Letter

Helicity is a topological invariant of massless particles: $C = -2h$

Eric Palmerduca* and Hong Qin†
Department of Astrophysical Sciences, Princeton University, Princeton, New Jersey 08544, USA
and Princeton Plasma Physics Laboratory, Princeton, New Jersey 08543, USA

(Received 11 July 2024; accepted 11 February 2025; published 2 April 2025)

There is an elementary but indispensable relationship between the topology and geometry of massive particles. The geometric spin s is related to the topological dimension of the internal space V by $\dim V = 2s + 1$. This breaks down for massless particles, which are geometrically characterized by their helicity h, but all have 1D internal spaces. We show that a subtler relation exists between the topology and geometry of massless particles. Wave functions of massless particles are sections of nontrivial line bundles over the light cone whose topology is completely characterized by their first Chern number C. We prove that in general $C = -2h$. In doing so, we also exhibit a method of generating all massless bundle representations via an Abelian group structure of massless particles.

DOI: 10.1103/PhysRevResearch.7.L022001

The presence of nontrivial topology is a unifying feature that ties together an array of important physical discoveries of the last 50 years, including the Aharonov-Bohm effect [1], the Berry phase [2], the quantum Hall effects [3–5], and the existence of topological insulators [6,7]. More recently, it has becomes apparent that nontrivial topology can emerge more generally in many systems supporting waves, such as fluids [8–13], plasmas [14–20], and even the vacuum [21–24]. The latter refers to the fact that massless particles with nonzero helicity are topologically nontrivial vacuum excitations. In Wigner's classification [25,26], elementary particles are characterized by their geometry rather than their topology, in particular, they are classified as unitary irreducible representations (UIRs) of the (proper orthochronous) Poincaré group $\mathrm{ISO}^+(3,1)$. In this Letter, we derive the general relationship between the topology and geometry of massless particles, showing that the geometric helicity h is related to the topological Chern number C via $C = -2h$; this result has not been previously reported. In doing so, we exhibit an Abelian group structure of massless particles with respect to the tensor product of line bundles which leads to a novel method of generating massless particle representations of all helicities from the photon representations. These representations are easier to construct and more explicit than those found by Asorey *et al.* [27] and Dragon [24]. They also provide a much simpler alternative construction of the R and L graviton bundles than the one we found in Ref. [23]. A number of implications of these results are discussed, including the nonexistence of spin (SAM) and orbital angular momentum (OAM) operators for massless particles and the description of the total angular momentum states in terms of spin-weighted spherical harmonics.

There is a simple and well-known relationship between the geometry and topology of *massive* particles. Conventionally, particles are represented by UIRs of $\mathrm{ISO}^+(3,1)$ on the space of n-component square-integrable wavefunctions over the four-momentum space M, denoted by $L^2(M, \mathbb{C}^n)$ [25,26]. Note that here we ignore quantum numbers arising from nonspacetime symmetries, such as color charge. For massive particles with $m > 0$, M is a mass hyperboloid. According to Wigner's little group method, these representations are classified by the UIRs of the little group $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ on the internal space $V \doteq \mathbb{C}^n$, which in turn are labeled by the nonnegative integer or half-integer spins s. The spin is related to $\dim V$ by

$$\dim V = 2s + 1. \tag{1}$$

Spin characterizes the geometry of the representation while the dimension of a vector space is a topological invariant. Thus Eq. (1) gives the elementary but highly useful relationship between the geometry and topology of massive particles.

The situation is very different for massless particles. The spacetime symmetry is again characterized by a UIR of the little group, but for a massless particle the little group is SO(2) rather than SO(3) [22,25,26]. The UIRs are labeled not by their spin, but by their (possibly negative) integer helicities h (assuming all elementary massless particles are bosons, as in the standard model). Thus helicity describes the geometry of massless particles. Unlike the spin representations of SO(3), the UIRs of SO(2) are all one-dimensional, so there is no relationship like Eq. (1) relating the helicity with the dimension of the internal space. For example, although R and L photons have helicities $h = \pm 1$ and R and L gravitons have $h = \pm 2$, each of these four particles have a single polarization degree of freedom [22,23]. What then is the relationship between the geometry and topology of massless particles?

