

Thermodynamic uncertainty relations in superconducting junctions

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It has recently been predicted that the thermodynamic uncertainty relation (TUR), which expresses a trade-off between the amount of dissipation and the absence of fluctuations, can be violated in the context of charge transport in normal quantum conductors. However, the required conditions are difficult to meet, and no experimental violation has been reported yet in mesoscopic quantum transport experiments. Here, we predict that TUR can be largely violated in superconducting (SC) contacts without introducing an energy dependency in the normal transmission coefficient. This violation originates from the coexistence of tunneling processes involving different numbers of transmitted charges, namely, quasiparticle tunneling and Andreev reflections. The SC junctions invoked here have been fabricated by many groups and constitute an ideal platform to demonstrate TUR breaking in charge transport experiments.

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Introduction. The implementation of superconducting (SC) materials has become a key ingredient to design better and more functional quantum thermal machines. They have been proposed to be efficient for cooling performances [1–5] as well as for pumping or generating power [6,7]. Additionally, SCs can exhibit remarkable thermoelectric properties in the nonlinear transport regime when electron-hole symmetry is broken [8–11].

Another advantageous aspect when dealing with SC elements is that they overcome the fundamental bound imposed by the thermodynamic uncertainty relation (TUR) [5,12–19]. The TUR establishes a trade-off relation between uncertainty of a current traversing the system and the associated entropy production or dissipation. The TUR is a nonequilibrium relation derived from the Markovian evolution of a system and sets a more restrictive bound for the entropy production in terms of the noise-to-current ratio. In the case of charge transport, the TUR dictates $S/I^2 \geq 2k_B/\dot{\Sigma}$, where k_B is the Boltzmann constant, $\dot{\Sigma}$ the entropy production, I the average charge current, and S the corresponding current fluctuations.

TUR violations have been predicted in quantum systems attributed to either quantum coherence or the breakdown of the local detailed balance [5,20–25]. Large deviations from

the TUR are desirable for the quantum-enhanced performance of the engine operation, i.e., less dissipative and more stable machines [19,21,26,27]. Additionally, quite recently, there has been a notable effort toward finding quantum systems exhibiting a prominent breakdown of the TUR [5,23,24,24,28,29]. Thus, for instance, a chain of quantum dots with a rectangular transmission was proposed to violate the standard TURs [28]. Additionally, moderate deviations in the TUR relation have been predicted in hybrid normal-SC quantum dots in the weak tunnel coupling regime [5]. However, all these proposals require stringent conditions, and all attempts to observe a violation of the TUR in quantum conductors have failed thus far, see, e.g., Ref. [30].

Here, we predict that the TUR can be largely violated in few-channel quantum point contacts involving SC leads, which have already been realized, e.g., with the help of SC atomic-size contacts. We show that the strong violations of the TUR in these junctions are due to the coexistence of different transport processes, namely, single-quasiparticle (QP) tunneling and (multiple) Andreev reflections (ARs). We find that the largest TUR violations occur for highly transmissive contacts, and no energy dependence of the normal state transmission is required, contrary to what happens in normal quantum point contacts [26,28].

Quantifying TUR deviations. In what follows, we present a general discussion to explain what makes SCs special in the context of the TUR. For this purpose, we consider a generic isothermal two-terminal quantum device that may contain normal metals and/or SCs. If we apply a bias voltage V , entropy production due to the Joule heating is given by $\dot{\Sigma} = IV/T$, where I is the charge current and T the temperature

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of the electrodes. Deviations from the TUR, i.e., $\frac{S}{I^2} \geq 2k_B$, with S being the current fluctuations, can be quantified by the TUR-breaking coefficient

$$\mathcal{F} \equiv F^* - \frac{2k_B T}{V}, \quad (1)$$

where $F^* = S/I$ is the Fano factor, and we have assumed that the electron charge is unity. The coefficient $\mathcal{F}$ is negative for a broken TUR and positive (or zero) for an unbroken TUR, where $\mathcal{F} = 0$ corresponds to the classical limit. Below, we shall present a quantitative analysis of $\mathcal{F}$ valid for arbitrary range of parameters, but to get insight into the conditions required to break the TUR, we now follow Ref. [26] and focus on the low-bias regime in which it is possible to make analytical progress. With this idea in mind, we make a Taylor expansion in the bias of the current and noise

