^{(m)}\, \frac{k^m}{m!}. \tag{5}$$

These regularity assumptions define the RLD class studied within this work. Generalizations to arbitrary, singular, or nonanalytic LD structures, which are in fact natural candidates in critical settings, are only briefly discussed in the conclusions and left for future studies. From Eq. (5), we deduce that in the RLD regime the mth cumulant of X_t scales as

$$\kappa_{X_t}^{(m)}(t) = (t/t_0)^\theta\, \kappa_\lambda^{(m)} \quad \text{(RLD regime)}. \tag{6}$$

Contrary to SS, in the RLD regime each cumulant has the same scaling in time, defined by the speed exponent θ.

A first concern is whether Eqs. (1) and (4) are formally compatible, since Eqs. (3) and (6) point out very different time evolutions. For the excess kurtosis, an indicator of the PDF's "tailedness" that reads

$$\mathrm{Kurt}_{X_t}(t) - 3 \equiv \frac{\mathbb{E}\left[X_t^4\right]}{\left(\mathbb{E}\left[X_t^2\right]\right)^2} - 3 = \frac{\kappa_{X_t}^{(4)}(t)}{\left[\kappa_{X_t}^{(2)}(t)\right]^2}, \tag{7}$$

Eqs. (3) and (6) imply, for instance, that in an SS phase $\mathrm{Kurt}_{X_t}(t) - 3$ is constant, whereas in an RLD regime $\mathrm{Kurt}_{X_t}(t) - 3 \propto 1/t^\theta$ [27]. Although the SS and RLD regimes imply precise scaling behavior for all moments and not only the second and the fourth ones, in what follows we use the excess kurtosis as a compact *diagnostic*, once the asymptotic regimes have been identified independently. It has been recently pointed out [36–38] that an interesting class of scaling functions that may reconcile SS and LD regimes is that of stretched exponentials $g(x) = \mathcal{N}\, \mathrm{e}^{-c\,|x|^\zeta}$, with $\mathcal{N} > 0$ and $c > 0$ a normalization and a scale factor, respectively, and $\zeta > 0$. In the Supplemental Material [27], we demonstrate that our derivation of Eq. (6) applies to stretched exponentials uniquely in the Gaussian case with $\zeta = 2$. As a matter of fact, within the regular setting in which both the CGF and the scaled CGF of X_t admit a Taylor expansion around the origin, compatibility between the SS and RLD cumulant scalings arises only for a centered Gaussian process (zero excess kurtosis). Indeed, SS implies $\kappa_{X_t}^{(m)}(t) \sim t^{Hm}$, whereas an RLD principle implies $\kappa_{X_t}^{(m)}(t) \sim t^\theta$ for all m. For these two behaviors to be consistent for $m = 2$ and for any nonzero $m > 2$, all cumulants above the second order must vanish, which is only true for Gaussian processes. In the conventional linear-speed case $\theta = 1$, one has $H = 1/2$. More generally, Gaussian compatibility requires $\theta = 2H$ and the sole nonzero cumulant is the second one, a circumstance that implies the identity $\kappa_{X_t}^{(2)}(t) = (t/t_0)^\theta\, \kappa_{X_{t_0}}^{(2)} = (t/t_0)^{2H}\, \kappa_\lambda^{(2)}$, with $\kappa_{X_{t_0}}^{(2)} = \kappa_\lambda^{(2)}$ and $\theta = 2H = 1$. For a centered process with a regular CGF and regular scaled CGF, monofractal self-similarity and a regular LD principle can coexist only if the scaling distribution is Gaussian. This effectively acts as a compatibility/exclusion criterion, providing a useful test: A robust non-Gaussian exact SS regime rules out a simultaneous analytic scaled-CGF RLD description, while an established RLD regime with nonzero higher cumulants rules out exact monofractal SS.

