

Cat states in one- and two-mode $\mathbb{Z}_3$ Rabi modelsAnatoliy I. Lotkov^{1,*}, Denis V. Kurlov,¹ Valerii K. Kozin¹, Jelena Klinovaja¹ and Daniel Loss^{1,2,3,4}¹Department of Physics, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland²Physics Department, King Fahd University of Petroleum and Minerals, 31261 Dhahran, Saudi Arabia³Quantum Center, KFUPM, 31261 Dhahran, Saudi Arabia⁴RDIA Chair in Quantum Computing, 31261 Dhahran, Saudi Arabia

(Received 22 September 2025; accepted 25 March 2026; published 11 May 2026)

We investigate one- and two-mode variants of the $\mathbb{Z}_3$ -symmetric quantum Rabi model, which describe the interaction of a qutrit with one or two bosonic modes and are directly relevant for circuit-QED and spin-qudit platforms. We find a canonical transformation that allows one to obtain the spectrum of the $\mathbb{Z}_3$ Rabi model using perturbation theory in a magnetic field. We show that in a certain parameter regime (deep-strong coupling and a small magnetic field), the three lowest eigenstates become $\mathbb{Z}_3$ qutrit-boson cat states. In order to characterize these states, we introduce a joint qutrit-boson Wigner function and derive its closed-form expression for the qutrit-boson cat states. Numerical calculations across a wide range of coupling strengths show that the proposed Wigner function is a useful tool that allows one to unambiguously identify the states of the $\mathbb{Z}_3$ qutrit-boson cat states.

DOI: [10.1103/bvk8-kxmy](https://doi.org/10.1103/bvk8-kxmy)

I. INTRODUCTION

Cat states have been proposed as a platform for quantum computing [1–4]. Since encoding qubits into the cat states passively suppresses the phase-flip errors and therefore biases the noise toward bit flips, quantum error correction can be significantly simplified. Experimentally, cat states are typically implemented in a Kerr resonator with two-photon driving and dissipation [5–8]. Similarly, replacing the two-photon drive with an n -photon drive leads to more general $\mathbb{Z}_n$ -symmetric cat states. The n -dimensional qudits encoded in these $\mathbb{Z}_n$ -symmetric cat states also possess passive suppression of the phase noise. Although qudit-based quantum computing is less common than qubit-based approaches, it can offer certain advantages. Specifically, the qudit platform can reduce the number of entangling gates for certain algorithms [9–15]. This can relax the requirements on the entangling-gate fidelity, which is particularly relevant since the entangling-gate fidelity remains imperfect [16,17].

However, the experimental realization of the driven-dissipative Kerr resonator requires the engineering of a controlled dissipation, which is rather complicated. On the other hand, the cat states obtained in the unitary setting circumvent this problem. For example, the Rabi model is known to host cat states in the deep-strong-coupling limit. Its natural $\mathbb{Z}_n$ -symmetric generalization, i.e., a qudit coupled to a boson [18–21] can produce $\mathbb{Z}_n$ -symmetric cat states.

This was shown semiclassically in the deep-strong-coupling limit in Ref. [20]. Here, we obtain cat states in the $\mathbb{Z}_n$ Rabi model directly by applying a canonical transformation that simplifies the perturbation theory in the deep-strong-coupling limit. Our second contribution is an introduction of the joint boson-qudit Wigner function as a tool for studying cat states in hybrid systems. For boson cat states, the phase space methods are particularly powerful: Across diverse contexts, Wigner functions provide an intuitive and quantitative diagnostic of nonclassicality and interference [22–24]. However, the boson Wigner functions cannot fully characterize the qudit-boson (QB) cat hybrid states of the $\mathbb{Z}_n$ Rabi model. The proposed joint boson-qudit Wigner function, on the other hand, completely captures the structure of the qudit-boson cat states. Specifically, this Wigner function allows us to separate the qutrit-boson cat states from the boson cat states and mixed states.

In this paper, we mostly focus on the $n = 3$ case, i.e., the $\mathbb{Z}_3$ -symmetric Rabi model and cat states, since this simplifies the expressions and makes the key ideas more transparent. However, the results can be easily generalized to arbitrary $n > 2$. We place particular emphasis on the *two-mode* $\mathbb{Z}_3$ Rabi model, motivated by an optomechanical implementation proposed in Ref. [20]. In a companion paper [25], we further show that this two-mode model can be simulated by a superconducting circuit.

The rest of the paper is organized as follows. In Sec. II, we briefly discuss the general $\mathbb{Z}_n$ Rabi model, introduce the one-mode $\mathbb{Z}_3$ Rabi model, and analyze its low-energy behavior. In Sec. III, we carry out the corresponding analysis for the two-mode $\mathbb{Z}_3$ Rabi model. In Sec. IV, we introduce the joint qutrit-boson Wigner functions and use them to characterize the eigenstates obtained in the previous sections. Finally, Sec. V contains our conclusions. Technical details are deferred to Appendixes A–C.

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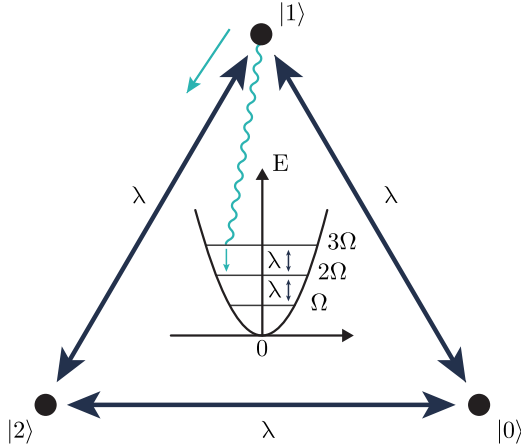


FIG. 1. Schematic diagram of the $\mathbb{Z}_3$ Rabi model [Eq. (6)]. The vertices of the outer triangle depict the qutrit states $|i\rangle$, $i = 0, 1, 2$. The levels inside the inner parabola illustrate the bosonic states $|n\rangle$ with energies $n\Omega$, $n = 0, 1, 2, 3, \dots$. The navy blue arrows show the action of the interaction term $\lambda(a^\dagger X + a X^\dagger)$ on the boson and qutrit, while the teal arrows illustrate a specific matrix element transferring an excitation from the boson mode to the qutrit.

II. ONE-MODE RABI MODEL

In this section, we first briefly discuss the general case of the $\mathbb{Z}_n$ Rabi model for the sake of completeness. Then, we focus on the special case of $\mathbb{Z}_3$ symmetry. We show that, in the regime of deep-strong coupling and weak magnetic field, the lowest eigenstates of the $\mathbb{Z}_3$ Rabi model correspond to the $\mathbb{Z}_3$ cat states. In order to demonstrate this, we employ a canonical transformation that allows for a transparent perturbative treatment in the relevant regime.

A. $\mathbb{Z}_n$ -symmetric case

The one-mode $\mathbb{Z}_n$ Rabi model describes the interaction between an n -level qutrit and a bosonic mode [18,19]. The Hamiltonian reads

$$H = \Omega a^\dagger a + \sum_{j=1}^{n-1} B_j Z^j - \lambda(a X^\dagger + a^\dagger X), \quad (1)$$

where a ($a^\dagger$) is a canonical boson annihilation (creation) operator, X and Z are the generalized Pauli matrices (commonly referred to as the shift and clock matrices, respectively), Ω is the bosonic mode frequency, λ is the boson-qutrit coupling strength, B_j are parameters that define the qutrit energy levels, and the integer n is larger than one. To ensure Hermiticity, we require $B_j = B_{n-j}^*$. The $n \times n$ clock and shift matrices X and Z obey the following algebraic relation:

$$Z^n = 1, \quad X^n = 1, \quad ZX = \omega XZ, \quad (2)$$

where $\omega = \exp[2\pi i/n]$ is the n th principal root of unity. The X and Z matrices are unitary, and they generalize the Pauli matrices σ_x and σ_z , correspondingly. However, they are not Hermitian. Hereafter, we call the purely qutrit term $\sum_j B_j Z^j$ a “magnetic” term, since its $\mathbb{Z}_2$ analog is the Zeeman term. For $n = 3$, we schematically illustrate the model in Fig. 1.

We also define two bases in the qutrit space: the computational basis diagonalizing the Z matrix $Z|j\rangle = \omega^j|j\rangle$, $j = 0, \dots, n-1$, and the Fourier basis diagonalizing the X matrix $X|\omega^k\rangle = \omega^{-k}|\omega^k\rangle$, $k = 0, \dots, n-1$. These two bases are related to each other by the discrete Fourier transform:

$$|\omega^k\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega^{kj} |j\rangle. \quad (3)$$

In the computational basis, the X and Z matrices have the following form:

$$X_{ij} = \delta_{(i-j-1 \bmod n)}, 0, \quad Z_{ij} = \delta_{ij} \omega^{j-1}. \quad (4)$$

The model has the $\mathbb{Z}_n$ symmetry, which counts the number of bosonic modes and qutrit excitations modulo n . One can think of the system as if there were an excitation hopping between bosonic and qutrit degrees of freedom. The explicit form of the $\mathbb{Z}_n$ symmetry generators is

$$\mathcal{P} = \exp\left[\frac{2\pi i}{n} a^\dagger a\right] Z. \quad (5)$$

More general $\mathbb{Z}_n$ -symmetric variants of Eq. (1) are possible. In particular, terms like $a^2(X^\dagger)^2$, $(a^\dagger)^2 X^2$ are also allowed since they conserve the total number of excitations modulo n . However, these terms make the system more complicated, and we do not consider them in this paper.