$$I = G_1 V + \frac{1}{2!} G_2 V^2 + \frac{1}{3!} G_3 V^3 + O(V^4), \quad (2)$$

$$S = S_0 + S_1 V + \frac{1}{2!} S_2 V^2 + O(V^3). \quad (3)$$

Here, G_1 , G_2 , and G_3 are linear and nonlinear conductances; and S_0 , S_1 , and S_2 are the equilibrium and nonequilibrium noise terms. Using the nonlinear fluctuation relations [20,31–33] together with the Johnson-Nyquist relation for the equilibrium noise (i.e., $S_0 = 2k_B T G_1$), we obtain $\mathcal{F} = V k_B T / (G_1) C_{\text{neq}} + O(V^2)$, where the coefficient [19]

$$C_{\text{neq}} \equiv \frac{3S_2 - 2k_B T G_3}{6k_B T} \quad (4)$$

quantifies TUR deviations. Namely, when $C_{\text{neq}} < 0$ (in the limit of small voltages), the TUR is violated.

The stochastic nature of charge transfer in mesoscopic junctions allows us to express the current in an arbitrary mesoscopic junction as follows [34,35]:

$$I = \int \frac{dE}{h} \left(\sum_n n p_n \right), \quad (5)$$

where $p_n = p_n(E, V, T)$ are charge-resolved probabilities of a transport process transferring n charges across the junction that depends on the energy E , voltage V , and temperature T . The corresponding current fluctuations are given by [34,35]

$$S = \int \frac{dE}{h} \left[\sum_n n^2 p_n - \left(\sum_n n p_n \right)^2 \right]. \quad (6)$$

These formulas for I and S hold for junctions containing normal metals and/or SCs. For normal contacts, the transport is solely due to single-electron tunneling ($|n| = 1$), while with SC electrodes, multiple electron ($|n| > 1$) tunneling processes can coexist. This is what makes SCs special, as we are about to show.

To get further insight, we expand the charge-resolved probabilities in the applied voltage

$$p_n = p_n^{(0)} + p_n^{(1)} V + \frac{p_n^{(2)} V^2}{2} + \frac{p_n^{(3)} V^3}{6} + O(V^4). \quad (7)$$

By using the detailed balance condition $p_{-n} = p_n e^{-n\beta V}$ [32], where $\beta = 1/(k_B T)$, and inserting the expanded probabilities

[see Eq. (7)] in the current and noise formulas [see Eqs. (5) and (6)], we obtain the following result for the coefficient C_{neq} in Eq. (4):

$$\frac{6C_{\text{neq}}}{\beta^3} = \int \frac{dE}{h} \left\{ \sum_{n>0} n^4 p_n^{(0)} [1 - 6p_n^{(0)}] - \sum_{m>n>0} 12n^2 m^2 p_n^{(0)} p_m^{(0)} \right\}, \quad (8)$$

which notably only depends on the zeroth-order approximation of the transmission probabilities $p_n^{(0)}$.

To exemplify Eq. (8) in the context of SC junctions, we now consider a single-channel normal metal–superconductor (NS) contact. The SC lead is assumed to be a conventional BCS SC with a temperature-dependent gap $\Delta \equiv \Delta(T)$. In this case, there are two transport processes, as illustrated in Figs. 1(a) and 1(b). For zero temperature, QPs can only be transferred for energies above the gap ($V \geq \Delta$) and transfer one charge, which is shown in Fig. 1(a). In the subgap region ($V \leq \Delta$), transport can take place via AR, where two charges are transferred in one instance, see Fig. 1(b). For nonzero temperatures, QP transfer can also occur for in-gap voltages because of the thermal broadening of the corresponding Fermi functions. The two processes are associated with different transmission probabilities, namely, $p_{\pm 1}$ corresponds to QP tunneling and $p_{\pm 2}$ to AR, and all other probabilities are zero [36]. With this information at hand, the coefficient C_{neq} reads

$$\frac{6C_{\text{neq}}^{\text{NS}}}{\beta^3} = \int \frac{dE}{h} [p_1^{(0)}(1 - 6p_1^{(0)}) + 2^4 p_2^{(0)}(1 - 6p_2^{(0)}) - 48p_1^{(0)} p_2^{(0)}]. \quad (9)$$