*Contact author: ep11@princeton.edu
†Contact author: hongqin@princeton.edu

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

The topology of massless particles is far more subtle than that of massive particles. In fact, this topology is essentially obscured in Wigner's conventional representations. The momentum space of massless particles is the forward light cone

$$\mathcal{L}_+ = \{k^\mu = (\omega, \boldsymbol{k}) : k^\mu k_\mu = 0, \omega > 0\} \tag{2}$$

$$\cong \mathbb{R}^3 \setminus \{0\}. \tag{3}$$

In Wigner's representations [25], massless particles are all represented on the same Hilbert space $L^2(\mathcal{L}_+, \mathbb{C})$ because the internal spaces are all one-dimensional. However, unlike the massive representations, when $h \neq 0$ these representations are not smooth, meaning that Poincaré transformations typically map a smooth wave function to a nonsmooth wave function [22,28]. To avoid this problem, one instead considers massless particles as UIRs on $L^2(E)$, the space of L^2-integrable sections of a complex Hermitian line bundle $\pi : E \to \mathcal{L}_+$ [22,27–29]. Since $\mathcal{L}_+$ has a hole at the origin (as massless particles have no rest frame), it is noncontractible and thus E can be topologically nontrivial (unlike in the massive case when M is a mass hyperboloid). Such lines bundles are topologically characterized by their (first) Chern number C [22,30]; when $C \neq 0$, these particles are topologically nontrivial. We recently showed that photons [22] and gravitons [23] are topologically nontrivial; R and L photons have Chern numbers ∓ 2 and helicities ± 1, while R and L gravitons have Chern numbers ∓ 4 and helicities ± 2. However, the method we previously used to construct the graviton bundles becomes increasingly difficult for higher helicity particles, and does not easily generalize to arbitrary helicity. In this Letter, we use an alternative construction of massless particle bundles to prove that in general $C = -2h$, giving the precise relationship between the topology and geometry of massless particles. This relationship is the massless analog of Eq. (1). That helicity can be viewed as a topological charge was also recently discovered by Dragon [24], but the explicit relationship $C = -2h$ was not given.

We first review the vector bundle formalism for massless particles; see Refs. [22,29] for additional details. The UIRs of $\mathrm{ISO}^+(3, 1)$ on the wave functions $L^2(E)$ arise from vector bundle representations on E. The notion of a representation of a Lie group G on a vector bundle is the natural generalization of representation of G on a vector space. Let $\pi : E \to \mathcal{L}_+$ be a complex Hermitian vector bundle over the forward light cone $\mathcal{L}_+$ with rank r and Hermitian product $\langle \cdot, \cdot \rangle$. We will assume all vector bundles are smooth and have nonzero rank. The fiber at k is denoted by E_k and a vector $v \in E$ may be written as (k, v) to indicate that $v \in E_k$. E is a bundle representation of a Lie group G if there are smooth G-actions on $\mathcal{L}_+$ and E,

$$k \mapsto gk \tag{4}$$

$$(k, v) \mapsto (gk, gv), \tag{5}$$

such that the action of g linearly maps E_k onto E_{gk}. In particular, the action commutes with projection: $\pi\, g(k, v) = gk$. Note that throughout we write group actions of representations multiplicatively, e.g., the action of g on v is denoted by gv; it will always be stated or clear from context which representation is being refereed to. The representation is unitary if $g \in G$ maps E_k to E_{gk} unitarily, that is, if

$$\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle \tag{6}$$

for any v_1, v_2 in the same fiber. A subrepresentation of E is a subbundle F of E such that F is a representation of G under the same G-action. E is irreducible if it has no proper subbundles. Two bundle representations E_1 and E_2 of G are unitarily equivalent if there is a unitary vector bundle isomorphism $\psi : E_1 \to E_2$ which is equivariant with respect to the group action:

$$\psi(gv) = g\psi(v) \tag{7}$$

for any $g \in G$ and $v \in E_1$. Note in particular that this implies E_1 and E_2 have the same topology as vector bundles.

We will primarily be concerned with the case when $G = \mathrm{ISO}^+(3, 1) = \mathbb{R}^{3+1} \rtimes \mathrm{SO}^+(3, 1)$, where $\rtimes$ denotes the semidirect product and $\mathrm{SO}^+(3, 1)$ the connected component of the Lorentz group. A generic element of $\mathrm{ISO}^+(3, 1)$ can be written as $T_a \circ \Lambda$ where T_a is a translation by $a \in \mathbb{R}^{3+1}$ and $\Lambda \in \mathrm{SO}^+(3, 1)$. We assume that $\mathrm{SO}^+(3, 1)$ acts on $\mathcal{L}_+$ by the standard 4-vector action $k \mapsto \Lambda k$.

Massless particles can be considered as unitary irreducible UIRs of $\mathrm{ISO}^+(3, 1)$ on vector bundles over $\mathcal{L}_+$ in which translations T_a act via [22,29]

$$T_a(k, v) = e^{ik^\mu a_\mu}(k, v). \tag{8}$$

We will call such a representation a massless bundle UIR. Condition (8) ensures that (k, v) has momentum k. As in the case of Wigner's representations [25,26], the massless bundle UIRs of $\mathrm{ISO}^+(3, 1)$ are classified by their actions of the little group, which are in turn labeled by the integer helicities h [22]. In particular, every such bundle E is a line bundle ($r = 1$) such that every vector in E is an eigenvector of the helicity operator $\hat{\boldsymbol{k}} \cdot \boldsymbol{J}$ with eigenvalue h:

$$(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})v = hv \tag{9}$$

Here, $\boldsymbol{J}$ is the generator of the $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3, 1)$ action on E, meaning that if $R_{\hat{\boldsymbol{k}}}(\theta)$ is a rotation by θ about $\hat{\boldsymbol{k}}$ then

$$R_{\hat{\boldsymbol{k}}}(\theta)(k, v) = e^{-i\hat{\boldsymbol{k}} \cdot \boldsymbol{J}\theta}(k, v) = (k, e^{-ih\theta}v). \tag{10}$$

