

Phase transitions in full counting statistics of free fermions and directed polymers

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We consider directed polymers in 1+1 spatial dimension under action of an external repulsive potential along a line. Using the exact mapping onto imaginary time evolution of free fermions we find that for sufficiently strong potential the system of polymers undergoes a continuous configurational phase transition. The transition corresponds to merging empty regions in the dominant limit shape.

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Introduction. Limit shapes, i.e., the emergence of dominant configurations with sharp boundaries in fluctuating statistical systems, are spectacular phenomena playing an important role in studying universal macroscopic behavior in statistical physics [1,2]. Recently, this phenomenon was discovered in directed polymers or vicious walkers where it was shown that the mutual repulsion between polymers gives rise to emergent long-range entropic forces that combine with rigid topological constraints to produce sharp features of the coarse-grained polymer density [3,4].

In quantum many-body systems limit shape configurations were reported in studies of the so-called emptiness formation probability (EFP) [5–9], the probability $P_R(0)$ that a macroscopic (on the scale of interparticle separation) interval $I = [-R/2, R/2]$ is completely devoid of particles. The quantum hydrodynamic approach [8] has elucidated the relationship between EFP and a limit shape—the optimal density profile with an astroid-shaped boundary in Euclidean space-time. In recent works similar limit shapes were found for weakly interacting bosonic liquids [10] and polytropic gases [11].

It is known since the work of de Gennes [12] that noncrossing directed polymers are exactly mapped to world lines of noninteracting fermions propagating in imaginary time. Thus the combination of the Pauli principle and the emptiness constraint leads naturally to essentially the same limit shape phenomenon. In the following, we focus on the limit shape problem in the fermionic formulation referring the reader to Refs. [3,4] for details of the mapping to corresponding polymer problems.

EFP is a particular case of full counting statistics (FCS), defined via the generating function

$$\chi_R(s) = \langle e^{-sN_R} \rangle = \sum_{N=0}^{\infty} P_R(N) e^{-sN}. \quad (1)$$

FCS is defined as the probability $P_R(N)$ that the number of polymers crossing the interval $N_R = N$. EFP is obtained by taking a $s \rightarrow +\infty$ limit in (1).

FCS was extensively studied in the context of quantum noise for mesoscopic condensed matter [13,14], ultracold atomic systems [15–17], and one-dimensional magnetism [18–20]. The cumulants of FCS are universally related to quantum entanglement entropy [21–24]. Recently, the generating function Eq. (1) appeared naturally in the context of continuously measured quantum systems [25]. It was proposed in Ref. [20] to use analytical properties of the FCS generating function to characterize the thermodynamic phases and to probe nonlocal correlations in many-body quantum systems.

In these works FCS was studied in the regime of *typical* fluctuations where the corresponding probability distribution $P_R(N)$ is almost Gaussian with higher cumulants suppressed by inverse powers of R [9,26]. In contrast, EFP represents a rare event in which the density of particles in the interval deviates macroscopically from its average value $\nu_\infty = \langle N_R \rangle / R$. The corresponding optimal instanton configuration in 1+1 dimensions has a peculiar astroid shape with typical dimensions $R \times R$. This motivates our study of FCS in the large deviation form

$$\chi_R(s) \sim e^{-R^2 f(s/R)}, \quad (2)$$

in terms of the rescaled parameter $\alpha = s/R$. In this Letter we show that the transition from typical to rare fluctuations is accompanied by a structural change of the dominant limit shape.

Model. We consider a statistical ensemble of nonintersecting paths $\{x_i(\tau)\}$, $x_i(\tau) < x_{i+1}(\tau)$ representing directed polymers. The polymers are parametrized by the continuous longitudinal coordinate τ , $\tau \in [-\beta/2, \beta/2]$ and lateral coordinates $x_i \in \mathbb{Z}$, which take values on a one-dimensional lattice and obey $x_i(\beta/2) = x_i(-\beta/2)$. Their density is $\nu(x, \tau) = \sum_i \delta_{x, x_i(\tau)}$, where $\delta_{x,y}$ is the Kronecker delta. In terms of the polymers, FCS is the distribution of the observable N_R equal to the number of polymers with $|x_i(0)| < R/2$.

