

Spatio-spectral quantum networks in nonlinear photonic lattices

Natalia Costas,¹ Nadia Belabas², and David Barral^{1,*}

¹*Galicja Supercomputing Center (CESGA), Avda. de Vigo S/N, 15705 Santiago de Compostela, Spain*

²*Centre de Nanosciences et de Nanotechnologies, CNRS, Université Paris-Saclay, 91120 Palaiseau, France*



(Received 17 June 2025; accepted 17 November 2025; published 22 December 2025)

Multiplexing information in different degrees of freedom is a natural solution to the scalability bottleneck in optical quantum communications and computing. However, for bulk-optics systems, size, cost, stability, and reliability can make scalability either impractical or highly challenging to implement. We present a framework to engineer continuous and discrete-variable entanglement produced through nondegenerate spontaneous parametric down-conversion in $\chi^{(2)}$ nonlinear photonic lattices in spatial and spectral degrees of freedom that can alleviate the scalability challenge. We show how spatio-spectral pump shaping produces cluster states that are natively distributable in quantum communication networks and a resource for measurement-based quantum computing.

DOI: [10.1103/bgls-md94](https://doi.org/10.1103/bgls-md94)

I. INTRODUCTION

As the world becomes increasingly interconnected through technology, there is a growing need to enhance the capacity of communication channels. To support emerging technologies such as artificial intelligence, internet of things, and cloud computing, communication networks must handle greater data volumes. In optical networks, when a channel reaches its capacity limit, multiplexing different degrees of freedom (DOF) becomes essential. Recent advancements in space-division multiplexing, combined with existing wavelength-division multiplexing, have enabled transmission rates exceeding few petabytes per second through a single optical fiber [1].

Similarly, the future of quantum networks, which underpin emerging technologies like the quantum internet, distributed quantum sensing, and distributed quantum computing, faces a similar challenge [2–4]. As the number of nodes in a quantum network or the complexity of quantum algorithms increases, so does the demand for more physical resources to process and transmit quantum information. Multiplexing DOF offers a practical solution to this challenge. In the optical domain, quantum information is encoded in space, path, frequency, time, polarization, and angular momentum, among others. Recent advances in multiplexing these DOF have shown remarkable progress in the quantum domain, demonstrating that quantum networks can scale more efficiently and handle growing demands like their classical counterparts [5].

The transition from laboratory experiments to real-world applications is another critical aspect of quantum technologies. Small-footprint photonic circuits offer a promising solution, enabling the integration of various quantum

operations—such as generation, processing, distribution, and measurement of quantum information—into a single, compact device [6,7]. These integrated optics solutions not only reduce the size but also improve stability, reliability, and scalability, making them ideal for building large, functional quantum networks [8–10].

Quantum information may be encoded in variables that exhibit a continuous spectrum of eigenvalues, known as continuous variables (CV). In photonic systems, this encoding is realized through the fluctuations of field quadratures [11]. Numerous tabletop experiments have successfully demonstrated CV quantum networks in the spatial, frequency, and temporal domains [12]; and recently, integrated experiments in one DOF [13–16]. However, extending and adapting bulk-optics-based methods, such as sequential squeezing and entanglement, to systems involving a larger number of modes remains a highly demanding task. Schemes based on multiplexing spatial and spectral DOF have been proposed for bulk-optics systems [17,18]. In this paper, we show the potential of spatio-spectral multiplexed encoding of quantum information in the propagating modes generated in a nonlinear $\chi^{(2)}$ photonic lattice without bulk-optics analogous. The distributed simultaneous nonlinearity and evanescent coupling configuration of the photonic lattice, together with the spectral properties of both the waveguides and the interacting fields, parallelize multimode spatio-spectral transformations. Previous works have analyzed the potential of nonlinear photonic lattices in discrete variables [19–24]. We propose a general framework for nonlinear waveguide arrays in CV—which includes discrete variables (DV) as a limit case—that allows engineering two- and three-dimensional graph and cluster states naturally distributable in quantum networks and that are a resource for quantum computing.

II. DYNAMICS OF A NONLINEAR PHOTONIC LATTICE

A nonlinear photonic lattice consists of N identical $\chi^{(2)}$ waveguides in which spontaneous down-conversion (SPDC)

*Contact author: dbarral@cesga.es

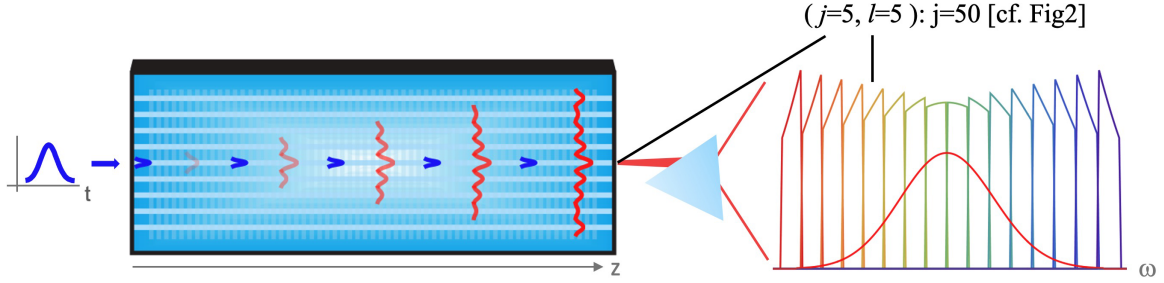


FIG. 1. Sketch of a nonlinear photonic lattice and measurement basis in a given spatio-spectral mode (fixel j) associated with a given waveguide (pixel j) and frequency band (frexel l). Left: a pump pulse (blue) is coupled to the center waveguide of a $N = 9$ lattice producing SPDC (red) that spreads accordingly to a coupling profile. Right: orthonormal homodyne measurement basis—frexels (rainbow)—for a Gaussian-shaped local oscillator (decomposed in frequency bins, normalized in power—rainbow slices), and, in red, the signal marginal of the joint spatio-spectral amplitude (JSSA) (equally for idler) projected on the frexel basis. This example shows the measurement of 16 frexels corresponding to one pixel, hence $9 \times 16 = 144$ fixels. In general, the system gives access to $N \times L = N$ fixels with the notations of Table I.

and nearest-neighbor evanescent coupling between the generated fields take place (see Fig. 1) [25–27]. We consider type O/I down-conversion, where in each waveguide an input harmonic field at frequency ω_h is down-converted into signal (s) and idler (i) fields, respectively, at frequency ω_s and $\omega_i = \omega_h - \omega_s$ with identical polarization. Other strategies as type II down-conversion can be equally implemented adding polarization as an extra DOF [28]. The buildup of the nonlinear interaction is driven by the ability of propagating the interacting waves with the same velocity or wave vector phase-matching. Birefringence is not always applicable, or not to the highest second-order tensor component. A possible solution is the periodic modulation of the nonlinear coefficient to quasi phase match the propagation constants: $\Delta\beta(\omega_s, \omega_i) \equiv \beta(\omega_h) - \beta(\omega_s) - \beta(\omega_i) - 2\pi/\Lambda = 0$, with β the propagation constant and Λ the poling period. For instance in lithium niobate, where birefringence does not give access to its highest second-order tensor component d_{33} , periodic poling enables access to it. State-of-the-art poled waveguides have demonstrated up to 8 dB of squeezing [29]. Below, we consider that phase-matching is just produced in the region with evanescent coupling between generated modes [30]. The energy of the signal/idler modes propagating in each waveguide is exchanged between the coupled waveguides through evanescent waves, whereas the interplay of the second-harmonic waves is negligible for the considered propagation lengths due to their high confinement into the guiding region. We set our calculation in the regime of pump undepletion.

The physical processes taking place in $\chi^{(2)}$ waveguides can be described by a dynamical operator $\hat{\mathcal{M}}$ obtained by quantizing the flux of momentum of the electromagnetic fields [31]. Particularly, we tackle the problem of broadband down-

conversion with an arbitrary pump; i.e., we assume that the down-converted spectrum Δ_{SPDC} is much broader than the pump bandwidth such that we can neglect the dependence of the nonlinear susceptibility $\chi^{(2)}$ on frequency [32]. Thus, the following Heisenberg equation is obtained for an array of N evanescently coupled nonlinear waveguides with an arbitrary pump waveform in the SPDC regime:

$$\frac{d\hat{A}_j(\omega_s, z)}{dz} = iC_M[f_{j-1}(\omega_s)\hat{A}_{j-1}(\omega_s, z) + f_j(\omega_s)\hat{A}_{j+1}(\omega_s, z)] + ig \int \alpha_j(\omega_h) e^{i\Delta\beta(\omega_s, \omega_i)z} \hat{A}_j^\dagger(\omega_i, z) d\omega_i, \quad (1)$$

where $\hat{A}_0 = 0$ and $\hat{A}_{N+1} = 0$, $f_0 = f_N = 0$, and $j = 1, \dots, N$ is the individual mode index corresponding to each waveguide. C_M is the largest linear coupling strength at the waveguide-array design operating frequency $\omega_h/2$, $f_j(\omega)$ are the elements of the coupling profile $\vec{f}$, and z is the coordinate along the direction of propagation. $\hat{A}_j(\omega_s, z) \equiv \hat{A}_j^s$ and $\hat{A}_j(\omega_i, z) \equiv \hat{A}_j^i$ are monochromatic slowly varying amplitude annihilation operators of signal and idler photons corresponding to the j th waveguide—that we denote as pixel basis (Table I and Fig. 2 summarize the different bases introduced in the paper)—fulfilling local commutation relations $[\hat{A}_j(\omega, z), \hat{A}_{j'}^\dagger(\omega', z)] = \delta(\omega - \omega')\delta_{j,j'}$. We consider the nonlinear coupling constant g —proportional to $\chi^{(2)}$ and to the spatial overlap of the signal and harmonic fields in each waveguide—equal for all waveguides. We work at the degeneracy point of the phase-matching $\omega_h = 2\omega_s$. $\alpha_j(\omega_h) \equiv \langle \alpha | \hat{A}_j(\omega_h) | \alpha \rangle = \sqrt{P_h} \eta_j \Omega$ ($\omega_h = \omega_s + \omega_i$) is the complex amplitude of a strong coherent undepleted pump

TABLE I. Notation used to designate the mode bases and the annihilation operators related to spatial, spectral and spatio-spectral individual modes, linear supermodes, and nonlinear supermodes. The number of modes per DOF is given in the last row. The elements with * are not addressed in this article but detailed in the mentioned Refs. [33,34]. n.a. stands for not applicable.