Since E is a bundle representation of $\mathrm{ISO}^+(3, 1)$, it is necessary that a $\theta = 2\pi$ rotation acts trivially, so h must be an integer. Note that if one considers the generalization to projective bundle representations, then half-integer helicities can also be obtained. Since we are assuming all massless particles are bosons, we can ignore the projective case. Two massless bundle UIRs are unitarily equivalent if and only if they have the same helicity (Ref. [22], Thm. 35). In this way, h fully characterizes the geometry of the these representations. However, a priori, it is not obvious that particles with different helicities have different topologies. We will prove that this is indeed the case.

The examples of the R and L photon bundles are exceptionally important, as we will show that all other massless particles can be constructed from them. We will review the photon bundles here; additional details can be found in Ref. [22]. The total photon bundle γ_T is defined as the collection of all eigenmodes (photons) in Fourier space of the vacuum Maxwell's equations, which form a vector bundle

over $\mathcal{L}_+$. In particular, the solutions are labeled by the Fourier transformed electric field $\boldsymbol{E}$, so the fiber at $k^\mu = (|\boldsymbol{k}|, \boldsymbol{k})$ is the 2D complex vector space defined by $\{\boldsymbol{E} \in \mathbb{C}^3 | \boldsymbol{k} \cdot \boldsymbol{E} = 0\}$. It is a massless unitary representation of $\mathrm{ISO}^+(3,1)$ defined by the standard Lorentz transformations of $\mathcal{L}_+$ and the electric field. That is, $\boldsymbol{E}$ transforms under rotations like a 3-vector, under translations T_a by $\boldsymbol{E} \to \boldsymbol{E} e^{ik^\mu a_\mu}$, and under boosts by the well-known relativistic electromagnetic boost transformation [22,31]. γ_T is, however, a reducible bundle representation:

$$\gamma_T = \gamma_1 \oplus \gamma_{-1}, \tag{11}$$

where $\gamma_{\pm 1}$ are the line subbundles consisting of R- and L-polarized photons, respectively. More explicitly, $\gamma_{\pm 1}$ contains all vectors of the form $(k, \boldsymbol{E})$ where $\boldsymbol{E} = \alpha(\boldsymbol{e}_1 \pm i\boldsymbol{e}_2)$, $\alpha \in \mathbb{C}$, and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{k})$ is a right-handed coordinated system of $\mathbb{R}^3$. That $\gamma_{\pm 1}$ are also representations of $\mathrm{ISO}^+(3,1)$ reflects the fact that R- and L-polarizations are preserved under Poincaré transformations. These bundles have helicity $h = \pm 1$ and Chern number $C = \mp 2$. The former comes from

$$R_{\hat{\boldsymbol{k}}}(\theta)(k, \boldsymbol{E}_\pm) = e^{-i(\hat{\boldsymbol{k}}\cdot\boldsymbol{J})\theta}(k, \boldsymbol{E}_\pm) \tag{12}$$

$$= (k, e^{\mp i\theta}\boldsymbol{E}_\pm), \tag{13}$$

and the latter can be explicitly computed from the Berry connection or the clutching construction [22].

