

Mutual information bounded by Fisher information

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We derive a general upper bound to mutual information in terms of the Fisher information. The bound may be further used to derive a lower bound for the Bayesian quadratic cost. These two provide alternatives to other inequalities in the literature (e.g., the van Trees inequality) that are useful also for cases where the latter ones give trivial bounds. We then generalize them to the quantum case, where they bound the Holevo information in terms of the quantum Fisher information. We illustrate the usefulness of our bounds with a case study in quantum phase estimation. Here, they allow us to adapt to mutual information (useful for global strategies where the prior plays an important role), the known and highly nontrivial bounds for the Fisher information in the presence of noise. The results are also useful in the context of quantum communication, both for continuous and discrete alphabets.

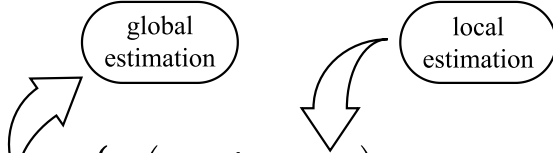
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Introduction. The Fisher information (FI) $F(\phi)$ measures how much information a conditional probability distribution $p(x|\phi)$ contains on some parameter ϕ . Instead, the mutual information (MI) $I(X, \Phi)$ measures how much information two random variables X and Φ have on one another. Then, it is clear that the two quantities must be related if one considers the parameter ϕ as realization of an unknown random variable Φ , and indeed many such relations have appeared in the literature [1–6]. Yet, most of them work only in the asymptotic limit and/or require additional assumptions about the regularity of probability distributions. The most relevant one is the Efroimovich inequality [7–9], valid for a finite number of samples without additional assumptions. It is a generalization of the van Trees inequality (or Bayesian Cramér-Rao bound) [10–12]. However, both of them might fail to give significant bounds, e.g., when the prior probability on ϕ has sharp edges, such as the important case of a uniform prior on a finite interval.

Here we derive two universal upper bounds to the mutual information in terms of the Fisher information which do not suffer from these issues: a simple one, valid in the useful case where the prior distribution $p(\phi)$ for ϕ has finite support, and a general one valid for any prior. Our bounds provide a bridge between *local* and *global* estimation (Fig. 1). For local estimation when a large number of probes is under consideration the FI is a sufficient tool for meaningful analysis. If, instead, one needs to take into account also a nontrivial

prior information $p(\phi)$, then *global* estimation approaches, such as the one based on mutual information [13–17] or the Bayesian approach [18–21], or the minimax cost [22–24] are more useful. Since an upper bound on the entropy of a probability distribution imposes a lower bound on its moments, our bound to MI immediately implies the bounds for any Bayesian cost [25]. Therefore, the bounds for MI are more meaningful than the bounds on Bayesian cost, in the same way as the entropic uncertainty relations are stronger than the standard ones [26–28]. The most famous example is the relation between MI and averaged mean square error [13,14]. Using this relationship, our result also sets a bound on the squared error, operating for a broader class of problems than the standard van Trees inequality.

In quantum mechanics, where conditional probability $p(x|\phi)$ comes from performing measurements on a quantum state, MI maximized over the choice of measurements is known to be bounded by Holevo information [29]. Therefore,



$$I(X, \Phi) \leq \begin{cases} \ln \left(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F(\phi)} \right), & \text{supp } p(\phi) \subseteq [a, b] \\ \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi) p(\phi)^2 + \dot{p}(\phi)^2} \right) + H(\Phi) \end{cases}$$

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FIG. 1. The bound for the mutual information $I(X, \Phi)$ in terms of the Fisher information $F(\phi)$ allows for the transfer of the results obtained for local estimation to global estimation. Here the parameter ϕ is considered as realization of a random variable Φ .

we also present the quantum version of the bound, which connects the Holevo information with the quantum Fisher information (QFI).

To show the usefulness of our bounds, we apply them to a case study in quantum metrology. Quantum metrology [30–38] is the study of how quantum effects such as entanglement [30] or squeezing [39] can be used to enhance the precision of the measurement of a parameter ϕ of a quantum system. In the noiseless case, it is easy to show that the ultimate limit in precision when the system is sampled N times, the Heisenberg scaling $\Delta\phi \propto 1/N$, can be achieved only using quantum effects, except in the trivial case in which a single probe samples the system repeatedly and is measured only once. Instead, classical strategies are limited to the standard quantum limit (SQL) $\Delta\phi \propto 1/\sqrt{N}$ of the central limit theorem. In the noisy case [40–48] the situation is much more complicated, and, even though typically in the asymptotic regime of large N the Heisenberg scaling cannot be achieved anymore, one can still obtain quantum enhancements by large factors. Most of these results have been obtained using Fisher information techniques, such as the quantum Cramer-Rao bound [40,42–48]. As we show below, our new bounds allow us to transfer the highly sophisticated results on FI in the presence of noise to the MI in a simple way. In the noisy case, this provides a nontrivial bridge between local and global quantum estimation strategies. Besides quantum metrology, our result may be used to bound channel capacity in quantum communication.

Bounding mutual information by Fisher information. The estimation of a parameter ϕ from measurements is described by two probabilities: the prior distribution of the parameter $p(\phi)$ and the conditional probability $p(x|\phi)$ of the measurement results x . Together they constitute the joint probability $p(x, \phi) = p(x|\phi)p(\phi)$. The MI between random variables X and Φ is $I(X, \Phi) := H(\Phi) - H(\Phi|X)$, where $H(\Phi) = -\int d\phi p(\phi) \ln p(\phi)$ is the entropy of $p(\phi)$ and $H(\Phi|X)$ is the conditional entropy, given x [49]. MI tells us the amount of information (in nats) about ϕ obtained from x . The FI in given point ϕ is $F(\phi) = \sum_x \frac{1}{p(x|\phi)} \left(\frac{dp(x|\phi)}{d\phi} \right)^2$ and tells us how much information on ϕ is contained in $p(x|\phi)$. The following two theorems relate MI to FI.

Theorem 1. If the parameter is guaranteed to lie inside of a finite-size set, namely $\text{supp } p(\phi) \subseteq [a, b]$, then

$$I(X, \Phi) \leq \ln \left(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F(\phi)} \right). \quad (1)$$

Note especially, that there are no further constraints on $p(\phi)$ —it is an arbitrary probability density, continuous or discrete (or a combination of these two), as long as it takes value 0 outside of $[a, b]$. This inequality holds for any $p(x|\phi)$, so in the context of quantum metrology, by replacing the FI with the quantum Fisher information, one obtains a bound valid for any possible measurement. Moreover the integral is invariant for reparametrization of the probabilities $\phi \rightarrow \phi'$ (with appropriately modified limits of integration), as $F(\phi') = F(\phi) \left(\frac{d\phi}{d\phi'} \right)^2$, so $\sqrt{F(\phi')} d\phi' = \sqrt{F(\phi)} d\phi$. In a broader context, this feature is crucial in defining noninformative prior distributions, the Jeffreys' prior [50]. Yet another way to understand reparametrization-invariance of Eq. (1) is to notice,

that the FI is known to define a metric corresponding to an Euclidean distance between $\sqrt{p(x|\phi)}$ and $\sqrt{p(x|\phi + d\phi)}$ [51], so the above integral is just the size of the region $[a, b]$ in this metric.