This expression reduces to the result in Ref. [19] for normal contacts [$p_2^{(0)} = 0$]. The new contributions in the NS case, the second and third terms, are related to the AR and the coexistence of the two processes, respectively. The sign of $C_{\text{neq}}^{\text{NS}}$ depends on the precise energy dependence of $p_{1,2}^{(0)}$. One can gain some intuition by analyzing an NS contact with an energy-independent normal transmission coefficient. In this case, the $p_{1,2}^{(0)}$ coefficients can be computed, e.g., with the well-known BTK model [37]. As we show in Ref. [38], such an analysis suggests that two requirements must be simultaneously met to significantly break the TUR: (i) coexistence of the two processes at a given bias and temperature and (ii) high transmissions. The coexistence of the two processes is responsible for a large negative contribution of the cross term $-48p_1^{(0)} p_2^{(0)}$ in Eq. (9), while high transmissions are necessary to have high probabilities and make the terms $n^4 p_n^{(0)} [1 - 6p_n^{(0)}]$ negative. The dependence of the zero-bias probabilities on transmission and temperature as well as a discussion on the role of the different terms in Eq. (9) can be found in Ref. [38]. This intuition will be further confirmed below by our numerically exact results at arbitrary bias, and we shall use those results to test the validity of Eq. (9).

In the case of superconductor–superconductor (SS) contacts, new transport processes in the form of multiple ARs (MARs) can take place. Their corresponding probabilities $p_{\pm n}$ for $n > 2$ contribute to the current and noise (see Refs. [39,40]) and therefore appear in Eq. (8). The lowest-

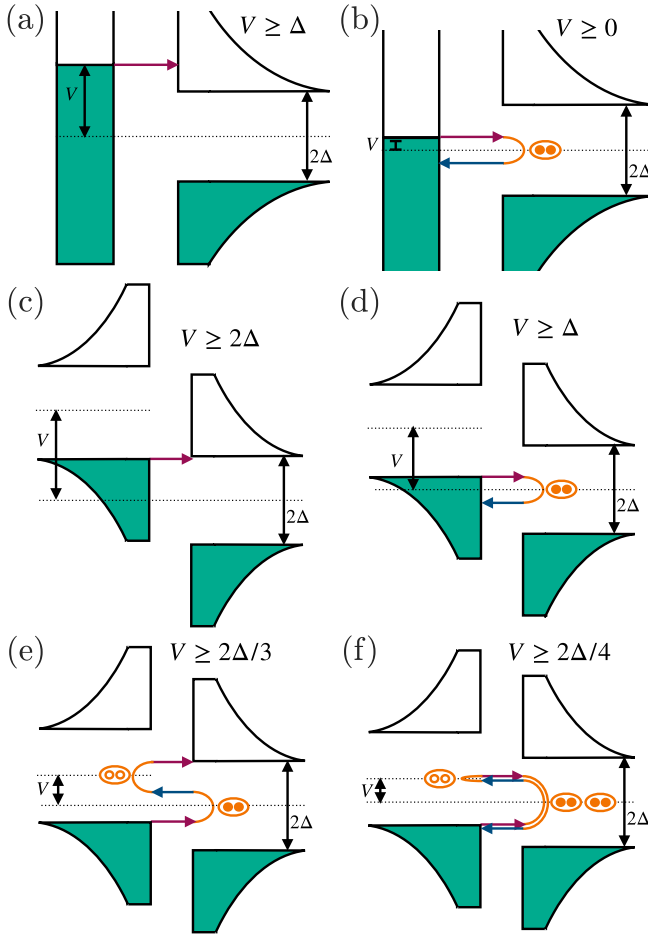


FIG. 1. (a) Single-quasiparticle (QP) tunneling into the empty density of states of the superconductor (SC) transferring one electron. The applied voltage has to exceed the SC gap, $V \geq \Delta$. (b) Andreev reflection (AR) in which an electron is reflected as a hole inside the SC gap transferring two electrons. The voltage is arbitrary $V \geq 0$. (c) QP tunneling between two SCs at voltages $> 2\Delta$. (d) AR from a voltage-biased SC to another SC transferring two electrons. The voltage has to exceed the SC gap $V \geq \Delta$. (e) First multiple ARs (MARs), where an incoming electron is reflected as a hole and that hole is retroreflected as another electron, transferring a total of three electrons. The voltage has to exceed $V \geq 2\Delta/3$. (f) Higher-order MARs involve a process that transfers four electrons with onset voltage $V \geq 2\Delta/4$. The voltage thresholds correspond to the case of zero temperature.

