

Measuring non-Hermitian topological invariants directly from quench dynamics

Xiao-Dong Lin^{1,2} and Long Zhang^{1,3,*}¹*School of Physics and Institute for Quantum Science and Engineering,
Huazhong University of Science and Technology, Wuhan 430074, China*²*Hefei National Research Center for Physical Sciences at the Microscale and Department of Modern Physics,
University of Science and Technology of China, Hefei 230026, China*³*Hefei National Laboratory, Hefei 230088, China*

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While non-Hermitian (NH) topological phases and phenomena have been observed across various quantum systems, directly measuring NH topological invariants remains a significant challenge. In this study, we present a generic and unified framework for the direct measurement of various NH topological invariants in odd-dimensional systems through quench dynamics. We demonstrate that in one-dimensional (1D) NH systems with sublattice symmetry, the line-gap winding number and point-gap braiding degree can be extracted from the winding patterns of a dynamically constructed field based on postquench spin textures. Specifically, line-gap topology is characterized by integer-valued winding, whereas point-gap complex-band braiding is revealed by half-integer or integer winding with abrupt jumps. We also extend our approach to higher-dimensional winding numbers and non-Bloch topological invariants under open-boundary conditions. Additionally, we propose a practical cold-atom setup to realize and detect 1D NH topological phases, showing that our dynamical measurement scheme is feasible in current experimental settings. This work paves the way for the direct measurement of NH topological invariants in quantum systems.

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Introduction. The interplay between non-Hermiticity and topology gives rise to a range of fascinating phenomena that have no Hermitian counterparts [1–6], such as exceptional points (EPs) [2,7,8] and non-Hermitian (NH) skin effects [3–5,9]. These phenomena have garnered significant attention and have been extensively explored both theoretically [9–55] and experimentally [56–86]. In classifying NH topological bands, two types of complex-energy gaps, namely line and point gaps, are typically employed [20–24]. NH Hamiltonians exhibiting line gaps can be continuously deformed into a Hermitian limit, allowing the topological invariants from Hermitian systems to be naturally extended to characterize line-gap topology [16,21]. Under open boundary conditions, non-Bloch topological invariants are defined within the framework of the generalized Brillouin zone (GBZ) to restore the bulk-boundary correspondence [9,13–15]. In contrast, point-gap topology is intrinsic to NH systems and leads to unconventional boundary behaviors [20,28–34]; notably, the NH skin effects in one dimension arise from the spectral winding associated with point gaps [30,31]. Beyond these conventional classifications, braid and knot theories have been developed for NH Hamiltonians with separable bands [37–39].

Despite the observation of NH topological phenomena across various quantum platforms [70–86], direct measurement of NH topological invariants remains a challenge. Here, we propose a generic scheme to measure NH topological invariants through quench-induced nonunitary dynamics. Unlike previous works [76–78,87–91], our scheme is capable of characterizing both line-gap and point-gap topology, applicable to one-dimensional (1D) and higher-dimensional systems with sublattice or chiral symmetry, and it accommodates both periodic and open boundary conditions. Importantly, all proposed measurements are defined solely within the basis of right eigenvectors, making them straightforward to implement experimentally.

In this Letter, we introduce the winding number W_d of a dynamical field $\mathbf{g}(\mathbf{k})$, derived from quench measurements of spin textures, to characterize the NH topology in odd dimensions. Our key findings are as follows: (i) In 1D systems exhibiting sublattice symmetry (SLS), i.e., $SH(k)S^{-1} = -H(k)$ [21], the topology of real or imaginary line gaps is characterized by an integer-valued W_1 [see Eq. (4)]. This result extends seamlessly to higher-dimensional line-gapped systems with SLS, where the topology is defined by W_d [see Eq. (8)]. (ii) The point-gap topology in 1D systems is identified by a half-integer or integer-valued W_1 , with abrupt π jumps in the winding of $\mathbf{g}(k)$, highlighting the braiding structure of complex-energy bands [see Eq. (6)]. This characterization also applies to higher-dimensional systems, such as NH topological semimetals. (iii) Open-boundary systems with complex-energy bands can be similarly characterized, with all measurements projected onto the GBZ. For systems featuring a real open-boundary

*Contact author: lzhangphys@hust.edu.cn

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spectrum, some modifications are required. (iv) We propose a practical cold-atom setup to realize a 1D NH topological system, with detailed studies of its realization and detection using ^{40}K atoms as a specific example. Additionally, a comprehensive discussion on NH systems with chiral symmetry [satisfying $\Gamma H^\dagger(k)\Gamma^{-1} = -H(k)$ [21]] is included in the Supplemental Material [92].

