

Relaxation dynamics and long-time tails explain shear-induced diffusion of soft athermal particles near jamming

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We numerically study shear-induced diffusion of soft athermal particles in two dimensions. The Green-Kubo (GK) formula is applicable to the shear-induced diffusion coefficient, where both mean-squared transverse velocity and relaxation time included in the GK formula are well described by critical scaling near jamming. We show that the autocorrelation function of transverse velocities is stretched exponential if the system is below jamming or shear rate is large enough. However, if the system is above jamming and the shear rate is sufficiently small, the autocorrelation function exhibits a long-time tail such that time integral in the GK formula diverges in two dimensions. We propose an empirical scaling relation for the critical exponents and show that the long-time tail is consistent with the divergence of the shear-induced diffusion coefficient.

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Soft athermal particles such as foam, emulsions, colloidal suspensions, and granular materials are ubiquitous, and a better understanding of their transport properties is of central importance to technology, e.g., for manufacturing of food, pharmaceutical, and personal care products [1,2]. It has been well recognized that soft athermal particles exhibit a *jamming transition* at critical packing fraction ϕ_J [3–5], where their mechanical [5,6] and rheological properties [7–13] can be understood in terms of critical phenomena. For instance, flow curves are nicely explained by critical scaling: Both viscosity divergence [14–18] and scaling of yield stress [7,8,19] are described by power laws of the proximity to jamming, $\Delta\phi \equiv \phi - \phi_J$, where ϕ is the packing fraction of the particles.

In contrast, *shear-induced diffusion*, which is relevant to mixing and segregation of soft athermal particles [20], has been studied with the focus on its dependence on a shear rate. Using natural length scales and timescales, i.e., the particle diameter d_0 and inverse of shear rate $\dot{\gamma}^{-1}$, one can expect the scaling of shear-induced diffusion coefficient as $D \sim d_0^2 \dot{\gamma}$ [21–25]. However, increasing the shear rate, researchers observed a crossover from the linear scaling to sublinear scaling, i.e., from $D \propto \dot{\gamma}$ to $\dot{\gamma}^{0.8}$ [26–29]. Another sublinear scaling, $D \propto \dot{\gamma}^{0.78}$, was also found at the onset of jamming transition, $\phi \simeq \phi_J$ [30]. Moreover, the scaling of shear-induced diffusion coefficient was modified as $D \sim d_0^2 \dot{\gamma} / \sqrt{I}$ with the inertia number I in order to take account of the influence of collective motions of the particles [31–33].

Recently, we demonstrated a scaling data collapse of the shear-induced diffusion coefficient near jamming [34], predicting not only the crossover but also the critical divergence

of D (that is consistent with the inverse proportionality with the pressure p , i.e., $D \propto p^{-1}$ [24]). It was shown that the diffusion coefficient scales as $D \sim d_0 \xi \dot{\gamma}$ with a characteristic size of collective motions, ξ [31,35]. If the system is above jamming ($\phi > \phi_J$) and sheared at a sufficiently small shear rate, the size of collective motions extends to the system length L , i.e., $\xi \sim L$, so that the diffusion coefficient over the shear rate is linear in the system length as $D/\dot{\gamma} \propto L$ [34]. This agrees with the previous studies of sheared amorphous solids [36–39] though the finite-size effect on $D/\dot{\gamma}$ was explained as a result of spatially correlated flip events. In recent years, (slightly) different finite-size scaling, $D/\dot{\gamma} \propto L^{1.05}$ [40,41] and $L^{1.5}$ [42], was also reported by numerical simulations of elastoplastic model.

The previous studies have revealed the relationship between the shear-induced diffusion coefficient and characteristic length scales. However, transport properties of thermal systems such as classical fluids and glasses are explained by relaxation dynamics [43–45], and little is known about the role of relaxation dynamics in the shear-induced diffusion of soft athermal particles. Notice that the relaxation dynamics of soft athermal particles near jamming have been extensively investigated [10–12,46–54].

