

Competing shape evolution, crossing configurations, and single particle levels in nuclei

N. Gavrielov **GANIL, CEA/DSM-CNRS/IN2P3, Boulevard Henri Becquerel, BP 55027, 14076 Caen Cedex 5, France
and Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel* (Received 16 August 2024; revised 28 October 2024; accepted 4 January 2025; published 23 April 2025)

The evolution of shape in the even-even zirconium (Zr) isotopes has been the subject of study for many years. However, the odd-mass isotopes have not been investigated as extensively due to limited experimental accessibility and computational challenges. This work, employing the interacting boson-fermion model with configuration mixing, examines the effect of rapid shape evolution and normal-intruder configuration crossing, both identified as quantum phase transitions, alongside evolution in single particle energies, on the positive-parity spectrum of odd-mass $^{93-103}\text{Zr}$ isotopes. Calculated energy levels, magnetic moments, $B(E2)$ values, and quadrupole moments are compared to experimental data, showing good agreement. This agreement emphasizes the unique role of the odd neutron in the data. The special case of ^{99}Zr , which lies near the critical point of both quantum phase transitions, is also addressed, offering a new interpretation of the $7/2_1^+$ isomeric state and the occurrence of the type II shell evolution, in light of recent debates.

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The study of atomic nuclei reveals a diverse array of phenomena, from single particle excitations to collective motions and shape coexistence. These phenomena are crucial for understanding the evolution of nuclear structure across different isotopes. In particular, the evolution of effective single particle energies (ESPEs) and the resulting formation of collective bands [1] provide insight into the interplay between nucleon interactions and nuclear shape changes [2].

Often, the evolution in shapes discloses a quantum phase transition (QPT) [3] that is apparent in the spectrum, where the shape is the phase that changes as one varies the control parameter, the number of nucleons (A). Furthermore, coexisting shapes have been studied extensively and are shown to occur due to particle-hole excitations across multiple shell model configurations [4]. In certain cases, these configurations can cross, leading to a different kind of QPT [5] with the phase being the dominant shell model configuration in the ground-state wave function.

The scenario of configuration crossing QPT is rather unique and was identified in the even-even zirconium (Zr) isotopes, attributed to proton-neutron interactions within spin-orbit partner orbitals [6], i.e., orbitals with $j_\pi = \ell \pm 1/2$ and $j_\nu = \ell \mp 1/2$. Subsequent studies found that such spin-orbit partners can cause a type II shell evolution of the ESPEs, which reorganizes the shell structure and influences the QPT [7].

The effect of single particle degrees of freedom is even more accentuated in odd-mass nuclei, where the nucleus can be viewed as an even-even core coupled to an extra fermion. Nevertheless, theoretical investigations of odd-mass nuclei remain limited. This gap is particularly evident in the study of odd Zr isotopes, where previous works mentioned the need for incorporating configuration mixing [8–14].

Recent studies have demonstrated that the configuration crossing QPT in the even-even Zr isotopes is intertwined with a shape-evolution QPT, leading to a scenario termed intertwined QPTs (IQPTs) [15–17]. Subsequently, using the new interacting boson-fermion model with configuration mixing (IBFM-CM) [18], the spectrum of odd-mass niobium (Nb) isotopes also revealed the occurrence of IQPTs [18,19]. This underscored the necessity of incorporating multiple configurations to fully understand the structure of odd-mass nuclei.

The odd $^{93-103}\text{Zr}$ isotopes exhibit a rapid change in ground-state total angular momentum: $5/2^+$, $5/2^+$, $1/2^+$, $1/2^+$, $3/2^+$, and $5/2^-$. This change, caused by neutron orbit occupations that shift the Fermi level, is key to understanding the behavior of these isotopes. In heavier Zr isotopes, the evolving single quasiparticle energies (SqPEs) affect the spectrum. The SqPEs evolution, driven also by the configuration crossing QPT and shape evolution QPT within the even-even Zr core [17], contrasts with the Nb isotopes study [18,19], where the Fermi level remains unchanged. It is therefore the goal of this paper to unravel the effect on the spectrum and the interplay between the triad evolution of shapes, configurations, and single quasiparticle energies in the odd $^{93-103}\text{Zr}$ isotopes.