We now turn to applications. We consider, in particular, subordinated processes, a broad, physically motivated class

of stochastic processes in which cumulants, correlation times, and crossover regimes can be obtained explicitly. Their exact solutions allow us to show how the incompatibility between SS and RLD regimes is resolved through temporal crossover, convergence to a Gaussian fixed point, or loss of regularity at criticality. Consider X_t arising from Brownian motion with a diffusion coefficient D_t that fluctuates in time. This could stem from conformational changes of the diffusing particle (examples are provided below) or may effectively represent heterogeneity of the environment. The overdamped Langevin description of such a motion becomes

$$\mathrm{d}X_t = \sqrt{2\,D_t}\,\mathrm{d}W_t, \tag{8}$$

where $\mathrm{d}W_t$ is an infinitesimal Wiener increment. With $X_0 = 0$, the trajectory end point is an accumulated stochastic observable, $X_t = \int_0^t \sqrt{2D_s}\,\mathrm{d}W_s$. Introducing the random operational time (or subordinator) $S_t \equiv 2\int_0^t \mathrm{d}t'\,D_{t'}$, in free diffusion with natural boundary conditions, one gets the following subordination formula connecting the subordinator PDF, $p_{S_t}(s,t)$, to the PDF of the subordinated process X_t [39,40]:

$$p_{X_t}(x,t) = \int_0^\infty \mathrm{d}s\, p_{S_t}(s,t)\,\frac{\mathrm{e}^{-\frac{x^2}{2s}}}{\sqrt{2\pi\,s}}. \tag{9}$$

This equation allows one to express the moments of X_t in terms of those of S_t. In a similar fashion, the cumulant-generating functions are connected. Indeed, from Eq. (9) we have

$$K_{X_t}(k,t) = \ln \mathbb{E}\big[\mathrm{e}^{k^2 S_t/2}\big]. \tag{10}$$

Note the presence of the square of k in the expected value above, which implies that a cumulant of order m of X_t is associated with a cumulant of order $m/2$ of S_t. In particular, if S_t is SS with Hurst exponent H_S, then X_t is SS with Hurst exponent $H_X = H_S/2$. In a similar fashion, correlations among the fluctuations of the process X_t are induced by those of the process D_t, which in particular identifies the timescale τ_c over which these correlations vanish [27]. In Fig. 1, we summarize our findings for the broad class of stochastic processes under consideration. In short, the only point of contact between regular large deviations and self-similarity is the Gaussian process; in all other cases (which include examples of SS, $\zeta \neq 2$ stretched-exponential PDFs), we show below that there exists a timescale that separates the RLD and SS regimes. Allowing different exponents for the scaling of the variable and for the RLD speed does not alter this incompatibility: A short derivation is reported in the Supplemental Material [27].

A first practical example of a subordinated process is a polymer that undergoes spontaneous transitions between a folded state "A" and an unfolded state "B", as commonly observed in RNA molecules and proteins [41,42]. By the Stokes-Einstein relation [25,43], the diffusion coefficient is inversely proportional to the polymer's hydrodynamic radius, which changes during the stochastic transition A $\leftrightharpoons$ B. The conformational change thus induces a transition for the diffusion coefficient among two values $D^{(\mathrm{A})}$ and $D^{(\mathrm{B})}$. A minimal model for D_t is a telegraph process [44,45] with forward and backward rates λ_{AB} and λ_{BA}, respectively. An exact calculation detailed in the Supplemental Material [27] yields the CGF of X_t. For times much shorter than the characteristic time

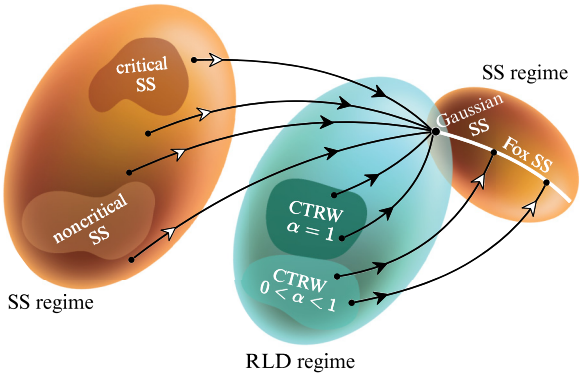


FIG. 1. Schematic dynamic phase diagram reporting dynamical trajectories of subordinated processes. In the critical SS surface, the CGF is singular, whereas in the noncritical SS surface it is regular. Both surfaces are characterized by an infinite correlation time τ_c. On the other hand, the Gaussian SS point and the Fox SS points [35] on the upper-right line have $\tau_c = 0$. To avoid possible confusion, throughout the article the large-deviation regime is understood as the *regular large-deviation* regime, namely, the one encoded by a regular scaled cumulant-generating function. Cumulants of trajectories starting close to an SS surface (finite τ_c) evolve approximately according to Eq. (3) for $t \ll \tau_c$ (white arrows) and then cross over to an RLD regime approximated by Eq. (6) for $t \gg \tau_c$ (black arrows). Trajectories for the CTRW originate within the RLD regime described by Eq. (6) (black arrows) and converge to the Gaussian SS point if the average waiting time is finite ($\alpha = 1$) and to the Fox SS points [35] if the average waiting time is infinite ($0 < \alpha < 1$).