In this Letter, we investigate the shear-induced diffusion of soft athermal particles by molecular dynamics (MD) simulations. Employing the Green-Kubo (GK) relation for the diffusivity, we clarify the link between diffusion coefficient D and relaxation time extracted from the velocity autocorrelation function of the particles. We show that (1) mean-squared particle velocities obey critical scaling near jamming, (2) the relaxation time is also explained by critical scaling only if the system is not in a quasistatic regime above jamming, and (3) long-time tails are observed in the quasistatic regime above jamming such that D formulated by the GK relation diverges in the thermodynamic limit, $L \rightarrow \infty$. We use the reduced GK formula to derive (4) an empirical scaling relation for the

critical exponents. In contrast to the previous works, where D is connected to the length scales [31–34] and finite-size effects on D are interpreted as the system spanning flip events [36–42], our analysis is based on the timescale, where the divergence of D is explained by the long-time tail.

Numerical methods. We perform MD simulations of soft athermal particles in two dimensions. Our system consists of an equal number of small and large particles with diameters d_0 and $1.4d_0$, respectively [5]. The total number of particles is $N = 8192$ (we examine finite-size effects in latter), and their packing fraction ϕ is controlled around the jamming transition density, $\phi_J \simeq 0.8433$ [7]. A repulsive force between the particles, i and j , in contact is modeled by elastic force $\mathbf{f}_{ij}^e = k\delta_{ij}\mathbf{n}_{ij}$, where k is the stiffness and $\mathbf{n}_{ij} \equiv \mathbf{r}_{ij}/|\mathbf{r}_{ij}|$ with the relative position $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ is the normal unit vector. The elastic force is linear in the overlap $\delta_{ij} \equiv R_i + R_j - |\mathbf{r}_{ij}| > 0$, where R_i (R_j) is the radius of the particle i (j). We also add a damping force to every particle as $\mathbf{f}_i^d = -\zeta\{\mathbf{v}_i - \mathbf{u}(\mathbf{r}_i)\}$, where ζ , $\mathbf{v}_i$, and $\mathbf{u}(\mathbf{r})$ are the damping coefficient, particle velocity, and external flow field, respectively. Note that the stiffness and damping coefficient determine a microscopic timescale as $t_0 \equiv \zeta/k$. To simulate simple shear flows of the particles, we impose the external flow field in the x direction as $\mathbf{u}(\mathbf{r}) = (\dot{\gamma}y, 0)$ under the Lees-Edwards boundary condition [55], where $\dot{\gamma}$ is introduced as a shear rate. Due to the external flow field, the particles are driven in the x direction and diffuse in the y direction. In our simulations, motions of the particles are described by *overdamped dynamics* [7,8,56], i.e., $\sum_{j \neq i} \mathbf{f}_{ij}^e + \mathbf{f}_i^d = \mathbf{0}$, where we numerically integrate the particle velocity, $\mathbf{v}_i = \mathbf{u}(\mathbf{r}_i) + \zeta^{-1} \sum_{j \neq i} \mathbf{f}_{ij}^e$, with a small time increment, $\Delta t = 0.1t_0$.

In the following, we only analyze the data in a steady state, where the shear strain applied to the system exceeds unity, and scale every time and length by t_0 and d_0 , respectively. In addition, we vary ϕ and $\dot{\gamma}$ in the ranges $0.8 \leq \phi \leq 0.9$ and $10^{-7} \leq \dot{\gamma}t_0 \leq 10^{-2}$, respectively.

Diffusion coefficient and the Green-Kubo relation. We quantify shear-induced diffusion of the particles by the transverse component of mean-squared displacement (MSD) [34],

$$\Delta(\tau)^2 = \left\langle \frac{1}{N} \sum_{i=1}^N \Delta y_i(t + \tau)^2 \right\rangle_t. \quad (1)$$

Here, $\Delta y_i(t + \tau) \equiv y_i(t + \tau) - y_i(t)$ is the y component of particle displacement for the duration τ , and we take the ensemble average $\langle \cdots \rangle_t$ by changing the initial time t in a steady state. The MSD is linear in τ if the amount of shear strain (for the duration τ) exceeds unity, i.e., if $\gamma \equiv \dot{\gamma}\tau > 1$. Thus, a diffusion coefficient can be defined as $D = \lim_{\tau \rightarrow \infty} \Delta(\tau)^2/2\tau$.