B. $\mathbb{Z}_3$ -symmetric case

For simplicity, from now on we restrict ourselves to the case of the $\mathbb{Z}_3$ Rabi model, although most of our results can be straightforwardly generalized to $\mathbb{Z}_n$ symmetry with arbitrary integer $n > 1$. The Hamiltonian (1) then reduces to

$$H_{R1} = \Omega a^\dagger a + B(e^{i\phi} Z + e^{-i\phi} Z^\dagger) - \lambda(a^\dagger X + a X^\dagger), \quad (6)$$

where we defined $B_1 = B e^{i\phi}$ and $B_2 = B e^{-i\phi}$, with B and ϕ being the absolute value and phase. The subscript “R1” on the Hamiltonian (6) emphasizes that it describes the one-mode $\mathbb{Z}_3$ Rabi model, distinguishing it from the two-mode $\mathbb{Z}_3$ Rabi model discussed below. In Fig. 1, we show the schematic diagram of the $\mathbb{Z}_3$ Rabi model. In the $\mathbb{Z}_3$ case, the X and Z matrices read

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (7)$$

Thus, the Hamiltonian (6) can be written as a 3×3 matrix:

$$H_{R1} = \begin{pmatrix} \Omega a^\dagger a + \epsilon_0 & -\lambda a^\dagger & -\lambda a \\ -\lambda a & \Omega a^\dagger a + \epsilon_1 & -\lambda a^\dagger \\ -\lambda a^\dagger & -\lambda a & \Omega a^\dagger a + \epsilon_2 \end{pmatrix}, \quad (8)$$

where $\epsilon_k = 2B \cos(\phi + 2\pi k/3)$, with $k = 0, 1, 2$. The $\mathbb{Z}_3$ symmetry generator follows directly from Eq. (5):

$$\mathcal{P}_{R1} = \exp\left[\frac{2\pi i}{3} a^\dagger a\right] Z. \quad (9)$$

We are interested in obtaining the ground state of the Hamiltonian (6). In the next subsection, we describe a canonical transformation that simplifies the perturbation theory in

the regime of deep-strong coupling $\lambda \gtrsim \Omega$ and weak magnetic field $\Omega \gg B$.

C. Canonical transformation

Several ways exist to demonstrate the cat-state structure of the ground state in the (ordinary) Rabi model, including a variational method [26] and the Schrieffer-Wolff transformation [27]. However, we find an approach based on a canonical transformation that decouples qubit degrees of freedom [28] to be the most straightforward. This approach can be extended to the $\mathbb{Z}_3$ Rabi model, as we demonstrate in this section. We introduce a canonical transformation that decouples qutrit degrees of freedom from the bosonic ones:

$$b = aX^\dagger, \quad b^\dagger = a^\dagger X. \quad (10)$$

Intuitively, the operator b annihilates a boson while simultaneously lowering the qutrit state. The newly introduced boson creation/annihilation operators satisfy the canonical commutation relations:

$$[b, b^\dagger] = X[a, a^\dagger]X^\dagger = 1. \quad (11)$$

However, they no longer commute with qutrit operators. Then, we eliminate the qutrit degrees of freedom altogether. First of all, the harmonic oscillator term and the interaction term can be easily expressed in terms of the operators:

$$\begin{aligned} \Omega a^\dagger a - \lambda(a^\dagger X + aX^\dagger) \\ = \Omega(b^\dagger - \lambda/\Omega)(b - \lambda/\Omega) - \lambda^2/\Omega, \end{aligned} \quad (12)$$

which is simply a shifted harmonic oscillator.

However, we are still left with the “magnetic” term $B(e^{i\phi}Z + e^{-i\phi}Z^\dagger)$. It can also be expressed in terms of the operators b and $b^\dagger$. To do so, we decompose the full Hilbert space into three subspaces with a fixed $\mathbb{Z}_3$ parity $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. Here, $\mathcal{H}_k$ is an eigenspace of $\mathcal{P}_{R1}$ [see Eq. (9)] with a fixed eigenvalue:

$$\mathcal{P}_{R1}|_{\mathcal{H}_k} = \omega^k. \quad (13)$$

Here, we used the vertical line to denote the restriction of the operator to the subspace, i.e., for an operator $\mathcal{O} : \mathcal{H} \rightarrow \mathcal{H}$ and a linear subspace $\mathcal{H}' \subset \mathcal{H}$, we define $\mathcal{O}|_{\mathcal{H}'}$ to be a part of the operator $\mathcal{O}$ mapping $\mathcal{H}'$ to $\mathcal{H}'$.

After substituting the definition of the $\mathbb{Z}_3$ symmetry generator $\mathcal{P}_{R1}$ into Eq. (13), we can express the qutrit clock matrix as

$$Z|_{\mathcal{H}_k} = \omega^k \exp\left(-\frac{2\pi i}{3}b^\dagger b\right)|_{\mathcal{H}_k}. \quad (14)$$

Note that the subspace index k only enters the scalar phase factor ω^k . Using the relation (14), one can now express the “magnetic” term in the Hamiltonian (6) using the boson ladder operators:

$$(e^{i\phi}Z + e^{-i\phi}Z^\dagger)|_{\mathcal{H}_k} = 2\cos\left[\frac{2\pi}{3}(b^\dagger b - k) - \phi\right]|_{\mathcal{H}_k}. \quad (15)$$

Finally, we can write the full $\mathbb{Z}_3$ Rabi model Hamiltonian in terms of the operators b and $b^\dagger$:

$$\begin{aligned} H_{R1}|_{\mathcal{H}_k} = & \left\{ \Omega(b^\dagger - \lambda/\Omega)(b - \lambda/\Omega) - \lambda^2/\Omega \right. \\ & \left. + 2B\cos\left[\frac{2\pi}{3}(b^\dagger b - k) - \phi\right] \right\}|_{\mathcal{H}_k}. \end{aligned} \quad (16)$$

It is now straightforward to diagonalize the transformed Hamiltonian using perturbation theory in the limit of deep-strong coupling and a weak magnetic field, as we demonstrate below.

D. Cat states in the $\mathbb{Z}_3$ one-mode Rabi model

After the canonical transformation (10), the Hamiltonian (16) is conveniently divided into the unperturbed part and the perturbation:

$$\begin{aligned} H_{R1}|_{\mathcal{H}_k} &= H_{R1}^{(0)}|_{\mathcal{H}_k} + V_{R1}(k)|_{\mathcal{H}_k}, \\ H_{R1}^{(0)} &= \Omega(b^\dagger - \lambda/\Omega)(b - \lambda/\Omega) - \lambda^2/\Omega, \\ V_{R1}(k) &= 2B\cos\left[\frac{2\pi}{3}(b^\dagger b - k) - \phi\right], \end{aligned} \quad (17)$$

where the perturbation $V_{R1}(k)$ depends on the $\mathbb{Z}_3$ parity k . As we have already mentioned, the unperturbed Hamiltonian $H_{R1}^{(0)}$ is just a shifted harmonic oscillator.

Its ground state is defined by the following requirement:

$$(b - \lambda/\Omega)|\psi_k^{\text{QB}}\rangle = 0, \quad (18)$$

where the superscript “QB” emphasizes that this is a qutrit-boson state and the subscript k indicates that the ground state belongs to the symmetry sector $\mathcal{H}_k$, i.e., $\mathcal{P}_{R1}|\psi_k^{\text{QB}}\rangle = \omega^k|\psi_k^{\text{QB}}\rangle$. As a result, the ground state $|\psi_k^{\text{QB}}\rangle$ is a coherent state of the boson b and $b^\dagger$ with a coherent parameter λ/Ω [29]:

$$|\psi_k^{\text{QB}}\rangle = |\lambda/\Omega\rangle_{\hat{b},k}. \quad (19)$$

In terms of the original degrees of freedom, the state $|\psi_k^{\text{QB}}\rangle$ turns out to be the cat state $\mathbb{Z}_3$:

$$\begin{aligned} |\psi_k^{\text{QB}}\rangle &= \frac{1}{\sqrt{3}}[|\lambda/\Omega\rangle_{\hat{a}}|\omega^0\rangle + \omega^k|\omega\lambda/\Omega\rangle_{\hat{a}}|\omega\rangle \\ &+ \omega^{2k}|\omega^2\lambda/\Omega\rangle_{\hat{a}}|\omega^2\rangle]. \end{aligned} \quad (20)$$

Here, we can again see a parallel with the ordinary $\mathbb{Z}_2$ Rabi model, whose ground states are the usual cat states [30,31].

Next, we use the perturbation V_{R1} to lift the degeneracy of the states $|\psi_k^{\text{QB}}\rangle$. Although the three states $|\psi_k^{\text{QB}}\rangle$ with $k = 0, 1, 2$ are degenerate eigenstates of the unperturbed Hamiltonian, they still constitute the natural degenerate-perturbation-theory basis: Each $|\psi_k^{\text{QB}}\rangle$ is an eigenstate of the $\mathbb{Z}_3$ symmetry operator $\mathcal{P}_{R1}$ with different eigenvalues. Since $[V_{R1}, \mathcal{P}_{R1}] = 0$, the perturbation cannot mix these states, making them the correct basis choice for evaluating the perturbative corrections of the first order. We take the limit $B \ll \Omega$ to make the perturbative correction to $|\psi_k^{\text{QB}}\rangle$ negligible. The first-order correction yields the following approximate

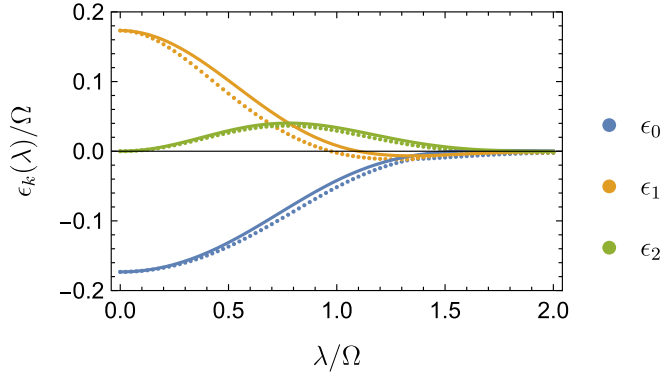


FIG. 2. Eigenvalues ϵ_k/Ω of the one-mode $\mathbb{Z}_3$ Rabi model Hamiltonian H_{R1} in Eq. (6) as functions of the qutrit-boson coupling λ/Ω . The dotted line represents the exact numerical diagonalization, while the solid line corresponds to the first-order perturbation theory (21). One can clearly see an excellent agreement between the numerical and the perturbative results. Parameters used: $B/\Omega = 0.1$ and $\phi = 7\pi/6$.