If the support of $p(\phi)$ is not bounded, then the following holds:

Theorem 2. For arbitrary differentiable probability distribution $p(\phi)$, we have

$$I(X, \Phi) \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi) p(\phi)^2 + \dot{p}(\phi)^2} \right) + H(\Phi), \quad (2)$$

where $\dot{p} = dp/d\phi$ and the $\ln$ part is obviously an upper bound to $-H(\Phi|x)$. In contrast to the previous bound, this is not reparametrization invariant (indeed, it is well known that the differential entropy $H(\Phi)$ is defined modulo an arbitrary additive constant). So, the upper bound, and also its tightness, will depend on the parametrization.

Proof of the two bounds. The mutual information may be written as

$$\begin{aligned} I(X, \Phi) &= \sum_x \int_{-\infty}^{+\infty} d\phi p(x, \phi) \ln \frac{p(x, \phi)}{p(x)p(\phi)} \\ &= \sum_x \int_{-\infty}^{+\infty} d\phi p(x, \phi) \ln \frac{p(x|\phi)f(\phi)}{p(x)} \\ &\quad - \sum_x \int_{-\infty}^{+\infty} d\phi p(x, \phi) \ln f(\phi), \end{aligned} \quad (3)$$

where $f(\phi)$ is an arbitrary nonnegative function satisfying $\text{supp } f(\phi) \supseteq \text{supp } p(\phi)$. The first element of the above may be bounded by:

$$\begin{aligned} &\leq \sum_x p(x) \ln \max_{\phi} \frac{p(x|\phi)f(\phi)}{p(x)} \\ &\leq \max_{q_x: \sum_x q_x = 1} \left(\sum_x q_x \ln \max_{\phi} \frac{p(x|\phi)f(\phi)}{q_x} \right) \\ &\leq \ln \sum_x \max_{\phi} p(x|\phi)f(\phi), \end{aligned} \quad (4)$$

where the last inequality comes from direct maximization over q_x , which is obtained for $q_x = \frac{p(x|\phi)f(\phi)}{\sum_x \max_{\phi} p(x|\phi)f(\phi)}$ (see Appendix A). If $f(\phi)$ goes to 0 for $\phi \rightarrow \pm\infty$, then we have

$$\sum_x \max_{\phi} p(x|\phi)f(\phi) \leq \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sum_x \left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right|. \quad (5)$$

From Cauchy's inequality for vectors $\sqrt{p(x|\phi)}$ and $\sqrt{\left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right|^2 \frac{1}{p(x|\phi)}}$ (treated as a functions of single parameter x , with fixed ϕ) we have (see Appendix A)

$$\sum_x \left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right| \leq \sqrt{F(\phi)f(\phi)^2 + \dot{f}(\phi)^2}. \quad (6)$$

Combining all above one may get

$$I(X, \Phi) \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi) f(\phi)^2 + \dot{f}(\phi)^2} \right) + \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi), \quad (7)$$

which may be further optimized over choice $f(\phi)$ (going to zero for $\phi \rightarrow \pm\infty$). Note especially, that for $f(\phi) = p(\phi)$, we get Eq. (2). Alternatively, for $f(\phi) = 1$ on $[a, b]$ and 0 outside (so $\dot{f}(\phi) = \delta(\phi - a) - \delta(\phi - b)$), we obtain Eq. (1), see Appendix B for more details. ■

From local to global estimation. The Fisher information, via the Cramér-Rao inequality, constitutes a lower bound for the variance for any locally unbiased estimator and it is a meaningful tool in the situation when the number of repetitions of the estimation protocol is large. However, if the number of measurement repetitions is finite, then properly including prior knowledge about the parameter may lead, when averaged, to a smaller variance (as the optimal estimator may not satisfy the local unbiasedness condition). It may be quantified by the general relation between averaged Bayesian cost $\overline{\Delta^2 \tilde{\phi}} := \int d\phi p(\phi) \int d\tilde{\phi} p(\tilde{\phi}|\phi) (\tilde{\phi} - \phi)^2$, the entropy $H(\Phi)$ of the prior distribution, and MI [13,52]:

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{e^{2H(\Phi|X)}}{2\pi e} = \frac{e^{2H(\Phi)} e^{-2I(X, \Phi)}}{2\pi e}, \quad (8)$$

Note that the above is tight iff all *a posteriori* distributions $p(\phi|x)$ are Gaussian with the same variance. The above inequality in a clear manner separates the impact of the initial knowledge $-H(\Phi)$ from the knowledge obtained from the experiment $I(X, \Phi)$. Note, that while $I(X, \Phi)$ is reparametrization invariant, both $\overline{\Delta^2 \tilde{\phi}}$ and $e^{2H(\Phi)}$ behave under reparametrization in a similar way, making all inequality consistent. By applying the bound Eq. (1) to the above we obtain

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{e^{2H(\Phi)}}{2\pi e} \frac{1}{\left(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F(\phi)}\right)^2}. \quad (9)$$

Note that the prior information affects not only $H(\Phi)$, but also the range of the integral in the denominator. The better the parameter is known from the beginning, the more difficult it is to acquire significant new information from measurements. In particular, for the relevant case of a rectangle prior of width d with FI constant over the prior, one still obtains a significant bound, contrary to the van Trees case (see next section for discussion):

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{2}{\pi e} \frac{1}{(2/d + \sqrt{F})^2}. \quad (10)$$

As expected, in the limit of many repetitions N of the estimation procedure, the prior knowledge becomes irrelevant, since F typically grows linearly with N . A disadvantage of this bound is that it is not asymptotically tight, because of the multiplicative factor $2/\pi e$.

Of course, a further bound for the Bayesian cost can be obtained also for unbounded-support $p(\phi)$ by inserting Eq. (2)

into Eq. (8):

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{2}{\pi e} \frac{1}{\left(\int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi) p(\phi)^2 + \dot{p}(\phi)^2}\right)^2}, \quad (11)$$

which, for example, for Gaussian priors leads to a bound qualitatively similar to the van Trees one (see Appendix C). Note that for specific case of rectangle prior both bounds Eqs. (9) and (11) leads to the same results (see Appendix B).

Our results (9) and (11) provide bounds to the quadratic Bayesian cost. But a relation similar to Eq. (8) may be derived for any moment (see Lemma B.1. in Ref. [25]). Therefore, our bounds to MI imply a bound for the Bayesian cost with an arbitrary cost function, assuming that it may be expanded in Taylor series. These show that the bound for the MI is more informative than any bound to Bayesian cost.

