

Extending Kolmogorov theory to polymeric turbulence

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We present a formalism that reconciles polymeric turbulence with the classical Kolmogorov phenomenology. By relying on an appropriate form of the Kármán-Howarth-Monin-Hill equation, we define extended velocity increments and structure functions that also incorporate the non-Newtonian, polymeric contribution. The p th-order extended structure functions exhibit a power-law behavior in the elastoinertial range of scales, with exponents deviating from the analytically predicted value of $p/3$. These deviations are readily accounted for by considering local averages of the total dissipation rather than global averages. We also demonstrate the scale invariance of multiplier statistics of the extended velocity increments, whose distributions collapse for a wide range of scales.

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Turbulence remains a very long-standing problem in classical physics [1,2]. A fundamental breakthrough by Kolmogorov [3] (K41) showed that the skewness $S_3 \equiv \langle (\delta \mathbf{u} \cdot \mathbf{r}/r)^3 \rangle$ of the distribution of velocity increments $\delta \mathbf{u} = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})$ over a separation $\mathbf{r}$ is related to the average rate of energy dissipation $\langle \varepsilon \rangle$ by $S_3 = (-\frac{4}{5})\langle \varepsilon \rangle r$ in (statistically) stationary, homogeneous, and isotropic turbulence, where $r = |\mathbf{r}|$ [4,5]. This universal relation holds in the so-called inertial range of scales $\eta \ll r \ll L$, where L is the (large) scale of energy injection and η is the (small) dissipative Kolmogorov scale. This exact result is central to any theory of turbulence and emerges as a direct consequence of the Kármán-Howarth equation, which states that energy is, on average, transferred from large to small scales at a constant rate that equals $\langle \varepsilon \rangle$ [6]. A phenomenological extension of this relation via dimensional arguments suggests that a general p th-order structure function must scale as $S_p \equiv \langle (\delta \mathbf{u} \cdot \mathbf{r}/r)^p \rangle \sim \langle \varepsilon \rangle^{p/3} r^{p/3}$ [4,5]. The universality of this result was, however, questioned by Landau, who argued that the random nature of the energy transfer from large to small scales can not be disregarded [4,7]. This led Kolmogorov [8], following a suggestion by Oboukhov [9], to refine the similarity hypotheses (KO62) and account for the intermittent nature of turbulence by relaxing the global averaging of dissipation $\langle \varepsilon \rangle$ in favor of its locally averaged value over a scale r , $\langle \varepsilon_r \rangle$, so that now $S_p \sim \langle \varepsilon_r^{p/3} \rangle r^{p/3}$. Although a first-principles derivation of this revised relation remains an open problem, authors

of numerical and experimental studies have underlined its validity [10–16].

The Kolmogorov phenomenology describes the nature of turbulence in flows of Newtonian fluids. In general, however, turbulent flows can be multiphase and transport objects with various internal and external degrees of freedom. Such added phases, by virtue of their interaction with the carrier flow, modify the nature of turbulence in a nontrivial manner [17–19]. A particularly interesting situation arises when polymers are added to a carrier flow. A small concentration of polymers in a turbulent flow, indeed, modifies the way energy is distributed and transferred across scales [20–22] and results in a variety of intriguing, emergent phenomena that range from a chaotic state of elastic turbulence in otherwise laminar flows [23,24] to that of drag reduction in turbulent flows [25]. The nature of these polymeric flows is characterized by two dimensionless numbers, the Reynolds number Re and the Deborah number De . A large $Re \equiv Lu_{rms}/\nu$ typically implies a turbulent flow that is rough, chaotic, and unpredictable, where u_{rms} is the root mean square of the velocity fluctuations and ν the kinematic viscosity of the fluid. Here, $De \equiv \tau_p/\tau_L$ quantifies the elasticity of the polymers via a typical polymer relaxation time τ_p relative to the largest timescale of the flow $\tau_L \equiv L/u_{rms}$. A large De implies that polymers stretch more and take longer to relax back to their equilibrium configuration. Polymeric flows at large Re and unit De have recently been shown, both experimentally and numerically [21,22], to exhibit a self-similar scaling with $S_2 \sim r^{\xi_2}$, where $\xi_2 \approx 1.3$, which departs significantly from the Kolmogorov exponent of $\xi_2 = \frac{2}{3}$. We illustrate this in Fig. 1(a), where the $\frac{2}{3}$ scaling range of Newtonian turbulence (NT) is modified to a 1.3 scaling range in polymeric turbulence (PT). It is thus clear that the addition of polymers drives turbulence statistics far away from the classical Kolmogorov predictions.