eigenenergies for the cat states:

$$\begin{aligned} \epsilon_k &= \langle \psi_k^{\text{QB}} | V_{R1}(k) | \psi_k^{\text{QB}} \rangle \\ &= 2Be^{-\frac{3}{2}(\lambda/\Omega)^2} \cos \left[\frac{2\pi k}{3} + \phi - \frac{\sqrt{3}}{2} \left(\frac{\lambda}{\Omega} \right)^2 \right]. \end{aligned} \quad (21)$$

The matrix elements between $|\psi_k^{\text{QB}}\rangle$ and $|\psi_l^{\text{QB}}\rangle$ with $k \neq l$ are zero due to the symmetry of $\mathbb{Z}_3$. The comparison of the perturbative result (21) with the numerical results obtained by the exact diagonalization of the Hamiltonian (16) is shown in Fig. 2, which demonstrates excellent agreement between the two. In contrast, the variational method used in Ref. [20] yields the correct eigenvalues only for $\lambda > 0.61$. With the help of Fig. 2, we can also clarify why the cat states in Eq. (20) emerge in the deep-strong-coupling regime $\lambda \gtrsim \Omega$. In the limit of $\lambda/\Omega \rightarrow \infty$, the low-energy spectrum approaches a threefold degenerate manifold, consistent with an incipient breaking of the $\mathbb{Z}_3$ symmetry. Because true spontaneous symmetry breaking occurs only in the thermodynamic limit [20,27], the exact eigenstates remain $\mathbb{Z}_3$ symmetric. Consequently, the eigenstates are catlike superpositions of the coherent-product states that break the symmetry $|\omega^k \lambda/\Omega\rangle_{\hat{a}} |\omega^k\rangle$, with $k = 0, 1, 2$. These three product states are mapped to each other by the $\mathbb{Z}_3$ symmetry operation and differ by the discrete phase of the corresponding order parameter (equivalently, by the phase of the coherent amplitude), so each individual component corresponds to a spontaneously broken symmetry, while their superposition restores the symmetry.

III. TWO-MODE RABI MODEL

In this section, we explore several variations of the two-mode $\mathbb{Z}_3$ Rabi model and summarize their key properties. Then, we restrict our attention to the one that is most relevant experimentally. We also show that the latter model exhibits $\mathbb{Z}_3$ cat states in the limit of deep-strong coupling $\lambda \gtrsim \Omega$ and a weak magnetic field $\Omega \gg B$.

A. Hamiltonian of the model

There are several possible variants of the two-mode $\mathbb{Z}_3$ Rabi model, each featuring a slightly different qutrit-boson interaction. Here, we consider a particular model, whose possible experimental realization has recently been proposed in an optomechanical system [20]. Furthermore, in our companion paper [25], we demonstrate that this model can be realized on a superconducting-circuit platform. The Hamiltonian for this two-mode $\mathbb{Z}_3$ Rabi model is

$$\begin{aligned} H_{R2} &= \Omega(a_1^\dagger a_1 + a_2^\dagger a_2) + B(e^{i\phi} Z + e^{-i\phi} Z^\dagger) \\ &\quad - \lambda(a_1 + a_2^\dagger)X - \lambda(a_1^\dagger + a_2)X^\dagger, \end{aligned} \quad (22)$$

where a_1 and a_2 are canonical boson annihilation operators, X is the shift operator $\mathbb{Z}_3$, and the parameters Ω , B , and λ are defined in Eqs. (1) and (6). The Hamiltonian subscript “R2” distinguishes the two-mode Rabi model Hamiltonian from the one-mode Rabi model Hamiltonian H_{R1} in Eq. (6). In Appendix A, we show that the model of Ref. [20] is equivalent to that of Eq. (22).

The model (22) is a two-mode extension of the one-mode $\mathbb{Z}_3$ Rabi model (6), with the qutrit-boson interaction term differing for the two bosonic modes. Specifically, the qutrit-boson interaction $(a_1 + a_2^\dagger)X + \text{H.c.}$ couples the first mode through its annihilation operator and the second mode through its creation operator. A related two-mode variant of the one-mode $\mathbb{Z}_3$ Rabi model H_{R1} in Eq. (6) with the interaction term $(a_1^\dagger + a_2^\dagger)X + \text{H.c.}$ is discussed in Appendix B. However, this two-mode variant turns out to be too similar to the one-mode Rabi model (6) because they have the same eigenvalues, albeit with different degeneracies.

The $\mathbb{Z}_3$ symmetry generator for the Hamiltonian (22) has the following form:

$$\mathcal{P}_{R2} = \exp \left[\frac{2\pi i}{3} L_3 \right] Z, \quad (23)$$

where $L_3 = a_1^\dagger a_1 - a_2^\dagger a_2$ is a generator of the symmetry $\text{SU}(2)$ (see Appendix C). Hence, the boson part of the $\mathbb{Z}_3$ symmetry in the two-mode Rabi model (22) is a residual part of the $\text{SU}(2)$ symmetry present in the two-dimensional quantum harmonic oscillator (2D QHO).

B. Cat states in the $\mathbb{Z}_3$ two-mode Rabi model

To find the spectrum of the two-mode $\mathbb{Z}_3$ Rabi model, we apply the same method we used for the one-mode case. To avoid repetition, we briefly present the derivation steps. We again introduce boson operators $b_1 = a_1 X$ and $b_2 = a_2 X^\dagger$. After substituting them into the Hamiltonian (22), the boson term and the interaction term for each bosonic mode are transformed into a shifted harmonic oscillator, similarly to Eq. (12),

$$\begin{aligned} &\sum_{i=1}^2 \Omega a_i^\dagger a_i - \lambda(a_1^\dagger X^\dagger + a_1 X) - \lambda(a_2^\dagger X + a_2 X^\dagger) \\ &= \sum_{i=1}^2 \Omega(b_i^\dagger - \lambda/\Omega)(b_i - \lambda/\Omega) - 2\lambda^2/\Omega. \end{aligned} \quad (24)$$

Then, to completely eliminate the qutrit operator in the Hamiltonian (22), we treat the different sectors $\mathbb{Z}_3$ separately. This allows us to express the qutrit clock operator Z through the boson operators b_i and $b_i^\dagger$ using the definition of the $\mathbb{Z}_3$ symmetry operator (23). Consequently, the Hamiltonian (22), rewritten in terms of the boson operators, is naturally divided into an unperturbed part and a perturbation:

$$H_{R2}|_{\mathcal{H}_k} = H_{R2}^{(0)}|_{\mathcal{H}_k} + V_{R2}(k)|_{\mathcal{H}_k}, \quad (25)$$

where

$$H_{R2}^{(0)} = \Omega \sum_{i=1}^2 (b_i^\dagger - \lambda/\Omega)(b_i - \lambda/\Omega) - 2\lambda^2/\Omega, \quad (26)$$

$$V_{R2}(k) = 2B \cos \left[\frac{2\pi}{3}(L_3 - k) - \phi \right], \quad (27)$$

with $\mathbb{Z}_3$ parity taking values $k = 0, 1, 2$.

As a result, we have two shifted QHOs with a perturbation depending on the $\mathbb{Z}_3$ parity. The ground states of the unperturbed Hamiltonian in each $\mathbb{Z}_3$ sector are similarly easy to find. They are coherent states of the boson operator $b_i|\psi_k^{Q2B}\rangle = (\lambda/\Omega)|\psi_k^{Q2B}\rangle$, with a fixed $\mathbb{Z}_3$ parity $\mathcal{P}_{R2}|\psi_k^{Q2B}\rangle = \omega^k|\psi_k^{Q2B}\rangle$. Or, in terms of the original boson and qutrit degrees of freedom, the ground states are

$$|\psi_k^{Q2B}\rangle = \frac{1}{\sqrt{3}} \sum_{l=0}^2 \omega^{lk} |\omega^{-l}\lambda/\Omega\rangle_{\hat{a}_1} |\omega^l\lambda/\Omega\rangle_{\hat{a}_2} |\omega^{-l}\rangle, \quad (28)$$

where the subscript “Q2B” indicates that this is a joint state of a qutrit and two bosonic modes. Although states $|\psi_k^{Q2B}\rangle$ have a cat-state structure for any coupling strength, it becomes physically meaningful only once the interaction reaches the deep-strong-coupling regime [32–34] $\lambda \gtrsim \Omega$, where the individual coherent-state components in Eq. (28) become well separated in phase space. This is illustrated by the qubit-boson Wigner function introduced in Sec. IV. In particular, Fig. 7 shows how the coherent-state components move apart as the coupling constant λ increases.