In Ref. [34], we demonstrated a scaling data collapse of the diffusion coefficient near jamming and numerically confirmed its critical scaling as $D \sim |\Delta\phi|^{-\nu}\dot{\gamma}$ ($\phi < \phi_J$), $D \sim |\Delta\phi|^{0.3\lambda-\nu}\dot{\gamma}^{0.7}$ ($\phi > \phi_J$), and $D \sim \dot{\gamma}^{1-\nu/\lambda}$ ($\Delta\phi$ -independent critical regime). As shown in the Supplemental Material [57], the critical exponents are quantitatively estimated as $\nu \simeq 0.939$ and $\lambda \simeq 3.87$ [58], which agree well with the previous simulation [30]. Because the diffusion coefficient over the shear rate is proportional to the length scale as $D/\dot{\gamma} \propto \xi$ [34,57], our result below jamming implies the scaling,

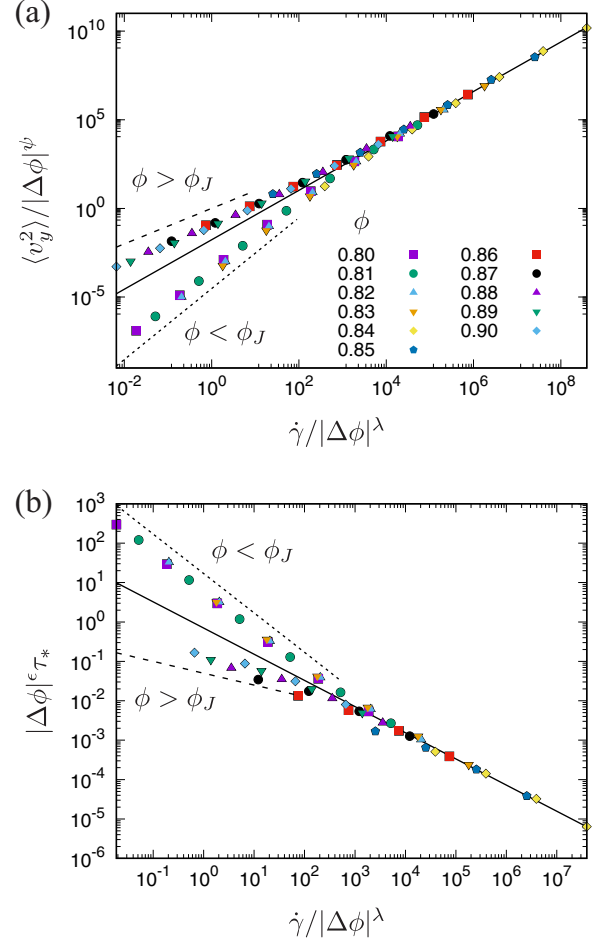


FIG. 1. Scaling data collapses of (a) the mean-squared transverse velocity $\langle v_y^2 \rangle$ and (b) relaxation time τ_* , where the scaling exponents are given by $\lambda = 3.87$, $\psi = 5.39$, and $\epsilon = 2.57$. The solid lines have the slopes (a) ψ/λ and (b) $-\epsilon/\lambda$, and the symbols represent the packing fraction ϕ as listed in the legend of panel (a).

$\xi \propto D/\dot{\gamma} \sim |\Delta\phi|^{-\nu} \simeq |\Delta\phi|^{-0.939}$, which is consistent with the divergence of correlation length at the jamming transition, i.e., $\xi \sim |\Delta\phi|^{-1}$ [35,59,60]. In addition, our scaling indicates a crossover from $D \propto \dot{\gamma}$ to $\dot{\gamma}^{1-\nu/\lambda} \simeq \dot{\gamma}^{0.757}$ with the increase of shear rate, which also agrees with the previous studies [26–30].