$\tau_c \equiv (\lambda_{\mathrm{AB}} + \lambda_{\mathrm{BA}})^{-1}$ and under equilibrium initial conditions for the subordinator, the behavior is SS with $H_X = 1/2$:

$$K_{\overline{X}_{t,1/2}}(k,t)_{t \lessapprox \tau_c} \ln\left(\frac{\mathrm{e}^{k^2 D^{(\mathrm{A})} t_0}}{1 + r^{-1}} + \frac{\mathrm{e}^{k^2 D^{(\mathrm{B})} t_0}}{1 + r}\right), \tag{11}$$

with $r \equiv \lambda_{\mathrm{BA}}/\lambda_{\mathrm{AB}}$. Indeed, for $t \ll \tau_c$, D_t can be approximated as a static random variable D with values $D^{(\mathrm{A})}$, $D^{(\mathrm{B})}$, and the subordinator then takes the SS form $S_t = 2Dt$ with Hurst exponent $H_S = 1$. On the contrary, for $t \gg \tau_c$, correlations fade away leading to an RLD regime with

$$\lambda(k)_{t \gtrapprox \tau_c} \frac{D^{(\mathrm{A})} t_0}{1 + r^{-1}}\,k^2 + \frac{D^{(\mathrm{B})} t_0}{1 + r}\,k^2. \tag{12}$$

With $D^{(\mathrm{A})} \neq D^{(\mathrm{B})}$, this is a case in which the short-time, SS rescaled CGF is non-Gaussian but regular, while the scaled CGF in the long-time RLD regime is regular and Gaussian. Hence, SS and RLD cannot coexist in the same regime. The time evolution of the excess kurtosis shown in Fig. 2(a) confirms the *sequence* of the two dynamical regimes.

The second subordinated process we address is the diffusing-diffusivity model, a widely used framework for representing transport in heterogeneous environments [22–24]. To capture environmental variability, the particle's diffusion coefficient is taken to be the square of an Ornstein-Uhlenbeck process, $D_t \equiv \boldsymbol{Y}_t^2$, where $\boldsymbol{Y}_t$ follows an Ornstein-Uhlenbeck

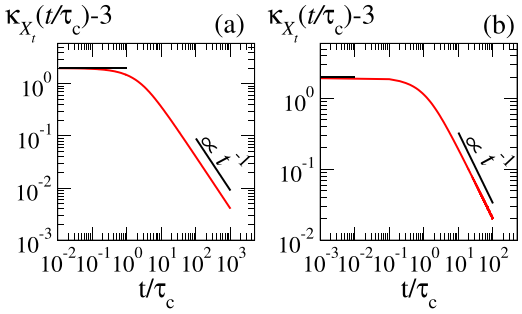


FIG. 2. Exact time evolution of the excess kurtosis for two-state polymers [panel (a)] and the diffusing-diffusivity model [panel (b)]—see the Supplemental Material [27] for the explicit derivations. In both cases, the subordinator is under stationary (equilibrium) conditions and guidelines are added to emphasize the SS regime in which $\mathrm{Kurt}_{X_t}(t)-3$ is constant and the RLD one during which $\mathrm{Kurt}_{X_t}(t)-3\propto 1/t$. In panel (a), $D^{(\mathrm{B})}=10\,D^{(\mathrm{A})}$ and $r=1$; in panel (b), $d_{\boldsymbol{Y}}=3$.