Similarly to the one-mode $\mathbb{Z}_3$ Rabi model, the cat states $|\psi_k^{Q2B}\rangle$ serve as a unique basis in the ground state of the shifted quantum harmonic oscillator with a specific $\mathbb{Z}_3$ parity: $\mathcal{P}_{R2}|\psi_k^{Q2B}\rangle = \omega^k|\psi_k^{Q2B}\rangle$. For later convenience, we denote the subspace spanned by $|\psi_k^{Q2B}\rangle$ as $\mathcal{R} = \text{Span}\{|\psi_k^{Q2B}\rangle\}$. Because the perturbation (27) commutes with $\mathcal{P}_{R2}$ (23), the symmetry prevents mixing between sectors with different k , making $|\psi_k^{Q2B}\rangle$ the natural basis for the degenerate perturbation theory. Similarly as in Sec. IID, we consider the limit $B \ll \Omega$ to make the eigenstate corrections negligible. Using first-order perturbation theory, we find the energy corrections:

$$\begin{aligned} \epsilon_k &= \langle \psi_k^{Q2B} | V_{R2}(k) | \psi_k^{Q2B} \rangle \\ &= 2B e^{-3(\lambda/\Omega)^2} \cos \left[\frac{2\pi k}{3} + \phi \right]. \end{aligned} \quad (29)$$

The matrix elements between $|\psi_k^{Q2B}\rangle$ and $|\psi_l^{Q2B}\rangle$ with $k \neq l$ are zero due to the symmetry of $\mathbb{Z}_3$. Note that the cosine phase does not depend on the coherent-state parameter λ/Ω here. The comparison of the perturbative result (21) with the numerical results obtained by the exact diagonalization of the Hamiltonian (16) is shown in Fig. 3, which demonstrates excellent

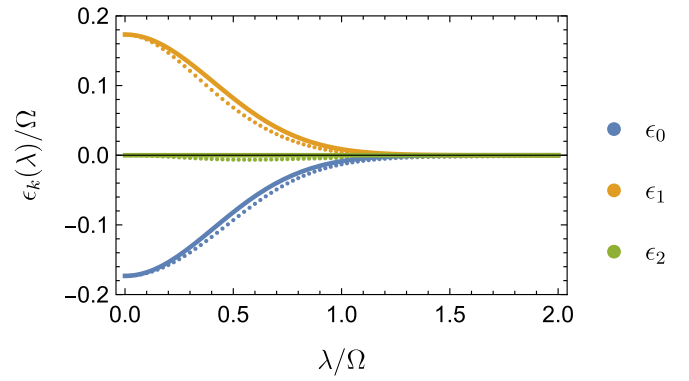


FIG. 3. Eigenvalues ϵ_k/Ω of the two-mode $\mathbb{Z}_3$ Rabi model H_{R2} in Eq. (22) as functions of coupling λ/Ω . The dotted line represents the exact numerical diagonalization, while the solid line corresponds to the first-order perturbation theory (29). The figure shows that the numerical and the perturbation theory results are very close. Parameters used: $B/\Omega = 0.1$ and $\phi = 7\pi/6$.

agreement between the two. Similarly to the one-mode Rabi model (cf. Fig. 2 and the discussion thereof), the eigenstates become threefold degenerate in the limit of $\lambda/\Omega \rightarrow \infty$ (see Fig. 3). Since the system is zero dimensional (finite size), true spontaneous symmetry breaking does not occur, and the exact eigenstates remain $\mathbb{Z}_3$ symmetry eigenstates; consequently, the eigenvectors are catlike superpositions of the three coherent-product states breaking coherent-product states. The residual splitting becomes exponentially small at strong coupling and disappears only in the strict $\lambda/\Omega \rightarrow \infty$ limit.

One of the reasons why cat states attract significant attention is that they are resistant to phase noise [1]. In qubit systems, this protection can be extended to bit-flip errors by switching from the usual two-component cats to four-component cats. The same idea carries over to the $\mathbb{Z}_3$ setting: The three-component cat states already suppress phase noise, and embedding the logical qutrit in a six-component cat (as produced, for example, by a $\mathbb{Z}_6$ Rabi model) should also guard against flip errors.

In this section, we showed that the two-mode $\mathbb{Z}_3$ model also has $\mathbb{Z}_3$ cat states as its lowest eigenstates in the deep-strong-coupling regime. We now proceed to investigate the properties of these cat states.

IV. WIGNER FUNCTION

The Wigner function [35–37] has proven valuable for analyzing cat states in bosonic systems. At the same time, there is extensive literature on qudit Wigner functions. The qudit Wigner function based on the finite Heisenberg group is distinct from the $SU(2)$ -based coherent-state construction. The finite Heisenberg group is particularly natural in our context because it already contains the $\mathbb{Z}_n$ symmetry featured throughout this paper. Consequently, to make the structure of the $\mathbb{Z}_3$ -symmetric cat states clearer, we introduce a Wigner function for a system of a qutrit and Q2B.

A. Boson Wigner function

As a reminder, let us briefly review the Wigner function for a single bosonic mode. In the literature on Wigner functions,

this system is usually referred to as a continuous-variable (CV) system, and one usually speaks of the CV Wigner function. Here, we use the term boson Wigner function to stay consistent with the rest of the paper.

For a bosonic mode, we introduce exponentials of the coordinate and momentum operators $e^{ip\hat{q}}$ and $e^{iq\hat{p}}$ (we use hats to distinguish between parameters q, p and operators $\hat{q}, \hat{p}$ if it is not clear from the context). From the canonical commutation relation, it follows that

$$e^{ip\hat{x}}e^{iq\hat{p}} = e^{iqp}e^{iq\hat{p}}e^{ip\hat{x}}. \quad (30)$$

These two operators can be interpreted as the coordinate and momentum translation operators, respectively. Combining them yields the displacement operator $D(q, p) = e^{-iqp/2}e^{ip\hat{x}}e^{-iq\hat{p}} = e^{z\hat{a}^\dagger - z^*\hat{a}}$, where we introduce a complex coordinate $z = (q + ip)/\sqrt{2}$. In what follows, we predominantly use complex coordinates z to make the notation more compact. Using the complex coordinate z also simplifies the following statement: The displacement operator $D(z)$ upon acting on the vacuum creates a coherent state with a coherent state parameter equal to z , i.e., $D(z)|0\rangle = |z\rangle$.

For a boson system with a density matrix ρ , the Wigner function is defined through the displacement operator as

$$W_B(\rho; z) = \frac{1}{\pi} \text{Tr}[\rho D(2z) \Pi], \quad (31)$$

where $\Pi = (-1)^{\hat{a}^\dagger \hat{a}}$ is the inversion operator. In the coordinate basis, it acts as $\Pi|x\rangle = |-x\rangle$.

Since $W_B(\rho; z)$ can take negative values, it is called a quasiprobability distribution. While being distinct from the ordinary probability distribution, it retains some of its key properties, i.e., the quasiprobability distribution is normalized, and its marginal distributions reproduce the correct quantum probabilities,

$$\int d^2z W_B(\rho; z) = 1, \quad (32)$$

$$\int dq W_B(\rho; (q + ip)/\sqrt{2}) = \langle p|\rho|p\rangle, \quad (33)$$

$$\int dp W_B(\rho; (q + ip)/\sqrt{2}) = \langle q|\rho|q\rangle, \quad (34)$$

where we reverted to the coordinates (q, p) for convenience.

Because cat states are central to this work, we recall that the Wigner functions of the purely boson cat states $|\varphi_\pm\rangle = (|+\alpha\rangle \pm |-\alpha\rangle)/\sqrt{2}$ are

$$W_B(|\varphi_\pm\rangle\langle\varphi_\pm|; z) = \frac{1}{\pi\mathcal{N}_2} [e^{-2|\alpha-z|^2} + e^{-2|\alpha+z|^2} \pm 2e^{-2|z|^2} \cos(4\text{Im}\{\alpha z^*\})], \quad (35)$$

where $\mathcal{N}_2$ is a normalization factor and α is a coherent-state parameter of cat states $|\varphi_\pm\rangle$. We do not provide an exact expression for the normalization factor $\mathcal{N}_2$, since for our purposes it is not important. The resulting Wigner function is depicted in Fig. 4(a). It consists of two purely exponential terms, which are the Wigner functions of the coherent states $|\pm\alpha\rangle$, and a term with a cosine, which describes the interference between $|+\alpha\rangle$ and $|-\alpha\rangle$.

Taking into account that our final goal is to calculate the Q2B Wigner function for the eigenstates of the two-mode $\mathbb{Z}_3$

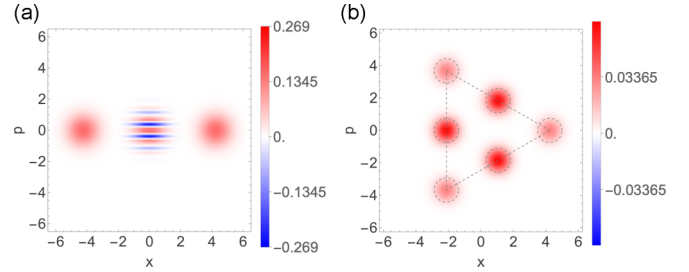


FIG. 4. Wigner function of the one-boson $\mathbb{Z}_2$ cat state $W_B(|\varphi_\pm\rangle\langle\varphi_\pm|; z)$ [Eq. (35)] in panel (a), and Wigner function of the two-boson $\mathbb{Z}_3$ cat state $W_{2B}(|\psi_0^{2B}\rangle\langle\psi_0^{2B}|; z_1, z_2)$ [Eq. (37)] in panel (b). The coherent parameter is $\alpha = 3$. For the two-boson $\mathbb{Z}_3$ cat state (a four-dimensional Wigner function), we show a two-dimensional section along the plane $(z_1 = w/\sqrt{2}, z_2 = w^*/\sqrt{2})$, with $w \in \mathbb{C}$.