On the other hand, the diffusion coefficient is formulated by the GK relation as [30,43]

$$D = \langle v_y^2 \rangle \int_0^\infty C(\tau) d\tau. \quad (2)$$

Here, $\langle v_y^2 \rangle \equiv \langle N^{-1} \sum_{i=1}^N v_{iy}(t)^2 \rangle_t$ is the y component of mean-squared particle velocity, where $\langle \cdots \rangle_t$ is the time average in a steady state. On the right-hand side of Eq. (2), $C(\tau) = \langle v_y(t + \tau)v_y(t) \rangle_t / \langle v_y^2 \rangle$ is the normalized velocity autocorrelation function.

Critical scaling of the mean-squared velocity. As the diffusion coefficient D exhibits critical behavior near jamming [30,34], the mean-squared transverse velocity $\langle v_y^2 \rangle$ in the GK formula [Eq. (2)] is described by critical scaling. Figure 1(a) displays a scaling data collapse of $\langle v_y^2 \rangle$, where the packing

fraction ϕ increases as listed in the legend. In this figure, the scaling exponent is given by $\psi \simeq 5.39$ (λ is the same as that for D). The solid line has the slope ψ/λ , representing the $\Delta\phi$ -independent critical regime. In the Supplemental Material [57], we show how to estimate ψ quantitatively and confirm that the x component of mean-squared velocity fluctuation, i.e., $\langle \delta v_x^2 \rangle$ with $\delta v_x \equiv v_x - \dot{\gamma}y$, exhibits almost the same critical behavior. This means that the critical scaling of *effective temperature* [19,61–63] of our system is similar to that of $\langle v_y^2 \rangle$ [Fig. 1(a)].

In our MD simulations, the energy injected by shear is given by $\dot{\gamma}\sigma$, with σ the shear stress of the system. The injected energy is dissipated by the damping force f_i^d such that the rate of energy dissipation χ is approximated as $\chi \sim \langle v_y^2 \rangle$ [8,64,65]. Therefore, due to the energy balance ($\dot{\gamma}\sigma \sim \chi$), the mean-squared velocity is quadratic in the shear rate as $\langle v_y^2 \rangle \sim \eta\dot{\gamma}^2$, where $\eta \equiv \sigma/\dot{\gamma}$ is defined as shear viscosity. The shear viscosity is constant ($\eta = \text{const}$) if the system is in the Newtonian regime, i.e., if the system is below jamming, $\phi < \phi_J$, and the shear rate is sufficiently small, $\dot{\gamma}t_0 \ll 1$. In contrast, if the system is above jamming, $\phi > \phi_J$, the shear stress exhibits yield stress σ_Y in the quasistatic limit, $\dot{\gamma} \rightarrow 0$ [2]. In this case, the energy balance equation tells us that the mean-squared velocity is linear in the shear rate as $\langle v_y^2 \rangle \sim \dot{\gamma}\sigma_Y$.

Taking account of the dependence of $\langle v_y^2 \rangle$ on $\dot{\gamma}$, we describe our results in Fig. 1(a) as

$$\frac{\langle v_y^2 \rangle}{|\Delta\phi|^\psi} = \mathcal{F}\left(\frac{\dot{\gamma}}{|\Delta\phi|^\lambda}\right), \quad (3)$$

with a scaling function $\mathcal{F}(x)$. In the Newtonian regime, the scaling function is quadratic, $\mathcal{F}(x) \sim x^2$ [dotted line in Fig. 1(a)], so that $\langle v_y^2 \rangle \sim |\Delta\phi|^{\psi-2\lambda}\dot{\gamma}^2 \simeq |\Delta\phi|^{-2.35}\dot{\gamma}^2$. Since $\langle v_y^2 \rangle \sim \eta\dot{\gamma}^2$ in the Newtonian regime, this result indicates $\eta \sim |\Delta\phi|^{-2.35}$, which agrees well with the viscosity divergence near jamming [7,9,10]. If the system is in the quasistatic regime above jamming, the scaling function is linear, $\mathcal{F}(x) \sim x$ [dashed line in Fig. 1(a)], such that $\langle v_y^2 \rangle \sim |\Delta\phi|^{\psi-\lambda}\dot{\gamma} \simeq |\Delta\phi|^{1.52}\dot{\gamma}$. Because $\langle v_y^2 \rangle \sim \dot{\gamma}\sigma_Y$ above jamming, this result means that the yield stress scales as $\sigma_Y \sim |\Delta\phi|^{1.52}$, which is consistent with the previous result, $\sigma_Y \sim |\Delta\phi|^{3/2}$ [8,19]. In the $\Delta\phi$ -independent critical regime, the scaling function is $\mathcal{F}(x) \sim x^z$ with the exponent z . From Eq. (3), we find $\langle v_y^2 \rangle \sim |\Delta\phi|^{\psi-\lambda z}\dot{\gamma}^z$, which should not depend on $\Delta\phi$. Therefore, the exponents satisfy $\psi - \lambda z = 0$, i.e., $z = \psi/\lambda$, as indicated by the solid line in Fig. 1(a).