Relation to the Efroimovich and van Trees inequalities. We will now compare our results with the bounds that exist in the literature. MI may be related to FI via Efroimovich relation [7] (see also Ref. [8] for some generalizations and Ref. [[9], Corollary 3] for an alternative proof):

$$I(X, \Phi) \leq \frac{1}{2} \ln \left[\frac{1}{2\pi e} \left(\int_{-\infty}^{+\infty} d\phi F(\phi) p(\phi) + P \right) \right] + H(\Phi),$$

with $P = \int_{-\infty}^{+\infty} d\phi \frac{1}{p(\phi)} \left(\frac{dp(\phi)}{d\phi} \right)^2,$ (12)

where P can be interpreted as the information included in the prior. While for Gaussian prior the quantity P is a reasonable measure of information (the inverse of the variance), it is completely unreasonable for the rectangle distribution, where P diverges. In general, P depends more on the sharpness of the prior distribution on its edges than on its actual width, which is more an artifact of the derivation of the bound, rather than a well-motivated feature. Overcoming this problem was also discussed in Refs. [8,9].

Introducing Efroimovich's inequality into Eq. (8) we find the van Trees one [11,53]:

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{1}{\int_{-\infty}^{+\infty} d\phi F(\phi) p(\phi) + P}, \quad (13)$$

which again suffers sharp $p(\phi)$. See also Refs. [11,54,55] for more advanced versions of the inequality that solve this problem.

This issue does not appear in our bounds (1) and (2). Indeed, the impact of $\dot{p}(\phi)$ can be bounded by $\frac{1}{2} \int_{-\infty}^{+\infty} d\phi |\dot{p}(\phi)|$, as discussed in Appendix B. In particular, for any prior concentrated around one region (more formally: the prior where the derivative $\dot{p}(\phi)$ changes its sign only once), it may be bounded by constant $\max_{\phi} p(\phi)$, no matter how sharply the prior changes.

At last, let us discuss the asymptotical behavior of the bounds for MI. We start with showing asymptotic saturability of Efroimovich's inequality (12) in the case when newly obtained data dominate initial knowledge. Assuming that P does not diverge, and that the probability $p(x, \phi)$ is not degenerate [i.e., $\forall_{\phi' \neq \phi} \exists_x p(x, \phi') \neq p(x, \phi)$], then Eq. (12) is tight asymptotically in the number of repetitions N of the estimation, and saturable using the maximum likelihood estimator. Indeed, first, the FI increases linearly with N , so asymptotically the

impact of P in the upper bound becomes negligible. Second, from the asymptotic normality of the maximum likelihood estimator, the difference between $\tilde{\phi}_{\text{ML}}$ and the true value of the parameter ϕ converge to normal distribution of variance $1/F(\phi)$, so we have [1]

$$H(\Phi|\tilde{\Phi}_{\text{ML}}) \rightarrow -\frac{1}{2} \ln \left[\frac{1}{2\pi e} \int_{-\infty}^{+\infty} d\phi F(\phi) p(\phi) \right]. \quad (14)$$

Third, assigning the estimator's value to the measurement result $\tilde{\phi}_{\text{ML}}(x)$ may only decrease the MI, namely $I(\tilde{\Phi}_{\text{ML}}, \Phi) \leq I(X, \Phi)$. This lower bound, through Eq. (14) converges to the upper bound Eq. (12) in the limit, where P is negligible.

From this, we see that for regular priors (finite P) the bound Eq. (2) is not asymptotically tight, because of the multiplicative factors inside the $\ln$.

Bound for Holevo information. Now we consider the quantum case, where the conditional probability distribution of the measurement output is given via the Born rule $p(x|\phi) = \text{Tr}(\rho_\phi M_x)$, where ρ_ϕ is density matrix of the quantum state, M_x is a POVM element ($M_x \geq 0$, $\sum_x M_x = \mathbb{1}$). Further, we assume that the state ρ_ϕ is drawn with probability distribution $p(\phi)$. The mutual information $I(X, \Phi)$, maximized over the choice of the measurement $\{M_x\}_x$, is bounded by Holevo information [29]. The following quantum version of Theorem 1 and Theorem 2 holds.

Theorem 3. Given a family of quantum states ρ_ϕ (differentiable with respect to ϕ), appearing with probability distribution $p(\phi)$ with $\text{supp } p(\phi) \subseteq [a, b]$, the Holevo information χ is bounded by

$$\chi \leq \ln \left(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F_Q[\rho_\phi]} \right), \quad (15)$$

with F_Q the quantum Fisher information of ρ_ϕ and $\chi := S(\int p(\phi) \rho_\phi d\phi) - \int p(\phi) S(\rho_\phi) d\phi$, with $S(\rho) := -\text{Tr}(\rho \log \rho)$ the von Neuman entropy.

For any differentiable probability distribution $p(\phi)$, the Holevo information is bounded by

$$\chi \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F_Q[\rho_\phi] p(\phi)^2 + \dot{p}(\phi)^2} \right) + H(\Phi). \quad (16)$$

Proof. See Appendix E ■

Applications to quantum metrology. To show the usefulness of the newly derived bounds, we now show how they can be used to give bounds to noisy quantum metrology for mutual information.

Consider the ϕ -dependent CPTP map $\Lambda_\phi[\cdot]$ that noisily encodes the parameter ϕ onto a quantum probe. The phase estimation problem is a typical example. Consider the simple case of qubit probes, where the parameter ϕ is encoded through a unitary $U_\phi = |0\rangle\langle 0| + e^{i\phi} |1\rangle\langle 1|$, followed by some kind of noise (e.g., dephasing, erasing) which together constitute the map $\Lambda_\phi[\cdot]$, which we will further treat as a quantum gate. The question is: what maximal MI may be obtained with the usage of N gates in an optimal way?

In Ref. [16] the Heisenberg scaling (HS) for MI has been defined as $I \propto \ln N$ and the standard quantum limit as $I \propto \frac{1}{2} \ln N$ (in analogy to the scaling of RMSE $\Delta\phi \propto 1/N$ and

$\Delta\phi \propto 1/\sqrt{N}$, respectively). In the case of noiseless estimation, it is known that using gates separately allows at most standard quantum limits, while more complicated schemes, including entanglement or multipassing, allow for obtaining HS (for example, via QPEA algorithm [16]). However, the noisy case has not been discussed up to now while using MI as a figure of merit.

Note that to obtain HS it is necessary to use all available resources jointly in a single experiment realization, therefore the argument based on the asymptotic efficiency of the maximum likelihood estimator cannot be applied here. Therefore, even if intuitively we expect that the inability to obtain HS in FI should imply an inability to obtain HS also for MI, it is not trivial to prove it formally in a general case. Note also, that in this model $\phi \in [0, 2\pi[$, and the Efroimovich's inequality or van Trees inequality cannot be applied to a uniform prior.