Rabi model, which contains one qutrit and two-boson (2B) degrees of freedom, let us first look at the Wigner function for a 2B $\mathbb{Z}_3$ cat state:

$$|\psi_k^{2B}\rangle = \frac{1}{\sqrt{\mathcal{N}_3}} \sum_{j=0}^2 \omega^{jk} |\omega^{-j}\alpha\rangle |\omega^j\alpha\rangle. \quad (36)$$

The superscript “2B” emphasizes that this is a cat state of two bosons. We once again omit an explicit expression for the normalization factor $\mathcal{N}_3$, because it is not important. The 2B Wigner function is a simple generalization of the one-boson Wigner function: $W(\rho; z_1, z_2) = \frac{1}{\pi^2} \text{Tr}[\rho D_1(z_1) D_2(z_2) \Pi]$, where $D_i(z_i)$ acts only on the i th boson, while Π acts as an inversion operator on both bosons. Applying this definition to $|\psi_k^{2B}\rangle$, we obtain the following:

$$\begin{aligned} W_{2B}(|\psi_k^{2B}\rangle\langle\psi_k^{2B}|; z_1, z_2) &= \frac{1}{\mathcal{N}_3} \sum_{i,j=0}^2 \omega^{k(j-i)} W_B(|\omega^{-i}\alpha\rangle |\omega^i\alpha\rangle \langle\omega^{-j}\alpha| \langle\omega^j\alpha|; z_1, z_2) \\ &= \frac{1}{\pi^2 \mathcal{N}_3} \sum_{b=0}^2 e^{-2|\alpha - \omega^b z_1|^2 - 2|\alpha - \omega^{-b} z_2|^2} \\ &\quad + \frac{1}{\pi^2 \mathcal{N}_3} \sum_{b=0}^2 e^{-|\alpha + 2\omega^b z_1|^2/2 - |\alpha + 2\omega^{-b} z_2|^2/2} \\ &\quad \times 2 \cos[2\sqrt{3} \text{Re}(\alpha[\omega^b z_1^* - \omega^{-b} z_2^*]) + 2\pi k/3], \end{aligned} \quad (37)$$

where we introduced a summation index $b = -i - j \pmod{3}$. In the last expression, the first sum corresponds to the case $i = j$. These terms describe the Wigner functions of the three coherent states $|\omega^{-i}\alpha\rangle |\omega^i\alpha\rangle$, while the second sum contains interference terms between the different coherent states $|\omega^{-i}\alpha\rangle |\omega^i\alpha\rangle$ and $|\omega^{-j}\alpha\rangle |\omega^j\alpha\rangle$, with $i \neq j$. The latter is a counterpart of the fringe-pattern term for the ordinary cat state (35). All six terms in Eq. (37) are four-dimensional Gaussians, whose centers all lie on the plane $(z_1 = w\alpha/(\sqrt{2}|\alpha|), z_2 = w^*\alpha/(\sqrt{2}|\alpha|))$, with $w \in \mathbb{C}$ and form a triangle [see Fig. 4(b)]. The dashed triangle highlights the locations of the coherent and interference blobs. We notice that for $k = 0$ the interference blob is twice as bright as the coherent term. For illustration, we refer to Fig. 5,

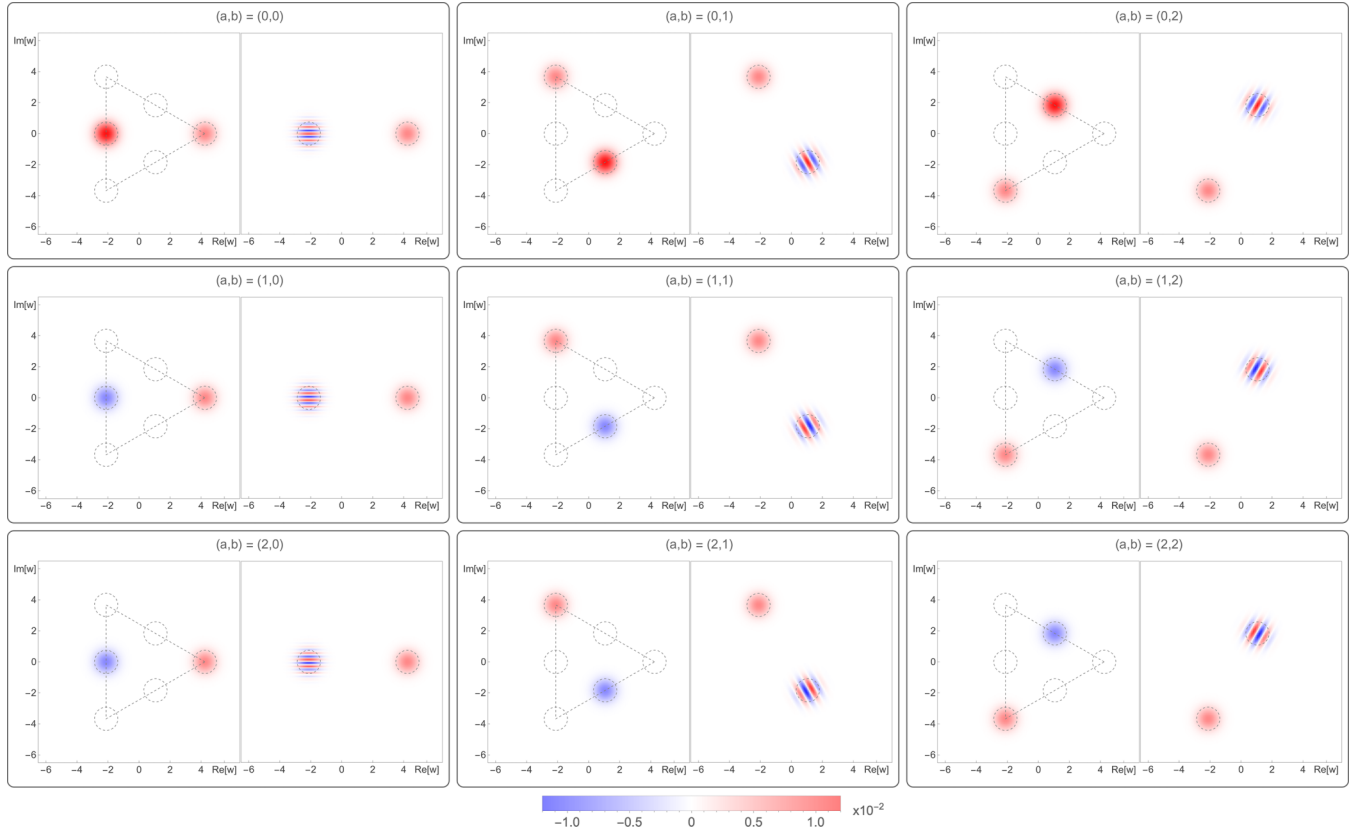


FIG. 5. Qutrit-boson Wigner function of the qutrit-boson cat state $W_{Q2B}(|\psi_k^{Q2B}\rangle \langle \psi_k^{Q2B}|; z_1, z_2, a, b)$ [given by Eq. (46)] with the coherent-state parameter $\alpha = 3$ (here, $k = 0$). The 3×3 grid enumerates the two discrete qutrit variables (a, b) . For each qutrit variable value (a, b) , we show two different two-dimensional sections of the four-dimensional bosonic phase space $(z_1, z_2) \in \mathbb{C}^2$. The left plot in each pair uses $(z_1 = w/\sqrt{2}, z_2 = w^*/\sqrt{2})$, $w \in \mathbb{C}$; the right plot uses a b -dependent section chosen to make the interference fringes visible.

which shows the joint-qubit boson Wigner function along the two planes. The Wigner function section along the transverse section plane explicitly shows the interference terms' oscillations. For the purely boson $\mathbb{Z}_3$ cat states (37), the interference terms have the same behavior.

The same recipe is sometimes applied to hybrid qudit-boson (or qubit-boson) cat states. One first takes the partial trace over the finite-dimensional subsystem and then plots the boson Wigner function of the reduced state. However, this procedure discards all information about the qudit-boson entanglement. We therefore advocate using the full qudit-boson Wigner function instead.

B. Qutrit Wigner function

We now explain how the Wigner functions can be applied to qutrit systems. The qutrit shift (X) and clock (Z) operators are analogous to the boson translation operators $e^{ip\hat{q}}$ and $e^{-iq\hat{p}}$, respectively. Indeed, the boson operators satisfy the commutation relation (30), and the qutrit operators satisfy the same algebra $ZX = \omega XZ$. Consequently, qutrit displacement operators can be introduced in a manner similar to the boson Wigner function: $D_Q(a, b) = \omega^{-ab/2} Z^b X^a$. Therefore, the qutrit Wigner functions can be defined as

$$W_Q(\rho; q, p) = \frac{1}{3} \text{Tr}[\rho D_Q(2q, 2p) \Pi_Q], \quad (38)$$

where ρ is a qutrit density matrix and Π_Q is the qutrit inversion operator. Concretely, on the computational basis, we define

$$\begin{aligned} \Pi_Q &= \sum_{j=0}^2 |j\rangle \langle -j| \\ &= |0\rangle \langle 0| + |2\rangle \langle 1| + |1\rangle \langle 2|, \end{aligned} \quad (39)$$

where the index $-j$ is understood modulo 3 (so that $|-1\rangle \equiv |2\rangle$ and $|-2\rangle \equiv |1\rangle$). This is an analog of the boson inversion operator Π_B . If we interpret the qutrit states $|j\rangle$, $j = 0, 1, 2$ as coordinate states (as opposed to momentum states), then Π_Q switches $|1\rangle$ and $|2\rangle$ while keeping $|0\rangle$ intact. This operation can be interpreted as a discrete reflection around the coordinate origin $|0\rangle$. This operator is Hermitian and unitary $\Pi_Q^\dagger = \Pi_Q$ and $\Pi_Q^2 = \text{Id}$ and should not be confused with the symmetry generator $\mathbb{Z}_3 Z$ (which has order 3). Here, q and p are discrete phase-space coordinates that take values 0, 1, 2, which correspond to the eigenvalues of the operators Z and X , respectively.

The qudit Wigner functions have been extensively studied [38–42]. In particular, the definition of the Wigner function for even- and odd-dimensional qudits differs significantly, although a unifying framework has recently been proposed [43]. For our purposes, such generality is unnecessary, and we restrict ourselves only to the qutrit case. However, all our

results can be easily generalized to arbitrary odd-dimensional qudits. While a similar construction is also possible for the even-dimensional case, it is considerably more involved.