We present a novel construction of massless bundle UIRs of arbitrary helicity which is motivated by observations about the little group representations. In both Wigner's formalism [25,26] and the vector bundle formalism [22,23,29], irreducible representations of $\mathrm{ISO}^+(3,1)$ are related to representations of the little group. The little group H_k at a fixed k in the momentum space M is defined as the subset of $\mathrm{SO}^+(3,1)$ which leaves k invariant. The little groups for different k are related by $H_{\Lambda k} = \Lambda H_k \Lambda^{-1}$ for $\Lambda \in \mathrm{SO}^+(3,1)$. Since any two k in M are related by some Lorentz transformation, the isomorphism type of the group H_k is independent of k; the isomorphism type of H_k is called the little group H of the representation. For massive particles M is a mass hyperboloid, and $H = \mathrm{SO}(3)$. However, for massless particles with $M = \mathcal{L}_+$, $H = \mathrm{ISO}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, the group of (inhomogeneous) isometries of $\mathbb{R}^2$ [25]. Representations in which the $\mathbb{R}^2$ translations act nontrivially on E correspond to particles with a continuum of internal degrees of freedom. As there are no known particles exhibiting this property, we make the conventional assumption that the $\mathbb{R}^2$ translations act by the identity on E [26,32]. Thus the little group essentially acts like SO(2). The finite-dimensional vector space UIRs of SO(3) and SO(2) are the well-known spin and helicity representations, respectively. The spin representations of SO(3) are labeled by the nonnegative integers s, and have complex dimension $2s+1$. In contrast, the helicity representations are labeled by the complete set of integers h, but always have complex dimension 1. If $R(\theta) \in \mathrm{SO}(2)$ is a rotation by angle θ, then in the helicity h representation $R(\theta)v = e^{-ih\theta}v$ for any vector v. One can ask the following question: given two UIRs, can one produce another UIR? Suppose U_1 and U_2 are two UIRs of a group G on complex vector spaces $V_1 \cong \mathbb{C}^{n_1}$ and $V_2 \cong \mathbb{C}^{n_2}$. The direct sum $U_1 \oplus U_2$ and direct product $U_1 \otimes U_2$ are well-defined representations of G on $V_1 \oplus V_2$ and $V_1 \otimes V_2$ with dimensions $n_1 + n_2$ and $n_1 n_2$, respectively [33]. For $v_i \in V_i$, they are defined by

$$g(v_1 + v_2) = gv_1 + gv_2, \tag{14}$$

$$g(v_1 \otimes v_2) = gv_1 \otimes gv_2. \tag{15}$$

These representations are unitary under the Hermitian products

$$\langle (v_1 + v_2), (w_1 + w_2) \rangle = \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle, \tag{16}$$

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle \tag{17}$$

where the Hermitian products on the rhs are those on V_1 and V_2. $U_1 \oplus U_2$ is obviously reducible. Typically, $U_1 \otimes U_2$ is also reducible; the problem of decomposing $U_1 \otimes U_2$ into UIRs is the Clebsch-Gordon problem [33]. The most well-known cases are those of SO(3) and SU(2), which satisfy:

$$S_l \otimes S_m \cong S_{l-m} \oplus S_{l-m+1} \oplus \ldots \oplus S_{l+m}, \tag{18}$$

where S_k is the spin k representation, $l \geqslant m$, and $\cong$ denotes isomorphism of representations [33]. In particular,

$$S_1 \otimes S_1 \cong S_0 \oplus S_1 \oplus S_2. \tag{19}$$

However, if U_1 and U_2 both have dimension 1, then $U_1 \otimes U_2$ also has dimension 1 and is therefore irreducible. Thus if W_h is the helicity h representation, then $W_{h_1} \otimes W_{h_2}$ is irreducible. In fact, in this representation we have

$$R(\theta)(v_1 \otimes v_2) = e^{-i(h_1+h_2)\theta}(v_1 \otimes v_2), \tag{20}$$

so

$$W_{h_1} \otimes W_{h_2} \cong W_{h_1+h_2}. \tag{21}$$

Since $W_1 \otimes W_{-1} \cong W_0$, we define $W_1^{-1} = W_{-1}$. Then for $h \neq 0$,

$$W_h \cong \bigotimes_{i=1}^{|h|} W_1^{\mathrm{sgn}(h)} \doteq W_1^h. \tag{22}$$

Defining $W_1^0 = W_0$, the equation $W_h \cong W_1^h$ holds for all integers h. Thus the helicity representations are not just in bijective correspondence with $\mathbb{Z}$, but they are actually isomorphic to $\mathbb{Z}$ as an Abelian group, with the tensor product as the group operation. As such, they are all generated by W_1 (or W_{-1}). This is a distinctly massless phenomena, as Eq. (19) shows that a similar result does not apply to the massive spin representations. Indeed, the spin representations are labeled by nonnegative integers, which do not even form a group.

We now show that we can construct all the bundle UIRs of massless particles in a similar way, by taking tensor products of $\gamma_{\pm 1}$. Suppose we have two massless bundle UIRs E_1 and E_2 with helicities h_1 and h_2. Let $E_1 \otimes E_2$ be the tensor product of the bundles, which is a line bundle over $\mathcal{L}_+$ in which the fiber at k is the vector space $E_{1,k} \otimes E_{2,k}$ [34]. It carries a Hermitian product given on each fiber by Eq. (17). We can define a unitary representation of $\mathrm{ISO}^+(3,1)$ on $E_1 \otimes E_2$ as

follows:

$$\Lambda(k, v_1 \otimes v_2) = (\Lambda k, \Lambda v_1 \otimes \Lambda v_2), \tag{23}$$

$$T_a(k, v_1 \otimes v_2) = e^{ik^\mu a_\mu}(k, v_1 \otimes v_2), \tag{24}$$

$$(T_a \circ \Lambda)(k, v_1 \otimes v_2) = T_a[\Lambda(k, v_1 \otimes v_2)]. \tag{25}$$