	Spatial (guided)	Spectral	Spatio-spectral
Individual modes	Pixel: $\hat{A}_j$	Frexel: $\hat{A}_j^l$	Fixel: $\hat{A}_j^l \equiv \hat{A}_j$
Linear supermodes	L -pixel: $\hat{B}_k$	n.a.	L -fixel: $\hat{B}_k^l \equiv \hat{B}_k$
Nonlinear supermodes	*[34]	*[33]	N -fixel: $\hat{C}_m$
Number of modes	N	L	$N = N \times L$

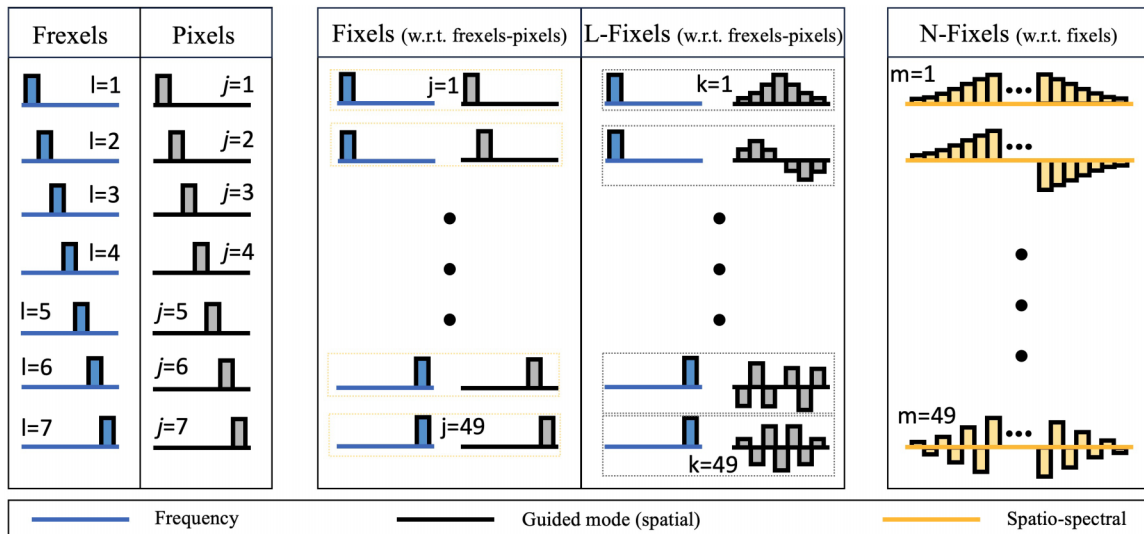


FIG. 2. Sketch of mode bases appearing in the text. On the left, we display spectral and spatial nonoverlapping (individual) mode bases: frexels and pixels, respectively. In the center, we show two families of spatio-spectral modes: fixels and L -fixels. Fixels are nonoverlapping modes separable in spectral and spatial DOF. L -fixels are nonoverlapping modes in frequency and overlapping modes (supermodes) in space, separable in spectral and spatial DOF. The dotted boxes represent single spatio-spectral modes (yellow for fixels, black for L -fixels). On the right, we introduce a third family of spatio-spectral modes: N -fixels. They are overlapping modes in frequency and space, in general inseparable in spectral and spatial DOF, thus only describable in terms of fixels (or L -fixels). This basis is not accessible experimentally unless spectral and spatial DOF are decoupled. Fixels and L -fixels are independent of z (but L -fixels depend on the coupling profile), while N -fixels depend on coupling and pump profiles, and on z . The nonlinear photonic lattice produces quantum correlations in fixels and L -fixel bases, and independent squeezing in the N -fixel basis. In the figure, we have set as an example $L = 7$ spectral and $N = 7$ spatial modes—thus $N = 49$ spatio-spectral modes. Blue, black, and yellow abscissas represent frequency, guided mode, and fixels, respectively. Rectangles represent relative amplitudes with respect to other basis. w.r.t. stands for with respect to.

field propagating in the j th waveguide, where P_h is the total pump power in the array, η_j is the normalized complex amplitude of pump power directed to each waveguide j , and Ω is the normalized spectral pump complex amplitude that feeds the production of a pair of photons in frequencies ω_s and ω_i . The spatial and spectral pump profiles can be tuned by spatial and temporal pulse shapers that independently manipulate the relative phases and amplitudes of propagating waveguide modes and frequency modes, respectively [35,36]. Experimentally, distributable frequency modes or frexels (see Fig. 2) are measured in bands of a given bandwidth either by shaping a local oscillator (LO) or in a multipixel homodyne detector [33,37,38]. These modes can be spatially separated by means of a dispersive element such as a grating or a prism and a microlens array, or fiber-based wavelength-division multiplexing [36,39]. The orthogonality of these modes is guaranteed as they do not overlap spectrally. We thus discretize the down-converted spectrum in L bands centered at frequencies ω_l such that each signal (idler) frexel mode is labeled with l (l'). The frexel-pixel operators $\hat{A}_j^l(z)$ —the fixel basis—are related to the monochromatic frequency-mode operators $\hat{A}_j(\omega, z)$ simply by a basis transformation between the discrete and the continuous basis $\hat{A}_j^l(z) = \int d\omega \xi^l(\omega) \hat{A}_j(\omega, z)$, fulfilling local commutation relations $[\hat{A}_j^l(z), \hat{A}_{j'}^{l'\dagger}(z)] = \delta_{j,j'} \delta_{l,l'}$, where $\xi^l(\omega)$ corresponds to normalized frexel modes of width Δ_F taken as real for simplicity. The extent and number of frexel modes are fixed by experimental constraints: they result from (1) the bandwidth of the local oscillator and (2) the resolution of an LO pulse shaper or a multipixel detector, the channel

bandwidth in a dense wavelength division multiplexer, etc. [36].

Considering coupling only between nearest-neighbor waveguides, a linear waveguide array—Eq. (1) with $g = 0$ —presents spatial linear supermodes (L -pixels) $\hat{B}_k$, i.e., propagation eigenvectors [40]. These eigenvectors form a basis and are represented by an orthogonal matrix $\mathbf{M} \equiv \mathbf{M}(\vec{f})$ with real elements $M_{k,j}$. The L -pixels are the same for signal and idler frequencies as long as the coupling profile is constant in the considered bandwidth $\vec{f}(\omega) \approx \vec{f}$. We consider a SPDC bandwidth Δ_{SPDC} , where this condition holds and discuss the validity of this assumption in the Appendix. Pixels and L -pixels, and their corresponding spatio-spectral versions fixels and L -fixels, are thus related by $\hat{B}_k^l = \sum_{j=1}^N M_{k,j} \hat{A}_j^l$ (see Table I and Fig. 2). These supermodes are orthonormal $\sum_{j=1}^N M_{k,j} M_{k',j} = \delta_{k,k'}$, with a spectrum of eigenvalues $\lambda_k \equiv \lambda_k(C_M, \vec{f})$. λ_k is the propagation constant of the k th L -pixel. Recent developments have broken the static structure of integrated optical lattices, enabling the modification of coupling constant profiles on demand [41,42].