These Wigner functions share the basic properties of their boson counterparts. They are normalized and their marginal distributions reproduce the physical probabilities,

$$\sum_{q,p=0}^2 W_Q(\rho; q, p) = 1, \quad (40)$$

$$\sum_{q=0}^2 W_Q(\rho; q, p) = \langle \omega^p | \rho | \omega^p \rangle, \quad (41)$$

$$\sum_{p=0}^2 W_Q(\rho; q, p) = \langle q | \rho | q \rangle. \quad (42)$$

As a simple illustration, the Wigner functions of the computational basis states (i.e., the eigenstates of Z) are given by $W_Q(|i\rangle\langle i|; q, p) = (1/3) \delta_{q,i}$. Similarly, for the eigenstates of X , we have $W_Q(|\omega^j\rangle\langle \omega^j|; q, p) = (1/3) \delta_{p,j}$. Thus, we see that the qutrit Wigner function yields the expected quasiprobability distributions.

C. Qutrit-boson Wigner function

Finally, we have all necessary ingredients to discuss a joint QB Wigner function. While its construction is straightforward (it combines the qudit and boson Wigner functions), it is useful because it keeps discrete and continuous degrees of freedom on equal footing. In particular, it allows one to distinguish entangled hybrid cat states from (1) classical mixtures and (2) product states with identical bosonic reduced density matrices, a distinction that is generally lost if one first traces out the qudit and plots only a bosonic Wigner function. Moreover, it provides a compact phase-space diagnostic of coherent and interference contributions while retaining the accompanying qudit information, making it a convenient visualization tool for hybrid nonclassicality and qudit-boson entanglement. In experimental platforms where bosonic Wigner functions are accessed via displaced-inversion-operator measurements (e.g., circuit QED), the joint QB Wigner function can, in principle, be reconstructed from expectation values of a displaced boson and qutrit inversion operator. In this subsection, we first calculate the QB Wigner function for the Q2B cat state. Then, we present the numerical calculations of the QB Wigner functions for the two-mode $\mathbb{Z}_3$ Rabi model, which clearly demonstrate the formation of Q2B cat states in the regime of deep-strong coupling.

1. Analytical derivation in the deep-strong-coupling regime

Since we are now dealing with both qutrit and boson degrees of freedom, it is convenient to define a joint displacement operator

$$D_{Q2B}(z_1, z_2, a, b) = D_{B1}(z_1) D_{B2}(z_2) D_Q(a, b), \quad (43)$$

where D_{Bi} , with $i = 1, 2$ is a displacement operator for the i th bosonic mode and D_Q is a displacement operator for the qutrit. Similarly, one can introduce a joint inversion operator

$$\Pi_{Q2B} = \Pi_{B1} \Pi_{B2} \Pi_Q, \quad (44)$$

where Π_{Bi} , with $i = 1, 2$ is the inversion operator for the i th bosonic mode and Π_Q is the qutrit inversion operator. Then, the Q2B Wigner function of a qutrit-boson density matrix ρ_{Q2B} can be defined in the standard way

$$W_{Q2B}(\rho_{Q2B}; z_1, z_2, a, b) = \frac{1}{3\pi} \text{Tr}[\rho_{Q2B} D_{Q2B}(2z_1, 2z_2, 2a, 2b) \Pi_{Q2B}]. \quad (45)$$

Again, Eq. (45) suggests that the Q2B Wigner function can be measured analogously to the boson Wigner function $W_B(\rho; z)$ [44]. Specifically, to measure the Q2B Wigner function one first applies the qutrit-boson displacement operator $D_{Q2B}(z_1, z_2, a, b)$ to the state ρ . One then measures the joint inversion operator Π_{Q2B} , which corresponds to the independent measurement of the boson inversion operators Π_{B1} and Π_{B2} and the qutrit inversion operator Π_Q . The Π_{Q2B} measurement is repeated several times for each set of values (z_1, z_2, a, b) to obtain its expectation value. This procedure outputs the value of the Wigner function $W_{Q2B}(\rho_{Q2B}; z_1, z_2, a, b)$. Since the Wigner function fully characterizes the quantum state, this procedure is a variation of quantum tomography.

In the deep-strong-coupling regime, the eigenstates of the $\mathbb{Z}_3$ Rabi model are the $\mathbb{Z}_3$ cat states given by Eq. (28). Substituting them into the Q2B Wigner function (45) yields

$$\begin{aligned} W_{Q2B}(|\psi_k^{Q2B}\rangle\langle \psi_k^{Q2B}|; z_1, z_2, a, b) &= \frac{1}{9\pi^2} \{ e^{-2|\alpha - \omega^{-b} z_1|^2 - 2|\alpha - \omega^b z_2|^2} \\ &\quad + e^{-|\alpha + 2\omega^{-b} z_1|^2/2 - |\alpha + 2\omega^b z_2|^2/2} \\ &\quad \times 2 \cos[2\sqrt{3} \text{Re}(\alpha[\omega^b z_1^* - \omega^{-b} z_2^*]) + 2\pi(a+k)/3] \}, \end{aligned} \quad (46)$$

where $\alpha = \lambda/\Omega$ is a coherent state parameter. The Q2B Wigner function (46) is shown in Fig. 5.

By comparing Eqs. (37) and (46), we see that the Wigner function of the 2B cat states (37) [see Fig. 4(b)] has six terms, whereas the Wigner function of the Q2B cat states (46) (see Fig. 5) has only two terms. It is to be expected because in the Q2B $\mathbb{Z}_3$ cat state the bosonic modes are entangled with the qutrit. As a result, the six terms of the 2B $\mathbb{Z}_3$ cat state are spread over different values of the qutrit variable b . Namely, the plots corresponding to $(a = 0, b = 0)$ consist of the coherent state Wigner function $|\alpha\rangle\langle \alpha|$ and the Wigner function of the interference term between the states $|\omega\alpha\rangle\langle \omega^2\alpha|$ and $|\omega^2\alpha\rangle\langle \omega\alpha|$. In a similar manner for other values of b , the Wigner function $W_{Q2B}(\rho_{Q2B}; z_1, z_2, a = 0, b)$ consists of a coherent state term $|\omega^{2b}\alpha\rangle$, located at the corners of the triangle in Fig. 5 and an interference term between the remaining two coherent states, located at the midpoints of the triangle's sides. We also remark that for $a + k = 0$ (the first row of Fig. 5) the interference-term blob amplitude is twice the coherent-term one. As a result, if we sum the Q2B Wigner function (46) over b while keeping $a = 0$, we will obtain the result identical to Eq. (37).

This section simultaneously shows both blobs present in the (a, b) part of the qutrit-boson Wigner function. The dashed circles schematically show the locations of the corresponding two-boson cat-state blobs in the same section. The

interference terms between different coherent states correspond to the blobs in the middle of the triangle's sides.

Next, we discuss the Q2B Wigner function plot of the QB cat state (46) shown in Fig. 5 more thoroughly. First of all, since the Q2B Wigner function depends on the discrete variables $a, b \in \{0, 1, 2\}$, Fig. 5 shows nine pairs of plots labeled by (a, b) instead of a single one. The left-hand plot in a pair corresponding to a specific (a, b) values shows the Q2B Wigner function (46) along the two-dimensional section defined by $(z_1 = w/\sqrt{2}, z_2 = w^*/\sqrt{2})$, $w \in \mathbb{C}$. In other words, it is a plot of $W_{\text{Q2B}}(|\psi_k^{\text{Q2B}}\rangle\langle\psi_k^{\text{Q2B}}|; w/\sqrt{2}, w^*/\sqrt{2}, a, b)$ as a function of w . Because all the exponents in Eq. (46) are centered on this plane, the left plots have the same section for all (a, b) values. The dashed circles schematically show the locations of the corresponding two-boson cat-state blobs in the same section. The interference terms between different coherent states correspond to the blobs in the middle of the triangle's sides. Unfortunately, the fringes of the interference term are not visible along this plane because the two-bosons' oscillations perfectly compensate each other. On the other hand, the section plane of the right-hand plot explicitly depends on the value of the qutrit variable b . Specifically, the section plane of the (a, b) right-hand plot is

$$\left(\omega^b \left\{ \cos\left(\frac{\pi}{4} + \frac{2\pi b}{3}\right) \text{Re}(w) + \sin\left(\frac{\pi}{4} + \frac{2\pi b}{3}\right) \text{Im}(w) \right\} \right. \\ \left. \omega^{-b} \left\{ \cos\left(\frac{\pi}{4} - \frac{2\pi b}{3}\right) \text{Re}(w) - \sin\left(\frac{\pi}{4} - \frac{2\pi b}{3}\right) \text{Im}(w) \right\} \right), \quad (47)$$

with $w \in \mathbb{C}$. This plane is chosen to show the fringes of the interference term since they are not visible on the left plot.

Both the qutrit variable a and the Q2B cat state's number k enter the Wigner function (46) together through the cosine phase as $2\pi(a+k)/3$. Consequently, the Q2B Wigner function plots for the cat states $|\psi_k^{\text{Q2B}}\rangle$ with different $k = 0, 1, 2$ differ only by a cyclic permutation of rows in Fig. 5. Moreover, as changing the value of a with fixed k only affects the phase of the fringe oscillation, inspecting the $a = 0$ row of the QB Wigner function suffices to verify the Q2B cat structure. The $a = 0$ Q2B Wigner function of the cat state (46) already shows that the qutrit and two bosons are entangled with each other: The six blobs are evenly distributed among the three plot pairs, and the right-hand plots clearly display the fringe oscillations. The $a \neq 0$ plots show the same picture, only with a different phase of the fringe oscillation. Therefore, in Figs. 6 and 7 we plot only the three panels with $(a = 0, b = 0, 1, 2)$.