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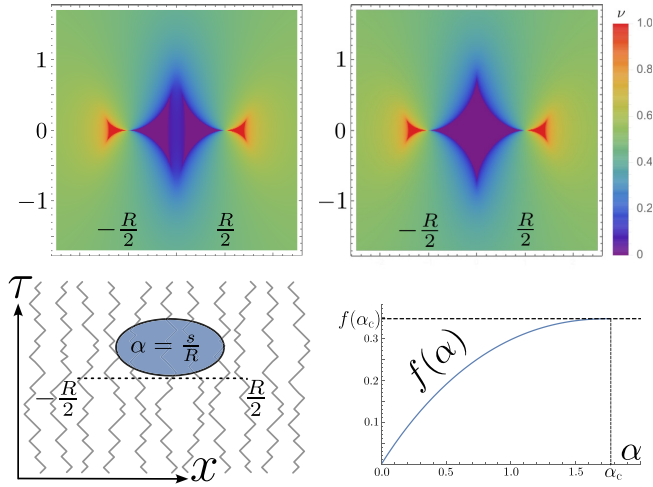


FIG. 1. Top: the dominant density configuration $\nu(x, \tau)$ is shown for $\alpha < \alpha_c$ (left) and $\alpha \geq \alpha_c$ (right) is shown for $\nu_\infty = 1/2$. Bottom left: a sketch of polymers' configuration. Each polymer passing through the interval $I = [-R/2, R/2]$ is penalized by weight $e^{-\alpha R}$. Bottom right: the rate function $f(\alpha)$; see Eq. (2).

Directed polymers can be mapped onto the world lines of noninteracting fermions hopping between nearest neighbor sites on a one dimensional lattice propagating in imaginary time τ [3,12,27]. Their evolution is governed by the tight-binding Hamiltonian H given by Eq. (11) below. In terms of this Hamiltonian, the generating function of FCS can then be represented as the many-body quantum expectation value

$$\chi_R(s) = \langle e^{-sN_R} \rangle = \frac{\text{Tr}(e^{-sN_R} e^{-\beta H})}{\text{Tr}(e^{-\beta H})}, \quad (3)$$

which is the main object of our study.

Results. The main result is a configurational transition of directed polymers at the critical value α_c , illustrated in Fig. 1 by plotting the typical density $\nu(x, \tau)$. The latter is obtained by solving

$$x + i\tau \sin k = Rg(e^{ik}), \quad (4)$$

for the real part of the complex momentum $k(x, \tau) = \pi\nu + i\nu$ at fixed x, τ . The function $g(z)$ is given by

$$g(z) = \frac{1}{2} \sqrt{\frac{(z - e^{ik_0})(z - e^{-ik_0})}{(z - e^{ik_F})(z - e^{-ik_F})}}. \quad (5)$$

Here $k_F = \pi\nu_\infty$ is proportional to the asymptotic density ν_∞ at $x, \tau \rightarrow \infty$ and the relation

$$\alpha = 2 \text{Re} \int_{\Gamma_\alpha} \frac{g(z)dz}{z} \quad (6)$$

fixes the parameter $k_0 = \pi\nu(0, 0) > 0$ proportional to the density in the center of the interval I . The contour of integration Γ_α runs from -1 to 1 connecting two branch cuts of $g(z)$ while avoiding the pole at $z = 0$. Equivalently, the right-hand side (RHS) of (6) can be written as a principal part integral along the interval $[-1, 1]$.

For moderate α , the optimal polymer density consists of two deltoid-shaped empty ($\nu = 0$) and two deltoid-shaped full ($\nu = 1$) regions, as shown in the top-left panel of Fig. 1.