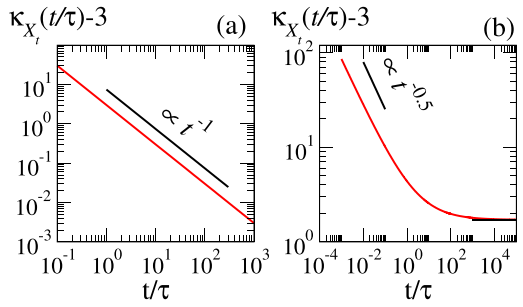


FIG. 3. Exact time evolution of the excess kurtosis for the CTRW model. Since $\tau_c=0$, the waiting-time PDF $w(t)$ introduces another characteristic time τ. (a) For $\alpha=1$ (exponential waiting-time PDF), the RLD regime lasts forever, asymptotically approaching the Gaussian SS point (zero excess kurtosis). (b) For $\alpha=0.5$, the initial RLD-like regime crosses over to a non-Gaussian self-similar fixed point of Fox type [35], with $\mathrm{Kurt}_{X_t}(t)-3=1.712\cdots$ for $t\gg\tau$.

dynamics [23] with

$$\mathrm{d}\boldsymbol{Y}_t=-\frac{\boldsymbol{Y}_t}{\tau_\mathrm{c}}\,\mathrm{d}t+\sigma\,\mathrm{d}\boldsymbol{W}_t. \tag{13}$$

The dimension of the vector $\boldsymbol{Y}_t$ is $d_{\boldsymbol{Y}}\in\mathbb{N}^*$, τ_c defines the autocorrelation timescale of $\boldsymbol{Y}_t$ and thus of D_t, and σ is the intensity of the fluctuations. It is possible [27] to obtain an exact expression for the time evolution of the excess kurtosis:

$$\mathrm{Kurt}_{X_t}(t)-3=\frac{3}{d_{\boldsymbol{Y}}}\,\frac{2t/\tau_\mathrm{c}+\mathrm{e}^{-2t/\tau_\mathrm{c}}-1}{(t/\tau_\mathrm{c})^2}. \tag{14}$$

Also in this case, the initial non-Gaussian SS behavior with $\mathrm{Kurt}_{X_t}(t)-3\,{}_{t\lessapprox\tau_\mathrm{c}}\,6/d_{\boldsymbol{Y}}$ is separated from the long-time RLD regime $\mathrm{Kurt}_{X_t}(t)-3\,{}_{t\gtrapprox\tau_\mathrm{c}}\,6\tau_\mathrm{c}/(d_{\boldsymbol{Y}}\,t)\propto 1/t$, as displayed in Fig. 2(b).

The CTRW model [17,18] with jump PDF $\psi(x,t)=\ell(x)\,w(t)$ and Gaussian jump length PDF $\ell(x)=\mathrm{e}^{-x^2/(2\,\sigma_0^2)}/\sqrt{2\pi\,\sigma_0^2}$ (where σ_0 is the characteristic single-step length) admits a natural interpretation as a subordination process [21]. Here, the operational time is simply the number of steps completed by time t, N_t. The end-point position is correspondingly the accumulated random sum

$$X_t=\sum_{j=1}^{N_t}\Delta x_j, \tag{15}$$

where Δx_j denotes the jth jump. Equations (9) and (10) recast into a discrete version:

$$p_{X_t}(x,t)=\sum_{n=0}^{\infty}p_{N_t}(n,t)\,\frac{\mathrm{e}^{-\frac{x^2}{2n\sigma_0^2}}}{\sqrt{2\pi\,n\,\sigma_0^2}}, \tag{16}$$

$$K_{X_t}(k,t)=\ln\mathbb{E}\left[\mathrm{e}^{k^2N_t\,\sigma_0^2/2}\right], \tag{17}$$

respectively. Because of the independence of the steps, the correlation timescale among fluctuations is zero, $\tau_\mathrm{c}=0$ [27]. Nevertheless, the waiting-time PDF introduces a characteristic time τ, which proves to be a relevant quantity. The subordinator PDF is conveniently expressed in Laplace space as $\widetilde{p}_{N_t}(n,u)=[\widetilde{w}(u)]^n\,(1-\widetilde{w}(u))/u$ (the tilde indicating the Laplace transform), from which the moments can be computed exactly [21]. When the waiting-time PDF $w(t)$ has a finite mean τ, X_t exhibits normal diffusion with $\mathbb{E}[X_t^2]\propto t$; in contrast, a power-law tail at large t in the waiting-time PDF, $w(t)\,{}_{t\gtrapprox\tau}\,\tau^\alpha/t^{1+\alpha}$ $(0<\alpha<1)$, leads to subdiffusion with $\mathbb{E}[X_t^2]\propto t^\alpha$. A standard example entails a waiting-time PDF characterized by a Mittag-Leffler cumulative function [21]. In such a case, the subordinator moments exhibit the following behavior:

$$\mathbb{E}[N_t]=A_\alpha\,(t/\tau)^\alpha, \tag{18}$$

$$\mathbb{E}\left[N_t^2\right]=A_\alpha\,(t/\tau)^\alpha+A_{2\alpha}\,(t/\tau)^{2\alpha}, \tag{19}$$

with $A_\alpha=1/\Gamma(1+\alpha)$ and $A_{2\alpha}=2/\Gamma(1+2\alpha)$ [27]. This result extends to finite-mean, exponential waiting-time PDFs taking $\alpha=1$, with $A_2=A_1=1$. Importantly, Eqs. (18) and (19) act as long-time attractors for any waiting-time PDF, whether the mean waiting time is finite or divergent. Combining Eqs. (17), (18), and (19), we obtain the exact expression for the time evolution of the excess kurtosis:

$$\mathrm{Kurt}_{X_t}(t)-3=3\left(\frac{1}{A_\alpha(t/\tau)^\alpha}+\frac{A_{2\alpha}-A_\alpha^2}{A_\alpha^2}\right). \tag{20}$$

It is well known [18] that, in the infinite time limit, the CTRW becomes SS with $H=\alpha/2$ and develops a stretch-exponential tail,

$$p_{X_t}(x,t)\,{}_{t\gtrapprox\tau}\,\mathrm{e}^{-c(t/t_0)^{\alpha/(2-\alpha)}|x|^{1/(1-\alpha/2)}}. \tag{21}$$

Indeed, from Eq. (20) one sees that the excess kurtosis at long times shows an SS behavior: $\mathrm{Kurt}_{X_t}(t)-3\,{}_{t\gtrapprox\tau}\,3\,(A_{2\alpha}/A_\alpha^2-1)$. On the contrary, at short times, we obtain $\mathrm{Kurt}_{X_t}(t)-3\,{}_{t\lessapprox\tau}\,3/[A_\alpha(t/\tau)^\alpha]\propto 1/t^\alpha$. Surprisingly, this produces a finite-time RLD-like scaling window with speed t^α $(\theta=\alpha)$ at short times, rather than an asymptotic LD principle in the usual $t\to\infty$ sense. Figure 3 highlights these findings. In particular, with an exponential waiting-time PDF $(\alpha=1)$, given the always nonzero excess kurtosis, we observe a non-Gaussian RLD regime that tends toward the Gaussian SS fixed point. By studying the PDF at any finite time, approximate self-similarity might appear in the bulk of the distribution;

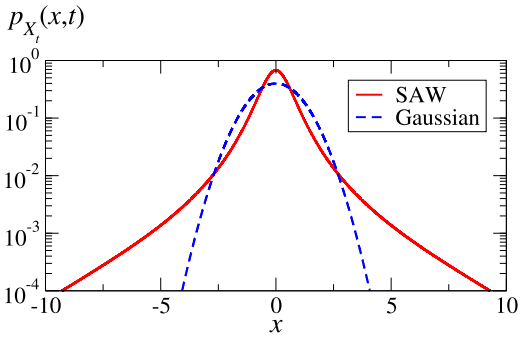


FIG. 4. Comparison of the PDF for the displacement of the center of mass of a self-avoiding walk (SAW) with that of a Gaussian process. The PDFs have been rescaled to unit variance. The three-dimensional self-avoiding walk is performed at a fugacity $z/z_{cr} = 0.997\,677$, where z_{cr} is the critical fugacity and the time corresponds to $t = 10^{-4}\,\tau_c$ [46].

however, exact self-similarity will only be observed at $t \to \infty$ along with pure Gaussian statistics. In this case, the RLD behavior lasts forever, as shown in Fig. 3(a), and it is the only instance in which the RLD and the SS regimes overlap because only the second cumulant survives for the Gaussian SS point.