The IBFM-CM is employed to study odd- A nuclei [18,19]. In the IBFM [20], a boson core represents the adjacent even-even nuclei and is coupled to a fermion. The core is described using the interacting boson (IBM) model with configuration mixing [21–23], where N_b monopole (s) and quadrupole (d) bosons represent $0p-0h$ excitations and $2p-2h$ core exci-

*Contact author: noam.gavrielov@mail.huji.ac.il

tations with $N_b + 2$ bosons. The total Hamiltonian can be written in the form

$$\hat{H} = \hat{H}_b + \hat{H}_f + \hat{H}_{bf} = \begin{bmatrix} \hat{H}^A(\eta^A) & \hat{V}(\omega) \\ \hat{V}(\omega) & \hat{H}^B(\eta^B) \end{bmatrix}. \quad (1)$$

In Eq. (1), the matrix form is for two Hamiltonians representing two shell model configurations, normal (denoted by A, the N_b boson space) and intruder (denoted by B, the $N_b + 2$ boson space), with respective parameters (η^A, η^B) and their coupling $\hat{V}$ with the mixing parameter ω . The diagonal matrix Hamiltonians are constructed from the boson ($\hat{H}_b$), fermion ($\hat{H}_f$), and Bose-Fermi ($\hat{H}_{bf}$) parts. Here $\hat{H}_b$ represents the boson Hamiltonian, with a normal configuration A of N_b bosons and an intruder configuration B of $N_b + 2$ bosons. The $\hat{H}_f$ represents the fermion Hamiltonian. The $\hat{H}_{bf}$ represents the Bose-Fermi Hamiltonian and consists of monopole, quadrupole, and exchange terms with respective strengths $A_0^{(i)}$, $\Gamma_0^{(i)}$, and $\Lambda_0^{(i)}$ ($i = A, B$) and occupation probabilities (u_j, v_j), where j is the single particle orbit, following the microscopic interpretation of the IBFM. The mixing interacting $\hat{V}$ is a combination of s and d bosons that connect the two boson spaces N_b and $N_b + 2$ (for more details on the interactions, see [19]).

After diagonalization of the Hamiltonian (1), the resulting wave functions are of the form

$$|\Psi; J\rangle = \sum_{\alpha, L, j} C_{\alpha, L, j}^{(N, J)} |\Psi_A; [N_b], \alpha, L, j; J\rangle + \sum_{\alpha, L, j} C_{\alpha, L, j}^{(N+2, J)} |\Psi_B; [N_b + 2], \alpha, L, j; J\rangle. \quad (2)$$

Here N_b is the boson number, α the quantum numbers from the boson Hamiltonian, L (j) the boson (fermion) angular momentum, and J the total angular momentum. The occupation probabilities for the boson number N_b , the n_d d -boson number, and the single particle orbitals of the wave function can be determined from

$$v_{(N_i, J)}^2 = \sum_{n_d} v_{n_d; (N_i, J)}^2 = \sum_j v_{j; (N_i, J)}^2, \quad i = A, B \quad (3a)$$

$$v_{(j; N_i, J)}^2 = \sum_{\alpha, L} |C_{\alpha, L, j}^{(N_i, J)}|^2, \quad (3b)$$

$$v_{(n_d; N_i, J)}^2 = \sum_{\tau, n_\Delta, j, L} |C_{n_d, \tau, n_\Delta, j, L}^{(N_i, J)}|^2. \quad (3c)$$

For Eq. (3a), $v_{(N_A, J)}^2 + v_{(N_B, J)}^2 = 1$, where $N_A = N_b$ and $N_B = N_b + 2$. It provides the occupation probability of N_b , indicating the proportion of normal ($v_{(N_A, J)}^2$) and intruder ($v_{(N_B, J)}^2$) configurations. Equation (3b) represents the occupation probability of the single particle orbitals j . Equation (3c) represents the occupation probability of a single n_d component, which indicates the degree of deformation. A large $v_{(n_d; N_i, J)}^2$ suggests dominant U(5) boson symmetry, equivalent to a spherical shape (phase) in the geometrical interpretation of the interacting boson model [21]. If the wave function is distributed among several n_d values, it indicates deformation. Thus, using Eqs. (3a)–(3c), one can analyze the evolution of occupation in the configurational, single particle, and deformation content of the wave functions.