To simplify the analysis, we order the modes in L blocks composed of N spatial modes in given frequency bands centered at ω_l with the following indices, respectively, for the fixels and L -fixels:

$$j = j + (l - 1)N, \quad k = k + (l - 1)N,$$

with $\{j, k\} = 1, \dots, N$ and $N \equiv N \times L$ [43]. The indices are ordered using l : for $l = 1$ we have $j(k) = 1, \dots, N$, for $l = 2$ we have $j(k) = N + 1, \dots, 2N$, etc. The L -fixel

transformation is then

$$\hat{B}_k = \sum_{j=1}^N M_{k,j} \hat{A}_j,$$

with M a block diagonal matrix with elements $M_{k,j} \equiv M_{k,j}$. Using slowly varying L -fixel amplitudes $\hat{B}_k = \hat{B}_k e^{-i\lambda_k z}$, the following propagation equation is obtained straightforwardly:

$$\frac{d\hat{B}_k}{dz} = ig\sqrt{P_h} \sum_{k'=1}^N \mathcal{L}_{k,k'}(z) \hat{B}_{k'}^\dagger, \quad (2)$$

where k and k' are two L -fixels coupled by a function $\mathcal{L}(z)$ with elements given by $\mathcal{L}_{k,k'}(z) = \tilde{\mathcal{L}}_{k,k'}(z) \tilde{\mathcal{L}}^{l,l'}(z)$, where

$$\tilde{\mathcal{L}}_{k,k'}(z) = \sum_{j=1}^N M_{k,j} M_{k',j} \eta_j e^{-i(\lambda_k + \lambda_{k'})z}, \quad (3)$$

$$\tilde{\mathcal{L}}^{l,l'}(z) = \iint d\omega_s d\omega_i \xi^l(\omega_s) \xi^{l'}(\omega_i) \Omega(\omega_s + \omega_i) e^{i\Delta\beta(\omega_s, \omega_i)z}. \quad (4)$$

$\mathcal{L}(z)$ is a complex matrix that gathers all the information about the spatio-spectral shape of the pump and the phase-matching. The symmetric complex matrices $\tilde{\mathcal{L}}_{k,k'}$ and $\tilde{\mathcal{L}}^{l,l'}$ couple, respectively, L -pixels and frexels, and, as we show below in Eq. (8), propagation couples both. Note the outstanding symmetry between the two expressions with a change of basis, a pump function, and a phase-matching function.

Remarkably, unlike broadband frequency modes, frexels preserve local multiplication. If the resolution of the frexel basis Δ_F^{-1} is large enough, the frequency-dependent functions are approximately constant within each frexel, and we can approximate $\Omega(\omega_s + \omega_i)$ and $\Delta\beta(\omega_s, \omega_i)$ by their frexel versions

$$\Omega^{l,l'} = \iint d\omega_s d\omega_i \xi^l(\omega_s) \xi^{l'}(\omega_i) \Omega(\omega_s + \omega_i), \quad (5)$$

$$\Delta\beta^{l,l'} = \iint d\omega_s d\omega_i \xi^l(\omega_s) \xi^{l'}(\omega_i) \Delta\beta(\omega_s, \omega_i), \quad (6)$$

such that $\tilde{\mathcal{L}}^{l,l'}(z) \approx \Omega^{l,l'} e^{i\Delta\beta^{l,l'}z}$. This coarse-grained description of the spectral functions can be used to get insight about the dynamics of the full system, although analytical calculations rigorously hold only for Eq. (4).

The symmetries of the L -fixels enable solving Eq. (2) analytically for any gain regime with suitable pump profiles [34,35,44]. However, in the general case, we can solve it in the low-gain regime, where space-ordering effects can be neglected [45]. Hence, the formal solution to Eq. (2) at given z is written as

$$\begin{pmatrix} \vec{B}(z) \\ \vec{B}^\dagger(z) \end{pmatrix} = \exp \left\{ \Gamma(z) \begin{pmatrix} 0 & \mathbf{f}(z) \\ \mathbf{f}^*(z) & 0 \end{pmatrix} \right\} \begin{pmatrix} \vec{B}(0) \\ \vec{B}^\dagger(0) \end{pmatrix}, \quad (7)$$

with $\vec{B} \equiv (\hat{B}_1, \dots, \hat{B}_k, \dots, \hat{B}_N)^T$, $\Gamma(z) = g\sqrt{P_h}z$ the total nonlinear amplitude, and $\mathbf{f}(z)$ the normalized joint spatio-spectral amplitude (JSSA), with elements given by

$$\begin{aligned} \mathbf{f}_{k,k'}(z) &= \eta_{k,k'} \iint d\omega_s d\omega_i \xi^l(\omega_s) \xi^{l'}(\omega_i) \Omega(\omega_s + \omega_i) \\ &\quad \times \text{sinc} \left(\frac{\Delta\beta(\omega_s, \omega_i) - (\lambda_k + \lambda_{k'})}{2} z \right) e^{i \frac{\Delta\beta(\omega_s, \omega_i) - (\lambda_k + \lambda_{k'})}{2} z}, \end{aligned} \quad (8)$$

with $\eta_{k,k'} = \sum_{j=1}^N M_{k,j} M_{k',j} \eta_j$. $\mathbf{f}_{k,k'}(z)$ couples pairs of L fixels k and k' and, in particular, the sinc function couples spatial and spectral DOF. Thus, the spectral features of the waveguides, the evanescent-coupling profile, and the spatial and spectral shape of the pump enable engineering the JSSA. This matrix contains all information about individual-mode correlations in the DV regime ($\Gamma(z) \ll 1$) through a simple change of basis in the spatial DOF [24].

Under the high-resolution approximation for frexels of Eqs. (5) and (6), Eq. (8) is separable into spatio-spectral pump and phase-matching functions as

$$\mathbf{f}_{k,k'}(z) \approx \alpha_{k,k'} \Phi_{k,k'}(z), \quad (9)$$

with $\alpha_{k,k'} = \eta_{k,k'} \Omega^{l,l'}$, $\Phi_{k,k'}(z) = \text{sinc}(\frac{\Delta\beta_{k,k'} z}{2}) e^{i \frac{\Delta\beta_{k,k'} z}{2}}$, and $\Delta\beta_{k,k'} = \Delta\beta^{l,l'} - (\lambda_k + \lambda_{k'})$.

The JSSA of Eqs. (8) and (9) is symmetric; i.e., it is invariant under the change of indices $k \leftrightarrow k'$. Using this property, the solution to Eq. (7) can be simplified through a full nonlinear supermode (N -fixel) basis $\hat{\mathcal{C}}$, given by $\hat{\mathcal{C}}_m = \sum_{k=1}^N \Upsilon_{m,k}^\dagger(z) \hat{B}_k$, where $\Upsilon(z)$ is a unitary matrix that diagonalizes $\mathbf{f}(z)$ by a congruence transformation—the Autonne-Takagi transformation—obtaining a real diagonal matrix with non-negative entries $\Lambda(z)$ [46]. Equation (7) in terms of N -fixels is thus simply given by

$$\hat{\mathcal{C}}_m(z) = \cosh[r_m(z)] \hat{\mathcal{C}}_m(0) + \sinh[r_m(z)] \hat{\mathcal{C}}_m^\dagger(0), \quad (10)$$

with $m = 1, \dots, N$, and where $r_m(z) = \Gamma(z) \Lambda_{m,m}(z)$ are the down-conversion gains at a propagation distance z . Each spatio-spectral N -fixel thus appears as a broadband nonlocal single-mode squeezed state.

In terms of fixels, the solution to the nonlinear system is

$$\hat{A}_j(z) = \sum_{j'} U_{j,j'}(z) \hat{A}_{j'}(0) + V_{j,j'}(z) \hat{A}_{j'}^\dagger(0), \quad (11)$$

with

$$\begin{aligned} U_{j,j'}(z) &= \sum_{k,m} M_{j,k} \Upsilon_{k,m}(z) M_{m,j'} e^{i\lambda_k z} \cosh[r_m(z)], \\ V_{j,j'}(z) &= \sum_{k,m} M_{j,k} \Upsilon_{k,m}(z) M_{m,j'} e^{i\lambda_k z} \sinh[r_m(z)]. \end{aligned}$$

This is one of the main results of our contribution: The full diagonalization of the system of Eq. (1) yields the general solution in spatio-spectral fixels j (localized, single band) given by Eq. (11). Indeed, the solution for any pump configuration and geometry of the lattice is obtained by calculating $\Upsilon_{k,m}(z)$ and $r_m(z)$ from Eq. (8). Importantly, the modes j can be distributed in a quantum network. In contrast, the N -fixel basis $\mathbf{m}$ (delocalized, broadband) with solution given by Eq. (10) is, in general, not directly accessible experimentally as the local oscillator in a homodyne measurement stage should indeed be shaped in a specific spatio-spectral N -fixel. This shaping is only possible if spatial and spectral DOF are decoupled. In general, Eq. (8) cannot be decoupled into a spatial and a spectral part, as the sinus cardinal couples spectral and spatial modes [20]. Nevertheless, the system might be decoupled in specific cases when the sinus cardinal can be approximated by a separable function like a Gaussian function [47–49].

Equation (11) can be rewritten in terms of CV amplitude and phase quadratures, respectively, given by $\hat{x}_j = (\hat{A}_j + \hat{A}_j^\dagger)$

and $\hat{y}_j = i(\hat{A}_j^\dagger - \hat{A}_j)$ (shot noise normalized to 1). This results in a propagator $\mathbf{S}(z)$ for the quadrature vector $\hat{\xi} = (\hat{x}_1, \dots, \hat{x}_N, \hat{y}_1, \dots, \hat{y}_N)^T$, such that $\hat{\xi}(z) = \mathbf{S}(z)\hat{\xi}(0)$ [34]. Gaussian states produced by SPDC are fully characterized by the second-order moments of their quadrature operators. These moments are properly arranged in a covariance matrix Σ given by

$$\Sigma(z) = \mathbf{S}(z)\mathbf{1}\mathbf{S}^T(z), \quad (12)$$

with $\mathbf{1}$ the identity matrix standing for vacuum shot noise. Moreover, the effect of propagation losses can be easily included in our model. Considering uniform Gaussian losses for all fixels, we get [50]

$$\Sigma_\eta(z) = \eta\Sigma(z) + (1 - \eta)\mathbf{1}, \quad (13)$$

where $\eta \in [0, 1]$ is the effective optical transmittance that takes into account both optical losses and efficiency of detection. Typically, the transmittance incorporates the propagation losses, the LO-signal mode overlap, the electronic noise of the homodyne detector, and the quantum efficiencies of the photodiodes [51]. Equation (13) shows how losses couple each mode to a vacuum mode, resulting in an increase of noise—diagonal of Σ_η —that degrades the squeezing, and in a decrease of quantum correlations.