Critical scaling of relaxation time. Next, we analyze the normalized velocity autocorrelation function $C(\tau)$ in the GK formula [Eq. (2)]. Figure 2(a) displays our numerical results of $C(\tau)$ (symbols), where $\dot{\gamma}t_0 = 10^{-6}$ is used in MD simulations and ϕ varies as listed in the legend. To describe our numerical results, we introduce a stretched exponential function as $C(\tau) = \exp[-(\tau/\tau_*)^\alpha]$, where α and τ_* are the stretching exponent and relaxation time, respectively [27,28]. Adjusting the two parameters (α and τ_*), we can see good agreements between the data below jamming, $\phi < \phi_J$, and stretched exponential functions [solid lines in Fig. 2(a)]. Note that α converges to unity if $\phi < \phi_J$ and $\dot{\gamma} \rightarrow 0$ so that $C(\tau)$

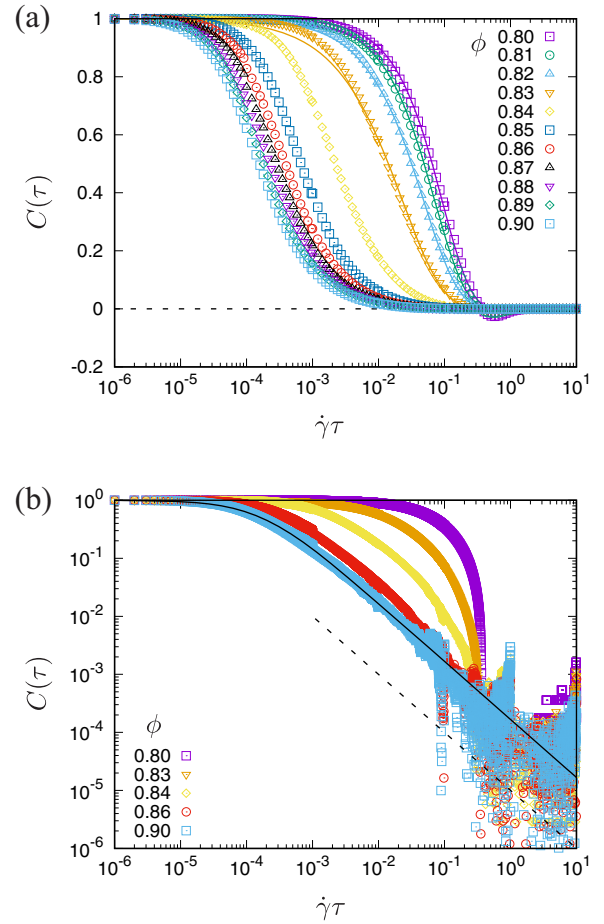


FIG. 2. (a) Semilogarithmic and (b) double-logarithmic plots of the normalized velocity autocorrelation function $C(\tau)$ (symbols), where $\dot{\gamma}t_0 = 10^{-6}$ and ϕ varies as listed in the legends. The horizontal axes are scaled by $\dot{\gamma}$, and the solid lines are the fits to $C(\tau)$ (see the text for the details). The dashed line in panel (a) indicates zero and that in panel (b) shows the power-law decay, $C(\tau) \sim (\dot{\gamma}\tau)^{-1}$.

exponentially decays to zero in the Newtonian regime [57]. We also find that the data above jamming, $\phi > \phi_J$, are well fitted to the stretched exponential function if the shear rate is large enough, i.e., $\dot{\gamma}t_0 \geq 10^{-5}$. However, $C(\tau)$ cannot be fitted to the stretched exponential function in a quasistatic regime above jamming, i.e., $\phi > \phi_J$ and $\dot{\gamma}t_0 \leq 10^{-6}$.