The number of polymers in the interval I is macroscopically depleted with average density

$$\nu_I = f'(\alpha) = \frac{1}{\pi} \text{Im} \int_{e^{-ik_F}}^{e^{ik_F}} \frac{g(z)dz}{z}. \quad (7)$$

The value $k_0 = 0$ marks the onset of the transition to a macroscopically empty interval, $\nu_I = 0$. At this point the integral (6) reaches its maximum value α_c given by

$$\cosh \frac{\alpha_c}{2} \cos \frac{k_F}{2} = 1. \quad (8)$$

The top-right panel of Fig. 1 shows the empty regions having merged into one astroid-shaped empty region which includes the whole interval I . This situation, where polymers avoid the interval completely, persists for $\alpha > \alpha_c$ with the constant rate function given by the EFP result of Ref. [8],

$$f(\alpha) = f(\alpha_c) = -\ln \cos \frac{k_F}{2}, \quad \alpha > \alpha_c. \quad (9)$$

Just below the transition $\alpha \lesssim \alpha_c$ the behavior of the rate function is found to be

$$f(\alpha_c - \delta) = f(\alpha_c) + \frac{\delta^2}{2 \ln \delta}, \quad \delta \ll 1. \quad (10)$$

Thus the transition is continuous, featuring a divergent third derivative of the rate function similar to the transitions reported earlier in [28–30].

Derivation. To generate the configurations of the polymers via fermionic evolution we use the tight-binding second quantized Hamiltonian

$$H = \sum_{xy} (h)_{xy} c_x^\dagger c_y \quad (11)$$

expressed via creation/annihilation operators $c_x^\dagger, c_x$ obeying standard anticommutation relations $\{c_x, c_{x'}^\dagger\} = \delta_{xx'}$. The nearest neighbor hopping matrix

$$h_{xy} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(x-y)} (-\cos k - \mu) \quad (12)$$

contains the tight-binding dispersion $\varepsilon(k) = -\cos k$ and a chemical potential μ fixing the equilibrium density. The operator of particle number in an interval I is

$$N_R = \sum_{xy} (n_I)_{xy} c_x^\dagger c_y = \sum_{x \in I} c_x^\dagger c_x, \quad (13)$$

where $(n_I)_{xy} = \delta_{xy}$, for $x \in I$ and zero otherwise. It is the one-particle projector, $n_I = n_I^2$, on the interval I . Due to the particle-hole symmetry, the FCS obeys $\chi_R(s, \beta, \mu) = e^{-sR} \chi_R(-s, \beta, -\mu)$, so in what follows we focus on $s \geq 0$.

To find the generating function Eq. (3) we express, following [31], the traces in many-particle Fock space as determinants of operators in one-particle space:

$$\chi_R(s) = \frac{\det(1 + e^{-sn_I} e^{-\beta h})}{\det(1 + e^{-\beta h})}. \quad (14)$$

The fact that n_I projects into R lattice sites allows us to rewrite $\chi_R(s)$ as a determinant of $R \times R$ Toeplitz matrix with the elements

$$[1 + (e^{-s} - 1)n_F]_{xy} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(x-y)} e^{-sv(k)}, \quad (15)$$

where $n_F = (1 + e^{\beta h})^{-1}$ is the Fermi occupation operator.

We consider the zero temperature limit, $\beta \rightarrow \infty$, corresponding to infinitely long polymers. In this limit n_F becomes the projector on the single interval in the momentum space, $|k| < k_F$, with $\mu = -\cos k_F$. The so-called Toeplitz symbol $e^{-sv(k)}$ becomes singular since

$$v(k) = -\frac{1}{s} \ln \frac{1 + e^{-s} e^{-\beta h(k)}}{1 + e^{-\beta h(k)}} \rightarrow \theta(k_F - |k|). \quad (16)$$