2. Visualizing entanglement in hybrid qutrit-boson systems

One of the common applications of the ordinary Wigner functions is to distinguish between cat states and maximally mixed states. The Q2B Wigner function generalizes this to the hybrid Q2B system. First, consider the ordinary boson system: The cat state ρ_{cat} and the maximally mixed state ρ_{mix} are given by

$$\rho_{\text{cat}} = \frac{1}{\mathcal{N}_2} \left(|+\alpha\rangle\langle+\alpha| + |-\alpha\rangle\langle-\alpha| \right) \left(\langle+\alpha| \pm \langle-\alpha| \right), \quad (48)$$

$$\rho_{\text{mix}} = \frac{1}{2} (|+\alpha\rangle\langle+\alpha| + |-\alpha\rangle\langle-\alpha|). \quad (49)$$

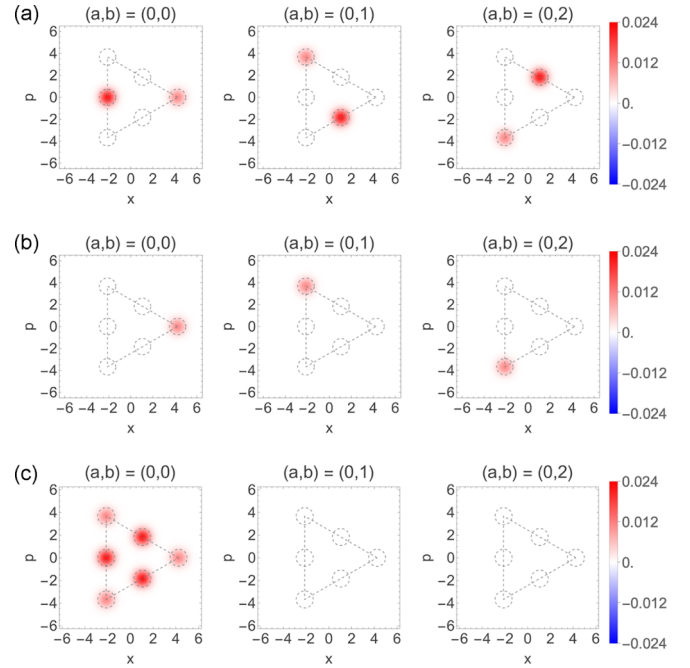


FIG. 6. Comparison of the Q2B Wigner functions W_{Q2B} [given by Eq. (45)] for three states: (a) Q2B cat state $\rho_{\text{Q2B cat}}$ with $k = 0$ defined in Eq. (50), (b) maximally mixed state ρ_{mix} in Eq. (51), and (c) 2B cat state unentangled with the qutrit $\rho_{2\text{B cat}}$ in Eq. (52). The coherent-state parameter is chosen to be the same for all three states, $\alpha = 3$. For each state the three plots $(a = 0, b = 0, 1, 2)$ are shown. With the two-boson-mode Wigner function being four dimensional, we show its section along the two-dimensional plane $(z_1 = w/\sqrt{2}, z_2 = w^*/\sqrt{2})$, with $w \in \mathbb{C}$, since all nonzero regions are centered on this plane. The dashed triangle schematically shows the Wigner function of the 2B cat state (37) to enhance the plot's readability.

The Wigner function can easily discern between the two. If the fringe pattern is present, then it is a cat state. Otherwise, it is a maximally mixed one.

Conceptually, for the Q2B systems the situation is very similar. However, the number of different options is larger. The Q2B Wigner function (46) allows us to distinguish between all possible cases. Specifically, the three plots with $(a = 0, b = 0, 1, 2)$ look drastically different for a Q2B cat state, a maximally mixed state, and a 2B cat state unentangled with the qutrit as follows.

- (1) The Q2B Wigner function plots of the *Q2B cat state*,

$$\rho_{\text{Q2B cat}} = |\psi_0^{\text{Q2B}}\rangle\langle\psi_0^{\text{Q2B}}|, \quad (50)$$

are shown in Fig. 6(a). Every plot has two blobs: one in the triangle's corner, i.e., the coherent state Wigner function, and one in the midpoint of the opposite side, i.e., the Wigner function of the interference term.

- (2) The Q2B Wigner function plots of the *maximally mixed state*,

$$\rho_{\text{mix}} = \frac{1}{3} \sum_{i=0}^2 |\omega^{-i}\alpha\rangle\langle\omega^i\alpha| \langle\omega^i\alpha| \langle\omega^{-i}\alpha|, \quad (51)$$

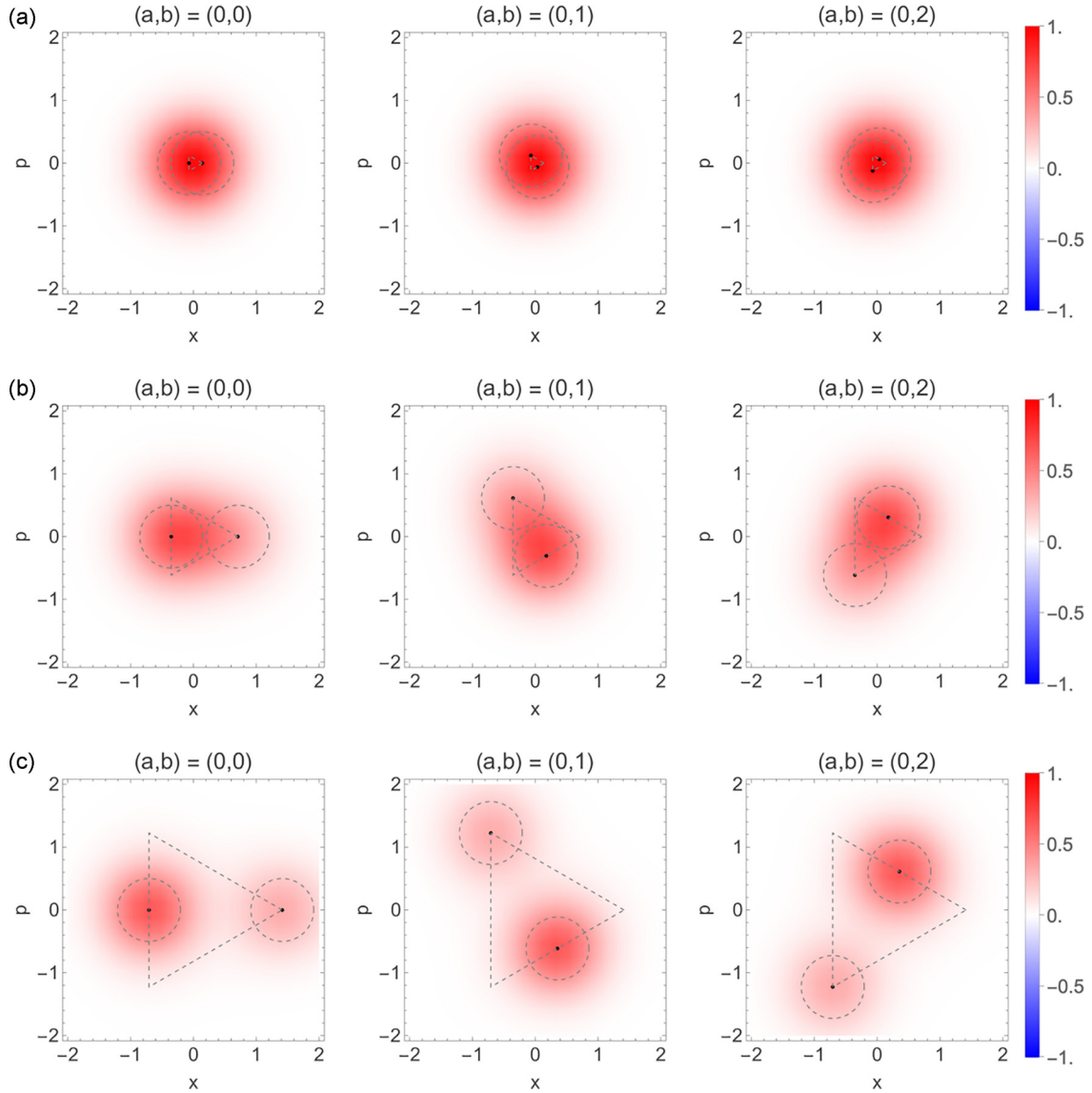


FIG. 7. Numerically calculated qutrit-boson Wigner function for the ground state of the two-mode $\mathbb{Z}_3$ Rabi model H_{R2} defined in Eq. (22) with coupling constants: (a) $\lambda/\Omega = 0.1$, $B/\Omega = 0.1$, (b) $\lambda/\Omega = 0.5$, $B/\Omega = 0.1$, and (c) $\lambda/\Omega = 1$, $B/\Omega = 0.1$. To save space, we plot only the first row ($a = 0$) of the 3×3 grid of the qutrit-boson Wigner function for each value of the coupling constant λ/Ω . The four-dimensional two-boson Wigner functions are plotted along the two-dimensional plane ($z_1 = w$, $z_2 = w^*$), $w \in \mathbb{C}$. In numerical calculations, the boson Hilbert spaces are truncated at $|n = 50\rangle$ in the occupational basis.

are shown in Fig. 6(b). Every plot has a single blob in the triangle's corner, i.e., the Wigner function for the coherent state. The blobs corresponding to the interference terms are absent since ρ_{mix} is a classical mixture of the three coherent states.

(3) The Q2B Wigner function plots of the *2B cat state* (36) *unentangled with the qutrit*,

$$\rho_{2B \text{ cat}} = |\psi_k^{2B}\rangle\langle\omega^0| \langle\psi_k^{2B}| \langle\omega^0|, \quad (52)$$

are shown in Fig. 6(c). Here, the ($a = 0, b = 0$) plot shows the Wigner function of the 2B cat state [identical to the one in Fig. 4(b)], while the rest of the plots are empty. It shows that the bosonic modes and the qutrit are not entangled.

As one can see, the Q2B Wigner function clearly distinguishes between the three states. The Q2B Wigner functions can also be applied to detect entanglement in the hybrid

systems. Since it can be experimentally measured similarly to the boson Wigner function, the Q2B Wigner function can be applied in actual physical devices. Consequently, we see that the W2B Wigner function is a convenient tool to study the hybrid qudit-boson system akin to the ordinary Wigner function in purely boson systems.