order transport processes in an SS junction are depicted in Figs. 1(c)–1(f) and correspond to QP ($p_{\pm 1}$), AR ($p_{\pm 2}$), MARs ($p_{\pm 3}$), and fourth-order MARs ($p_{\pm 4}$) transferring 1, 2, 3, and 4 electrons, respectively.

Quantum point contact of constant transmission. In the following, we focus on single-channel NS and SS contacts characterized by an energy-independent normal state transmission coefficient τ . These junctions have been realized with the help of SC atomic-size contacts in which both the current and noise have been thoroughly studied [41–46]. We make use of the framework of full counting statistics to obtain current, noise, and charge resolved probabilities using Ref. [47] to compute the full result for the TUR-breaking coefficient in Eq. (1).

Our purpose is to investigate whether the TUR can be broken in NS and SS systems and portray the phase diagram of the system. We assign the phase of an unbroken TUR if $\mathcal{F} \geq 0$ and the second phase of a broken TUR if $\mathcal{F} < 0$. To illustrate the results, we fix the voltage and temperature and vary the transmission τ of the quantum point contact. We define τ_c as the critical transmission for which the lower bound of the TUR inequality is satisfied ($\mathcal{F} = 0$). It holds that, for any $\tau < \tau_c$, the TUR is fulfilled, whereas it is violated for $\tau > \tau_c$. In Fig. 2(a), we show τ_c as a function of the dimensionless temperature $k_B T / \Delta$ and voltage V / Δ for the NS case. Notice that the TUR only breaks at high transmissions ($\tau_c \approx 0.91$), as anticipated in the discussion of Eq. (9). Note also that τ_c decreases upon increasing the temperature, which is due to the larger overlap between the two processes induced by the finite temperature. We want to stress that no energy dependence of the transmission is needed for breaking the TUR. The source of nonlinearity of the transport characteristics (I and S) as a function of the bias that is required to violate the TUR arises from the coexistence of QP tunneling and AR in addition to the nonconstant density of states of the SC. This is in contrast with the results in Ref. [19], where an energy dependence of the transmission is needed. For low temperatures, the TUR is only broken for $V \approx \Delta$ and only for $\tau \approx 1$. This indicates that the breaking of the TUR is rooted in the coexistence of the QP and AR (for more details, see Ref. [38]).

To quantify the violation of the TUR, we consider the voltage at which the TUR is maximally broken and define the minimal TUR-breaking coefficient $\mathcal{F}_{\min} \equiv \min_V \mathcal{F}$, which is positive (or zero) for an unbroken TUR and negative for a broken TUR. In addition, the more negative $\mathcal{F}_{\min}$, the more the TUR is broken. In Fig. 2(b), this coefficient is plotted as a function of the transmission and temperature. The solid line indicates the phase boundary between the unbroken and broken TUR phases. Notice that the TUR is more broken the higher the transmission is. These findings can be understood with the help of the low-bias analysis above, see Eq. (9). The probabilities $p_{1,2}$ are analytical in the NS (see Refs. [38,47]), and $C_{\text{neq}}^{\text{NS}}$ can be easily calculated. The phase boundary obtained with this approximation is shown as a dotted line in Fig. 2(b). It qualitatively reproduces the exact results, especially at high temperatures, where the energy dependence of $p_{1,2}$ is mainly determined by the smearing of the Fermi functions (for more details, see Ref. [38]). On the other hand, the fact that high transmissions are required to have a stronger violation of the TUR can be understood from the contribution $2^4 p_2^{(0)} [1 - 6p_2^{(0)}]$ in Eq. (9). This term scales quartically with the charge, which renders a stronger violation for highly probable AR processes coexisting with low-probability QP processes. Furthermore, an important contribution to $C_{\text{neq}}^{\text{NS}}$ also comes from the cross-term $-48p_1^{(0)} p_2^{(0)}$.