III. SPATIO-SPECTRAL CLUSTER STATES

In the following, we illustrate this framework using a relevant example. We analyze the following case: a pump with a flat spatial distribution such that $\eta_j = |\eta|e^{i\phi}$ and $\eta_{k,k'} = |\eta|\delta_{k,k'}e^{i\phi}$, spectrally Gaussian

$$\Omega(\omega_s + \omega_i) = ((2\pi)^{1/2}\Delta_p)^{-1/2}e^{-\frac{(\omega_h - (\omega_s + \omega_i))^2}{4\Delta_p^2}},$$

and narrowband—its linewidth (full width at half maximum) in intensity $2\sqrt{2\ln(2)}\Delta_p$ is much lower than the frexel resolution Δ_F . This pump distribution is coupled in each waveguide, producing pairs of photons spectrally symmetric with respect to $\omega_h/2$. We set $|\eta| = 1/\sqrt{N}$ and for simplicity we choose $\phi = -\pi/2$ —this is a global phase that will just change the squeezing angle such that $\tilde{\mathcal{L}}_{k,k'}(z) = -i\delta_{k,k'}e^{-2i\lambda_k z}/\sqrt{N}$. In type 0 (or I) down-conversion the wavevector phase-mismatch can be approximated at first order in frequency by $\Delta\beta(\omega_s, \omega_i) \approx \Delta\beta(\omega_h/2, \omega_h/2) + \gamma(\omega_h - \omega_s + \omega_i)$ with $\gamma = (\partial\beta/\partial\omega|_{\omega_h} - \partial\beta/\partial\omega|_{\omega(s/i)})$ [28]. The phase-match at the degenerate frequency can be achieved by, for instance, quasi-phase-matching and suitable temperature setting. Thus, for a narrowband pump $\Omega(\omega_s + \omega_i) = \delta(\omega_h - \omega_s - \omega_i)$ and $\tilde{\mathcal{L}}^{i,l'}(z) = \delta_{L+1-l,l'}$ in the bandwidth of interest. The joint spatio-spectral distribution is thus

$$\mathcal{L}_{k,k'}(z) = -ig\sqrt{p_h}\delta_{k,k'}\delta_{L+1-l,l'}e^{-2i\lambda_k z},$$

where p_h is here the pump power per waveguide $p_h \equiv P_h/N$, such that the spatial modes are decoupled (squeezed) and the spectral modes are coupled two by two (entangled). Equation (2) can then be written as

$$\frac{d\hat{B}_k}{dz} = g\sqrt{p_h}e^{-2i\lambda_k z}\hat{B}_{k'}^\dagger, \quad (14)$$

with $k \equiv k(k, l)$ and $k' \equiv k'(k, L+1-l)$. This is the equation of a two-mode squeezer between L -fixels k and k' with gain $G \equiv r_k z = [(g\sqrt{p_h})^2 - \lambda_k^2]^{1/2}z$. In the individual mode basis, the solution of Eq. (14) for each value of l is given by

$$\hat{\mathcal{A}}_{s,j'}(z) = \sum_{j'}(\tilde{U}_{j,j'}(z)\hat{\mathcal{A}}_{s,j'}(0) + \tilde{V}_{j,j'}(z)\hat{\mathcal{A}}_{i,j'}^\dagger(0)), \quad (15)$$

$$\hat{\mathcal{A}}_{i,j}^\dagger(z) = \sum_{j'}(\tilde{V}_{j,j'}(z)\hat{\mathcal{A}}_{s,j'}(0) + \tilde{U}_{j,j'}^*(z)\hat{\mathcal{A}}_{i,j'}^\dagger(0)), \quad (16)$$

with

$$\tilde{U}_{j,j'}(z) = \sum_k M_{j,k}M_{k,j'} \left\{ \cosh[r_k z] + i\frac{\lambda_k}{r_k} \sinh[r_k z] \right\},$$

$$\tilde{V}_{j,j'}(z) = \sum_k M_{j,k}M_{k,j'} \left\{ \frac{g\sqrt{p_h}}{r_k} \sinh[r_k z] \right\},$$

and where we have taken $l \equiv s$ (signal) and $L+1-l \equiv i$ (idler). Note that the different shapes of Eqs. (11), (15), and (16) are related to the change of basis back to the fixel basis from a fully decoupled N -fixel basis and from a partially decoupled L -fixel basis, respectively.

The solution of Eqs. (15) and (16) showing only entanglement between symmetric frexels around the central frequency is the limit of a pump spectrally Gaussian when its linewidth is much lower than the frexel resolution [52]. A broader pump would include terms modulated by the Gaussian spectral distribution, entangling symmetric frexels around $l \pm 1$, $l \pm 2$, etc.; generating correlations mainly between the central frexels with a strength that follows a Gaussian distribution [15,16].

Remarkably, in the limit case of one spectral mode (degeneracy in frequency, $L = 1$) the state given by Eqs. (15) and (16) is a linear cluster state in the spatial domain over a wide range of values of the governing parameters [34,35]. Hence, the state given by Eqs. (15) and (16) can form $L/2$ copies of a $(N \times 2)$ -grid cluster state for an even number L of spectral modes, or $(L-1)/2$ copies if L is odd. An example is shown in Fig. 3 for a homogeneous array of nonlinear waveguides with $N = 7$ waveguides and two spectral modes $l \equiv s$ and $L+1-l \equiv i$. The horizontal and vertical edges represent, respectively, spatial and spectral entanglement. For instance, for $L = 16$ frequency bands we would have eight spectrally shifted copies of the 7×2 grid state of Fig. 3. All copies are generated in the same temporal mode and can be easily distributable in a quantum network by means of suitable spectral demultiplexing.

We can demonstrate that the array produces this family of states using the graph calculus for Gaussian pure states [53]. Using Eqs. (15) and (16), we can calculate the covariance matrix Σ given by Eq. (12), but in this case related to the quadrature vector $\hat{\xi} = (\hat{x}_{s,1}, \hat{x}_{s,2}, \dots, \hat{x}_{i,1}, \hat{x}_{i,2}, \dots, \hat{y}_{s,1}, \hat{y}_{s,2}, \dots, \hat{y}_{i,1}, \hat{y}_{i,2})^T$. Notably, Σ can be written in block form in terms of two symmetric matrices $\mathbf{U}$ and $\mathbf{V}$ as [53]

$$\Sigma(z) = \begin{pmatrix} \mathbf{U}^{-1} & \mathbf{U}^{-1}\mathbf{V} \\ \mathbf{V}\mathbf{U}^{-1} & \mathbf{U} + \mathbf{V}\mathbf{U}^{-1}\mathbf{V} \end{pmatrix}.$$

From the two upper blocks of Σ , we can obtain the complex-weighted adjacency matrix $\mathbf{Z} = \mathbf{V} + i\mathbf{U}$ that fully

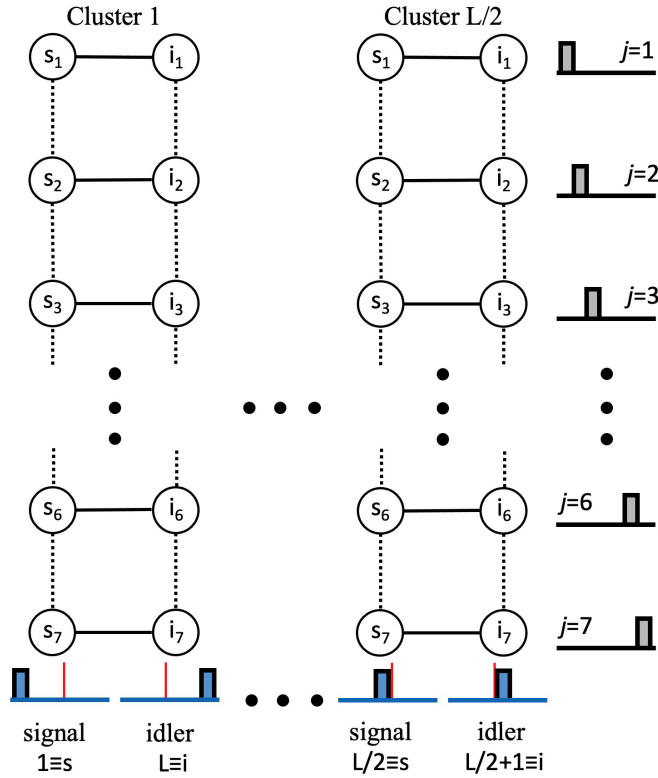


FIG. 3. Set of 7×2 -grid cluster states obtained with a monochromatic pump spectrum and a flat spatial pump distribution in a weakly coupled photonic lattice. Each grid state is composed by $N = 14$ fixels: a pair of frexels $l \equiv s, L + 1 - l \equiv i$ —symmetric with respect to $\omega_h/2$ (in red)—for each of the seven pixels $j = 1, \dots, 7$. Solid lines stand for spectral entanglement (horizontal) and dotted lines stand for spatial entanglement (vertical). $L/2$ ($(L - 1)/2$) spectrally shifted copies of this state are generated for an even (odd) number of frexels L . Each node of the cluster state is in a given spatio-spectral mode represented by gray (spatial) and blue (spectral) rectangles (cf. Fig. 2). Blue and black abscissas represent frequency and guided mode, respectively.