The relaxation time τ_* extracted from $C(\tau)$ exhibits critical behavior near jamming. Figure 1(b) shows a scaling data collapse of τ_* , where the scaling exponent is estimated as $\epsilon = 2.57$ (see the Supplemental Material [57]). The solid line with the slope $-\epsilon/\lambda$ represents the $\Delta\phi$ -independent critical regime. In this figure, τ_* cannot be defined for quasistatic flows above jamming ($\phi > \phi_J$ and $\dot{\gamma}t_0 \leq 10^{-6}$).

To describe the data in Fig. 1(b), we introduce a scaling function $\mathcal{G}(x)$ as

$$|\Delta\phi|^\epsilon \tau_* = \mathcal{G}\left(\frac{\dot{\gamma}}{|\Delta\phi|^\lambda}\right). \quad (4)$$

Based on our numerical results, we empirically determine the scaling function as follows. In the Newtonian regime, the scaling function is $\mathcal{G}(x) \sim x^{-1}$ [dotted line in Fig. 1(b)] so that

the relaxation time scales as $\tau_* \sim |\Delta\phi|^{\lambda-\epsilon} \dot{\gamma}^{-1}$. On the other hand, if the system is above jamming, the scaling function is $\mathcal{G}(x) \sim x^{-0.3}$ (dashed line), where the relaxation time scales as $\tau_* \sim |\Delta\phi|^{0.3\lambda-\epsilon} \dot{\gamma}^{-0.3}$. In the $\Delta\phi$ -independent critical regime, the scaling function is $\mathcal{G}(x) \sim x^w$ (solid line), where the exponents satisfy $\lambda w + \epsilon = 0$, i.e., $w = -\epsilon/\lambda \simeq 0.664$. This result is reminiscent of the sublinear scaling of relaxation time predicted by the nonequilibrium mode-coupling theory, $\tau_{\text{MCT}} \sim \dot{\gamma}^{-0.8}$ [66].

The reduced GK formula and a scaling relation. We validate the GK relation and show that it is independent of the system size as long as the system is *not* in the quasistatic regime above jamming ($\phi > \phi_J$ and $\dot{\gamma} t_0 \leq 10^{-6}$). Because the velocity autocorrelation function is well described by the stretched exponential function, we substitute $C(\tau) = \exp[-(\tau/\tau_*)^\alpha]$ into the GK formula. Calculating the time integral on the right-hand side of Eq. (2), we find that the diffusion coefficient is given by

$$D = \frac{\tau_* \langle v_y^2 \rangle}{\alpha} \Gamma\left(\frac{1}{\alpha}\right), \quad (5)$$

where $\Gamma(x)$ is the gamma function [57]. As shown in Fig. 3(a), D extracted from the transverse MSD [Eq. (1)] and the right-hand side of Eq. (5) coincide. In addition, Eq. (5) holds even if we change the number of particles N [Fig. 3(b)].

Substituting the critical scaling of D , $\langle v_y^2 \rangle$, and τ_* into the reduced GK formula [Eq. (5)], we derive a relationship between their critical exponents. In the Newtonian regime, we found the critical scaling as $D \sim |\Delta\phi|^{-\nu} \dot{\gamma}^2$, $\langle v_y^2 \rangle \sim |\Delta\phi|^{\psi-2\lambda} \dot{\gamma}^2$, and $\tau_* \sim |\Delta\phi|^{\lambda-\epsilon} \dot{\gamma}^{-1}$. Neglecting the weak dependence of α on the control parameters (ϕ and $\dot{\gamma}$), we substitute the critical scaling into $D \sim \tau_* \langle v_y^2 \rangle$ [Eq. (5)] and find $|\Delta\phi|^{-\nu} \dot{\gamma} \sim |\Delta\phi|^{\psi-\lambda-\epsilon} \dot{\gamma}$. Equating the exponents of $|\Delta\phi|$ on both sides, we obtain

$$\nu + \psi - \epsilon = \lambda. \quad (6)$$

Because this relation [Eq. (6)] connects the critical exponents for different quantities, we consider Eq. (6) to be a *scaling relation* (by analogy with critical phenomena). Notice that the exponents of $\dot{\gamma}$ on both sides of Eq. (5) coincide.