However, in this letter, we use an appropriate form of the Kármán-Howarth-Monin-Hill (KMH) equation (see Refs. [29–32] for past derivation and use of the KMH equation in NT and Refs. [20,26] for its use in PT) to show that,

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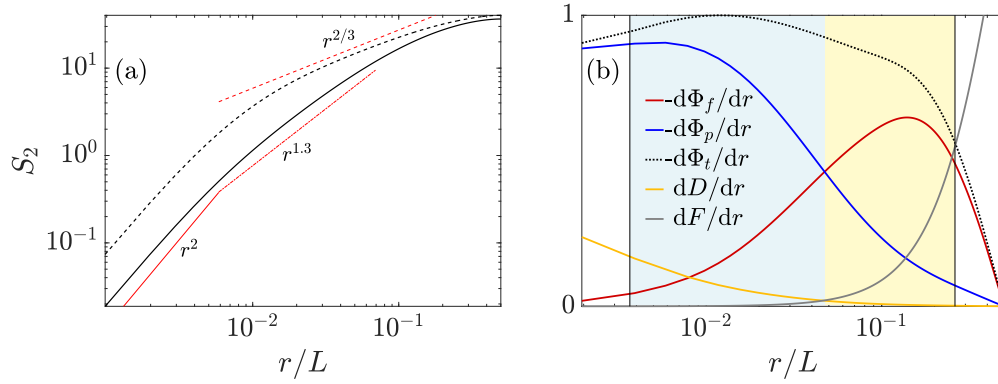


FIG. 1. (a) Velocity structure functions S_2 in polymeric turbulence (solid curve) scale as $S_2 \sim r^{1.3}$ [21,22], which in Newtonian turbulence (dashed curve) scales as $S_2 \sim r^{2/3}$. (b) This $r^{1.3}$ scaling holds in an intermediate range where fluid and polymeric flux contributions sum to a constant $d\Phi/dr = d\Phi_f/dr + d\Phi_p/dr$. We show in blue (yellow) the range of scales dominated by Φ_p (Φ_f). The dissipative D and forcing F contributions remain small in this entire range. For a characterization of the different contributions to the energy balance in turbulent polymer solutions at different Re and De , we refer to Refs. [22,26–28].

with suitable modifications, turbulence statistics of polymeric flows can be cast within a K41-like phenomenology. We also show that the refined KO62 phenomenology accounts for appropriate corrections to the deviations from a K41-like behavior, similar to classical NT. To this end, we first set up the analytical framework and then substantiate it with data from numerical simulations.

The dynamics of polymeric flows is governed by the modified Navier-Stokes equations that model how the polymers react back on the carrier flow, coupled with a simultaneous evolution of polymer stresses themselves

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{\rho} \frac{\partial T_{ij}}{\partial x_j} + f_i, \quad (1)$$

$$\frac{\partial R_{ij}}{\partial t} + u_k \frac{\partial R_{ij}}{\partial x_k} = \frac{\partial u_i}{\partial x_k} R_{kj} + R_{ik} \frac{\partial u_k}{\partial x_j} - \frac{\mathcal{P} R_{ij} - \delta_{ij}}{\tau_p}. \quad (2)$$

Here, u_i is the incompressible fluid velocity $\partial u_j/\partial x_j = 0$, p is the pressure, ν is the kinematic fluid viscosity, $\rho = 1$ is the fluid density, f_i is the external forcing that sustains turbulence, and the Peterlin function $\mathcal{P} = 1$ for the purely elastic Oldroyd-B model and $\mathcal{P} = (\ell_{\max}^2 - 3)/(\ell_{\max}^2 - R_{jj})$ for the finitely extensible nonlinear elastic (FENE-P) model [33], where $\ell_{\max}$ is the maximum length of the polymers. Polymer stresses are given by the symmetric tensor T_{ij} which is related to polymer conformation as $T_{ij} = \mu_p(\mathcal{P} R_{ij} - \delta_{ij})/\tau_p$, where μ_p is the polymeric viscosity. Equations (1) and (2) are supplemented with periodic boundary conditions for the following analysis, and we assume a statistically stationary state of fully developed PT.