We can now apply the bounds derived above, e.g., Eq. (1), to transfer the highly nontrivial theory of noisy-channel estimation theory from the FI to the MI formalism. For FI, the general necessary and sufficient conditions for the obtainability of HS in the estimation of a given channel are known, expressed in terms of Kraus operators [45,47,48] or Lindblad operators [46,56]. Moreover, the resulting bounds are known to be saturable in the asymptotic limit $N \rightarrow \infty$ of many gates [47]. For example, for a mentioned model of phase estimation, it was shown that in the presence of dephasing or erasure noise, the FI obtainable by any protocol involving N gates is bounded from above by $F \leq N\eta/(1-\eta)$ [45], where η is the noise parameter ($\eta \rightarrow 1$ in the noiseless case, $\eta \rightarrow 0$ for maximum noise). The bound remains valid even for the most general adaptive scheme, including entanglement with arbitrary large ancilla and applying unitary control after each action of gate. Applying this bound to Eq. (1) we immediately find $I(X, \Phi) \leq \ln(1 + \pi\sqrt{N}\sqrt{\eta/(1-\eta)})$ and using Eq. (10) we have $\Delta^2\tilde{\phi} \geq 2/(\pi e)(1/\pi + \sqrt{N}\sqrt{\eta(1-\eta)})^{-2}$.

Even if the bound is not tight (there is an additive constant gap), it implies a standard quantum limit asymptotic scaling for all $\eta < 1$, and also gives the scaling with the intensity of the noise (see Appendix D for more details). Similar results can be obtained easily using different bounds to FI [42–44]. Naturally, this does not rule out quantum enhancements: for sufficiently large η one might still find an arbitrarily large constant gain over the SQL.

These will provide only the upper bounds to MI and so they can only be used to exclude the possibility of attaining the HS, but never to prove its achievability. While there exist lower bounds on the MI expressed in terms of the FI in the literature, they all require additional assumptions and a general discussion is impossible. In the noiseless case, it may be exemplified by the N00N state which allows the HS for the FI, but not for the MI, due to its periodicity with period $1/N$ [indeed, irrespective of the value of ϕ , the N00N state belongs to a two-dimensional subspace of the full Hilbert space, so it can carry at most one bit of information $I \leq \ln(2)$]. This demonstrates that, while the N00N state is highly effective for local estimation, it presents significant challenges for global estimation and requires careful consideration in its application [57].

Applications to communication. Beyond the problems of estimation and metrology, our bound is also useful for transferring the theorems derived in the metrology context to

communication problems. Consider a situation where Alice employs the parameter $\phi \in [a, b]$ as an alphabet for communicating with a state ρ_ϕ . Alice has the freedom to choose if she will use all values of ϕ or restrict herself to a discrete subset. From Eq. (1) the MI between Alice and Bob is then bounded by $I(A, B) \leq \ln(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F_Q[\rho_\phi]})$. The bound works independently of what messages (and with what probabilities) Alice is sending. Moreover, one can use Eq. (1) to give bounds to the classical capacity of a quantum or classical channel when the decoding strategy is fixed and its FI is known.

Conclusions. In conclusion, we have derived two bounds, Eqs. (1) and (2), for the MI in terms of the FI. We used these bounds to give two extensions of the van Trees inequality, Eqs. (9) and (11). We discussed the relation of all these bounds to the Efroimovich inequality (12). We also derived the quantum version of these bounds, Eqs. (15) and (16), where the Holevo information is bounded in terms of QFI. To prove the usefulness of these bounds, we have shown that they can be used to extend to the MI-based quantum metrology many highly nontrivial results known for the FI case. They also constitute the bounds on communication in terms of FI. See also Ref. [58] for alternative bounds linking MI with FI.

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Appendix A: Derivation of inequalities (4) and (6). First we derive inequality (4), i.e., we perform direct maximization of

$$\max_{q_x: \sum_x q_x = 1} \left(\sum_x q_x \ln \max_\phi \frac{p(x|\phi)f(\phi)}{q_x} \right) \quad (\text{A1})$$

by using the Lagrange multiplier method. The condition for local extremum with constraint $\sum q_x = 1$ is

$$\begin{aligned} \frac{\partial}{\partial q_x} \left[\sum_x q_x \ln \max_\phi \frac{p(x|\phi)f(\phi)}{q_x} - \lambda \left(\left(\sum_x q_x \right) - 1 \right) \right] \\ = \ln \max_\phi \frac{p(x|\phi)f(\phi)}{q_x} - 1 - \lambda = 0, \end{aligned} \quad (\text{A2})$$

which imposes that $\ln \max_\phi \frac{p(x|\phi)f(\phi)}{q_x}$ are the same for all x . Together with $\sum q_x = 1$ it gives $q_x = \frac{p(x|\phi)f(\phi)}{\sum_x \max_\phi p(x|\phi)f(\phi)}$. Since it is the only stationary point and the function takes smaller values outside of this point, this is a global maximum, which was to be proven.

Next, we derive Eq. (6), using Cauchy's inequality for vectors $\sqrt{\left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right|^2 \frac{1}{p(x|\phi)}}$ and $\sqrt{p(x|\phi)}$:

$$\begin{aligned} \sum_x \left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right| \\ = \sum_x \left[\sqrt{\left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right|^2 \frac{1}{p(x|\phi)}} \sqrt{p(x|\phi)} \right] \\ \leq \sqrt{\sum_x \left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right|^2 \frac{1}{p(x|\phi)}} \sqrt{\sum_x p(x|\phi)} \\ = \sqrt{\sum_x \left| \frac{d(p(x|\phi)f(\phi))}{d\phi} \right|^2 \frac{1}{p(x|\phi)}} \\ \sqrt{\sum_x \left[\frac{\dot{p}(x|\phi)^2}{p(x|\phi)} f(\phi)^2 + 2\dot{p}(x|\phi)f(\phi)\dot{f}(\phi) + p(x|\phi)\dot{f}(\phi)^2 \right]} \\ = \sqrt{F(\phi)f(\phi)^2 + \dot{f}(\phi)^2}, \end{aligned} \quad (\text{A3})$$

where we used $\sum_x p(x|\phi) = 1$ and $\sum_x \dot{p}(x|\phi) = 0$.

Appendix B: Obtaining Eq. (1). Here we show how to obtain Eq. (1) from Eq. (7):

$$\begin{aligned} I(X, \Phi) \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi)f(\phi)^2 + \dot{f}(\phi)^2} \right) + \\ - \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi). \end{aligned} \quad (\text{B1})$$

First, note that in general $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, so the above expression implies

$$\begin{aligned} I(X, \Phi) \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi)f(\phi)^2} + \frac{1}{2} \int_{-\infty}^{+\infty} d\phi |\dot{f}(\phi)| \right) \\ - \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi). \end{aligned} \quad (\text{B2})$$

Next, choosing $f(\phi) = 1$ on $[a, b]$ and 0 outside (so $\dot{f}(\phi) = \delta(\phi - a) - \delta(\phi - b)$), we have

$$I(X, \Phi) \leq \ln \left(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F(\phi)} \right). \quad (\text{B3})$$

Suppose one would prefer a more rigorous approach (avoiding taking the derivative of the rectangle function). In that case, one may always consider a family of functions $f_\epsilon(\phi)$ satisfying: $f_\epsilon(\phi) = 1$ on $[a, b]$, $f_\epsilon(\phi) = 0$ outside of $[a - \epsilon, b + \epsilon]$ and $f_\epsilon(\phi)$ is monotonic and differentiable on both $[a - \epsilon, a]$ and $[b, b + \epsilon]$. Then, assuming that $F(\phi)$ does not diverge for any ϕ , in the limit $\epsilon \rightarrow 0$, Eq. (B2) converge to Eq. (B3).