For completeness, we also address in Ref. [38] the case of an NS contact featuring an energy-dependent transmission coefficient. We have studied the case in which the leads are coupled via a quantum dot. The transmission coefficient in this case corresponds to a Breit-Wigner resonance centered in the energy of the dot level ϵ_0 with a tunneling broadening given by Γ , i.e., $\tau(E) = \Gamma^2 / [(E - \epsilon_0)^2 + \Gamma^2]$. However, we find that this added energy dependence does not further

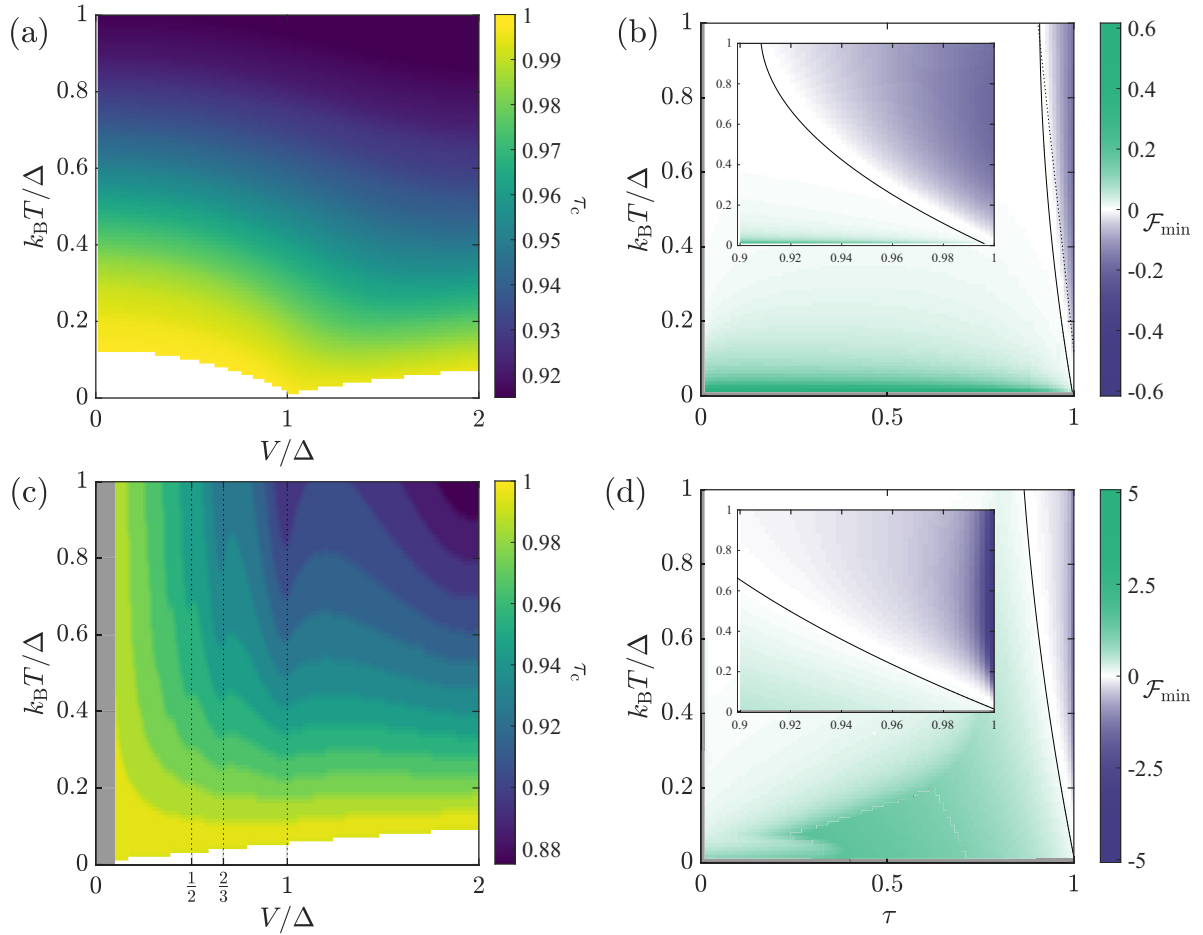


FIG. 2. (a) Critical transmission τ_c for breaking the thermodynamic uncertainty relation (TUR) for a normal metal–superconductor (NS) junction as a function of the bias voltage V/Δ and the temperature $k_B T/\Delta$. For areas with no color, the TUR is not broken. (b) The minimal TUR-breaking coefficient $\mathcal{F}_{\min}$ as a function of the dimensionless temperature $k_B T/\Delta$ and the transmission τ for a NS junction. A negative coefficient indicates a broken TUR. The black solid line indicates the phase boundary. The dotted line indicates the approximate phase boundary from the coefficient C_{neq} [see Eq. (9)]. The inset shows a zoom into the region where the TUR is broken. (c) Same as in (a) for a superconductor–superconductor (SS) junction. Dotted lines indicate special voltages of onsets of different transport processes, namely, $V/\Delta = 1, \frac{2}{3}, \frac{1}{2}$ for Andreev reflection (AR), multiple ARs (MARs), and fourth-order MARs. Gray area shows a parameter realm that was not analyzed. (d) Same as in (b) for an SS junction with the phase boundary as a solid line. The inset shows a zoom into the region where the TUR is broken.

break the TUR, see Ref. [38], which we attribute to the fact that the resonant level tends to reduce the high effective transmission at some energies, which is needed to break the TUR.

We turn now to the analysis of the SS case, where the presence of MAR processes, which are shown in Figs. 1(e) and 1(f), adds additional complexity. Again, we refer to Refs. [39,40,47] for the current and noise calculation in the framework of the full counting statistics. The critical transmission τ_c for the SS case is shown in Fig. 2(c). It is noticeable that the TUR is broken for generally smaller transmissions in comparison with the NS case [see Fig. 2(a)]. For higher temperatures, the critical transmission decreases because the processes overlap more, which benefits the TUR breaking. As compared with the NS case, the critical transmission depends more drastically on the voltage even for higher temperatures. The dependence of the voltage is more pronounced in the vicinity of voltages corresponding to onsets of MAR