characterizes Gaussian pure states [53]. The real part $\mathbf{V}$ is the graph of the ideal CV cluster state approximated by $\mathbf{Z}$. Its nullifiers are $\hat{\delta} = \hat{\mathbf{y}} - \mathbf{V}\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are vectors of amplitude and phase quadratures, respectively. If the imaginary part $\mathbf{U}$ is small, it can be interpreted as the covariance matrix of the nullifiers $\hat{\delta}$. The elements in its diagonal $\text{Var}(\hat{\delta}_j) = \mathbf{U}_{j,j}$ thus represent the variances of the nullifiers, with its trace $\text{Tr}(\mathbf{U})$ being the error of approximating the ideal cluster state $\mathbf{V}$ with $\mathbf{Z}$ [53]. Figure 4 shows an example of the real (upper) part and the trace of the imaginary part (lower) of $\mathbf{Z}$ obtained using Eqs. (15) and (16) for $N = 7$ and $L = 2$. We obtain a nonunit weight matrix $\mathbf{V}$ with negative entries and the weight of the spectral vertices ($s_j : i_j$) four times that of the spatial vertices ($s_j : s_{j\pm 1}$), ($i_j : i_{j\pm 1}$) (Fig. 4, upper panel). For the parameters used in our simulation, there are also negligible spatio-spectral correlations ($s_j : i_{j\pm 1}$) and ($i_j : s_{j\pm 1}$) more than ten times lower than the spatial ones (not shown for the sake of clarity). From this $\mathbf{V}$, we can write down an approximation of the nullifiers of the state

$$\hat{\delta}_{s,j} = \hat{y}_{s,j} + \frac{1}{4}(\hat{x}_{s,j-1} + \hat{x}_{s,j+1}) + \hat{y}_{i,j}, \quad (17)$$

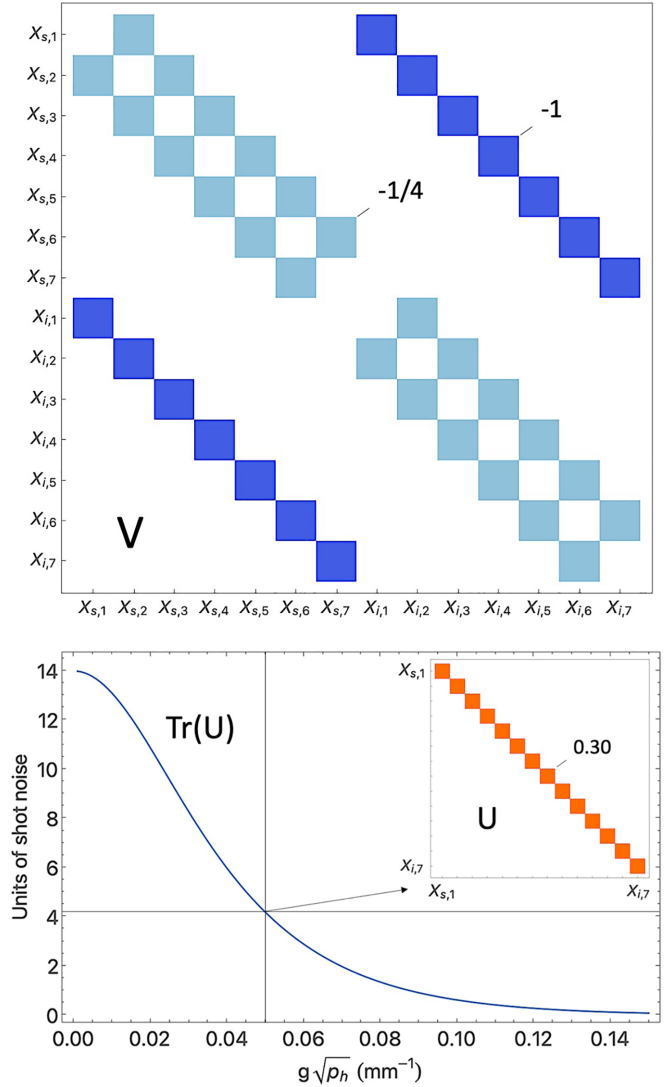


FIG. 4. Real part $\mathbf{V}$ (upper panel) of the complex-weighted adjacency matrix $\mathbf{Z} = \mathbf{V} + i\mathbf{U}$ obtained from Eqs. (15) and (16) for $L = 2, N = 7$. $\mathbf{V}$ is the canonical graph of the state, whereas the trace of the imaginary part $\mathbf{U}$ accounts for the error of the approximation. The value of $\text{Tr}(\mathbf{U})$ for different values of nonlinearity is shown in the lower panel (blue curve). The inset shows the matrix $\mathbf{U}$ for a specific pump power. As it is diagonal and small, it represents the variances of the graph nullifiers. $x_{s(i),j}$ is the amplitude quadrature of a fixel j (pixel j , frexel l). The vacuum shot noise is set as 1. We have applied a $\pi/2$ rotation in the idler-mode phase space (exchange of labels of quadratures for the idler-modes), and used a homogeneous coupling profile $\vec{f} = \vec{1}$ with $C_M = 0.01 \text{ mm}^{-1}$, $g\sqrt{p_h} = 0.05 \text{ mm}^{-1}$ (upper panel, lower panel inset) and $z = 20 \text{ mm}$.

$$\hat{\delta}_{i,j} = -\hat{x}_{i,j} + \frac{1}{4}(\hat{y}_{i,j-1} + \hat{y}_{i,j+1}) + \hat{x}_{s,j}, \quad (18)$$

for $j = 1, \dots, 7$, and with a $\pi/2$ rotation in the idler-mode phase space. The inset in Fig. 4 (lower panel) shows the matrix $\mathbf{U}$, that can be interpreted as the covariance matrix of the nullifiers for $\mathbf{U}$ small (here $g\sqrt{p_h}$ large). Here, the variances are approximately equal and squeezed—well below their respective shot noise level for both 3- and 4-node nullifiers, $33/16$ and $17/8$, respectively. Hence, for a given

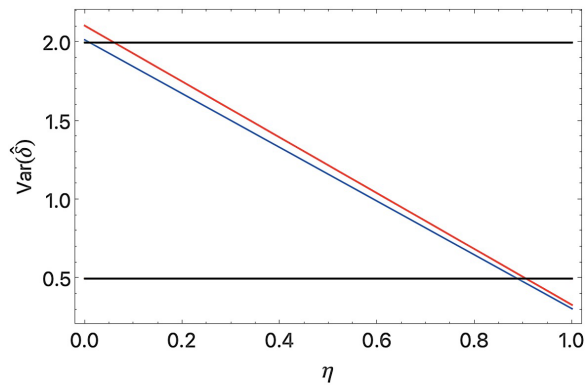


FIG. 5. Effect of losses on the variances of the nullifiers. Dependence of 3- and 4-node variances with η are in blue and red, respectively. Black horizontal lines represent the squeezing threshold per nullifier for spatial ($\text{Var}(\hat{\delta}) < 1/2$) and spectral ($\text{Var}(\hat{\delta}) < 2$) full inseparability.

length of the array, in the limit of weak evanescent coupling and large nonlinearity, the state given by Eqs. (15) and (16) resembles the grid state of Fig. 3 with the error vanishing for infinite squeezing (Fig. 4 lower). This result is expected since for weak evanescent coupling SPDC light produced in each waveguide *spreads* only into nearest-neighbor waveguides, but not beyond. In the limit of high coupling, the state will present a different entanglement geometry [44]. Note that this analysis is applicable to any number of spatial and spectral modes, N and L , respectively.

IV. FEASIBILITY AND OUTLOOK

We analyze now the effect of losses on the generation and measurement of the grid cluster state. The analysis of the previous section only applies for Gaussian pure states, so we cannot apply it in the present case. What we can do, however, is to calculate the variance of the nullifier given by Eqs. (17) and (18) directly from the elements of a lossy covariance matrix as in Eq. (13). In Fig. 5, we show how $\text{Var}(\hat{\delta})$ varies with respect to an effective optical transmittance η that characterizes uniform Gaussian losses. The first figure of merit is nullifier squeezing. The calculation shows that 3- and 4-node nullifiers, in blue and red, respectively, are squeezed while $\eta \neq 0$. A second figure of merit is nullifier entanglement, usually certified experimentally by the violation of van Loock-Furusawa inequalities [54]. Here, we get two bounds of full separability, one for spectral entanglement and the other for spatial entanglement. They are given, respectively, by

$$\text{Var}(\hat{\delta}_{s,j}) + \text{Var}(\hat{\delta}_{i,j}) \geq 4, \quad (19)$$

$$\text{Var}(\hat{\delta}_{s(i),j}) + \text{Var}(\hat{\delta}_{s(i),j+1}) \geq 1. \quad (20)$$

Thus, spectral entanglement appears between any pair of nodes minimally squeezed ($\text{Var}(\hat{\delta}) < 2$) whereas spatial entanglement requires a large squeezing ($\text{Var}(\hat{\delta}) < 1/2$) to be certified. These bounds are displayed in Fig. 5, showing that spatial entanglement is lost above $\approx 10\%$ losses. This value is challenging but feasible, as detection optical losses as low as 8% have been recently reported [29]. Interestingly, this threshold remains independent of the total number of spatio-spectral

modes, ensuring that scalability is not affected. Moreover, this limitation can be alleviated by building up the initial squeezing by increasing the value of $g\sqrt{p_h}$. Note that this is a consequence of the weak coupling regime set in this specific example. Other regimes with larger coupling—producing a different type of cluster states—will not suffer from this issue.