Note that the action of translations involves a factor of $e^{ik^\mu a_\mu}$ rather than a factor of $e^{2ik^\mu a_\mu}$ which one might expect from analogy with Eqs. (15) and (23). This reflects the fact that we are constructing new single-particle representations, in accord with Eq. (8), rather than two-particle representations with the tensor product. To verify that (23)–(25) is actually a representation of $\mathrm{ISO}^+(3, 1)$, it is necessary to check that

$$(\Lambda \circ T_a)(k, v_1 \otimes v_2) = \Lambda[T_a(k, v_1 \otimes v_2)], \tag{26}$$

which follows immediately from the fact that

$$\Lambda \circ T_a \circ \Lambda^{-1} = T_{\Lambda a}.$$

Since $E_1 \otimes E_2$ has rank 1, this representation is irreducible. Moreover, by Eq. (23)

$$e^{i(\hat{\boldsymbol{k}}\cdot\boldsymbol{J})\theta}(k, v_1 \otimes v_2) = e^{i(h_1+h_2)\theta}(k, v_1 \otimes v_2), \tag{27}$$

showing that $E_1 \otimes E_2$ has helicity $h_1 + h_2$. Define $\gamma_1^{-1} \doteq \gamma_{-1}$ and γ_0 to be the trivial line bundle over $\mathcal{L}_+$ on which T_a acts by $e^{ik^\mu a_\mu}$ and $\Lambda \in \mathrm{SO}^+(3, 1)$ acts by the identity. Then

$$\gamma_h \doteq \gamma_1^h \doteq \bigotimes_{i=1}^{h} \gamma_1^{\mathrm{sgn}(h)} \tag{28}$$

is a massless bundle UIR with helicity h. In this way we obtain smooth bundle representations for massless particles with any integer helicity. By theorem 35 of Ref. [22], every massless particle is unitarily equivalent to one of these representations.

Due to the tensor product structure, determining the topology of γ_h is also a simple matter. Recall that the topology of a line bundle over $\mathcal{L}_+$ is completely determined by its Chern number C. The Chern number can be considered as a measure of how "twisted" or nontrivial a vector bundle is; in particular, $C = 0$ corresponds to the trivial bundle [35]. For line bundles, the Chern number is additive with respect to the tensor product (Ref. [35], Eq. (20.1)):

$$C(E_1 \otimes E_2) = C(E_1) + C(E_2).$$

Note that this is a special property of line bundles which does not generally hold for higher rank bundles. As the R and L photon bundles have Chern numbers $C(\gamma_{\pm 1}) = \mp 2$, we obtain

$$C(\gamma_h) = C\left(\gamma_1^h\right) = -2h. \tag{29}$$

This equation describes the fundamental relationship between the geometry and topology of massless particles; it has not previously been reported.

Before discussing the implications of this relationship, we remark on the alternative vector bundle descriptions of massless particles given by Asorey *et al.* [27] and Dragon [24]. Asorey *et al.* give a method to construct massless bundle representations of helicity h in which the fibers are represented by symmetric traceless tensors of rank $|h|$ subject to a series of gauge conditions. On the other hand, in Dragon's formalism, particle states are described as sections of vector bundles, but the fibers (polarizations) of those vector bundles are never explicitly defined. Instead, the vector bundle structures are described implicitly through the transition functions of the sections. As massless particles are fully characterized by their helicity, the representations γ_h and those of Dragon and Asorey *et al.* are equivalent, but differ in their mathematical presentation. Nonetheless, we contend that the construction of γ_h is much simpler than the previous methods described. Indeed, Dragon's construction [24] requires rather messy calculations involving the action of boosts and rotations on sections. In Asorey *et al.*'s method, the construction gets increasingly difficult as $|h|$ grows, as one must impose more and more gauge constraints on symmetric traceless rank $|h|$ tensors. In contrast, the γ_h representations are equally simple to explicitly construct regardless of h. We note it is possible to derive the result (29) using the alterative formalisms of Refs. [24,27], but this has never been carried out.

One consequence of (29) is that the helicity can be considered as a topological invariant, despite being defined purely in terms of Poincaré geometry; this result was also established by Dragon [24]. Furthermore, massless particles with different helicity are topologically distinct, with the helicity measuring how globally twisted the particle space is. This is not readily detectable using the (nonsmooth) Wigner representations of massless particles [25,26], in which the space of wavefunctions of all massless particles are represented on the same Hilbert space $L^2(\mathcal{L}_+, \mathbb{C})$. Note in particular that Eq. (29) shows that massless particles with $h \neq 0$ are topologically nontrivial, and it immediately follows that it is not possible to smoothly choose polarization vectors in all directions $\boldsymbol{k}$ for such particles. Thus it is not possible to have massless particles of a fixed nonzero helicity with flux in all directions simultaneously, generalizing results we previously obtained for photons and gravitons [22,23].