We write Eq. (1) at two distinct points $\mathbf{x} = \mathbf{X} + \mathbf{r}/2$, $\mathbf{x}' = \mathbf{X} - \mathbf{r}/2$, separated by $\mathbf{r}$ and centered at $\mathbf{X}$. The difference of the two equations yields the evolution for the velocity increment $\delta u_i(\mathbf{X}, \mathbf{r}, t) = u_i(\mathbf{x}', t) - u_i(\mathbf{x}, t)$ over a separation $\mathbf{r}$ about the point $\mathbf{X}$, where $i = 1, 2, 3$. The equation for energy δu_i^2 then results upon multiplication by δu_i and captures the instantaneous rate of change of energy at any point in both space and scales, resulting from the transfers due to fluid-fluid and fluid-polymer interactions and the viscous dissipation. The KHM equation is then obtained by averaging $\langle \cdot \rangle$ over the ensemble of realizations and the three homogeneous

directions. Periodicity in space implies that the spatial derivatives $\partial/\partial X_j$ result in a zero contribution and that only the scale derivatives $\partial/\partial r_j$ survive. Moreover, statistical stationarity implies $\partial \langle \cdot \rangle / \partial t = 0$. The KHM relation for homogeneous PT [26] thus reads

$$\begin{aligned} \frac{\partial}{\partial r_j} \langle \delta u_j \delta u_i^2 \rangle - 4 \frac{\partial}{\partial r_j} \langle \delta u_i T_{ij}^* \rangle - 2\nu \frac{\partial^2}{\partial r_j^2} \langle \delta u_i^2 \rangle \\ = -4 \langle \varepsilon_f \rangle - 4 \langle \varepsilon_p \rangle + 2 \langle f_i \delta u_i \rangle, \end{aligned} \quad (3)$$

where repeated indices are summed, and $\cdot^*$ denotes the average of any quantity evaluated at $\mathbf{x}$ and $\mathbf{x}'$, i.e., $T_{ij}^* = [T_{ij}(\mathbf{x}) + T_{ij}(\mathbf{x}')]/2$. Here, $\langle \varepsilon_f \rangle \equiv \nu \langle \partial u_i / \partial x_j \rangle^2$ is the local fluid (pseudo)dissipation and $\langle \varepsilon_p \rangle \equiv \langle T_{ij} \partial u_i / \partial x_j \rangle = \langle T_{jj} \rangle / 2\tau_p = (\mu_p / 2\tau_p^2) \langle R_{jj} - 3 \rangle$, where $T_{jj} / 2\tau_p$ is the local, positive-definite polymeric dissipation. Integrating Eq. (3) into a spherical volume Ω of radius r results in a scalar relation for the net outward flux of energy from scale r . Dividing by the area $\Sigma(r) = 4\pi r^2$ described by the surface $\partial\Omega$ bounding this volume, we further have

$$\begin{aligned} \underbrace{\frac{1}{\Sigma(r)} \int_{\partial\Omega} d\Sigma n_j \langle \delta u_j (\delta u_i)^2 \rangle}_{\Phi_f(r)} + \underbrace{\frac{1}{\Sigma(r)} \int_{\partial\Omega} d\Sigma n_j \langle -4 \delta u_i T_{ij}^* \rangle}_{\Phi_p(r)} \\ = \underbrace{\frac{2\nu}{\Sigma(r)} \int_{\partial\Omega} d\Sigma n_j \frac{\partial}{\partial r_j} \langle (\delta u_i)^2 \rangle}_{D(r)} + \underbrace{\frac{2}{\Sigma(r)} \int_{\Omega} d\Omega \langle f_i \delta u_i \rangle}_{F(r)} \\ - \underbrace{\frac{4}{\Sigma(r)} \int_{\Omega} d\Omega \langle \varepsilon_f \rangle}_{(4/3) \langle \varepsilon_f \rangle r} - \underbrace{\frac{4}{\Sigma(r)} \int_{\Omega} d\Omega \langle \varepsilon_p \rangle}_{(4/3) \langle \varepsilon_p \rangle r}, \end{aligned} \quad (4)$$

which is the generalization of the Kármán-Howarth equation for homogeneous, turbulent, polymeric flows.