Although the theory of Toeplitz determinants with singular symbols is well developed, see, e.g., Ref. [32], it is not directly applicable in our case where the parameter s scales with the size of the matrix R . Here we use an alternative approach based on the exact representation of the generating function as the R -dimensional integral,

$$\chi_R(s) = \frac{1}{R!} \int \frac{d^R k}{(2\pi)^R} |\Delta(e^{ik})|^2 \prod_{l=1}^R e^{-sv(k_l)}, \quad (17)$$

which is a configurational integral of a classical two-dimensional Coulomb gas of R charges [33]. In addition to a repulsive interaction given by the logarithm of the Vandermonde determinant, $\Delta(e^{ik}) = \prod_{1 \leq m < n \leq R} (e^{ik_n} - e^{ik_m})$, the charges on the interval $|k| < k_F$ are subject to a constant electrostatic repulsive potential of magnitude $s = \alpha R$. This scaling makes the potential comparable to the interparticle repulsion, with both terms being proportional to R^2 justifying the scaling in Eq. (2).

Due to the presence of large parameter R the integral (17) is dominated by a stationary configuration of charges. Their positions, $z_l = e^{ik_l}$, are encoded in the poles of the resolvent

$$g(z) + \frac{1}{2} = \frac{1}{R} \sum_{l=1}^R \frac{z}{z - e^{ik_l}} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{z \sigma(k)}{z - e^{ik}}. \quad (18)$$

Here $\sigma(k) = \frac{1}{R} \sum_{l=1}^R \delta(k - k_l)$ is the density of the poles, which becomes the smooth function, for $R \rightarrow \infty$. In this limit the charges are distributed along the branch cuts of $g(z)$, those being the two disjoint arcs of unit circle $z = e^{ik}$, $|k| < k_0$, and $k_F < |k| < \pi$.

Introducing the complex potential, $g(z) + 1/2 = z \partial_z \varphi(z)$, the stationarity condition can be expressed as the electrostatic equilibrium of the charges: the real part of $\varphi(z)$ is constant along each of the branch cuts. The parameter α then plays the role of the voltage between the cuts—hence the condition (6) fixing the parameter v_0 . Conversely, the imaginary part of the potential counts the number of charges and leads to Eq. (7). The density corresponding to the resolvent $g(z)$ in Eq. (5) is

$$\sigma(k) = 2 \operatorname{Re} g(e^{ik}) = \sqrt{\frac{\sin^2 \frac{k}{2} - \sin^2 \frac{k_0}{2}}{\sin^2 \frac{k}{2} - \sin^2 \frac{k_F}{2}}} \quad (19)$$

and is shown in the left panel of Fig. 2. The calculations leading to these results are standard and can be found in the Supplemental Material [33].

Hydrodynamic interpretation. The solution of the Coulomb gas model contains information about the *state* of the system at $\tau = 0$ which, using the hydrodynamic evolution of fermions, allows one to extract full information about the real space configurations of constrained directed polymers. This

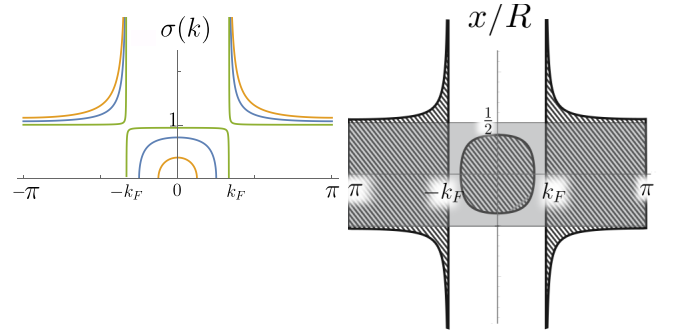


FIG. 2. Left: quasiparticle momentum distribution corresponding to the solution of the Coulomb gas problem Eq. (19) for $k_F/\pi = 1/3$ and α corresponding to $k_0/\pi = 0.332, 0.25, 0.125$. Right: quasiparticle Wigner function $f(x, k)$ for $k_0/\pi = 0.25$. The striped areas correspond to regions where $f(x, k) = 1$. The light gray rectangle represents regions where the quasiparticle vacuum ($s = 0$) Wigner function $f_{qp}^{\text{vac}}(x, k) = 1$.

can be traced back to interpreting Eq. (3) as an imaginary time path integral with the source term proportional to the counting parameter s acting at $\tau = 0$ and affecting R sites of the interval I .