Note that Eq. (B2) implies also, that an impact of $\dot{p}(\phi)$ in Eq. (2) may be bound by $\frac{1}{2} \int_{-\infty}^{+\infty} |\dot{p}(\phi)|$. Especially, for a specific case of rectangle prior ($p(\phi) = 1/(b - a)$ on $[a, b]$

and 0 outside), Eq. (2) therefore leads to Eq. (1), as

$$\begin{aligned}
 I(X, \Phi) &\leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F(\phi) p(\phi)^2 + \dot{p}(\phi)^2} \right) + H(\Phi) \\
 &\leq \ln \left(\frac{1}{b-a} + \frac{1}{2(b-a)} \int_a^b d\phi \sqrt{F(\phi)} \right) + \ln(b-a) \\
 &= \ln \left(1 + \frac{1}{2} \int_a^b d\phi \sqrt{F(\phi)} \right). \quad (\text{B4})
 \end{aligned}$$

Appendix C: Applying the bound for gaussian distribution. Consider an *a priori* distribution $p(\phi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{\phi^2}{2\sigma^2})$. Substituting Eq. (2) into Eq. (8) we obtain

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{2}{\pi e} \frac{1}{(\int d\phi \sqrt{F(\phi) + \phi^2/\sigma^4} p(\phi))^2}. \quad (\text{C1})$$

Further, assuming that $F(\phi)$ does not change with ϕ , the integral may be calculated analytically as $\frac{\sqrt{2}}{\sigma} U[-\frac{1}{2}, 0, \frac{F\sigma^2}{2}]$, where $U(\cdot)$ is Tricomi confluent hypergeometric function.

For a slightly less tight, but simpler bound for the case where FI does not depend on ϕ , note that after introducing the variable $\eta = \phi^2$, the quantity $\sqrt{F + \eta/\sigma^4}$ is concave as a function of η , so $\mathbb{E}[\sqrt{F + \eta/\sigma^4}] \leq \sqrt{F + \mathbb{E}[\eta]/\sigma^4}$ (with $\mathbb{E}[\cdot]$ denoting averaging), so the integral may be bounded from above by $\sqrt{F + 1/\sigma^2}$ (numerical results also show that this is a good approximation for the exact result for $F\sigma^2 \gg 1$). For any F, σ we have

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{2}{\pi e} \frac{1}{F + 1/\sigma^2}, \quad (\text{C2})$$

while from the van Trees inequality we have Eq. (13) (taken for constant F)

$$\overline{\Delta^2 \tilde{\phi}} \geq \frac{1}{F + 1/\sigma^2}. \quad (\text{C3})$$

Then, for this specific case, our bound is less tight, because of the $2/\pi e$ multiplicative factor.

Appendix D: Mutual information in occurrence of noise. We now give a global bound to the mutual information in terms of N , the number of calls to the phase gate $U_\phi = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|$, in the presence of dephasing and amplitude damping noise.

In our noise model, each unitary gate U_ϕ is replaced by the noisy gate acting on the density operator as

$$\Lambda_\phi : \rho \mapsto \sum_k K_k U_\phi \rho U_\phi^\dagger K_k^\dagger, \quad (\text{D1})$$

with K_k being the Kraus operators.

For dephasing noise the Kraus operators are

$$K_0 = \mathbb{1} \left(\frac{1 + \sqrt{\eta}}{2} \right)^{1/2}, \quad K_1 = \sigma_z \left(\frac{1 - \sqrt{\eta}}{2} \right)^{1/2}, \quad (\text{D2})$$

where $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, and η is the noise parameter. Also, the Kraus operators in the presence of

amplitude damping noise are

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{pmatrix}. \quad (\text{D3})$$

Finally, for erasure noise the Kraus operators are

$$\begin{aligned}
 K_0 &= \begin{pmatrix} \sqrt{\eta} & 0 & 0 \\ 0 & \sqrt{\eta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 K_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{1-\eta} & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{1-\eta} & 0 \end{pmatrix}, \quad (\text{D4})
 \end{aligned}$$

where a third dimension is added to indicate loss of phase information.

In this notation, an asymptotic upper bound to the Fisher information is [45,48]

$$F \leq NF_{\text{as}}, \quad (\text{D5})$$

where $F_{\text{as}} = \eta/(1-\eta)$ for both dephasing and erasure noise with the most general AD strategy, as well as amplitude damping noise with EN strategy [44]. Combining the Fisher information bound with Eq. (1), we get a global upper bound to the mutual information,

$$I(X, \Phi) \leq \ln \left(1 + \pi \sqrt{\frac{N\eta}{1-\eta}} \right). \quad (\text{D6})$$

For amplitude damping noise, the Fisher information bound is slightly different, and the mutual information bound can be derived similarly.

Especially, when limited to the EN strategy, a tighter bound to Fisher information for finite- N is given by [59]

$$F \leq \frac{NF_{\text{as}}}{1 + \frac{F_{\text{as}}}{N}}. \quad (\text{D7})$$

Combining the Fisher information bound with Eq. (1), we get a global upper bound to the mutual information

$$I(X, \Phi) \leq \ln \left(1 + \pi \sqrt{\frac{N\eta/(1-\eta)}{1 + \eta/N(1-\eta)}} \right), \quad (\text{D8})$$

for all these three noise models. We plot the right-hand side of Eq. (D8) in Fig. 2, in which we can see a transition from the HS to the SQL as N increases. The stronger the noise is, the earlier the transition happens.

Appendix E: Holevo information bounded by quantum fisher information—proof of Theorem 3. In this Appendix, we prove Theorem 3, using a similar way of reasoning as in the proof of Theorem 1, Theorem 2, adapted to the quantum version. Namely, we introduce function $f(\phi)$ and prove stronger statement:

Theorem E1. Given a family of quantum states ρ_ϕ (differentiable with respect to ϕ), appearing with probability distribution $p(\phi)$. For any nonnegative function $f(\phi)$ satisfying $\text{supp } f(\phi) \supseteq \text{supp } p(\phi)$ and vanishing at $\pm\infty$, the Holevo