On the other hand, if $\phi > \phi_J$ and $\dot{\gamma} t_0 \geq 10^{-5}$, the critical scaling is $D \sim |\Delta\phi|^{0.3\lambda-\nu} \dot{\gamma}^{0.7}$, $\langle v_y^2 \rangle \sim |\Delta\phi|^{\psi-\lambda} \dot{\gamma}$, and $\tau_* \sim |\Delta\phi|^{0.3\lambda-\epsilon} \dot{\gamma}^{-0.3}$. Substituting them into Eq. (5), we find $|\Delta\phi|^{0.3\lambda-\nu} \dot{\gamma}^{0.7} \sim |\Delta\phi|^{\psi-0.7\lambda-\epsilon} \dot{\gamma}^{0.7}$. Equating the exponents of $|\Delta\phi|$ on both sides, we find $\nu + \psi - \epsilon = \lambda$, which is exactly the same with Eq. (6). Note that the exponents of $\dot{\gamma}$ on both sides of Eq. (5) coincide. Moreover, $D \sim \dot{\gamma}^{1-\nu/\lambda}$, $\langle v_y^2 \rangle \sim \dot{\gamma}^{\psi/\lambda}$, and $\tau_* \sim \dot{\gamma}^{-\epsilon/\lambda}$ in the $\Delta\phi$ -independent critical regime. From Eq. (5), we find $\dot{\gamma}^{1-\nu/\lambda} \sim \dot{\gamma}^{(\psi-\epsilon)/\lambda}$ so that the exponents satisfy $\nu + \psi - \epsilon = \lambda$, i.e., the scaling relation, Eq. (6). We stress that our estimates of the critical exponents are in accord with Eq. (6) as $\nu + \psi - \epsilon \simeq 3.76$ and $\lambda \simeq 3.87$.

A long-time tail and divergence of the diffusion coefficient. If the system is in the quasistatic regime above jamming, i.e., $\phi > \phi_J$ and $\dot{\gamma} t_0 \leq 10^{-6}$, the data of $C(\tau)$ cannot be fitted to the stretched exponential function. Figure 2(b) shows double-logarithmic plots of $C(\tau)$, where $\dot{\gamma} t_0 = 10^{-6}$ and ϕ increases as listed in the legend. We observe power-law decay, $C(\tau) \sim (\dot{\gamma}\tau)^{-1}$ (dashed line), in the long-time limit, i.e., a