A novel and less immediate implication of Eq. (29) is explored in Ref. [36], where we prove a series of no-go theorems showing that it is not possible to split the angular momentum operator of $h \neq 0$ massless particles into spin SAM and OAM operators. The relationship $C = -2h$ reflects the fact that this obstruction can be understood either geometrically or topologically, with $C \neq 0$ and $h \neq 0$ obstructing the existence of SAM and OAM operators, respectively. This implies that in the case of general, nonparaxial light, it is not possible to define measurable SAM and OAM operators. This is consistent with the conclusions of many authors in the optics communities [37–40], but contradicts more recent claims to the contrary [41–44]. Rather, what are commonly referred to as SAM and OAM in quantum optics are mathematically useful artifacts of the paraxial approximation but not true angular momentum operators. They are scalar operators S_z and L_z which cannot be extended in a gauge-invariant way vector operators $\boldsymbol{S}$ and $\boldsymbol{L}$ which generate rotational symmetries (which is the defining feature of angular momentum operators). There have also been recent proposals to study the so-called OAM of gravitational waves [45–47], however, our theory similarly indicates that there is no gauge-invariant way to define such a gravitational OAM.

In deriving Eq. (29), we have illustrated a particularly simple method of constructing smooth bundle representations for any massless particle, namely, by taking repeated tensor products of the photon bundles $\gamma_{\pm 1}$. To illustrate some of the practical advantages of this construction, we will use the example of the R and L graviton bundles. These bundles are given by $\Xi_{\pm} \doteq \gamma_{\pm 1} \otimes \gamma_{\pm 1}$ with the plus and minus signs corresponding to the R and L graviton bundles, respectively. At $k = (|\boldsymbol{k}|, \boldsymbol{k})$, choose any right-handed basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{k})$ of $\mathbb{R}^3$. The photon fiber $\gamma_{\pm}(k)$ is spanned by $v_{\pm} = \boldsymbol{e}_1 \pm i\boldsymbol{e}_2$, so the graviton fiber $\Xi_{\pm}(\boldsymbol{k})$ is spanned by $v_{\pm} \otimes v_{\pm}$. By expressing the tensor product as an outer product,

$$(v_{\pm} \otimes v_{\pm})^{ij} = v_{\pm}^i v_{\pm}^j, \tag{30}$$

$\Xi_{\pm}(k)$ is embedded in the vector space of three-by-three symmetric tensors. Furthermore, since $(v_{\pm} \otimes v_{\pm}) \cdot \boldsymbol{k} = 0$ and $\sum_{i=1}^{3} v_{\pm}^i v_{\pm}^i = 0$, the vectors of $\Xi_{\pm}(k)$ are automatically in the typical transverse-traceless gauge commonly used to describe gravitons [23,32,48]. In this construction it was never necessary to manually impose the transverse-traceless gauge conditions as was done in Refs. [23,27]—the gauge fixing one chooses for the photon bundle automatically propagates through to all higher helicity bundles. Furthermore, the technique we previously employed in Ref. [23] involved constructing the total graviton bundle, and then manually identifying Lorentz invariant R and L subbundles. The difficulty of that procedure increases with $|h|$ as one must embed the particle states in spaces of increasing dimension and identify the Lorentz invariant subbundles. The tensor product construction of γ_h avoids such difficulties.

Another application of the tensor product representations γ_h pertains to the problem of finding simultaneous eigenstates of energy, helicity, J^2, and J_z for massless particles. The corresponding massive states (with spin replacing helicity) are easily constructed via Clebesch-Gordan sums of tensor products of spherical harmonics with definite spin states. This problem is surprisingly tricky for massless particles due to the coupling of the polarization states and the momentum (cf. § 16, Ref. [49]). Dragon [24] recently showed a novel method of solving this problem using vector bundle methods. However, in Dragon's formalism, the eigenstates are difficult to interpret, given as a set of two unfamiliar functions with overlapping domains in stereographic coordinates (cf. Eqs. (81) and (84), Ref. [24]). Furthermore, explicit formulas are only given for the states with $l = m$, where $l(l+1)$ and m are the eigenstates of J^2 and J_z; the states with $m < l$ must be constructed via the lowering operator $J_- = J_x - iJ_y$. If one instead uses the tensor product representations γ_h developed here, it can be shown that the total angular momentum eigenstates of a massless helicity h particle are simply given by [50]

$$ {}_{-h}Y_{lm}(\theta, \phi) e_h(\theta, \phi). \tag{31}$$

In this equation, θ and ϕ are the spherical coordinates in k space, (e_θ, e_ϕ) are the spherical unit vectors (which exist everywhere except the poles),