The configurations contributing to the generating function Eq. (1) can be parametrized by degrees of freedom of R fermionic quasiparticles propagating in imaginary time with $R\sigma(k)$ playing the role of their momentum distribution at $\tau = 0$.

It is well known [8] that the large scale dynamics of such fermionic quasiparticles is captured by the hydrodynamic approach. In this approach one introduces two hydrodynamic fields, the density $v(x, \tau)$ and velocity $v(x, \tau)$ combined into complex momentum $k(x, \tau) = \pi v + i v$. The complex momenta satisfy the complex Burger's equation

$$i \partial_\tau k + \partial_x \varepsilon(k) = 0, \quad \tau \neq 0 \quad (20)$$

equivalent to hydrodynamic continuity and Euler equations after separation into real and imaginary parts.

As a result of time-reversal symmetry $k(x, -\tau) = \bar{k}(x, \tau)$ the density $v(x, \tau)$ and velocity $v(x, \tau)$ fields are even/odd functions of τ correspondingly. The initial condition $k_0(x) = k(x, 0)$ is, therefore, real. The inverse function $x_0(k)$ such that $x_0(k_0(x)) = x$ parametrizes the implicit solution to Eq. (20)

$$x + i \varepsilon'(k) \tau = x_0(k). \quad (21)$$

Below we argue that the right-hand side is given by the rescaled solution of the Coulomb gas problem as

$$x_0(k) = R\sigma(k)/2 = Rg(e^{ik}). \quad (22)$$

Indeed, for $s = 0$ the solutions of hydrodynamic Eq. (20) are given by the constant configuration $k(x, \tau) = \bar{k}(x, \tau) = k_F = \pi v_\infty$. For $s > 0$ the density is nonuniform and we write

$$v(x, 0) - v_\infty = \int \frac{dk}{2\pi} [f(x, k) - f^{\text{vac}}(x, k)]. \quad (23)$$

Here we represented the density fluctuation as a sum of contributions from different momenta, using the semiclassical Wigner function $f(x, k)$. The vacuum contribution $f^{\text{vac}}(x, k)$ corresponding to uniform density v_∞ at $s = 0$ is subtracted as

$v - v_\infty = 0$ for $s = 0$. The function $f(x, k)$ has the form of a local Fermi distribution, $f(x, k) = 1$ for $|k| < k_0(x)$ and zero outside this interval; see Fig. 2. The branches $\pm k_0(x)$ are thus the local “Fermi momenta” of the fermionic particles.

The same boundary can be represented using the inverse function $x_0(k)$. In terms of the Wigner function we have

$$2x_0(k) = \sum_x [f(x, k) - f^{\text{vac}}(x, k) + f_{\text{qp}}^{\text{vac}}(x, k)]. \quad (24)$$

Here we added the Wigner function $f_{\text{qp}}^{\text{vac}}(x, k) = \theta(R/2 - |x|)$ representing the vacuum ($s = 0$) distribution of *quasiparticles* as shown in Fig. 2. The sum in the RHS of Eq. (24) is nothing but the momentum distribution of the quasiparticles $R\sigma(k)$. This establishes the important link, Eq. (22), between the Coulomb gas charge density and the fluid dynamics.

The solution $k_0(x)$ of Eq. (21) at $\tau = 0$ shows that for $0 < \alpha < \alpha_c$ there is a nonzero density in some interval around $x = 0$. As one moves towards the ends of the interval the density drops to zero at $\pm x_- = Rg(1)$. At $x = \pm R/2$ the density jumps to unity and stays saturated up to $\pm x_+ = Rg(-1)$. Beyond this point the density decays to its asymptotic value $v_\infty = k_F/\pi$.