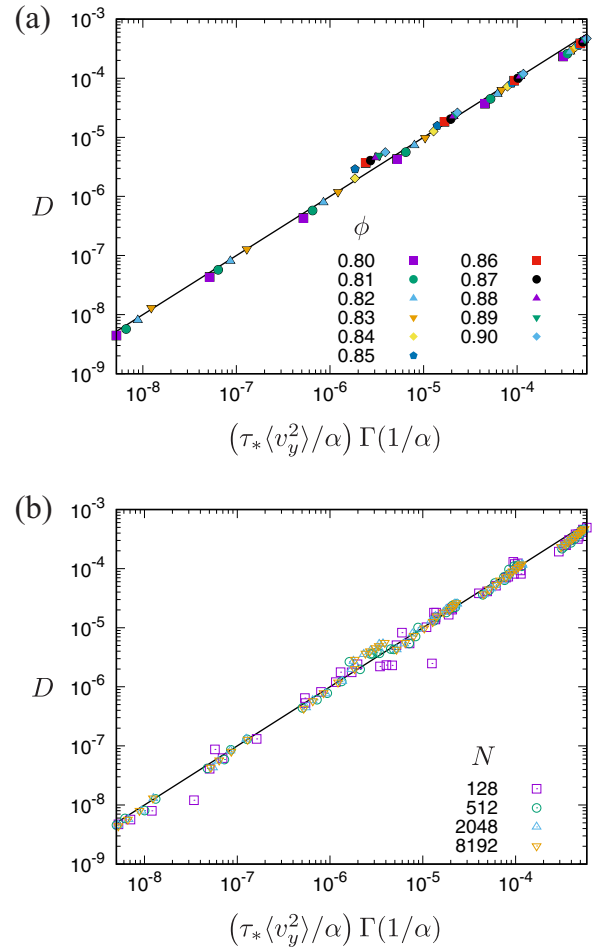


FIG. 3. (a) A scatter plot of the diffusion coefficient D extracted from the transverse MSD [Eq. (1)] and the right-hand side of Eq. (5), i.e., $(\tau_* \langle v_y^2 \rangle / \alpha) \Gamma(1/\alpha)$. The system size is $N = 8192$ and the packing fraction ϕ increases as listed in the legend. (b) System size dependence of the reduced GK formula [Eq. (5)], where the number of particles N increases as listed in the legend. In both panels (a) and (b), the solid lines represent $D = (\tau_* \langle v_y^2 \rangle / \alpha) \Gamma(1/\alpha)$.

long-time tail, if $\phi > \phi_J$. The data for $\phi = 0.90$ are well fitted to a function, $C(\tau) = (1 + c\dot{\gamma}\tau)^{-1}$ with $c \simeq 5.6 \times 10^3$ (solid line). Note that similar power-law decay of velocity autocorrelation function was found at the onset of jamming transition, $\phi \simeq \phi_J$ [30], and in MD simulations of granular particles sheared at constant pressure [67,68].

If the long-time tail is observed as $C(\tau) \sim (\dot{\gamma}\tau)^{-1}$ [dashed line in Fig. 2(b)], the time integral in the GK formula [Eq. (2)] diverges so that $D \rightarrow \infty$ [57]. In recent numerical studies of sheared amorphous solids, it was reported that the diffusion coefficient over the shear rate scales as $D/\dot{\gamma} \sim L^\beta$ with the system length L and exponent $\beta \geq 1$ [34,36–42]. This means that the diffusion coefficient diverges in the thermodynamic limit, $L \rightarrow \infty$, as is consistent with the long-time tail. However, the finite-size scaling, $D/\dot{\gamma} \sim L^\beta$, cannot be explained by our analysis and is left as a future work.

Discussion. We have studied shear-induced diffusion of two-dimensional soft athermal particles near jamming by MD

simulations. Many previous works focused on the relation between the shear-induced diffusion coefficient D and length scales such as the cluster size ξ [31–34] and system length L [36–42]. In contrast, we have made a link between D and the relaxation time τ_* through the reduced GK formula as $D \sim \tau_* \langle v_y^2 \rangle$. The critical scaling of the mean-squared transverse velocity $\langle v_y^2 \rangle$ is consistent with the viscosity divergence below jamming and the scaling of yield stress above jamming [7–12]. Except for quasistatic flows above jamming, we have numerically confirmed that the nontrivial dependence of τ_* on the control parameters, ϕ and $\dot{\gamma}$, can be described by the critical scaling. If the system is in the quasistatic regime above jamming, the velocity autocorrelation function $C(\tau)$ exhibits a long-time tail. In this case, the time integral in the GK relation diverges, implying finite-size effects on the diffusion coefficient [36–42]. Recently, it was reported that the shear-induced diffusion coefficient of three-dimensional particles does not diverge in the thermodynamic limit [39]. This means that the same scenario in classical fluids, where the GK relation is broken by a long-time tail in two dimensions but is *not* in three dimensions [43], can be applied to nonequilibrium athermal systems. Nevertheless, we need to examine whether $C(\tau)$

exhibits a long-time tail in three dimensions and also have to discuss the relation to the long-time tail observed in granular flows [69–73]. Moreover, we have proposed a scaling relation between the critical exponents, where our numerical results are in accord with it. In future, it is necessary to investigate theoretical background to these critical exponents. Note that our system sizes (typically $N = 8192$) are not sufficiently large, and the study of much larger system sizes is left as a future work.

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Data availability. Additional data are available in the Supplemental Material [57] and further inquiries can be directed to the corresponding author.

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