TABLE I. Neutron single quasiparticle occupations $v_{n\ell j}^2$, the Fermi energy λ_F , and parameters in MeV of the boson-fermion interactions, $\hat{H}_{bf}^{(i)}$ of Eq. (1), obtained from a fit assuming $A_0^{(i)} = A_0$, $\Gamma_0^{(i)} = \Gamma_0$, and $\Lambda_0^{(i)} = \Lambda_0$ for $i = A$ and B. Beneath each isotope the number of valence neutrons used in the shell model ($^{93,95}\text{Zr}$) and BCS ($^{97-103}\text{Zr}$) evaluations is listed.

Isotope	^{93}Zr	^{95}Zr	^{97}Zr	^{99}Zr	^{101}Zr	^{103}Zr
No. of particles	3	5	1	3	5	7
$v_{v1d5/2}^2$	3/6	5/6				
$v_{v2s1/2}^2$	0	0	0.085	0.351	0.567	0.697
$v_{v1d3/2}^2$	0	0	0.037	0.110	0.189	0.275
$v_{v0g7/2}^2$	0	0	0.033	0.091	0.152	0.219
$v_{v0h11/2}^2$	0	0	0.034	0.095	0.158	0.229
λ_F	0	0	−1.339	−0.281	0.122	0.385
A_0	−0.252	0.983	−0.367	−0.150	−0.146	0.133
Γ_0	0.037	0.086	0.059	0.455	0.464	0.757
Λ_0	1.191	1.063	1.121	1.910	1.910	1.824

To analyze electromagnetic transitions, the operator with multipolarity L for transitions of type σ is defined as

$$\hat{T}(\sigma L) = \hat{T}_b(\sigma L) + \hat{T}_f(\sigma L), \quad (4)$$

with boson $\hat{T}_b(\sigma L)$ and fermion $\hat{T}_f(\sigma L)$ parts (see [19] for more details). For $\sigma L = E2$ transitions, Eq. (4) defines the boson effective charges for configurations A and B as $e^{(A)}$ and $e^{(B)}$, respectively, and the fermion effective charge as e_f . For $\sigma L = M1$ transitions, Eq. (4) specifies the free value of the neutron spin g factor as $g_s = -3.8263\mu_N$, which is quenched by 30%, while the orbital angular momentum g factor is $g_l = 0\mu_N$.

The $^{A}_{40}\text{Zr}$ isotopes with mass number $A = 93$ –103 are described by coupling a neutron to the respective even-even $^{A}_{40}\text{Zr}$ cores with $A = 92$ –102. The parameters of $\hat{H}_b$ in Eq. (1) and the boson numbers are thus taken from a previous work [17], with the sign of χ is modified for $^{92-96}\text{Zr}$, as recent measurements of spectroscopic quadrupole moments of ^{94}Zr have been shown to be positive [24].

In the odd $^{A}_{40}\text{Zr}$ isotopes, the valence neutrons occupy different orbits within the $Z = 50$ –82 shell. Talmi [25] showed that neutrons in $^{90-96}\text{Zr}$ primarily reside in the $v1d5/2$ orbital, with ^{96}Zr representing a subshell closure. Therefore, for $^{93-95}\text{Zr}$, occupations of $n/(2j + 1) = 3/6$ and $5/6$, respectively, are assumed, where n is the number of valence neutrons. For $^{97-103}\text{Zr}$, a BCS calculation is performed [19] to evaluate the SqPEs ϵ_j and their occupations v_j^2 . As more neutrons are added, the changing Fermi energy affects these evaluations. According to the microscopic interpretation of the IBFM [26], the ϵ_j are required for $\hat{H}_f$, and v_j^2 impact $\hat{H}_{bf}$ of Eq. (1). The neutrons are distributed among the $v2s1/2$, $v1d3/2$, $v0g7/2$, and $v0h11/2$ orbitals, with the $v1d5/2$ considered closed, as done in a previous IBFM calculation [27]. For both shell model BCS regions, the single particle energies of ^{91}Zr are used [28], along with a pairing gap of $\Delta_F = 0.9$ MeV, calculated as the average of $\Delta_F = \frac{1}{4}[B(N - 2, Z) - 3B(N - 1, Z) + 3B(N, Z) - B(N + 1, Z)]$ [29]. Table I presents the resulting v_j^2 and the strengths A_0 , Γ_0 , and Λ_0 for the