Dynamical measurement scheme. We start with 1D NH systems with SLS to illustrate our scheme. The Hamiltonian is generally written as

$$H(k) = H_0(k) + iH_1(k),$$

$$H_0(k) = \sum_{i=x,y} h_i(k)\sigma_i, \quad H_1(k) = \sum_{i=x,y} h'_i(k)\sigma_i, \quad (1)$$

for which $S = \sigma_z$. Here $\sigma_{x,y,z}$ are the Pauli matrices and $h'_{x,y}$ can be constant including zero. The right eigenvectors of $H(k)$ are denoted as $|u_{n=\pm}^R(k)\rangle$ with eigenvalues $E_{\pm}(k) = \pm\varepsilon(k)$. Here $\varepsilon(k) = [\sum_{i=x,y} (h_i + ih'_i)^2]^{-1/2}$ for real-line-gapped $H(k)$. We investigate the time evolution governed by $H(k)$ starting from a chosen initial fully polarized state $|\psi_0(k)\rangle$, which can be realized experimentally by applying an additional Zeeman field and suddenly changing its strength from a large value to zero.

To measure the NH topology, we introduce an auxiliary Hermitian matrix $Q(k)$:

$$Q(k) = \frac{1}{2} \left[\frac{H(k)}{\varepsilon(k)} + \frac{H^\dagger(k)}{\varepsilon^*(k)} \right], \quad (2)$$

and define the *normalized* long-time spin textures (LTSTs)

$$\overline{\langle \sigma_i(k) \rangle} = \frac{\langle \sigma_i(k) \rangle_\infty}{\sqrt{\sum_{j=x,y} \langle \sigma_j(k) \rangle_\infty^2}}, \quad i = x, y, \quad (3)$$

where $\langle \sigma_i \rangle_\infty = \lim_{t \rightarrow \infty} \mathcal{N}_k^{-1} \text{Tr}(e^{-iHt} \rho_0 e^{iH^\dagger t} \sigma_i)$ with $\mathcal{N}_k \equiv \text{Tr}(e^{-iHt} \rho_0 e^{iH^\dagger t})$ and the density matrix of the initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$. After some derivation, we find that (i) $H(k)$ is topologically equivalent to $Q(k)$ in terms of a line (real or imaginary) gap [92], and (ii) regardless of the initial state ρ_0 [93], the topology of the Hermitian Q matrix is characterized by the winding number of a dynamical field $\mathbf{g}(k) = (g_x(k), g_y(k))$:

$$W_1 = \frac{1}{2\pi} \int_{\text{BZ}} dk [g_x(k) \partial_k g_y(k) - g_y(k) \partial_k g_x(k)], \quad (4)$$

where the integral is over the first Brillouin zone (BZ) and $\mathbf{g}(k)$ is defined either by $g_i(k) = \langle \sigma_i(k) \rangle$ for imaginary line gaps, or by the *deformed* LTSTs as

$$g_i(k) = \widetilde{\langle \sigma_i(k) \rangle} = \begin{cases} \overline{\langle \sigma_i(k) \rangle} & \text{Im}[\varepsilon(k)] > 0, \\ -\overline{\langle \sigma_i(k) \rangle} & \text{Im}[\varepsilon(k)] < 0, \end{cases} \quad i = x, y \quad (5)$$

for real line gaps [92]. Therefore, one can characterize the NH topology of $H(k)$ by measuring W_1 from LTSTs. Although it is introduced to detect the line-gap topology, W_d can, remarkably, also characterize the point-gap topology of $H(k)$, which will be detailed below. We emphasize that all the LTSTs introduced above are defined only in the basis of right eigenvectors $\{|u_n^R(k)\rangle\}$, and thus can be directly measured in experiment.

1D examples. We first consider the paradigmatic NH Su-Schrieffer-Heeger (SSH) model [18,19], which is described by the Hamiltonian (1) with $(h_x, h_y) = (t_1 + t_2 \cos k, t_2 \sin k)$ and $(h'_x, h'_y) = (0, \eta)$. Here t_1 (t_2) describes the tunneling within (between) unit cells, and η represents the asymmetric hopping. When $t_1 < t_2 - \eta$ or $t_1 > t_2 + \eta$ (assuming $t_2 > \eta > 0$), this system features a real line gap. An example is shown in Fig. 1(a). The normalized LTSTs $\overline{\langle \sigma_{x,y}(k) \rangle}$ are obtained by investigating the quench dynamics of a fully polarized initial state $\rho_0 = (\mathbb{1} - \sigma_z)/2$ [93] [left subfigure of Fig. 1(c)]. At $k = 0$, one can observe a discontinuity and a cusp in $\overline{\langle \sigma_{x,y}(k) \rangle}$, respectively. This result arises because after long-term dynamics, only the right eigenvectors with $\text{Im}(E_n) > 0$ persist and they belong to different energy bands, as highlighted in orange in the left subfigure of Fig. 1(a). To accurately characterize the band topology, we note that each right eigenvector $|u_n^R\rangle$ with eigenenergy E_n has a sublattice-symmetric counterpart $S|u_n^R\rangle$ with $-E_n$, which suggests the deformation defined in Eq. (5) [92]. The deformed LTSTs $\widetilde{\langle \sigma_{x,y}(k) \rangle}$ then capture the correct information of a single band [colored in magenta in the right subfigure of Fig. 1(a)], and the nonzero winding of the constructed dynamical field $\mathbf{g}(k)$ characterizes the line-gap topology with $W_1 = 1$ [Fig. 1(c)]. An example of characterizing imaginary-line-gapped topological phases is given in [92].