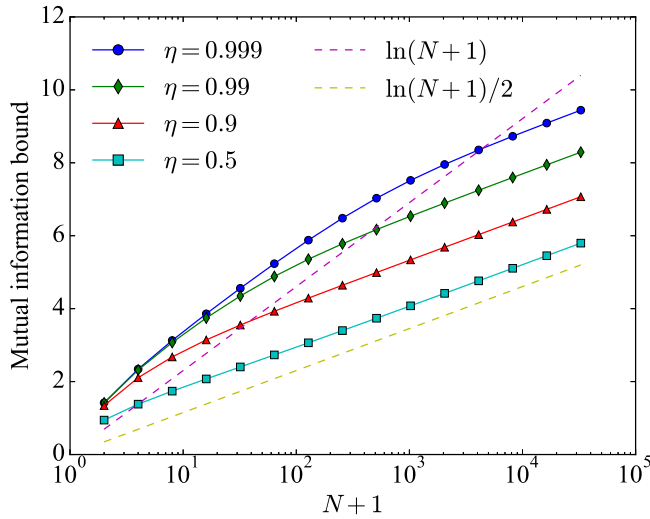


FIG. 2. The mutual information bound given by Eq. (1) with maximum Fisher information, for the dephasing channel and amplitude damping channel. The two dashed lines help to compare the bound with HS and SQL.

information χ is bounded by

$$\chi \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F_Q[\rho_\phi] f(\phi)^2 + \dot{f}(\phi)^2} \right) - \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi), \quad (\text{E1})$$

with F_Q the quantum Fisher information of ρ_ϕ and $\chi := S(\int p(\phi) \rho_\phi d\phi) - \int p(\phi) S(\rho_\phi) d\phi$, with $S(\rho) := -\text{Tr}(\rho \log \rho)$ the von Neuman entropy.

Then Theorem 3 comes as a collorary of Eq. (E1) with $f(\phi) = 1/(b-a)$ on $[a, b]$ and 0 outside or with $f(\phi) = p(\phi)$.

Before the proof we give the following lemmas.

Lemma E1. For any two vectors $|a\rangle, |b\rangle$,

$$\text{Tr} |a\rangle\langle b| + |b\rangle\langle a| \leq 2\sqrt{\langle a|a\rangle\langle b|b\rangle}, \quad (\text{E2})$$

where $|A| := \sqrt{A^\dagger A}$ for any operator A .

Proof of Lemma E1. The column space of the operator $|a\rangle\langle b| + |b\rangle\langle a|$ is $\mathcal{A} := \text{span}(|a\rangle, |b\rangle)$ of dimension at most 2. When $\dim \mathcal{A} \leq 1$ the left-hand side becomes $2|\text{Re}\langle a|b\rangle|$, making it a trivial result.

When $\dim \mathcal{A} = 2$, suppose $x|a\rangle + y|b\rangle \in \mathcal{A}$ is an eigenvector of $|a\rangle\langle b| + |b\rangle\langle a|$ with eigenvalue λ , then

$$\lambda = \langle b|a\rangle + \frac{y}{x} \langle b|b\rangle = \langle a|b\rangle + \frac{x}{y} \langle a|a\rangle. \quad (\text{E3})$$

By eliminating y/x we get

$$\lambda^2 - (\langle a|b\rangle + \langle b|a\rangle)\lambda - (\langle a|a\rangle\langle b|b\rangle - \langle a|b\rangle\langle b|a\rangle) = 0. \quad (\text{E4})$$

Note that $\langle a|b\rangle + \langle b|a\rangle \in \mathbb{R}$ and $-\langle a|a\rangle\langle b|b\rangle + \langle a|b\rangle\langle b|a\rangle \leq 0$, thus the equation has two real roots λ_1, λ_2

with opposite signs. Moreover,

$$\begin{aligned} \text{Tr} |a\rangle\langle b| + |b\rangle\langle a| &= |\lambda_1| + |\lambda_2| \\ &= |\lambda_1 - \lambda_2| \\ &= \sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2} \\ &= \sqrt{(\langle a|b\rangle + \langle b|a\rangle)^2 + 4(\langle a|a\rangle\langle b|b\rangle - \langle a|b\rangle\langle b|a\rangle)} \\ &= \sqrt{(\langle a|b\rangle - \langle b|a\rangle)^2 + 4\langle a|a\rangle\langle b|b\rangle} \\ &\leq 2\sqrt{\langle a|a\rangle\langle b|b\rangle}, \end{aligned} \quad (\text{E5})$$

where in the last inequality we use $\langle a|b\rangle - \langle b|a\rangle \in i\mathbb{R}$. ■

Lemma E2. Given positive operators A, B satisfying $0 < A \leq B$, i.e., A is positive definite and $(B - A)$ is positive semidefinite, then

$$\ln(A) \leq \ln(B). \quad (\text{E6})$$

Proof of Lemma E2. The condition $0 < A \leq B$ implies that $0 < A^{1/2} \leq B^{1/2}$ [60]. By iterating it we know $0 < A^{1/2^m} \leq B^{1/2^m}$ holds for any positive integer m . Finally, by the identity $\ln(a) = \lim_{x \rightarrow 0} (a^x - 1)/x$,

$$\ln(A) = \lim_{m \rightarrow +\infty} 2^m (A^{1/2^m} - I), \quad (\text{E7})$$

then Eq. (E6) is obtained. ■

Proof of Theorem E1. We first consider the pure state case where $\rho_\phi = |\psi_\phi\rangle\langle\psi_\phi|$. Write

$$\rho = \int_{-\infty}^{+\infty} p(\phi) \rho_\phi d\phi. \quad (\text{E8})$$

Then $\chi = S(\rho)$, and the quantum Fisher information is also simplified to

$$F_Q[\rho_\phi] = 4[\langle \dot{\psi}_\phi | \dot{\psi}_\phi \rangle - \langle \dot{\psi}_\phi | \psi_\phi \rangle \langle \psi_\phi | \dot{\psi}_\phi \rangle]. \quad (\text{E9})$$

Moreover, by adjusting the global phase of $|\psi_\phi\rangle$ we can force $\langle \psi_\phi | \dot{\psi}_\phi \rangle = 0$ and simplify Eq. (E9) to

$$F_Q[\rho_\phi] = 4\langle \dot{\psi}_\phi | \dot{\psi}_\phi \rangle. \quad (\text{E10})$$

To prove Eq. (E1), we first shift the last term to the left side. For pure state [using $\chi = S(\rho)$], we therefore have

$$\begin{aligned} \chi + \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi) &= S(\rho) + \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi) \\ &= S(\rho) + \text{Tr} \int_{-\infty}^{+\infty} d\phi p(\phi) \rho_\phi \ln[f(\phi) \rho_\phi]. \end{aligned} \quad (\text{E11})$$

Now, to bound it from above we will find the matrix bounding $f(\phi) \rho_\phi$ from above (in the sense of matrix inequality).

To do that, we define an unnormalized vector $|\Psi_\phi\rangle = \sqrt{f(\phi)} |\psi_\phi\rangle$ and the operator:

$$\tilde{\rho} = \frac{1}{2} \int_{-\infty}^{+\infty} \left| \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| \right| d\phi. \quad (\text{E12})$$

For any $\phi_0 \in \mathbb{R}$,

$$\begin{aligned}\tilde{\rho} &= \frac{1}{2} \int_{-\infty}^{\phi_0} \left| \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| \right| d\phi + \frac{1}{2} \int_{\phi_0}^{+\infty} \left| \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| \right| d\phi \\ &\geq \frac{1}{2} \int_{-\infty}^{\phi_0} \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| d\phi - \frac{1}{2} \int_{\phi_0}^{+\infty} \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| d\phi \\ &= |\Psi_{\phi_0}\rangle \langle \Psi_{\phi_0}| = f(\phi_0) \rho_{\phi_0},\end{aligned}\quad (\text{E13})$$

where we use $f(\phi)|_{\pm\infty} = 0$.