Although critical fixed points are SS, self-similarity does not imply criticality. In renormalization-group language, critical points are nontrivial fixed points characterized by an infinite correlation length ξ *and* a singular free energy; they differ from trivial fixed points, which are still self-similar but have $\xi = 0$ and a regular free energy. In the context of the stochastic processes considered here, ξ maps to the correlation time τ_c and the free energy to the CGF. In this mapping, a large but finite τ_c only indicates proximity to an SS fixed point and thus extended crossover regimes, whereas genuine criticality requires $\tau_c \to \infty$ together with a nonanalytic, singular, CGF. The examples we have provided so far are noncritical: Even if τ_c can be assumed to be arbitrarily large for two-state polymers and in the diffusing-diffusivity model, the CGF and the associated cumulants display nonsingular behavior. Things become different if we address the motion of the center of mass of a polymer in the grand-canonical ensemble [46–48]. Again, in virtue of the Stokes-Einstein relation [25,43], in the Zimm model the diffusion coefficient is proportional to the inverse of the hydrodynamic radius [25], which in turn depends on the polymer size M; namely, $D(M) \sim 1/M^{\nu}$, where ν is the critical metric exponent. In the grand-canonical ensemble, the polymer size M_t fluctuates during the dynamical evolution, rendering the diffusion coefficient a stochastic process. At criticality, τ_c becomes infinite, and the partition function becomes singular as does the CGF [47]. In Fig. 4, we report the PDF for the center of mass of a three-dimensional self-avoiding walk—a paradigmatic polymer model—close to criticality. The probability of a 3σ fluctuation is already about twice what a Gaussian process would predict, and for truly extreme events, the discrepancy becomes dramatic: a 5σ fluctuation is roughly three orders of magnitude more likely than in the Gaussian case. In such a critical setting, mistakenly assuming an RLD-like time evolution would rapidly drive any forecast away from the true non-Gaussian steady-state behavior and toward the Gaussian curve. The consequence is not a small bias but a catastrophic underestimation, by orders of magnitude, of the occurrence rate of the most extreme events. As a practical example, if events are taken to occur daily a fluctuation above 5σ occurs for a Gaussian PDF every 4780 years (once in recorded history), whereas according to the PDF of the critical polymer in Fig. 4 it happens each year!

The dramatic difference between the critical and Gaussian predictions highlighted above is a purely dynamical signature, i.e., rooted in the time evolution of the stochastic process. It should therefore be distinguished from equilibrium large deviations with respect to system size, where different considerations apply. Our claim is not that critical equilibrium systems generically fail to satisfy any large-deviation principle; rather, it is that the conventional RLD structure, with a smooth rate function, needs to be replaced by a nonstandard large-fluctuation behavior. This loss of regularity is precisely how the constraint is resolved in such cases: Non-Gaussian critical scaling need not fit inside the regular analytic scaled-CGF framework considered here.

In this work, we established a structural compatibility criterion between exact monofractal SS scaling and a regular dynamical LD principle. Whenever both hypotheses are required to hold for the same centered observable, coexistence is restricted to Gaussian statistics. We illustrated this statement with a broad class of subordinated and heterogeneous diffusion processes, including CTRWs, diffusing-diffusivity models, and polymer-inspired dynamics, whose explicit solutions reveal how the incompatible structures are separated in practice. Because the compatibility argument rests on the scaling of the observable and its scaled CGF, not on the subordination construction itself, it extends naturally to accumulated or time-integrated observables, and to other stochastic systems that satisfy the same hypotheses. We find that whenever the displacement statistics remain non-Gaussian, exact SS and RLD descriptions cannot hold simultaneously. Instead, non-Gaussian SS behavior and RLD-like scaling typically occupy distinct time windows separated by a characteristic timescale (such as a correlation time), producing observable crossovers in both the central part and the tails of the distribution. This separation is reversed in CTRWs, where an anomalous RLD-like regime can emerge at short times before the SS scaling dominates the long-time behavior.

We further argued that "criticality" should be understood as a distinct notion from self-similarity, marked by a singular cumulant-generating function and a diverging correlation time, which dramatically enhances the probability of extreme fluctuations. These results caution against a mechanical use of LD tools to quantify rare events in non-Gaussian and especially critical dynamics, where tail probabilities can be underestimated by orders of magnitude. Our framework provides a unified and operational way to identify the appropriate scaling regime and the relevant crossover scale from data. This paves the way to data-driven diagnostics in single-particle tracking experiments, where subordinated and heterogeneous diffusion processes are often observed, but where distinguishing genuine criticality from near-critical crossover behavior is a challenge. Extensions of this framework to higher

dimensions, interacting many-body systems, and nonequilibrium driving represent natural directions for future work.

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DATA AVAILABILITY

The data used to generate Fig. S1 of the Supplemental Material [27] are available from [51].

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