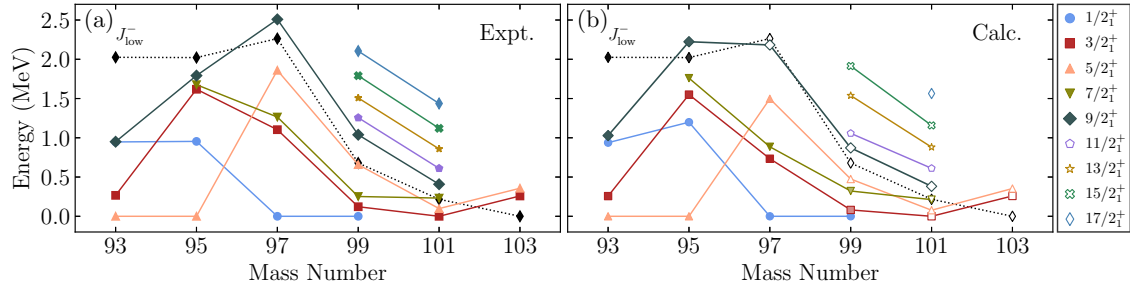


FIG. 1. Comparison between (a) experimental and (b) calculated lowest-energy levels in odd Zr isotopes. In (b) the character of the symbols indicates the dominance of the normal A configuration (closed) or intruder B configuration (open) with assignments based on Eq. (3a). For example, the $3/2_1^+$ is a normal A configuration for $A = 93-97$, mixed for $A = 99$, and an intruder B configuration for $A = 101-103$.

positive-parity states. A discussion of the strengths and an analysis of the negative-parity states are beyond the scope of the present work. The exchange strength Λ_0 increases gradually along the isotopic chain. The quadrupole strength Γ_0 remains relatively constant for $^{93-97}\text{Zr}$ and $^{99-103}\text{Zr}$, suggesting an increase in deformation between these regions. The monopole strength A_0 decreases from ^{93}Zr to ^{101}Zr , with exceptions at ^{95}Zr and ^{103}Zr , possibly due to the lack of experimental data. The occupation probabilities increase with the number of valence neutrons, which in turn influences the SqPEs. The evolution in SqPEs [shown in Fig. 2(a)] depicts a significant lowering of the $\nu 0h_{11/2}$, $\nu 0g_{7/2}$, and $\nu 1d_{3/2}$ orbitals from ^{93}Zr to ^{103}Zr .

The trends in strengths and SqPEs are reflected in the spectrum of $^{93-103}\text{Zr}$ isotopes, shown in Fig. 1, illustrating the energies of selected states. Figure 1(b), showing good agreement with the experimental data, depicts the normal-intruder mixing, where closed (open) symbols represent normal (intruder) states, as defined by Eq. (3a). For the lowest positive-parity state, the total angular momentum undergoes an intricate change: $5/2_1^+$ for $^{93,95}\text{Zr}$, $1/2_1^+$ for $^{97,99}\text{Zr}$, and $3/2_1^+$ for $^{101,103}\text{Zr}$. This change results from a combination of evolution in deformation and configuration mixing. However, it is most affected by the neutron single quasiparticle occupation. For $^{93,95}\text{Zr}$, the $5/2_1^+$ state results from the $\nu 1d_{5/2}$ orbital weakly coupled to the respective $^{92,94}\text{Zr}$ cores, particularly the spherical normal 0_1^+ . A similar situation occurs for $^{97,99}\text{Zr}$, where the $1/2_1^+$ state is associated with coupling the $\nu 2s_{1/2}$ orbital to the $^{96,98}\text{Zr}$ cores. For $^{101,103}\text{Zr}$, the $3/2_1^+$ state results from closely spaced orbitals, on top of the onsets of deformation and configuration crossing, where all the lowest lying states are intruder states.