An optical phase-locking system is essential to distribute the cluster nodes in a quantum communication network while keeping the relative phases between nodes constant. This can be accomplished using a classical frequency comb that is phase locked to the broadband SPDC: Classical-comb frequency bands will act as the dedicated pilot tone for their corresponding SPDC spectral band. Typically, a mode-locked laser could act as master laser, being split into a LO, an auxiliary (seed) field, and a field to be upconverted by second-harmonic generation to generate the pump [15,16,33,55]. In a hold-and-measure optical phase locking, the seed takes the role of the SPDC field along a chopping cycle, allowing one to retrieve phase information from the interaction of the seed with the pump and LO fields [56]. Since pump and SPDC phases are locked by the nonlinear interaction, setting the relative phases between seed and pump and seed and LO through suitable detection and phase feedback, enables locking of the LO-SPDC phase over time. Once the seed is blocked, the feedback loop keeps the phase stable for a period longer than the measurement time, time used to measure the correct quadrature at each node.

Grid cluster states are an important resource for measurement-based quantum computing [57]. The limit in the size of the grid of Fig. 3 depends mainly on the number of pixels—waveguides—in the lattice. Arrays with tens of modes (N) have been recently demonstrated [27,41]. In addition, shaping spatially the pump has been demonstrated in different substrates [13,30,58,59]. The low efficiency of the $\chi^{(2)}$ nonlinearity in waveguides could nonetheless hamper the scalability of the system due to the need for large amounts of pump power per waveguide. This issue can be alleviated with thin-film waveguides with efficiencies 20 times that of standard waveguides [60,61]. Notably, using an efficiency for second-harmonic generation of $2600\% \text{ W}^{-1} \text{ cm}^{-2}$ as reported in Ref. [60], only 10 mW of pump power per waveguide would be necessary to reach the value $g\sqrt{p_h} = 0.05 \text{ mm}^{-1}$ used for the simulation of Fig. 4. On the spectral side, the number of frexels depends mainly on the spectral width of the local oscillator—that can be coherently broadened using an all normal dispersion (ANDi) photonic crystal fiber producing low-noise self-phase modulation [62]—and the dispersive power of the diffractive element used to demultiplex frequencies [36]. Notably, homodyne detection with up to $L = 16$ frexels has been demonstrated over bandwidths of tens of nm [15,16,33,55]. Taking all the above into account, achieving graph states with $N = N \times L \approx 100$ spatio-spectral modes seems possible with current technology.

Larger grid cluster states can be created using temporal modes under suitable temporal multiplexing [63,64]. For instance, we can obtain grid states composed of $N \times L$ elements by multiplexing linear clusters in frequency and space encodings. This state is generated at the repetition rate of the pump laser and can be time multiplexed by applying delay lines. Thus, the ability to shape the pump field in both space

and frequency in a nonlinear photonic lattice opens a wide range of possibilities to create two- and three-dimensional cluster states, instrumental, respectively, for universal and fault-tolerant measurement-based quantum computing [65]. For instance, suitable spatial pump shaping can produce a closed linear cluster that multiplexed in frequency results in a torus in a single temporal mode [35].

Another option of practical interest is to use individual modes in the spatial domain and squeezed supermodes in the frequency domain with a flat spatial distribution of the pump fields. In this case, there is a number of independently squeezed spectral supermodes for each j spatial mode [38]. For instance, for both Gaussian phase-matching and pump spectrum, the spectral supermodes are close to Hermite-Gaussian modes [49]. This spectral basis is reachable through suitable LO pulse shaping in homodyne detection. Notably, in this case, the independently squeezed Hermite-Gauss spectral supermodes are entangled due to spatial correlations. Thus, in the example above we would obtain L copies of an N -mode linear cluster state, each one with a different Hermite-Gaussian spectrum.

De-Gaussifying quantum states generated in the optical lattice can be easily accomplished by photon subtraction in the fixel basis using just a dispersive element that isolates a specific fixel and a high-transmission beam splitter. This is in contrast with broadband-frequency modes that require a nonlinear interaction with a gate beam that selects the spectral mode of subtraction [66]. Fixel-based photon subtraction avoids crosstalks natively as is carried out in a frequency band occupied by just one spectral mode, unlike spectrally broadband photon subtraction, where each mode occupies simultaneously a number of bands, increasing the risk of crosstalk and thus of purity degradation. The non-Gaussianity then spreads over the cluster opening the possibility to engineer multimode non-Gaussian states [67], a crucial resource for quantum advantage [68].

In general, CV graph states of any size can be generated in the spatial domain via individual squeezers and universal optical circuits like a mesh of Mach-Zehnder interferometers [12]. In contrast, our system gives access to a limited subset of all possible graph states. What subset is an open question. As an outlook, a group-theory analysis would determine the set of graph states that is accessible through pump shaping in nonlinear optical lattices. However, the subset accessible via nonlinear optical lattices has a better scaling than optical linear circuits, as the size of the array will be similar to that of the individual waveguide squeezers section, saving the space corresponding to the interferometer section. Moreover, the tunability in the state generation is, in our case, inherited from the classical-pump pulse shaping in contrast to the linear optical circuit, which acts directly on the generated quantum states, that are more sensitive to propagation losses.

Finally, we note that in general type-0 and type-I SPDC in few-centimeter waveguides with periodic poling like PP-KTP and PPLN present a broad Δ_{SPDC} limiting the squeezing available per frequency band [16]. Group velocity matching in type-II and, notably, in type-I SPDC has been recently proposed as a possible solution for large squeezing in a limited bandwidth in centimeter-scale waveguides [28,69]. Moreover, poling engineering in $\chi^{(2)}$ nonlinear waveguides enables the

generation of tailored spatio-spectral modes that can improve resources like squeezing or entanglement [70,71].

V. CONCLUSIONS

As a summary, we have analyzed the spatio-spectral features of a $\chi^{(2)}$ nonlinear photonic lattice, presented the general solution of the system in the low-gain regime, and discussed the possibilities to generate large and distributable two-dimensional spatio-spectral cluster states with a single integrated source through suitable spatio-spectral pump shaping. Recent demonstrations of squeezing in single nonlinear waveguides in spatial and spectral modes [14–16,29], and of second-harmonic generation in arrays of nonlinear waveguides and in slab waveguides with arbitrarily reconfigurable two-dimensional distribution of nonlinearity [30,72], open the door to realizing fully integrated multimode spatio-spectral squeezing in optical lattices, paving the way for scalable quantum photonic technologies as quantum networks, distributed quantum sensing, and universal and fault-tolerant measurement-based quantum computing.

ACKNOWLEDGMENTS

This work was supported by MICINN through the European Union NextGenerationEU recovery plan (PRTR-C17.I1), the Galician Regional Government through “Planes Complementarios de I+D+I con las Comunidades Autónomas” in Quantum Communication, and the Paris Ile-de-France region in the framework of DIM SIRTEQ.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

We below analyze the approximation over which our model is valid. For that, we describe the dependence of the coupling with the wavelength following the model of Ref. [20], but making explicit the dependence with the distance between waveguides as in Ref. [30]. We write thus

$$C(\lambda, d) = \frac{C_0}{\lambda} e^{-\Gamma_0 \frac{n(\lambda)}{\lambda} d}, \quad (\text{A1})$$

where λ is the wavelength, $n(\lambda)$ the refractive index in the waveguides, d the distance between waveguides, and C_0 and Γ_0 are constants. This model is suited for distances d larger than a minimal distance where next-to-nearest neighbor effects are negligible. For instance, using the data of Ref. [30] for a nonlinear directional coupler in LN, we get $C_0 = 25.6 \mu\text{m}/\text{mm}$ and $\Gamma_0 = 0.19$ for $d \geq 13 \mu\text{m}$ and λ in μm .