We focus on the intermediate range of scales which is sufficiently separated from the limited band at large r where the external forcing f_i injects energy into the flow as well as from the smallest scales where the viscous effects are relevant. Consequently, forcing $F(r)$ and viscous $D(r)$ contributions can be neglected in this elastoinertial range so that Eq. (4)

further simplifies to

$$\Phi_t(r) \equiv \Phi_f(r) + \Phi_p(r) = -\frac{4}{3}\langle \varepsilon_t \rangle r, \quad (5)$$

where Φ_f , Φ_p are the fluid nonlinear (inertial) and the non-Newtonian (polymeric) fluxes, while $\varepsilon_t = \varepsilon_f + \varepsilon_p$ is the local total dissipation. Equation (5) reveals the clear distinction between NT and PT [22]. In turbulent flows of Newtonian fluids, dissipation $\langle \varepsilon_f \rangle$ results only due to the fluid viscosity, and the flux of energy through scales is only due to the fluid nonlinearity Φ_f . Turbulent polymeric flows, however, present an additional dissipation $\langle \varepsilon_p \rangle$ and an additional flux contribution Φ_p due to the polymeric microstructure. Equation (5) further hints that the rate of total energy transfer $d\Phi_t/dr$ is invariant in the entire elastoinertial range of scales and equals the total rate of energy dissipation $\langle \varepsilon_t \rangle = \langle \varepsilon_f \rangle + \langle \varepsilon_p \rangle$. A further assumption of local statistical isotropy means that we can write the left side of Eq. (4) as $1/\Sigma(r) \int_{\partial\Omega} d\Sigma n_j \langle \delta u_j (\delta u_i)^2 \rangle = \langle \delta u_{\parallel} (\delta u_{\parallel})^2 \rangle$ and $1/\Sigma(r) \int_{\partial\Omega} d\Sigma n_j \langle -4\delta u_i T_{ij}^* \rangle = -4\langle \delta u_i T_{i\parallel}^* \rangle$, where the subscript $\parallel$ denotes the direction parallel to $\mathbf{r}$. Equation (5) can be thus recast as

$$\underbrace{\langle \delta u_{\parallel} \delta u_{\parallel}^2 \rangle - 4\langle \delta u_i T_{i\parallel}^* \rangle}_{\tilde{S}_3(r)} = -\frac{4}{3}\langle \varepsilon_t \rangle r. \quad (6)$$

This relation establishes the statistical equivalence of two ergodic processes: The averaged total dissipation is directly related to the sum of the third moment of the velocity increment and a fluid-polymeric coupling term.

We are now in the position to extend the Kolmogorov phenomenology to PT. To illustrate this better, we use Eq. (6) to introduce an extended structure function (ESF) of order p as $\tilde{S}_p \equiv \langle \delta \tilde{V}_{\parallel}^p \rangle$, where $\delta \tilde{V}_{\parallel} = (\delta u_{\parallel} \delta u_{\parallel}^2 - 4\delta u_i T_{i\parallel}^*)^{1/3}$ is the extended velocity increment and the third root is taken to be the real signed root of the expression within the parenthesis. Clearly, from Eq. (6), we have $\delta \tilde{V}_{\parallel} = \delta u_{\parallel}$ (up to a constant factor) in the absence of any dissolved polymers [34], so that $\tilde{S}_p$ reduces to the classical S_p [35]. The extended observable $\delta \tilde{V}_{\parallel}$ captures both the inertial (Newtonian) and the elastic (non-Newtonian) contributions to the total flux. Notably, unlike the conventional velocity increments, $\delta \tilde{V}_{\parallel}$ is not simply a difference, as the non-Newtonian contribution also comprises an average between two points T_{ij}^* . Nevertheless, in the spirit of K41, Eq. (6) inspires us to use dimensional arguments and expect for any $\tilde{S}_p$ a power-law behavior as $\tilde{S}_p \sim \langle \varepsilon_t \rangle^{p/3} r^{p/3}$. Such a relation, indeed, holds exactly for $p = 3$, while the corrections for $p \neq 3$, as will be shown later, can be accounted for by invoking the revised KO62 phenomenology which considers only the scale-averaged dissipation $\varepsilon_{t,r}(\mathbf{x}) \equiv (3/4\pi r^3) \int \varepsilon_t(\mathbf{x} + \mathbf{r}) d\mathbf{r}$, to obtain $\tilde{S}_p \sim \langle \varepsilon_{t,r}^{p/3} \rangle r^{p/3}$. The average $\langle \cdot \rangle$ is now taken to be over the ensemble $\varepsilon_{t,r}(\mathbf{x})$. We now substantiate our analytical results using data from direct numerical simulations of homogeneous, isotropic, PT of a purely elastic Oldroyd-B polymeric fluid at a microscale Reynolds number of $\text{Re}_{\lambda} = u_{\text{rms}}\lambda/\nu \approx 460$ (λ is the Taylor length scale) and $\text{De} = 1$. The KMHM Eq. (5) implies that a constant rate of energy transfer $d\Phi_t/dr$ in the elastoinertial range results in a Kolmogorov-like scaling for our ESFs, i.e., $\tilde{S}_p \sim \langle \varepsilon_t \rangle^{p/3} r^{p/3}$. That such a prediction holds very well is seen from the plots of (the absolute of) odd order $\tilde{S}_p$ in Figs. 2(a) and 2(b) and