In Fig. 1 the lowest negative-parity level J_{low}^- , as well as its configuration content, is also shown. For $^{93,95}\text{Zr}$ it is $11/2^-$ normal, for ^{97}Zr it is $11/2^-$ intruder, for ^{99}Zr it is $7/2^-$ intruder, and for $^{101,103}\text{Zr}$ it is $5/2^-$ intruder. The analysis above is further stressed in Fig. 2, depicting the resulting SqPEs [Fig. 2(a)] and the different occupations of the lowest positive-parity state J_{low}^+ : the single quasiparticle occupation (3b) [Fig. 2(b)], n_d U(5) d -boson spherical symmetry occupation (3c) [Fig. 2(c)], and configuration content (3a) Fig. 2(d)].

For $^{93,95}\text{Zr}$, Fig. 2(b) shows a dominant $\nu 1d_{5/2}$ orbit, Fig. 2(c) shows a dominant $n_d = 0$ component, and Fig. 2(d) shows a dominant normal component. This indicates a weak-coupling scenario of a normal wave function with good U(5) boson symmetry coupled to a $\nu 1d_{5/2}$ orbital. A similar pattern

is observed for $^{97,99}\text{Zr}$, but with the $\nu 2s_{1/2}$ orbital instead. In contrast, for $^{101,103}\text{Zr}$, Fig. 2(b) shows a mixture of $\nu 2s_{1/2}$, $\nu 1d_{3/2}$, and $\nu 0g_{7/2}$ orbitals within the intruder part of the wave function, Fig. 2(c) shows multiple boson U(5) n_d occupations indicating a deformed intruder state, and Fig. 2(d) shows almost no normal occupation in the wave function, indicating an almost pure intruder state.

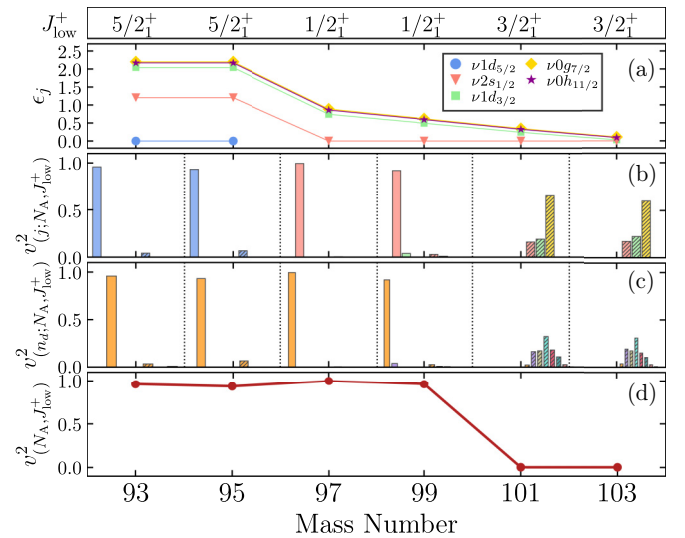


FIG. 2. Evolution of (a) single quasiparticle energies ϵ_j and (b) single quasiparticle occupation probabilities of the lowest positive-parity state J_{low}^+ [Eq. (3b)]. In both panels, each color indicates a different orbital: $\bullet$, $d_{5/2}$; $\blacktriangledown$, $s_{1/2}$; $\blacksquare$, $d_{3/2}$; $\blacklozenge$, $g_{7/2}$; and $\star$, $h_{11/2}$. In (b), note the transition from a singular single quasiparticle occupation for $^{93-99}\text{Zr}$ to a distribution of them for $^{101-103}\text{Zr}$ as the different ϵ_j values become closer in (a). (c) Evolution of n_d occupation probability of the lowest positive-parity state J_{low}^+ [Eq. (3c)], depicting the different possible normal ($n_d = 0, \dots, N_b$) and intruder ($n_d = 0, \dots, N_b + 2$) n_d values. Each color represents a different n_d number from $n_d = 0$ (orange) to $n_d = 8$ (purple). Note the transition from a single dominant bar (representing a spherical state) for $^{93-99}\text{Zr}$ to a distribution of them (representing a deformed state) for $^{101-103}\text{Zr}$. In both (b) and (c), bars from the left (right) of the mass number correspond to normal (intruder) occupations, with intruder bars filled with diagonal lines. (d) Evolution of the amount of normal occupation in the wave function [Eq. (3a)]. Note the transition from normal in $^{93-99}\text{Zr}$ to intruder in $^{101-103}\text{Zr}$. The values of J_{low}^+ are indicated in the uppermost panel.