One notable finding is that the point-gap topology of this NH SSH model can also be characterized by $W_1 = 1/2$, indicating the unknotted braiding of complex-energy bands [39,63,68,80,83,84]. As illustrated in Fig. 1(b), when $t_2 - \eta < t_1 < t_2 + \eta$, the eigenenergy trajectories $E_{\pm}(k)$ braid once to form a single loop in the $(\text{Re}(E), \text{Im}(E), k)$ space (an unknot [39]), resulting in a band structure featuring a point gap. This unknotted braiding implies that our measurement via $\widetilde{\langle \sigma_i(k) \rangle}$ represents a half-period measurement, effectively covering only half of the closed-loop spectrum, as highlighted in magenta in Fig. 1(b). Consequently, the dynamical field $\mathbf{g}(k)$, constructed using the same approach as in the line-gapped case, exhibits incomplete winding characterized by an abrupt π jump, yielding $W_1 = 1/2$ [Fig. 1(d)]. This half-integer winding number also reflects the vorticity of an EP [12,16,19].

In addition to the unknot discussed above, our dynamical measurement can unveil other types of complex-energy braiding. For illustration, we consider the twister Hamiltonian that retains the same form as Eq. (1): $H(k) = m_1 T_1 + m_2 T_2 + m_3 T_3$, where $T_n = |\uparrow\rangle\langle\downarrow| + e^{in\pi} |\downarrow\rangle\langle\uparrow|$ [39]. The phase diagram is depicted in Fig. 1(j), where three distinct phases are identified: the unknot, the Hopf link, and the trefoil [39]. Remarkably, we find that all these phases can be detected through our dynamical measurement scheme. As demonstrated in Figs. 1(h) and 1(i), the winding number W_1 , in conjunction with varying numbers of π jumps, can uniquely characterize the complex-energy braiding. Specifically, when N jumps occur (with $N > 0$), the winding of $\mathbf{g}(k)$ gives the braiding degree

$$|\nu| = N = 2|W_1|, \quad (6)$$

where ν counts how many times the complex bands braid in the $(\text{Re}(E), \text{Im}(E), k)$ space [63].