especially for the exact case of $p = 3$ (straight lines show the $p/3$ scaling). This is the first confirmation of our hypotheses: K41 scaling holds in PT for the extended observable $\delta \tilde{V}_{\parallel}$. The assumptions of forcing $F(r)$ and viscous $D(r)$ contributions being negligible in the elastoinertial range are justified from the corresponding curves in Fig. 1(b). Furthermore, in the intermediate range, a dominant fluid nonlinearity $d\Phi_f/dr$ marks the inertial range at large r , while a dominant polymeric contribution $d\Phi_p/dr$ defines the elastic range at relatively small r . In this entire elastoinertial range, the net rate of energy transfer $d\Phi_t/dr = -\frac{4}{3}\langle \varepsilon_t \rangle$ is approximately constant and is shown as a dotted black curve in Fig. 1(b). Unexpectedly, Fig. 2(a) also shows that the same $p/3$ scaling holds even for $r \rightarrow 0$, where dissipation clearly dominates and Eq. (5) no longer holds. Such behavior at small r is a consequence of the analytic nature of the velocity and the polymeric fields [34].

We now focus on the intermediate elastoinertial range of scales. Figure 2(a) shows that the ESFs deviate from the expected $p/3$ scaling (given by the straight lines) at intermediate r when $p \neq 3$. Deviations from such K41 predictions are well known to arise from the intermittent nature of turbulence in Newtonian flows, where the structure functions $S_p \sim r^{\xi_p}$ with $\xi_p < p/3$, for $p > 3$ [4]. In PT, we instead find that $\tilde{S}_p \sim r^{\zeta_p}$ with $\zeta_p > p/3$ for $p > 3$ —the $\tilde{S}_p$ curves are steeper than the guiding lines at intermediate r —hinting that $\delta \tilde{V}_{\parallel}$ is less rough than $\delta u_{\parallel}$ in NT. This is because $\delta \tilde{V}_{\parallel}$ comprises both differences and averages over a separation $\mathbf{r}$: Differences emphasize fluctuations, while averages tend to smoothen them. The inertial contributions therefore make $\delta \tilde{V}_{\parallel}$ rough, while the elastic contribution makes it smoother. The weakening fluid nonlinearity with decreasing r results in a smoother $\delta \tilde{V}_{\parallel}$ in PT than $\delta u_{\parallel}$ in NT, thus resulting in $\zeta_p > \xi_p$. The deviation of ζ_p 's from their predicted $p/3$ value hints that the relation $\tilde{S}_p \sim \langle \varepsilon_t \rangle^{p/3} r^{p/3}$ does not hold exactly for $p \neq 3$, and corrections must be accounted for. This can be achieved within the KO62 phenomenology by associating the statistics of $\delta \tilde{V}_{\parallel}$ with the scale-averaged total dissipation $\varepsilon_{t,r}$. That is, we relax the global average $\langle \varepsilon_t \rangle$ in favor of the scale average $\langle \varepsilon_{t,r} \rangle$ and write $\tilde{S}_p = A_p \langle \varepsilon_{t,r}^{p/3} \rangle r^{p/3}$. This means that hyperflatness $\tilde{H}_{p,q}$ defined as the compensated ratio

$$\tilde{H}_{p,q} \equiv \frac{\tilde{S}_p \langle \varepsilon_{t,r}^{q/3} \rangle^{p/q}}{\tilde{S}_q^{p/q} \langle \varepsilon_{t,r}^{p/3} \rangle} = \frac{A_p}{A_q^{p/q}} = \text{const.} \quad (7)$$