$$e_h = \bigotimes_{i=1}^{|h|} (e_\theta + \mathrm{sgn}(h) i e_\phi), \tag{32}$$

and the ${}_{-h}Y_{lm}$ are the spin-weighted spherical harmonics of weight $-h$ first introduced by Newman and Penrose for applications in general relativity [51]. While ${}_{-h}Y_{lm}(\theta, \phi)$ and $e_h(\theta, \phi)$ separately have singularities at the poles, the eigenstates obtained from their product in Eq. (31) are smooth sections of γ_h for all momenta $\boldsymbol{k}$. Note that by the properties of spin-weighted spherical harmonics, l and m are restricted such that $l \geqslant |h|$ and $|m| \leqslant l$, so it is immediate that particles with $h \neq 0$ do not possess zero angular momentum states. This gives a more concrete illustration of the nonexistence of SAM and OAM operators for such massless particles. If, for example, there existed SAM and OAM operators such that there existed states with $j_{SAM} = 1$ and $j_{OAM} = 1$, then by Eq. (19) there would exist states of total angular momentum $j = 0$. That no such $j = 0$ states exists gives strong evidence that SAM and OAM operators no not exist, a result which can be rigorously proven through more abstract methods [36].

It has long been known that the limit of zero particle mass is singular [25]. This is conventionally viewed as a geometric singularity, as the little group jumps abruptly from being SO(3) for massive particles to SO(2) for massless particles. However, there is an accompanying topological singularity that occurs too, as the momentum space jumps from being a contractible mass hyperboloid to the noncontractible light cone $\mathcal{L}_+$. This allows for massless particles to have nontrivial topology, as illustrated in the bundle representations. We have shown that this singular limit involves other nonobvious differences between massive and massless particles, such as a completely different relationship between the topology and geometry as well as the emergence of an Abelian group structure of massless particles. These in turn can be used to derive further idiosyncrasies of massless particles, including the nonexistence of SAM and OAM operators and the description of their angular momentum eigenstates in terms of spin-weighted spherical harmonics. The topological analysis of elementary particles is still fairly young, and we anticipate that other important implications of the topological nontriviality of massless particles will be discovered.

This work is supported by U.S. Department of Energy (DE-AC02-09CH11466).

[1] Y. Aharonov and D. Bohm, Significance of electromagnetic potentials in the quantum theory, Phys. Rev. **115**, 485 (1959).

[2] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. Lond. A **392**, 45 (1984).

[3] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall conductance in a two-dimensional periodic potential, Phys. Rev. Lett. **49**, 405 (1982).

[4] J. E. Avron, R. Seiler, and B. Simon, Homotopy and quantization in condensed matter physics, Phys. Rev. Lett. **51**, 51 (1983).

[5] M. Kohmoto, Topological invariant and the quantization of the Hall conductance, Ann. Phys. **160**, 343 (1985).

[6] B. A. Volkov and O. A. Pankratov, Two-dimensional massless electrons in an inverted contact, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 145 (1985) [Sov. Phys. JETP Lett. **42**, 178 (1985)].

[7] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. **82**, 3045 (2010).

[8] A. Souslov, B. C. Van Zuiden, D. Bartolo, and V. Vitelli, Topological sound in active-liquid metamaterials, Nat. Phys. **13**, 1091 (2017).

[9] P. Delplace, J. B. Marston, and A. Venaille, Topological origin of equatorial waves, Science **358**, 1075 (2017).

[10] C. Tauber, P. Delplace, and A. Venaille, A bulk-interface correspondence for equatorial waves, J. Fluid Mech. **868**, R2 (2019).

[11] A. Souslov, K. Dasbiswas, M. Fruchart, S. Vaikuntanathan, and V. Vitelli, Topological waves in fluids with odd viscosity, Phys. Rev. Lett. **122**, 128001 (2019).

[12] M. Perrot, P. Delplace, and A. Venaille, Topological transition in stratified fluids, Nat. Phys. **15**, 781 (2019).

[13] F. Faure, Manifestation of the topological index formula in quantum waves and geophysical waves, Ann. Henri Lebes. **6**, 449 (2023).

[14] B. Yang, M. Lawrence, W. Gao, Q. Guo, and S. Zhang, One-way helical electromagnetic wave propagation supported by magnetized plasma, Sci. Rep. **6**, 21461 (2016).

[15] W. Gao, B. Yang, M. Lawrence, F. Fang, B. Béri, and S. Zhang, Photonic Weyl degeneracies in magnetized plasma, Nat. Commun. **7**, 12435 (2016).

[16] J. B. Parker, J. B. Marston, S. M. Tobias, and Z. Zhu, Topological gaseous plasmon polariton in realistic plasma, Phys. Rev. Lett. **124**, 195001 (2020).

[17] Y. Fu and H. Qin, Topological phases and bulk-edge correspondence of magnetized cold plasmas, Nat. Commun. **12**, 3924 (2021).

[18] Y. Fu and H. Qin, The dispersion and propagation of topological Langmuir-Cyclotron waves in cold magnetized plasmas, J. Plasma Phys. **88**, 835880401 (2022).