In order to analyze the dependence of the coupling profile with the wavelength we need firstly to define it. The coupling profile is mapped to a set of distances between waveguides—or distance profile. Experimentally, we define a set of distances that map the coupling at a given wavelength λ_0 using Eq. (A1). The largest coupling of the profile C_M will

have associated the shortest interdistance d_m as

$$C_M = \frac{C_0}{\lambda_0} e^{-\Gamma_0 \frac{n(\lambda_0)}{\lambda_0} d_m}. \quad (\text{A2})$$

We calculate the coupling between waveguides j and $j + 1$, writing Eq. (A1) in terms of C_M and the j th interdistance d_j ,

$$C_j(\lambda, d_j) = C_M \frac{\lambda_0}{\lambda} e^{-\Gamma_0 \left(\frac{n(\lambda)}{\lambda} d_j - \frac{n(\lambda_0)}{\lambda_0} d_m \right)} \equiv C_M f_j(\lambda). \quad (\text{A3})$$

This equation defines a coupling profile wavelength dependent $\tilde{f}(\lambda)$. The (normalized) coupling profile of design is however $\tilde{f}^D = \tilde{f}(\lambda_0)$ with $f_j^D \in [0, 1]$. For example, for a Glauber-Fock array the normalized coupling profile is $f_j^D = \sqrt{j/(N-1)}$ for $j = 1, \dots, N-1$ [34]. The experimental set of distances corresponding to this coupling profile is

$$d_j = d_m - \frac{\lambda_0}{\Gamma_0 n(\lambda_0)} \ln(f_j^D). \quad (\text{A4})$$

The coupling profile is distorted at wavelengths λ off λ_0 and given by

$$f_j(\lambda) = \frac{\lambda_0}{\lambda} e^{\frac{\lambda_0}{\lambda} \frac{n(\lambda)}{n(\lambda_0)} \ln(f_j^D)} e^{-\Gamma_0 \left(\frac{n(\lambda)}{\lambda} - \frac{n(\lambda_0)}{\lambda_0} \right) d_m}.$$

The above equation shows that the profile distortion is more evident as the distance to the design wavelength $|\lambda - \lambda_0|$ increases and f_j^D decreases. For instance, for an array in LN with $d_0 = 13 \mu\text{m}$, $\Gamma_0 = 0.19$, and coupling profile elements at the design wavelength $f_j^D(1.55 \mu\text{m}) = 1$ (0.1), we get a slight increment of up to ± 5 (10)% in a bandwidth of 60 nm and up to ± 8 (16)% in 100 nm. This variation is lower than that produced by fabrication errors. In general, the distortion of the coupling profile—and thus of the supermodes—for this array would be negligible over a bandwidth above 60 nm and, for the particular case of an homogeneous coupling profile where $f_j^D(1.55 \mu\text{m}) = 1$ for all j , there would be no perceptible effect over a bandwidth above 100 nm. For larger/shorter wavelengths, the supermodes would retain its shape, but would experience different effective propagation lengths.

-
- [1] B. J. Puttnam, G. Rademacher, and R. S. Luis, Space-division multiplexing for optical fiber communications, *Optica* **8**, 1186 (2021).
 - [2] S. Wehner, D. Elkouss, and R. Hanson, Quantum internet: A vision for the road ahead, *Science* **362**, eaam9288 (2018).
 - [3] Z. Zhang and Q. Zhuang, Distributed quantum sensing, *Quantum Sci. Technol.* **6**, 043001 (2021).
 - [4] D. Barral, F. J. Cardama, G. Díaz, D. Faílde, I. F. Llovo, M. M. Juane, J. Vázquez-Pérez, J. Villasuso, C. Pieiro, N. Costas, J. C. Pichel, T. F. Pena, and A. Gómez, Review of distributed quantum computing. From single QPU to high performance quantum computing, *Comput. Sci. Rev.* **57**, 100747 (2025).
 - [5] L. G. Wright, W. H. Renninger, D. N. Christodoulides, and F. W. Wise, Nonlinear multimode photonics: Nonlinear optics with many degrees of freedom, *Optica* **9**, 824 (2022).
 - [6] E. Pelucchi, G. Fagas, I. Aharonovich, D. Englund, E. Figueroa, Q. Gong, H. Hannes, J. Liu, C.-Y. Lu, N. Matsuda, J.-W. Pan, F. Schreck, F. Sciarrino, C. Silberhorn, J. Wang, and K. D. Jons, The potential and global outlook of integrated photonics for quantum technologies, *Nat. Rev. Phys.* **4**, 194 (2021).
 - [7] G. Moody *et al.*, Roadmap on integrated quantum photonics, *J. Phys. Photon.* **4**, 012501 (2022).
 - [8] L. Labonté, O. Alibart, V. D'Auria, F. Dautre, J. Etesse, G. Sauder, A. Martin, E. Picholle, and S. Tanzilli, Integrated photonics for quantum communications and metrology, *PRX Quantum* **5**, 010101 (2024).
 - [9] PsiQuantum team, A manufacturable platform for photonic quantum computing, *Nature (London)* **641**, 876 (2025).
 - [10] H. Aghaee Rad *et al.*, Scaling and networking a modular photonic quantum computer, *Nature (London)* **638**, 912 (2025).
 - [11] S. L. Braunstein and P. van Loock, Quantum information with continuous variables, *Rev. Mod. Phys.* **77**, 513 (2005).
 - [12] W. Asavanant and A. Furusawa, Multipartite continuous-variable optical quantum entanglement: Generation and application, *Phys. Rev. A* **109**, 040101 (2024).
 - [13] F. Lenzini, J. Janousek, O. Thearle, M. Villa, B. Haylock, S. Kasture, L. Cui, H.-P. Phan, D. V. Dao, H. Yonezawa, P. K. Lam, E. H. Huntington, and M. Lobino, Integrated photonic platform for quantum information with continuous variables, *Sci. Adv.* **4**, eaat9331 (2018).
 - [14] R. Nehra, R. Sekine, L. Ledezma, Q. Guo, R. M. Gray, A. Roy, and A. Marandi, Few-cycle vacuum squeezing in nanophotonics, *Science* **377**, 1333 (2022).
 - [15] T. Kouadrou, F. Sansavini, M. Ansquer, J. Henaff, N. Treps, and V. Parigi, Spectrally shaped and pulse-by-pulse multiplexed multimode squeezed states of light, *APL Photon.* **8**, 086113 (2023).
 - [16] V. Roman-Rodriguez, D. Fainsin, G. L. Zanin, N. Treps, E. Diamanti, and V. Parigi, Multimode squeezed state for reconfigurable quantum networks at telecommunication wavelengths, *Phys. Rev. Res.* **6**, 043113 (2024).
 - [17] R. Yang, J. Zhang, I. Klich, C. González-Arciniegas, and O. Pfister, Spatiotemporal graph states from a single optical parametric oscillator, *Phys. Rev. A* **101**, 043832 (2020).
 - [18] R. F. Barros, G. B. Alves, O. Pfister, and A. Z. Khoury, Quantum-controlled cluster states, *Phys. Rev. A* **104**, 033713 (2021).
 - [19] A. S. Solntsev, A. A. Sukhorukov, D. N. Neshev, and Y. S. Kivshar, Spontaneous parametric down-conversion and quantum walks in arrays of quadratic nonlinear waveguides, *Phys. Rev. Lett.* **108**, 023601 (2012).
 - [20] R. Kruse, F. Katzschmann, A. Christ, A. Schreiber, S. Wilhelm, K. Laiho, A. Gábris, C. S. Hamilton, I. Jex, and C. Silberhorn, Spatio-spectral characteristics of parametric down-conversion in waveguide arrays, *New J. Phys.* **15**, 083046 (2013).
 - [21] A. Belsley, T. Pertsch, and F. Setzpfandt, Generating path entangled states in waveguide systems with second-order nonlinearity, *Opt. Express* **28**, 28792 (2020).
 - [22] C. S. Hamilton, R. Christ, S. Barkhofen, S. M. Barnett, I. Jex, and C. Silberhorn, Quantum-state creation in nonlinear-waveguide arrays, *Phys. Rev. A* **105**, 042622 (2022).