should be constant in r if the KO62 phenomenology must hold for the statistics of the extended observable in PT. We choose $p = 6$ and $q = 2$ and plot $\tilde{H}_{6,2}$ as a function of r in Fig. 2(c). Indeed, we see that $\tilde{H}_{6,2} \approx 5$ remains fairly flat in the entire elastoinertial range of scales. In comparison, the hyperflatness $H_{6,2} \equiv (S_6/S_2^3) \langle \varepsilon_{f,r}^{2/3} \rangle^3 / \langle \varepsilon_{f,r}^{6/3} \rangle$ of the purely Newtonian contribution is not constant and shows a strong scale dependence, especially in the elastic range. KO62 phenomenology thus works fairly well even in PT but for the suitably defined observable $\delta \tilde{V}_{\parallel}$. The curves for the uncompensated ratios $\tilde{S}_6/\tilde{S}_2^3$ and S_6/S_2^3 , expectedly, are not flat and are nontrivial functions of r . The weaker growth of $\tilde{S}_6/\tilde{S}_2^3$ with r than S_6/S_2^3 implies that $\delta \tilde{V}_{\parallel}$ is less intermittent than $\delta u_{\parallel}$, especially in the elastic range. Figure 2(d) shows the independence of our conclusions of the polymeric model and elasticity, with the hyperflatness

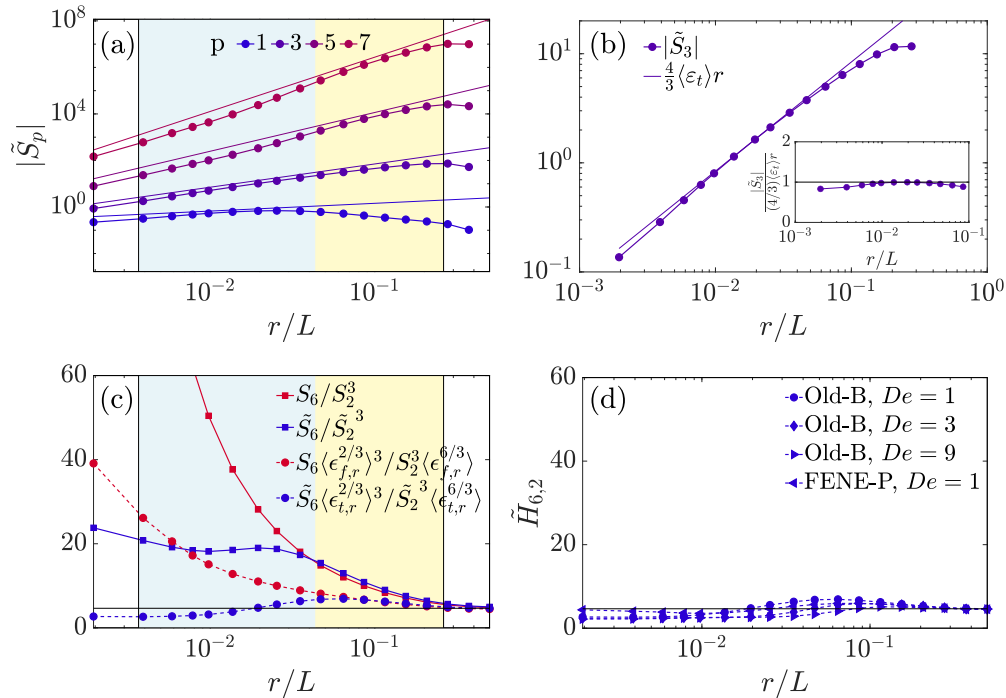


FIG. 2. (a) The extended structure functions $\tilde{S}_p$ in polymeric turbulence (PT) exhibit a close-to-Kolmogorov behavior $\tilde{S}_p \sim r^{p/3}$. Solid straight lines show $p/3$ scaling. (b) $|\tilde{S}_3|$ compared with $(\frac{4}{3})\langle\epsilon_t\rangle r$. The curves fall on top of each other in an intermediate range of scales, justifying the validity of the extended Kolmogorov theory to PT. Their ratio $\tilde{S}_3/(\frac{4}{3}\langle\epsilon_t\rangle r) = 1$ in the same range in the inset. (c) The flat compensated curve $\tilde{S}_6\langle\epsilon_{t,r}^{2/3}\rangle^3/\tilde{S}_2^3\langle\epsilon_{t,r}^{6/3}\rangle$ (blue circles) shows conformity to the refined Kolmogorov similarity hypothesis. In comparison, the Newtonian compensated (solid lines, red circles) and uncompensated (dashed line, red squares) ratios as well as the uncompensated extended ratio (dashed blue curve, square markers) are strongly dependent on r . Results in panels (a)–(c) are for Oldroyd-B and $De = 1$. (d) Dependence of the compensated extended structure function on the polymeric model and elasticity: blue circles are for Oldroyd-B and $De = 1$, blue diamonds for Oldroyd-B and $De = 3$, blue right-pointing triangles for Oldroyd-B and $De = 9$, and blue left-pointing triangles for FENE-P and $De = 1$.