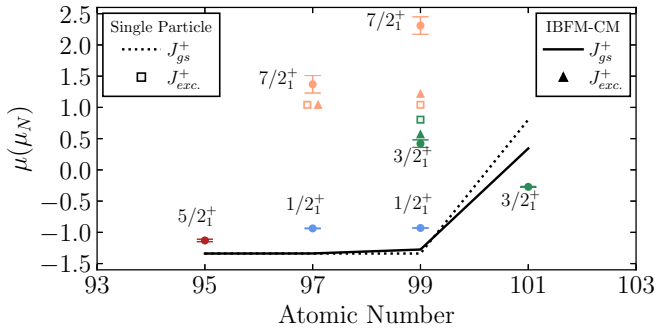


FIG. 3. Evolution of magnetic moments μ (in units of μ_N) along the odd Zr chain. Symbols denote experimental data for the ground state and excited states when available, taken from [32]. Each J value has a different color. The solid (dashed) line and black triangles (squares) denote calculated results, based on Eq. (4) (single particle estimation). See the text for more details.

The increase in j mixture observed in Fig. 2(b) corresponds clearly to the lowering of the SqPEs shown in Fig. 2(a). The evolution observed in Figs. 2(c) and 2(d), from spherical to deformed states and from normal to intruder configurations, indicates the presence of the two quantum phase transitions (IQPTs) similar to those seen in the even-even Zr and odd-mass Nb chains [30].

To further examine the calculated wave functions, it is essential to compare them with electromagnetic transition rates. For $E2$ transitions in Eq. (4), the boson effective charges $e^{(A)}$ and $e^{(B)}$ are taken from [19]. The fermion effective charge $e_f = 3.258 \sqrt{\text{W.u.}}$ (where W.u. denotes Weisskopf unit) is determined from a fit to the spectroscopic quadrupole moment of ^{95}Zr , $Q_{5/2^+} = 0.22(2) e b$.

The fitting procedure yields $Q_{3/2^+} = 0.27 e b$ for ^{101}Zr , compared to the experimental value of $Q = 0.81(6) e b$. The experimental $E2$ transitions (calculation in parentheses) are $B(E2; 3/2^+ \rightarrow 5/2^+) = 6.6(23) \text{ W.u. (1.1)}$ for ^{93}Zr , $B(E2; 7/2^+ \rightarrow 3/2^+) = 1.55(5) \text{ W.u. (4.80)}$ for ^{97}Zr , $B(E2; 7/2^+ \rightarrow 3/2^+) = 1.33(5) \text{ W.u. (1.91)}$ for ^{99}Zr , and $B(E2; 7/2^+ \rightarrow 3/2^+) > 130 \text{ W.u. (82)}$ for ^{101}Zr . The calculated values are reasonably close to the experimental ones and indicate the onset of deformation at ^{101}Zr , as evidenced by the strong increase in transition strength, with a possible need for more deformation.

Magnetic moments, which are strongly influenced by the single particle structure, provide further indicative observables. For the $M1$ transition operator (4), the same values for $g^{(A)}$, $g^{(B)}$, $\tilde{g}^{(A)}$, $\tilde{g}^{(B)}$ are used as in [19]. Figure 3 compares the calculated magnetic moments with the experimental ones, as well as the single particle estimations [31]. For $^{95-99}\text{Zr}$, there is good agreement between the calculation and experimental data for the ground state. Notably, the IBFM-CM result also coincides with the single particle estimations [31], further supporting the single particle character of the ground state. This interpretation extends to the $7/2^+$ state of ^{97}Zr , which exhibits an almost pure $\nu 0g_{7/2}$ occupation and spherical normal configuration in its wave function.

However, deviations from the single particle interpretation are observed for the excited states of ^{99}Zr and the ground state

of ^{101}Zr . For the $3/2^+$ and $7/2^+$ states in these isotopes, better agreement is achieved with the IBFM-CM calculation. The reason is that these states become more mixed in their orbitals, losing their single quasiparticle character due to the onset of deformation and configuration mixing. In ^{99}Zr , the $3/2^+$ and $7/2^+$ states are strongly mixed in configuration and n_d occupations. For the $7/2^+$ state, there is still some deviation for the IBFM-CM, which might suggest more deformation is needed in the ^{98}Zr core, as was suggested in [33]. For ^{101}Zr , the $3/2^+$ state is pure intruder deformed, with mixed orbitals, which explains the deviation from the single particle value.