Solving Eq. (21) away from $\tau = 0$ provides the full space-time profile of density and velocity in terms of the complex function $k(x, \tau)$. For $0 < \alpha < \alpha_c$ there is always a central fluctuating region between the deltoid-shaped frozen regions which evolve from the frozen intervals with $v = 0, 1$ at $\tau = 0$. The boundary of the frozen regions is given in the parametric form

$$x/R = G_\pm(v) - \tanh v G'_\pm(v), \quad (25)$$

$$\tau/R = \mp G'_\pm(v)/\cosh v, \quad (26)$$

which is obtained by substituting $k_+ = iv, k_- = \pi + iv$ into Eq. (4) and defining $G_\pm(v) = g(e^{ik_\pm}) = g(\pm e^{-v})$ for the boundary with an empty/full frozen region.

When α is increased from zero the parameter v_0 decreases monotonically from $v_0 = v_\infty$. The transition corresponds to disappearance of the cut, $v_0 = 0$, at the critical value α_c . Calculation of α_c , Eq. (8), and the corresponding value of free energy, Eq. (9), are outlined in the Supplemental Material [33].

To establish the critical behavior close to the transition, $v_0 \rightarrow 0$, we consider Eqs. (6) and (7). The density $\sigma(k)$ is zero everywhere in $[-k_F, k_F]$ except for a small interval $[-k_0, k_0]$, where it can be approximated by the Wigner semicircle. The standard calculation gives

$$v_I = \frac{k_0}{2 \sin \frac{k_F}{2}} \int_{-k_0}^{k_0} \frac{dk}{2\pi} \sqrt{1 - \left(\frac{k}{k_0}\right)^2} = \frac{k_0^2}{8 \sin \frac{k_F}{2}}. \quad (27)$$

Calculating α is more involved. One can show [33] that the corresponding integral contains a logarithmic nonanalyticity:

$$\alpha - \alpha_c = -\frac{k_0^2}{8 \sin \frac{k_F}{2}} \left[1 + \ln \left(\frac{64}{k_0^2} \tan^2 \frac{k_F}{2} \right) \right]. \quad (28)$$

Thus in the leading order $\delta = \alpha_c - \alpha \simeq v_I \ln(1/v_I)$, which can be inverted as $v_I = \delta/\ln(1/\delta)$. Integrating $f'(\alpha) = v_I$ from α_c to $\alpha = \alpha_c - \delta$ one gets Eq. (10).

Conclusions. In this work we have computed the full counting statistics of lattice fermions within a macroscopic interval in the previously unexplored regime of large deviations and found the presence of a continuous phase transition between zero and near zero occupancy of the interval. The mapping to Coulomb gas makes our findings in agreement with findings of Refs. [34].

We obtained an analytic form for the rate function associated with the large deviation using a mapping between the FCS generating function and a Coulomb gas model. In contrast to the previous studies [35–37] the Coulomb gas model is naturally formulated in momentum space of R quasiparticles leading to the existence of the following duality. It is convenient to rewrite (14) as

$$\chi_R(s) = \det(1 - n_F + e^{-sn_I} n_F) = \det[1 - (1 - e^{-s}) n_I n_F]. \quad (29)$$

In the limit of zero temperature, $\beta \rightarrow \infty$, the Fermi occupation operator n_F also becomes a projector on the negative energy states and the expression (29) is completely symmetric with respect to the interchange $n_I \leftrightarrow n_F$. One has a choice of computing (29) in the momentum basis where it becomes the Fredholm determinant with the sine-kernel or in the coordinate basis, where it becomes the Toeplitz determinant of the size $R \times R$. This change corresponds to switching between the original fermions and the quasiparticles used for the hydrodynamic description (see Fig. 2).