$\tilde{H}_{6,2} \approx 5$ being constant in r for $1 \leq De \leq 9$ and for both Oldroyd-B and FENE-P fluids [34].

Kolmogorov [8] also proposed an alternative similarity hypothesis without resorting to the locally averaged dissipation. The hypothesis states that, in the limit $Re \rightarrow \infty$, the distributions of the Kolmogorov multipliers $\alpha(\mathbf{X}, r, r') \equiv \delta u_i(\mathbf{X}, r\hat{e}_i)/\delta u_i(\mathbf{X}, r'\hat{e}_i)$ are universal functions of only the ratio r/r' and not of the absolute scales (r, r') themselves [8]. In NT, the distributions of such multipliers have indeed been shown to be scale invariant [36–39]. We now show that the distributions of (extended) multipliers in PT also exhibit scale-invariant behavior. To this end, we introduce the extended multipliers $\alpha_{\tilde{V}}(\mathbf{X}, r, r')$ as the ratio

$$\alpha_{\tilde{V}}(\mathbf{X}, r, r') \equiv \frac{\tilde{\delta V}_{\parallel}(\mathbf{X}, r)}{\tilde{\delta V}_{\parallel}(\mathbf{X}, r')}. \quad (8)$$

We fix the ratio $r'/r = 4$ to work with and plot the distributions of $\alpha_{\tilde{V}}$ in Fig. 3 for different $r(r')$, such that both lie in the elastoinertial range of scales. These distributions fall fairly well on top of each other for a wide range of r , showing universal power-law tails $\alpha_{\tilde{V}}^{-4}$. This demonstrates the scale invariance of $\tilde{\delta V}$ in the entire elastoinertial range of scales. As shown in the Supplemental Material [34], the same cannot be said for the Newtonian multipliers, whose distributions do not collapse and are not scale invariant.

We now provide an additional evidence of our arguments via a reformulation of KO62 hypothesis using scale-averaged fluxes by integrating over spherical shells $\partial\Omega$ of radius $r/2$ (note the absence of ensemble averaging compared with Eq. (4), see Ref. [34]):

$$\Phi_{f,r}(\mathbf{X}, t) = \frac{-3}{2r\Sigma(r/2)} \int_{\partial\Omega} d\Sigma n_j \delta u_i^2(\mathbf{X}, \mathbf{r}, t) \delta u_j(\mathbf{X}, \mathbf{r}, t)$$

$$\Phi_{p,r}(\mathbf{X}, t) = \frac{3}{2r\Sigma(r/2)} \int_{\partial\Omega} d\Sigma n_j 4\delta u_i(\mathbf{X}, \mathbf{r}, t) T_{ij}^*(\mathbf{X}, \mathbf{r}, t)$$

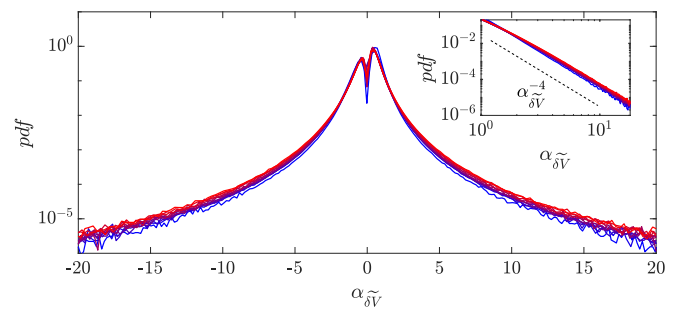


FIG. 3. The probability distribution functions of the extended Kolmogorov multipliers $\alpha_{\tilde{V}}$ collapse for a wide range of r, r' in the elastoinertial range, showing scale invariance. We set $r' = 4r$. Different colors refer to different values of r from the red curve at $r/L \approx 4 \times 10^{-3}$ to the blue curve at $r/L \approx 7.4 \times 10^{-2}$. The inset shows universal power-law decay of the tails with exponent -4 .