A closer examination of ^{99}Zr is intriguing, as it lies nearest to the critical points of the shape evolution and configuration crossing QPTs between ^{98}Zr and ^{100}Zr [17]. Recently, an IBFM calculation with a single configuration was presented in [11,12] for ^{99}Zr . In [12], the authors compared the $7/2^+$ state in ^{99}Zr to that in ^{97}Zr , noting that both states are isomers. However, for the $7/2^+$ state, the magnetic moment for ^{99}Zr , $\mu_{7/2^+} = 2.31(14)\mu_N$, deviates from the single particle prediction observed in ^{97}Zr . This led the authors in [12] to suggest the occurrence of a type II shell evolution, causing the *single particle* $\nu 2s_{1/2}$ orbital to become lower than the $\nu 1d_{5/2}$.

While Ref. [12] successfully reproduced this magnetic moment value, the calculated energies did not align well with experimental data. This discrepancy was later questioned in Ref. [34], leading the authors to propose an IBFM configuration mixing model for improvement. In [34] it was argued that there is no need for a change in the single particle energies, as suggested by previous works [8,27]. Furthermore, as was suggested in [7], only the resulting *effective* single particle energies are modified, leading to the type II shell evolution between the spin orbit partners $\nu 0g_{7/2}$ and $\pi 0g_{9/2}$.

This work confirms that lowering the $\nu 2s_{1/2}$ single particle orbital is unnecessary in ^{99}Zr . Instead, the resulting single quasiparticle energies account for the $1/2^+$ state as a normal single quasiparticle state of the $\nu 2s_{1/2}$ orbital, while the $3/2^+$ and $7/2^+$ states are mixed normal-intruder deformed states. Furthermore, the intruder character of the $7/2^+$ state in ^{99}Zr mainly results from coupling with the intruder 2^+ state of the ^{98}Zr core (unlike the less conventional choice of a ^{100}Zr boson core with a hole in [12]). Its isomeric nature is attributed to the dominant single quasiparticle component $\nu 0g_{7/2}$ in the normal-intruder components (total $v_{j=7/2^+}^2 = 0.86$) and for the $3/2^+$ state it is the $\nu 1d_{3/2}$ orbital ($v_{j=3/2^+}^2 = 0.73$), which are weakly connected via $\hat{T}_f(E2)$ in Eq. (4). In ^{97}Zr , the isomeric nature arises from the single particle character of the 2^+ state of ^{96}Zr .

In summary, a comprehensive analysis of the odd $^{93-103}\text{Zr}$ isotopes has been conducted using the interacting boson-fermion model with configuration mixing. This approach employed boson, fermion, and Bose-Fermi Hamiltonians with mixed configurations, focusing on positive-parity states. The good agreement with experimental data on energies, $E2$ transitions, and quadrupole and magnetic moments underscores the manifestation of shape, configuration, and single quasiparticle evolutions that influence the spectrum. Furthermore, the evolutions in single quasiparticle occupations and energies

demonstrate the important effect the odd neutron has on the spectrum. The special case of ^{99}Zr was further examined in light of recent studies [11,12] suggesting a type II shell evolution in the single particle $\nu 2s_{1/2}$ orbit. It was demonstrated that while the reorganization of the shell structure due to type II shell evolution occurs in the effective single particle orbits, as suggested in [7], it does not occur in the single particle orbits and does not lower the $\nu 2s_{1/2}$ orbit. Instead, the spectrum is shown to result from the interplay of shape, configuration, and single quasiparticle evolutions, offering a resolution to the recent debate [34,35]. This work may inspire further experimental and theoretical efforts to identify and

differentiate shape, configuration, and single particle effects in the structural evolution of atomic nuclei. As demonstrated here, these effects are particularly pronounced in odd-mass nuclei. Furthermore, they are of relevance to other physical quantum and open quantum systems, such as the recently suggested symmetries for the squeeze-driven Kerr oscillator and its Liouville superoperator [36,37].

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