At zero temperature FCS is, as expected, insensitive to the exact form of the fermion dispersion as long as a single interval of momenta is occupied in the ground state. This has to be contrasted with the space-time configurations of the hydrodynamic fields depending explicitly on the form of the dispersion; see, e.g., Eq. (21). Moreover, for a dispersion leading to crossing of the world lines of the fermions the hydrodynamic interpretation may even fail as some configurations would have negative Boltzmann weights [38]. These subtleties disappear in the continuous limit $k_F, k_0 \rightarrow 0$ with k_0/k_F fixed, in which $\epsilon(k) \sim k^2$ and Galilean symmetry is restored.

The momentum space representation used in this work can be extended to finite temperatures, by modifying the effective potential $v(k)$, Eq. (16), in the Coulomb gas model, Eq. (17).

In this work, we obtained the FCS generating function (1) in the fermion formulation and identified the dominant space-time configuration of fermion world lines. The latter are mapped onto the directed polymer configurations (see Fig. 1). In the limit of large R the statistically dominant contributions are coming from macroscopically large distances making our results universal. While the fermion problem is different from the polymer one at short scales, the microscopic details such as the lattice constant and polymer persistence length will affect only subleading asymptotics at large R .

It would be interesting to realize experimentally the phase transition described in this work. The transition must occur as a function of the strength of repulsion of polymers from the artificially created barrier (shown as a dashed segment in Fig. 1).