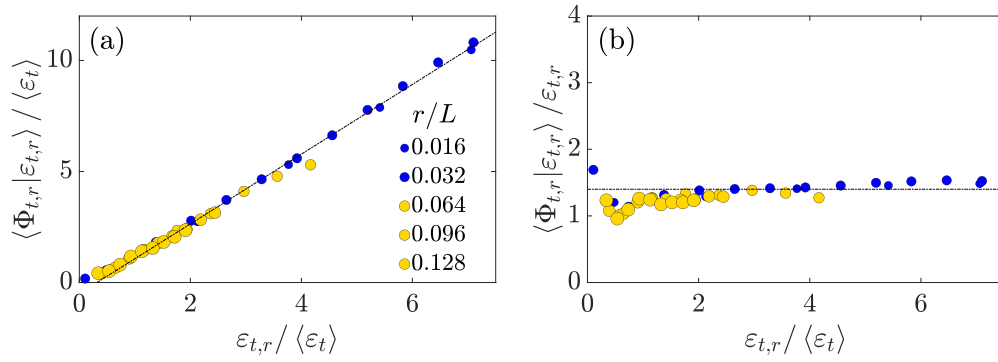


FIG. 4. (a) The values of the conditionally averaged total flux fall on a straight line showing $\langle \Phi_{t,r} | \varepsilon_{t,r} \rangle \propto \varepsilon_{t,r}$ for r in the elastoinertial range in conformity with the KO62 hypothesis. (b) The compensated values $\langle \Phi_{t,r} | \varepsilon_{t,r} \rangle / \varepsilon_{t,r} = C_\Phi$, therefore, lie on a horizontal line showing its independence with $\varepsilon_{t,r}$.

such that $\Phi_{t,r} = \Phi_{f,r} + \Phi_{p,r}$ is the local rate of net energy transfer out of scale r at a position $\mathbf{X}$ and time t . Now for any chosen r in the elastoinertial range, the KO62 hypothesis for PT implies that the statistics of $\Phi_{t,r}$ can only depend on the statistics of $\varepsilon_{t,r}(\mathbf{X})$. This means that the conditional average of $\Phi_{t,r}$ must obey $\langle \Phi_{t,r}(\mathbf{X}, t) | \varepsilon_{t,r} \rangle = C_\Phi \varepsilon_{t,r}$, where C_Φ is a constant independent of r and $\varepsilon_{t,r}$ (see Ref. [16] for NT). In Fig. 4(a), we plot $\langle \Phi_{t,r} | \varepsilon_{t,r} \rangle$ vs $\varepsilon_{t,r}$ for different r 's with two scales within the elastic range (blue) and three within the inertial range (yellow). As expected, the conditionally averaged flux values fall on a straight line showing $\langle \Phi_{t,r} | \varepsilon_{t,r} \rangle \propto \varepsilon_{t,r}$. In panel (b), we show the compensated conditional fluxes which clearly fall on a horizontal line. Evidently, our results show a very good collapse to the expected $\langle \Phi_{t,r} | \varepsilon_{t,r} \rangle = C_\Phi \varepsilon_{t,r}$ linear behavior for all r , confirming once again that the statistics of the total flux (and of the extended velocity increments) adhere fairly well with the refined Kolmogorov similarity hypothesis, even in PT.

In this letter, we have presented a reconciliation of turbulence statistics of non-Newtonian, polymeric flows with the Kolmogorov phenomenology. This is achieved by relying on the KMH equation tailored for turbulent polymeric flows, which naturally motivates the definition of extended velocity increments $\delta \tilde{V}_\parallel$ and extended structure functions

$\tilde{S}_p \equiv \langle \delta \tilde{V}_\parallel^p \rangle$. We show that $\tilde{S}_p$ exhibits self-similar, power-law behavior in the entire elastoinertial range of scales as $\tilde{S}_p \sim r^{\zeta_p}$ but where ζ_p deviates from the predicted value of $\zeta_p = p/3$ when $p \neq 3$. These deviations, in the spirit of the refined Kolmogorov similarity hypothesis, are well accounted for by considering local scale averages of the total dissipation rather than global averages. The scale invariance is further highlighted by the distributions of the multipliers of $\delta \tilde{V}_\parallel$, which indeed show a good collapse for a wide range of scales. The formalism presented in this letter is rather general and may be easily extended to inhomogeneous, anisotropic flows as well as flows comprising fluid mixtures.

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