3. Numerical calculation outside of the deep-strong-coupling regime

Now that we have introduced and discussed the Wigner functions for the hybrid Q2B system, we calculate them numerically for the eigenstates of the two-mode $\mathbb{Z}_3$ Rabi model outside of the deep-strong-coupling regime, where analytical results are not available. Our results clearly demonstrate how the $\mathbb{Z}_3$ cat states emerge on approach into the deep-strong-

coupling regime ($\lambda \gtrsim \Omega$ and $\Omega \gg B$). In Fig. 7, we plot the first row ($a = 0$) of the Q2B Wigner function for three different coupling constants $\lambda/\Omega = 0.1, 0.5, 1$. With accordance to Eq. (46), in case of $a + k = 0$, the interference-term blob is twice as bright as the coherent-term blob. Similarly to the behavior of the ordinary cat states, the blobs move apart when the coherent parameter $\alpha = \lambda/\Omega$ is increased. At $\lambda/\Omega = 1$, they become well separated. In Fig. 7, once again we demonstrate what we have already proved analytically in Sec. III: The three lowest eigenstates of the two-mode $\mathbb{Z}_3$ Rabi model become the Q2B $\mathbb{Z}_3$ cat states in the limit $\lambda \gtrsim \Omega$ and $\Omega \gg B$.

V. CONCLUSIONS

We have studied one-mode [given by Eq. (6)] and two-mode [given by Eq. (22)] variations of the $\mathbb{Z}_3$ quantum Rabi model. To find their spectrum, we proposed a canonical transformation that allows to find the spectrum of the Rabi models with the zero “magnetic” term. This in turn allowed a perturbative treatment of the Rabi model in the weak magnetic field parameter B/Ω in each $\mathbb{Z}_3$ symmetry sector. This approach yields the low-energy spectrum and eigenstates for arbitrary coupling λ/Ω (including the deep-strong-coupling regime), provided $B \ll \Omega$. In the deep-strong-coupling regime, the three lowest eigenstates become $\mathbb{Z}_3$ cat states. We then studied these states using joint qutrit-boson Wigner functions. In particular, we derived an explicit expression for the Wigner function of the qutrit-boson cat state and calculated the Wigner function of the $\mathbb{Z}_3$ Rabi model’s eigenstates numerically for different coupling strengths.

With the $\mathbb{Z}_n$ Rabi model recently attracting significant attention, our results should further the understanding of these models, in particular, their similarities and differences with the conventional $\mathbb{Z}_2$ Rabi model. Moreover, the proposed approach based on the joint qutrit-boson Wigner function improves on the ordinary Wigner function, allowing one to study the entanglement between the qutrit and boson subsystems. The approach readily generalizes to qubit-boson systems and thus promises to be generally useful for studies of the light-matter interaction.

While in this paper we extensively studied the regime of the deep-strong coupling and weak magnetic field, the understanding of the $\mathbb{Z}_n$ Rabi model in other coupling regimes remains an open problem. Compared to the ordinary Rabi model, various parameter regimes are still to be studied. This provides a direction for future work.

ACKNOWLEDGMENTS

We thank Daria Kalacheva, Henry Legg, Katharina Laubscher, and Ilia Luchnikov for fruitful discussions and useful comments. This work was supported as a part of NCCR SPIN, a National Centre of Competence in Research, funded by the Swiss National Science Foundation (Grant No. 225153). This work has received funding from the Swiss State Secretariat for Education, Research and Innovation (SERI) under Contract No. M822.00078. D.L. acknowledges the Deanship of Research and the Quantum Center at KFUPM for the support received under Grants No. CUP25102 and No. INQC2500, respectively.

DATA AVAILABILITY

The data used to generate Figs. 2–7 were obtained by numerical simulations of the corresponding models. The code for the numerical simulations and the data that support the findings of this article are openly available [45].

APPENDIX A: EQUIVALENCE BETWEEN THE HAMILTONIAN (22) AND THE MODEL OF REF. [20]

The two-mode $\mathbb{Z}_3$ Rabi model (22) is equivalent to the model considered in Ref. [20]. The latter reads

$$H_{R2'} = \Omega(a_1^\dagger a_1 + a_2^\dagger a_2) + B(e^{i\phi} Z + e^{-i\phi} Z^\dagger) - \lambda(x_1 + ix_2)X - \lambda(x_1 - ix_2)X^\dagger, \quad (A1)$$

where we defined $x_{1,2} = (a_{1,2} + a_{1,2}^\dagger)/\sqrt{2}$ and $p_{1,2} = (a_{1,2} - a_{1,2}^\dagger)/(\sqrt{2}i)$. In this Appendix, x_i and p_i with $i = 1, 2$ denote the canonical coordinate and momentum operators as opposed to the classical coordinates in Sec. IV.

This representation of the two-mode $\mathbb{Z}_3$ Rabi model can be transformed into our Hamiltonian H_{R2} in Eq. (22) by a canonical transformation that mixes the two bosonic modes. The $\mathbb{Z}_3$ symmetry generator in this case takes the form

$$\mathcal{P}_{R2'} = \exp\left[\frac{2\pi i}{3}(x_1 p_2 - x_2 p_1)\right] Z. \quad (A2)$$

Note that here the $\mathbb{Z}_3$ symmetry is realized in a peculiar way. The boson part of the $\mathbb{Z}_3$ symmetry generator (A2) is just a discrete rotation in the plane of the two-dimensional quantum harmonic oscillator.

APPENDIX B: ANOTHER VERSION OF THE TWO-MODE $\mathbb{Z}_3$ RABI MODEL

In this Appendix, we examine another two-mode extension of the $\mathbb{Z}_3$ Rabi model (6), which is more straightforward compared to the one used in the main text [see Eq. (22)]. Here, we simply double the number of bosonic modes without altering the Hamiltonian (6) in any other way:

$$H = \Omega(a_1^\dagger a_1 + a_2^\dagger a_2) + B(e^{i\phi} Z + e^{-i\phi} Z^\dagger) - \lambda[(a_1^\dagger + a_2^\dagger)X + (a_1 + a_2)X^\dagger]. \quad (B1)$$

Correspondingly, the $\mathbb{Z}_3$ symmetry generator is

$$\mathcal{P} = \exp\left[\frac{2\pi i}{3}N\right] Z, \quad \text{with } N = a_1^\dagger a_1 + a_2^\dagger a_2. \quad (B2)$$

Interestingly, unlike the two-mode $\mathbb{Z}_3$ Rabi model (22) discussed in the main text, the Hamiltonian (B1) possesses the $SU(2)$ symmetry of the 2D QHO. As a result, in this system we have the $\mathbb{Z}_3$ and $SU(2)$ symmetries combined into $G = \mathbb{Z}_3 \times SU(2)$. This leads to a degeneracy of the spectrum similar to the one we saw in Appendix C. In this sense, the addition of the second mode did not change anything but the multiplicities of the eigenenergies. As a consequence, we find the two-mode $\mathbb{Z}_3$ Rabi model discussed in Sec. III to be more interesting compared to the one in Eq. (B1).

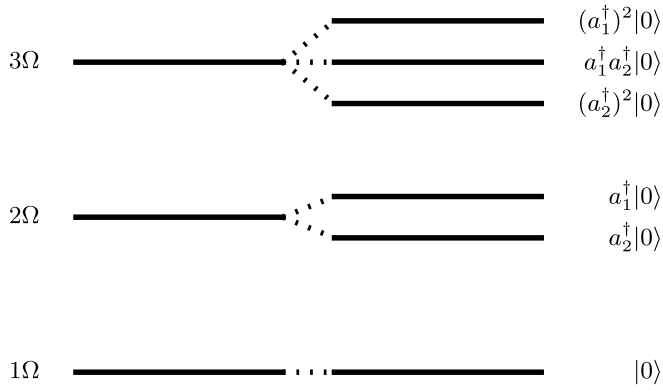


FIG. 8. Eigenvalue degeneracy of two-dimensional quantum harmonic oscillator with the Hamiltonian (C1).

APPENDIX C: SU(2) SYMMETRY IN TWO-DIMENSIONAL QUANTUM HARMONIC OSCILLATOR

While the 2D QHO is discussed in any introductory quantum mechanics course, for the sake of completeness, we would like to briefly review a certain fact relevant to our purposes. The Hamiltonian of the 2D QHO is

$$H = \Omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1). \quad (\text{C1})$$

Keeping in mind that the one-dimensional QHO eigenenergies are simply $\Omega(n + 1/2)$, $n = 0, 1, 2, 3, \dots$, we can easily

deduce the spectrum of the 2D QHO. Here, we have two modes in which we can put excitations. As a result, the n th eigenenergy, $n \geq 1$, becomes degenerate. Using the basic combinatorial fact that there are $(n + 1)$ ways to distribute n objects onto two groups, it follows that the n th energy level has $(n + 1)$ -fold degeneracy (see Fig. 8). This looks quite similar to the SU(2) group representations, i.e., there is exactly one SU(2) representation on a n -dimensional vector space. This is not a coincidence: The 2D QHO has a SU(2) symmetry, which “rotates” the excitations between two modes.

The explicit expressions for the SU(2) symmetry generators are given by

$$\begin{aligned} L_1 &= a_1^\dagger a_2 + a_2^\dagger a_1, \\ L_2 &= ia_1^\dagger a_2 - ia_2^\dagger a_1, \\ L_3 &= a_1^\dagger a_1 - a_2^\dagger a_2. \end{aligned} \quad (\text{C2})$$

Despite this being a simple and well-known fact, the presence of the SU(2) symmetry is relevant for our discussion of the two-mode $\mathbb{Z}_3$ Rabi model, which is a 2D QHO coupled with a three-level system (qutrit). The interaction between the 2D QHO and the three-level system can either break the SU(2) symmetry or preserve it.

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