[19] H. Qin and Y. Fu, Topological Langmuir-Cyclotron wave, Sci. Adv. **9**, eadd8041 (2023).

[20] H. Qin and E. Palmerduca, Oblique photons, plasmons, and current-plasmons in relativistic plasmas and their topological implications, arXiv:2404.05636.

[21] K. Y. Bliokh, D. Smirnova, and F. Nori, Quantum spin Hall effect of light, Science **348**, 1448 (2015).

[22] E. Palmerduca and H. Qin, Photon topology, Phys. Rev. D **109**, 085005 (2024).

[23] E. Palmerduca and H. Qin, Graviton topology, J. High Energy Phys. 11 (2024) 150.

[24] N. Dragon, Bundle structure of massless unitary representations of the poincaré group, Int. J. Theor. Phys. **63**, 149 (2024).

[25] E. Wigner, On unitary representations of the inhomogeneous Lorentz group, Ann. Math. **40**, 149 (1939).

[26] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, UK, 1995), Vol. 1.

[27] M. Asorey, L. J. Boya, and J. F. Cariñena, Covariant representations in a fibre bundle framework, Rep. Math. Phys. **21**, 391 (1985).

[28] M. Flato, D. Sternheimer, and C. Fronsdal, Difficulties with massless particles?, Commun. Math. Phys. **90**, 563 (1983).

[29] D. Simms, *Lie Groups and Quantum Mechanics*, Lecture Notes in Mathematics (Springer-Verlag, Berlin, 1968).

[30] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 3rd ed. (Oxford University Press, Oxford, UK, 2017).

[31] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, NY, 1999).

[32] M. Maggiore, *A Modern Introduction to Quantum Field Theory*, Oxford Master Series in Statistical, Computational, and Theoretical Physics (Oxford University Press, Oxford, UK, 2005).

[33] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, 2nd ed., Graduate Texts in Mathematics (Springer, Cham, Switzerland, 2015).

[34] L. Tu, *Differential Geometry: Connections, Curvature, and Characteristic Classes*, Graduate Texts in Mathematics (Springer, Cham, Switzerland, 2017).

[35] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics (Springer, New York, NY, 2013).

[36] E. Palmerduca and H. Qin, Four no-go theorems on the existence of spin and orbital angular momentum of massless bosons, arXiv:2407.06276.

[37] S. J. van Enk and G. Nienhuis, Spin and orbital angular momentum of photons, Europhys. Lett. **25**, 497 (1994).

[38] L. Allen, M. Babiker, and M. J. Padgett, Iv the orbital angular momentum of light, Prog. Opt. **39**, 291 (1999).

[39] J. Garrison and R. Chiao, *Quantum Optics* (Oxford University Press, Oxford, UK, 2008).

[40] I. Fernandez-Corbaton, X. Zambrana-Puyalto, and G. Molina-Terriza, On the transformations generated by the electromagnetic spin and orbital angular momentum operators, J. Opt. Soc. Am. B **31**, 2136 (2014).

[41] I. Bialynicki-Birula and Z. Bialynicka-Birula, Canonical separation of angular momentum of light into its orbital and spin parts, J. Opt. **13**, 064014 (2011).

[42] E. Leader and C. Lorcé, The angular momentum controversy: What's it all about and does it matter?, Phys. Rep. **541**, 163 (2014).

[43] L.-P. Yang, F. Khosravi, and Z. Jacob, Quantum field theory for spin operator of the photon, Phys. Rev. Res. **4**, 023165 (2022).

[44] P. Das, L.-P. Yang, and Z. Jacob, What are the quantum commutation relations for the total angular momentum of light? tutorial, J. Opt. Soc. Am. B **41**, 1764 (2024).

[45] I. Bialynicki-Birula and Z. Bialynicka-Birula, Gravitational waves carrying orbital angular momentum, New J. Phys. **18**, 023022 (2016).

[46] P. Baral, A. Ray, R. Koley, and P. Majumdar, Gravitational waves with orbital angular momentum, Eur. Phys. J. C **80**, 326 (2020).

[47] H. Wu and L. Chen, Twisting and entangling gravitons in high-dimensional orbital angular momentum states via photon-graviton conversion, Phys. Rev. D **107**, 125027 (2023).

[48] B. Schutz, *A First Course in General Relativity*, 2nd ed. (Cambridge University Press, Cambridge, UK, 2009).

[49] V. Berestetskii, E. Lifshitz, and L. Pitaevskii, *Quantum Electrodynamics*, 2nd ed. (Butterworth-Heinemann, Oxford, UK, 1982), Vol. 4.

[50] E. Palmerduca and H. Qin, Total angular momentum eigenfunctions of massless particles (unpublished).

[51] E. T. Newman and R. Penrose, Note on the Bondi-Metzner-Sachs Group, J. Math. Phys. **7**, 863 (1966).