processes, namely, $V = 2\Delta/n$, with n being an integer. Also notice that, for low temperatures, the TUR violation concentrates on only one point, namely, $V = T = 0$. This can be explained by considering that, in the limit $V, T \rightarrow 0$ for an SS junction with high transmission, all transmission probabilities up to infinite order contribute. In the case of perfect transmission, the current does not go to zero for $V, T \rightarrow 0$ and converges to a nonzero value [39,40], rendering the absolute limit of violation.

The minimal TUR-breaking coefficient is displayed in Fig. 2(d), where it is seen that the TUR is severely more broken than in the NS case, reaching absolute values of one order of magnitude higher than in the NS case. This enhanced breaking must originate from the MARs, which are exclusive to the SS. This large breaking results from highly probable, high-order MAR processes coexisting with lots of lower-probability MARs of lower order, granting an immensely strong violation of the TUR. As the voltage range

is restricted for $\mathcal{F}_{\min}$ in Fig. 2(d) due to computation time $V \geq 0.1\Delta$, an even stronger violation is achievable for lower voltages.

Conclusions. In this letter, we have studied the violation of the TUR in the coherent charge transport in junctions containing SCs. We have shown that the presence of multiple tunneling processes help break the TUR and that the degree of violation is clearly larger in the SS case. We have rationalized this by providing an analytical formula for the TUR-breaking coefficient for low voltages and show that it only depends on the zero-bias tunneling probabilities. We have also shown that, for junctions containing SCs, no energy dependence of the normal transmission coefficient is required to break the TUR, which is due to the intrinsic energy dependence of the probability of the different SC tunneling processes. This is a key advantage that makes it much more feasible to observe the violation of the TUR in SC junctions. Thus, in summary, in this letter, we show that SC junctions of simple quantum point contacts are promising candidates for the engineering of quantum thermal machines that are immune against fluctuations in a low-dissipation scenario. Our contribution also paves the way for the experimental observation of the TUR breaking

in the context of quantum electron transport, something that continues to be very elusive.

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