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- [1] R. Kenyon, Lectures on dimers, [arXiv:0910.3129](https://arxiv.org/abs/0910.3129).
 - [2] J.-M. Stéphan, Extreme boundary conditions and random tilings, *SciPost Phys. Lect. Notes* **2021**, 26 (2021).
 - [3] D. Z. Rocklin, S. Tan, and P. M. Goldbart, Directed-polymer systems explored via their quantum analogs: Topological constraints and their consequences, *Phys. Rev. B* **86**, 165421 (2012).
 - [4] D. Z. Rocklin and P. M. Goldbart, Directed-polymer systems explored via their quantum analogs: General polymer interactions and their consequences, *Phys. Rev. B* **88**, 165417 (2013).
 - [5] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, UK, 1993).
 - [6] A. G. Abanov and F. Franchini, Emptiness formation probability for the anisotropic XY spin chain in a magnetic field, *Phys. Lett. A* **316**, 342 (2003).
 - [7] F. Franchini and A. G. Abanov, Asymptotics of Toeplitz determinants and the emptiness formation probability for the XY spin chain, *J. Phys. A* **38**, 5069 (2005).
 - [8] A. G. Abanov, Hydrodynamics of correlated systems, in *Applications of Random Matrices in Physics*, edited by É. Brézin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin (Springer Netherlands, Dordrecht, 2006), pp. 139–161.
 - [9] J.-M. Stéphan, Emptiness formation probability, Toeplitz determinants, and conformal field theory, *J. Stat. Mech.* (2014) P05010.
 - [10] H.-C. Yeh and A. Kamenev, Emptiness formation probability in one-dimensional Bose liquids, *Phys. Rev. A* **101**, 023623 (2020).
 - [11] H.-C. Yeh, D. M. Gangardt, and A. Kamenev, Emptiness formation in polytropic quantum liquids, *J. Phys. A* **55**, 064002 (2022).
 - [12] P.-G. de Gennes, Soluble model for fibrous structures with steric constraints, *J. Chem. Phys.* **48**, 2257 (1968).
 - [13] L. S. Levitov and G. B. Lesovik, Charge-transport statistics in quantum conductors, *JETP Lett.* **55**, 555 (1992).
 - [14] Y. Nazarov, *Quantum Noise in Mesoscopic Physics* (Springer, New York, 2003).
 - [15] A. Polkovnikov, E. Altman, and E. Demler, Interference between independent fluctuating condensates, *Proc. Natl. Acad. Sci. USA* **103**, 6125 (2006).
 - [16] V. Gritsev, E. Altman, E. Demler, and A. Polkovnikov, Full quantum distribution of contrast in interference experiments between interacting one-dimensional Bose liquids, *Nat. Phys.* **2**, 705 (2006).
 - [17] M. Arzamasovs and D. M. Gangardt, Full counting statistics and large deviations in a thermal 1D Bose gas, *Phys. Rev. Lett.* **122**, 120401 (2019).
 - [18] V. Eisler, Z. Rácz, and F. van Wijland, Magnetization distribution in the transverse Ising chain with energy flux, *Phys. Rev. E* **67**, 056129 (2003).
 - [19] A. Lamacraft and P. Fendley, Order parameter statistics in the critical quantum Ising chain, *Phys. Rev. Lett.* **100**, 165706 (2008).
 - [20] D. A. Ivanov and A. G. Abanov, Characterizing correlations with full counting statistics: Classical Ising and quantum XY spin chains, *Phys. Rev. E* **87**, 022114 (2013).
 - [21] I. Klich and L. Levitov, Quantum noise as an entanglement meter, *Phys. Rev. Lett.* **102**, 100502 (2009).
 - [22] P. Calabrese, M. Mintchev, and E. Vicari, Entanglement entropy of one-dimensional gases, *Phys. Rev. Lett.* **107**, 020601 (2011).
 - [23] E. Vicari, Entanglement and particle correlations of fermi gases in harmonic traps, *Phys. Rev. A* **85**, 062104 (2012).
 - [24] P. Calabrese, P. Le Doussal, and S. N. Majumdar, Random matrices and entanglement entropy of trapped Fermi gases, *Phys. Rev. A* **91**, 012303 (2015).
 - [25] S. J. Garratt, Z. Weinstein, and E. Altman, Measurements conspire nonlocally to restructure critical quantum states, *Phys. Rev. X* **13**, 021026 (2023).
 - [26] D. A. Ivanov, A. G. Abanov, and V. V. Cheianov, Counting free fermions on a line: A Fisher–Hartwig asymptotic expansion for the Toeplitz determinant in the double-scaling limit, *J. Phys. A* **46**, 085003 (2013).
 - [27] M. E. Fisher, Walks, walls, wetting, and melting, *J. Stat. Phys.* **34**, 667 (1984).
 - [28] V. A. Kazakov and A. A. Migdal, Recent progress in the theory of noncritical strings, *Nucl. Phys. B* **311**, 171 (1988).
 - [29] S. N. Majumdar, C. Nadal, A. Scardicchio, and P. Vivo, Index distribution of Gaussian random matrices, *Phys. Rev. Lett.* **103**, 220603 (2009).
 - [30] R. Marino, S. N. Majumdar, G. Schehr, and P. Vivo, Index distribution of Cauchy random matrices, *J. Phys. A* **47**, 055001 (2014).
 - [31] I. Klich, An elementary derivation of Levitov’s formula, *Quantum Noise in Mesoscopic Physics*, edited by Yu. V. Nazarov (Springer Netherlands, Dordrecht, 2003), pp. 397–402.
 - [32] P. Deift, A. Its, and I. Krasovsky, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities, *Ann. Math.* **174**, 1243 (2011).
 - [33] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevResearch.7.L022008> for details of calculations.
 - [34] F. D. Cunden, P. Facchi, M. Ligabò, and P. Vivo, Third-order phase transition: Random matrices and screened

- Coulomb gas with hard walls, *J. Stat. Phys.* **175**, 1262 (2019).
- [35] R. Marino, S. N. Majumdar, G. Schehr, and P. Vivo, Phase transitions and edge scaling of number variance in gaussian random matrices, *Phys. Rev. Lett.* **112**, 254101 (2014).
- [36] R. Marino, S. N. Majumdar, G. Schehr, and P. Vivo, Number statistics for β -ensembles of random matrices: Applications to trapped fermions at zero temperature, *Phys. Rev. E* **94**, 032115 (2016).
- [37] N. Smith, P. Le Doussal, S. Majumdar, and G. Schehr, Full counting statistics for interacting trapped fermions, *SciPost Phys.* **11**, 110 (2021).
- [38] S. Bocini and J.-M. Stéphan, Non-probabilistic fermionic limit shapes, *J. Stat. Mech.* (2021) 013204.