By Lemma E2, $\ln[f(\phi)\rho_\phi] \leq \ln \tilde{\rho}$ in the support space of ρ_ϕ , thus we can further bound Eq. (E11) by

$$\begin{aligned}S(\rho) + \text{Tr} \int_{-\infty}^{+\infty} d\phi p(\phi) \rho_\phi \ln[f(\phi)\rho_\phi] \\ \leq S(\rho) + \text{Tr} \int_{-\infty}^{+\infty} d\phi p(\phi) \rho_\phi \ln \tilde{\rho} \\ = S(\rho) + \text{Tr}(\rho \ln \tilde{\rho}) \\ = -S(\rho || \tilde{\rho}^0) + \ln \text{Tr} \tilde{\rho} \\ \leq \ln \text{Tr} \tilde{\rho},\end{aligned}\quad (\text{E14})$$

where $\tilde{\rho}^0 := \tilde{\rho} / \text{Tr} \tilde{\rho}$ is the normalized $\tilde{\rho}$ and $S(\rho || \tilde{\rho}^0) := \text{Tr}[\rho(\ln \rho - \ln \tilde{\rho}^0)]$ is the quantum relative entropy, which is always nonnegative. While calculating $\text{Tr} \tilde{\rho}$ we can go with the trace under integral

$$\text{Tr} \tilde{\rho} = \frac{1}{2} \int_{-\infty}^{+\infty} \text{Tr} \left| \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| \right| d\phi, \quad (\text{E15})$$

so the right-hand side of Eq. (E14) may be further bounded using Lemma E1,

$$\begin{aligned}\text{Tr} \left| \frac{d}{d\phi} |\Psi_\phi\rangle \langle \Psi_\phi| \right| \\ = \text{Tr} | |\Psi_\phi\rangle \langle \dot{\Psi}_\phi| + |\dot{\Psi}_\phi\rangle \langle \Psi_\phi| | \\ \leq 2\sqrt{\langle \Psi_\phi | \Psi_\phi \rangle \langle \dot{\Psi}_\phi | \dot{\Psi}_\phi \rangle} \\ = \sqrt{f(\phi)^2 F_Q[\rho_\phi] + \dot{f}(\phi)^2},\end{aligned}\quad (\text{E16})$$

which, after integration and substitution to Eq. (E14) gives Eq. (E1).

Finally, in the general case where ρ_ϕ can be mixed states, we can find a purification of ρ_ϕ so that the proof above applies. The quantum Fisher information of ρ_ϕ is equal to the minimum quantum Fisher information of the purification [42]. Let $|\psi_\phi\rangle$ be a purification that reaches the minimum, i.e., $F_Q[\rho_\phi] = F_Q[|\psi_\phi\rangle \langle \psi_\phi|]$. Since $|\psi_\phi\rangle$ is pure, its Holevo quantity χ' satisfies

$$\begin{aligned}\chi' \leq \ln \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \sqrt{F_Q[|\psi_\phi\rangle \langle \psi_\phi|] f(\phi)^2 + \dot{f}(\phi)^2} \right) \\ - \int_{-\infty}^{+\infty} d\phi p(\phi) \ln f(\phi).\end{aligned}\quad (\text{E17})$$

Meanwhile, the Holevo quantity χ' is equal to the quantum mutual information between the two subsystems of $\int_{-\infty}^{+\infty} p(\phi) |\phi\rangle \langle \phi| \otimes |\psi_\phi\rangle \langle \psi_\phi|$ (where $|\phi\rangle$ is an orthonormal set), which does not increase under the partial trace operation to $|\psi_\phi\rangle \langle \psi_\phi|$, thus $\chi \leq \chi'$ (where χ is the Holevo information of ρ). Combining all the above, Eq. (E1) for general mixed states ρ_ϕ is obtained. ■

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- [1] N. Brunel and J.-P. Nadal, Mutual information, fisher information, and population coding, *Neural Comput.* **10**, 1731 (1998).
- [2] M. Bethge, D. Rotermund, and K. Pawelzik, Optimal short-term population coding: When fisher information fails, *Neural Comput.* **14**, 2317 (2002).
- [3] S. Yarrow, E. Challis, and P. Series, Fisher and Shannon information in finite neural populations, *Neural Comput.* **24**, 1740 (2012).
- [4] X.-X. Wei and A. A. Stocker, Mutual information, Fisher information, and efficient coding, *Neural Comput.* **28**, 305 (2016).
- [5] J. Czajkowski, M. Jarzyna, and R. Demkowicz-Dobrzański, Super-additivity in communication of classical information through quantum channels from a quantum parameter estimation perspective, *New J. Phys.* **19**, 073034 (2017).
- [6] L. P. Barnes and A. Özgür, Fisher information and mutual information constraints, in *2021 IEEE International Symposium on Information Theory (ISIT)*, Melbourne, Australia, (IEEE, 2021), pp. 2179–2184.
- [7] S. Y. Efroimovich, Information contained in a sequence of observations, *Probl. Inf. Transm.* **15**, 24 (1979).
- [8] E. Aras, K.-Y. Lee, A. Pananjady, and T. A. Courtade, A family of Bayesian Cramér-Rao bounds, and consequences for log-concave priors, in *Proceedings of the IEEE International Symposium on Information Theory (ISIT'19)* (IEEE, Piscataway, NJ, 2019), pp. 2699–2703.
- [9] K.-Y. Lee, *New Information Inequalities with Applications to Statistics* (University of California, Berkeley, 2022).
- [10] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I: Detection, Estimation, and Linear Modulation Theory* (John Wiley & Sons, New York, NY, 2004).
- [11] R. D. Gill and B. Y. Levit, Applications of the van trees inequality: A Bayesian Cramér-Rao bound, *Bernoulli* **1**, 59 (1995).
- [12] M. Schützenberger, A generalization of the Fréchet-Cramér inequality to the case of bayes estimation, *Bull. Am. Math. Soc.* **63**, 142 (1957).
- [13] M. J. W. Hall and H. M. Wiseman, Does nonlinear metrology offer improved resolution? Answers from quantum information theory, *Phys. Rev. X* **2**, 041006 (2012).
- [14] M. J. W. Hall, D. W. Berry, M. Zwierz, and H. M. Wiseman, Universality of the Heisenberg limit for estimates of random phase shifts, *Phys. Rev. A* **85**, 041802(R) (2012).
- [15] W. Rządowski and R. Demkowicz-Dobrzański, Discrete-to-continuous transition in quantum phase estimation, *Phys. Rev. A* **96**, 032319 (2017).
- [16] M. Hassani, C. Macchiavello, and L. Maccone, Digital quantum estimation, *Phys. Rev. Lett.* **119**, 200502 (2017).