- [23] Y. He, S. Xia, D. Leykam, and Z. Chen, Optimizing biphoton generation via reconfigurable nonlinear waveguide arrays based on scattering tensor, *Opt. Express* **32**, 32244 (2024).
- [24] J. Delgado-Quesada, D. Barral, K. Bencheikh, and E. A. Rojas-Gonzalez, Analytic solution to degenerate biphoton states generated in arrays of nonlinear waveguides, *Phys. Rev. A* **112**, 013703 (2025).
- [25] D. N. Christodoulides, F. Lederer, and Y. Silberberg, Discretizing light behaviour in linear and nonlinear waveguide lattices, *Nature (London)* **424**, 817 (2003).
- [26] A. S. Solntsev, F. Setzpfandt, A. S. Clark, C. W. Wu, M. J. Collins, C. Xiong, A. Schreiber, F. Katzschnmann, F. Eilenberger, R. Schiek, W. Sohler, A. Mitchell, C. Silberhorn, B. J. Eggleton, T. Pertsch, A. A. Sukhorukov, D. N. Neshev, and Y. S. Kivshar, Generation of nonclassical biphoton states through cascaded quantum walks on a nonlinear chip, *Phys. Rev. X* **4**, 031007 (2014).
- [27] A. Raymond, A. Zecchetto, J. Palomo, M. Morassi, A. Lematre, F. Raineri, M. Amanti, S. Ducci, and F. Baboux, Tunable generation of spatial entanglement in nonlinear waveguide arrays, *Phys. Rev. Lett.* **133**, 233602 (2024).
- [28] V. Roman-Rodriguez, B. Brecht, K. Srinivasan, C. Silberhorn, N. Treps, E. Diamanti, and V. Parigi, Continuous variable multimode quantum states via symmetric group velocity matching, *New J. Phys.* **23**, 043012 (2021).
- [29] T. Kashiwazaki, T. Yamashima, K. Enbutsu, T. Kazama, A. Inoue, K. Fukui, M. Endo, T. Umeki, and A. Furusawa, Over-8-dB squeezed light generation by a broadband waveguide optical parametric amplifier toward fault-tolerant ultra-fast quantum computers, *Appl. Phys. Lett.* **122**, 234003 (2023).
- [30] D. Barral, V. D'Auria, F. Dautre, T. Lunghi, S. Tanzilli, A. P. Rambau, S. Tascu, J. A. Levenson, N. Belabas, and K. Bencheikh, Supermode-based second harmonic generation in a nonlinear interferometer, *Opt. Express* **29**, 37175 (2021).
- [31] D. B. Horoshko, Generator of spatial evolution of the electromagnetic field, *Phys. Rev. A* **105**, 013708 (2022).
- [32] B. Dayan, Theory of two-photon interactions with broadband down-converted light and entangled photons, *Phys. Rev. A* **76**, 043813 (2007).
- [33] J. Roslund, R. Medeiros de Araujo, S. Jiang, C. Fabre, and N. Treps, Wavelength-multiplexed quantum networks with ultra-fast frequency combs, *Nat. Photon.* **8**, 109 (2014).
- [34] D. Barral, M. Walschaers, K. Bencheikh, V. Parigi, J. A. Levenson, N. Treps, and N. Belabas, Quantum state engineering in arrays of nonlinear waveguides, *Phys. Rev. A* **102**, 043706 (2020).
- [35] D. Barral, M. Walschaers, K. Bencheikh, V. Parigi, J. A. Levenson, N. Treps, and N. Belabas, Versatile photonic entanglement synthesizer in the spatial domain, *Phys. Rev. Appl.* **14**, 044025 (2020).
- [36] T. Michel, Optimization of the pump spectral shape in a parametric down conversion process to generate multimode entangled state, Thesis at Australian National University and Sorbonne Université, 2021.
- [37] F. Arzani, C. Fabre, and N. Treps, Versatile engineering of multimode squeezed states by optimizing the pump spectral profile in spontaneous parametric down-conversion, *Phys. Rev. A* **97**, 033808 (2018).
- [38] C. Fabre and N. Treps, Modes and states in quantum optics, *Rev. Mod. Phys.* **92**, 035005 (2020).
- [39] N. Liu, Y. Liu, J. Li, L. Yang, and X. Li, Generation of multi-mode squeezed vacuum using pulse pumped fiber optical parametric amplifiers, *Opt. Express* **24**, 2125 (2016).
- [40] E. Kapon, J. Katz, and A. Yariv, Supermode analysis of phase-locked arrays of semiconductor lasers, *Opt. Lett.* **9**, 125 (1984).
- [41] Y. Yang *et al.*, Programmable high-dimensional Hamiltonian in a photonic waveguide array, *Nat. Commun.* **15**, 50 (2024).
- [42] T. Onodera, M. M. Stein, B. A. Ash *et al.*, Arbitray control over multimode wave propagation for machine learning, *Nat. Phys.* (2025).
- [43] There is also the possibility of ordering by using N blocks composed of L spectral modes with indices $j = l + (j - 1)L$, $k = l + (k - 1)L$.
- [44] D. Barral, K. Bencheikh, J. A. Levenson, and N. Belabas, Scalable multimode entanglement based on efficient squeezing of propagation eigenmodes, *Phys. Rev. Res.* **3**, 013068 (2021).
- [45] A. Christ, B. Brecht, W. Maurer, and C. Silberhorn, Theory of quantum frequency conversion and type-II parametric down-conversion in the high-gain regime, *New J. Phys.* **15**, 053038 (2013).
- [46] G. Cariolaro and G. Pierobon, Bloch-Messiah reduction of Gaussian unitaries by Takagi factorization, *Phys. Rev. A* **94**, 062109 (2016).
- [47] W. P. Grice, A. U'Ren, and I. A. Walmsley, Eliminating frequency and space-time correlations in multiphoton states, *Phys. Rev. A* **64**, 063815 (2001).
- [48] W. Wasilewski, A. I. Lvovsky, K. Banaszek, and C. Radzewicz, Pulsed squeezed light: Simultaneous squeezing of multiple modes, *Phys. Rev. A* **73**, 063819 (2006).
- [49] G. Patera, N. Treps, C. Fabre, and G. J. Valcarcel, Quantum theory of synchronously pumped type I optical parametric oscillators: Characterization of the squeezed supermodes, *Eur. Phys. J. D* **56**, 123 (2010).
- [50] D. Barral, K. Bencheikh, V. D'Auria, S. Tanzilli, N. Belabas, and J. A. Levenson, Minimum resources for versatile continuous-variable entanglement in integrated nonlinear waveguides, *Phys. Rev. A* **98**, 023857 (2018).
- [51] F. Kaiser, B. Fredici, A. Zavatta, V. D'Auria, and S. Tanzilli, A fully guided-wave squeezing experiment for fiber quantum networks, *Optica* **3**, 362 (2016).
- [52] C. Roh, G. Gwak, Y.-D. Yoon, and Y.-S. Ra, Generation of three-dimensional cluster entangled state, *Nat. Photon.* **19**, 526 (2025).
- [53] N. C. Menicucci, S. T. Flammia, and P. van Loock, Graphical calculus for Gaussian pure states, *Phys. Rev. A* **83**, 042335 (2011).
- [54] P. van Loock and A. Furusawa, Detecting genuine multipartite continuous-variable entanglement, *Phys. Rev. A* **67**, 052315 (2003).
- [55] Y. Cai *et al.*, Multimode entanglement in reconfigurable graph states using optical frequency combs, *Nat. Commun.* **8**, 15645 (2017).
- [56] G. Roeland, Quantum non-gaussianity certification in continuous-variable optics, Thesis at Sorbonne Université, 2023.
- [57] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Universal quantum computation with continuous-variable cluster states, *Phys. Rev. Lett.* **97**, 110501 (2006).

- [58] H. Jin *et al.*, On-chip generation and manipulation of entangled photons based on reconfigurable lithium-niobate waveguide circuits, *Phys. Rev. Lett.* **113**, 103601 (2014).
- [59] J. Bao *et al.*, Very-large-scale integrated quantum graph photonics, *Nat. Photon.* **17**, 573 (2023).
- [60] C. Wang *et al.*, Ultrahigh-efficiency wavelength conversion in nanophotonic periodically poled lithium niobate waveguides, *Optica* **5**, 1438 (2018).
- [61] C. H. Stokowski *et al.*, Integrated quantum optical phase sensor in thin film lithium niobate, *Nat. Commun.* **14**, 3355 (2023).
- [62] P. Renault *et al.*, Experimental optical simulator of reconfigurable and complex quantum environment, *PRX Quantum* **4**, 040310 (2023).
- [63] M. V. Larsen, X. Guo, C. R. Breum, J. S. Neergaard-Nielsen, and U. L. Andersen, Deterministic generation of a two-dimensional cluster state, *Science* **366**, 369 (2019).
- [64] W. Asavanant, Y. Shiozawa, S. Yokoyama, B. Charoensombutamon, H. Emura, R. N. Alexander, S. Takeda, J. Yoshikawa, N. C. Menicucci, H. Yonezawa, and A. Furusawa, Generation of time-domain-multiplexed two-dimensional cluster state, *Science* **366**, 373 (2019).
- [65] J. E. Bourassa, R. N. Alexander, M. Vasmer, A. Patil, I. Tzitrin, T. Matsuura, D. Su, B. Q. Baragiola, S. Guha, G. Dauphinais, K. K. Sabapathy, N. C. Menicucci, and I. Dhand, Blueprint for a scalable photonic fault-tolerant quantum computer, *Quantum* **5**, 392 (2021).
- [66] Y. S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, Non-Gaussian quantum states of a multimode light field, *Nat. Phys.* **16**, 144 (2020).
- [67] M. Walschaers, S. Sarkar, V. Parigi, and N. Treps, Tailoring non-Gaussian continuous-variable graph states, *Phys. Rev. Lett.* **121**, 220501 (2018).
- [68] U. Chabaud and M. Walschaers, Resources for bosonic quantum computational advantage, *Phys. Rev. Lett.* **130**, 090602 (2023).
- [69] D. B. Horoshko, M. I. Kobaylov, V. Parigi, and N. Treps, Few-mode squeezing in type-I parametric downconversion by complete group velocity matching, *Opt. Lett.* **49**, 4078 (2024).
- [70] D. Barral, N. Belabas, K. Bencheikh, and J. A. Levenson, Coupling quasi-phase matching: Entanglement buildup in $\chi^{(2)}$ nonlinear-waveguide arrays, *Phys. Rev. A* **100**, 013824 (2019).
- [71] T. W. Weiss and A. Peruzzo, Nonlinear domain engineering for quantum technologies, *Appl. Phys. Rev.* **12**, 011318 (2025).
- [72] R. Yanagimoto, B. A. Ash, M. M. Sohoni, M. M. Stein, Y. Zhao, F. Presutti, M. Jankowski, L. G. Wright, T. Onodera, and P. L. McMahon, Programmable on-chip nonlinear photonics, *Nature (London)* (2025).