- [17] G. Chesi, A. Riccardi, R. Rubboli, L. Maccone, and C. Macchiavello, Protocol for global multiphase estimation, *Phys. Rev. A* **108**, 012613 (2023).
- [18] W. Górecki, R. Demkowicz-Dobrzański, H. M. Wiseman, and D. W. Berry, π -corrected Heisenberg limit, *Phys. Rev. Lett.* **124**, 030501 (2020).
- [19] S. Morelli, A. Usui, E. Agudelo, and N. Friis, Bayesian parameter estimation using Gaussian states and measurements, *Quantum Sci. Technol.* **6**, 025018 (2021).
- [20] V. Gebhart, A. Smerzi, and L. Pezzè, Bayesian quantum multiphase estimation algorithm, *Phys. Rev. Appl.* **16**, 014035 (2021).
- [21] H. T. Dinani, D. W. Berry, R. Gonzalez, J. R. Maze, and C. Bonato, Bayesian estimation for quantum sensing in the absence of single-shot detection, *Phys. Rev. B* **99**, 125413 (2019).
- [22] C. Butucea, M. Guță, and L. Artiles, Minimax and adaptive estimation of the Wigner function in quantum homodyne tomography with noisy data, *Ann. Stat.* **35**, 465 (2007).
- [23] M. Hayashi, Comparison between the Cramer-Rao and the mini-max approaches in quantum channel estimation, *Commun. Math. Phys.* **304**, 689 (2011).
- [24] W. Górecki and R. Demkowicz-Dobrzański, Multiple-phase quantum interferometry: Real and apparent gains of measuring all the phases simultaneously, *Phys. Rev. Lett.* **128**, 040504 (2022).
- [25] W.-N. Chen and A. Özgür, L_q lower bounds on distributed estimation via fisher information, *arXiv:2402.01895*.
- [26] I. Białynicki-Birula and J. Mycielski, Uncertainty relations for information entropy in wave mechanics, *Commun. Math. Phys.* **44**, 129 (1975).
- [27] M. J. Hall, Phase resolution and coherent phase states, *J. Mod. Opt.* **40**, 809 (1993).
- [28] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, *Rev. Mod. Phys.* **89**, 015002 (2017).
- [29] M. M. Wilde, From classical to quantum Shannon theory, *arXiv:1106.1445*.
- [30] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, *Phys. Rev. Lett.* **96**, 010401 (2006).
- [31] M. G. A. Paris, Quantum estimation for quantum technologies, *Int. J. Quantum Inf.* **07**, 125 (2009).
- [32] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photon.* **5**, 222 (2011).
- [33] G. Toth and I. Apellaniz, Quantum metrology from a quantum information science perspective, *J. Phys. A: Math. Theor.* **47**, 424006 (2014).
- [34] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, *Chapter Four—Quantum Limits in Optical Interferometry* (Elsevier, Amsterdam, 2015), pp. 345–435.
- [35] R. Schnabel, Squeezed states of light and their applications in laser interferometers, *Phys. Rep.* **684**, 1 (2017).
- [36] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, *Rev. Mod. Phys.* **89**, 035002 (2017).
- [37] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Quantum metrology with nonclassical states of atomic ensembles, *Rev. Mod. Phys.* **90**, 035005 (2018).
- [38] S. Pirandola, B. R. Bardhan, T. Gehring, C. Weedbrook, and S. Lloyd, Advances in photonic quantum sensing, *Nat. Photon.* **12**, 724 (2018).
- [39] L. Maccone and A. Riccardi, Squeezing metrology: A unified framework, *Quantum* **4**, 292 (2020).
- [40] A. Fujiwara and H. Imai, A fibre bundle over manifolds of quantum channels and its application to quantum statistics, *J. Phys. A: Math. Theor.* **41**, 255304 (2008).
- [41] J. Kołodyński and R. Demkowicz-Dobrzański, Phase estimation without *a priori* phase knowledge in the presence of loss, *Phys. Rev. A* **82**, 053804 (2010).
- [42] B. Escher, R. de Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, *Nat. Phys.* **7**, 406 (2011).
- [43] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guță, The elusive Heisenberg limit in quantum-enhanced metrology, *Nat. Commun.* **3**, 1063 (2012).
- [44] S. I. Knysh, E. H. Chen, and G. A. Durkin, True limits to precision via unique quantum probe, *arXiv:1402.0495*.
- [45] R. Demkowicz-Dobrzański and L. Maccone, Using entanglement against noise in quantum metrology, *Phys. Rev. Lett.* **113**, 250801 (2014).
- [46] R. Demkowicz-Dobrzański, J. Czajkowski, and P. Sekatski, Adaptive quantum metrology under general markovian noise, *Phys. Rev. X* **7**, 041009 (2017).
- [47] S. Zhou and L. Jiang, Asymptotic theory of quantum channel estimation, *PRX Quantum* **2**, 010343 (2021).
- [48] S. Kurdziałek, W. Górecki, F. Albarelli, and R. Demkowicz-Dobrzański, Using adaptiveness and causal superpositions against noise in quantum metrology, *Phys. Rev. Lett.* **131**, 090801 (2023).
- [49] Note that, as we are dealing with continuous variables, the entropies may be negative.
- [50] B. S. Clarke and A. R. Barron, Jeffreys' prior is asymptotically least favorable under entropy risk, *J. Stat. Plann. Inference* **41**, 37 (1994).
- [51] H. Chen, Y. Chen, and H. Yuan, Information geometry under hierarchical quantum measurement, *Phys. Rev. Lett.* **128**, 250502 (2022).
- [52] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (John Wiley & Sons, New York, NY, 2006).
- [53] Note, that van Trees' inequality has been derived first [12] and it has been noticed to be a consequence of Efroimovich's inequality only recently.
- [54] B.-Z. Bobrovsky, E. Mayer-Wolf, and M. Zakai, Some classes of global Cramer-Rao bounds, *Ann. Statist.* **15**, 1421 (1987).
- [55] M. Tsang, Physics-inspired forms of the Bayesian Cramér-Rao bound, *Phys. Rev. A* **102**, 062217 (2020).
- [56] D. Layden, S. Zhou, P. Cappellaro, and L. Jiang, Ancilla-free quantum error correction codes for quantum metrology, *Phys. Rev. Lett.* **122**, 040502 (2019).
- [57] B. Higgins, D. Berry, S. Bartlett, M. Mitchell, H. Wiseman, and G. Pryde, Demonstrating Heisenberg-limited unambiguous phase estimation without adaptive measurements, *New J. Phys.* **11**, 073023 (2009).
- [58] X. Lu, W. Górecki, C. Macchiavello, and L. Maccone, Number of bits returned by a quantum estimation, *Phys. Rev. A* **110**, 032405 (2024).
- [59] J. Kołodyński and R. Demkowicz-Dobrzański, Efficient tools for quantum metrology with uncorrelated noise, *New J. Phys.* **15**, 073043 (2013).
- [60] P. D. Lax, *Linear Algebra and Its Applications* (Wiley-Interscience, Hoboken, NJ, 2007).