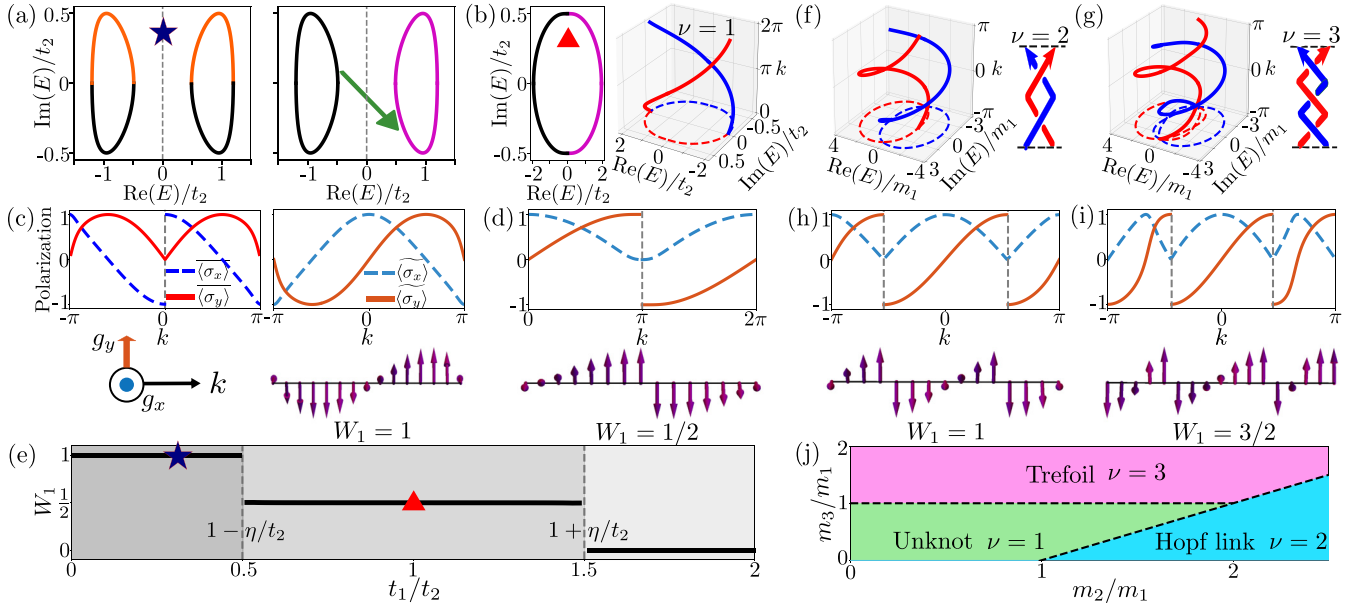


FIG. 1. Measuring the NH topology of 1D models. (a)–(e) Characterization of the NH SSH model. Complex-band structures in the real-line-gapped (a) and point-gapped (b) phases are shown, respectively. The corresponding dynamical measurements are illustrated in (c) and (d). In both (a) and (c), the left (right) subfigures represent the measurement before (after) the deformation defined in Eq. (5). The deformation maps the measured eigenvectors whose eigenenergies are colored in orange in the lower band to their sublattice-symmetric counterparts, as indicated by the green arrow. Through the mapping, the deformed measurement covers a single band (a) or only half of the band (b), as highlighted in magenta. The dynamical field $\mathbf{g}(\mathbf{k}) = (g_x, g_y)$ (purple arrows) constructed from the deformed LTSTs $\langle \sigma_{x,y} \rangle(\mathbf{k})$ exhibits the integer-valued winding (c) and the half-integer-valued winding with a π jump (d), respectively. Such measurements can precisely determine the phase diagram (e). Here, we set $\eta = 0.5t_2$ and take $t_1 = 0.3t_2$ (a) or t_2 (b). (f)–(j) Characterization of the twister Hamiltonian. Complex-band structures in the Hopf link phase (f) and the trefoil phase (g) are shown, respectively, with the corresponding braid diagrams. The dynamical measurements yield nonzero windings of $\mathbf{g}(\mathbf{k})$ with two (h) or three (i) jumps. The phase diagram as a function of m_2 and m_3 is plotted (j). Here, we take $(m_2/m_1, m_3/m_1) = (2, 0.5)$ in (f) and $(1, 1.5)$ in (g).

Higher dimensions. The present scheme can be extended to higher dimensions. For odd d -dimensional (d D) line-gapped systems, the Hamiltonian can generally take the form

$$H(\mathbf{k}) = H_0(\mathbf{k}) + \mathfrak{i}H_1(\mathbf{k}), \quad H_0(\mathbf{k}) = \sum_{i=0}^d h_i(\mathbf{k})\gamma_i, \\ H_1(\mathbf{k}) = h'_1(\mathbf{k})\gamma_1 + h'_2(\mathbf{k})\gamma_2 + \cdots, \quad l_i \in \{0, 1, \dots, d\}. \quad (7)$$

Here the γ matrices obey the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}$ ($i, j = 0, 1, \dots, d$) and are of dimensionality $2^{(d+1)/2}$ for an odd d , which is the minimal requirement to open a line gap. Such a system has the SLS $S = \mathfrak{i}^{(d+1)/2} \prod_{i=0}^d \gamma_i$. Similar to 1D systems, the topology of $H(\mathbf{k})$ can be dynamically investigated by introducing the auxiliary Q matrix. We find that when choosing $\rho_0 = (\mathbb{1} - S)/2$ at each $\mathbf{k}$, the line-gap topology is characterized by a d D winding number [92]

$$W_d = \frac{[(d-1)/2]!}{2\pi^{(d+1)/2}d!} \int_{\text{BZ}^d} \mathbf{g}(\mathbf{d}\mathbf{g})^d, \quad (8)$$

where the dynamical field $\mathbf{g} = (g_0, g_1, \dots, g_d)$ with $g_i(\mathbf{k}) = \langle \gamma_i(\mathbf{k}) \rangle$ [or $\langle \gamma_i(\mathbf{k}) \rangle$ as given in Eq. (5)] for imaginary (real) line gaps, $\mathbf{g}(\mathbf{d}\mathbf{g})^d \equiv \epsilon^{i_1 i_2 \dots i_{d+1}} g_{i_1} \mathbf{d}g_{i_2} \wedge \cdots \wedge \mathbf{d}g_{i_{d+1}}$ with $\epsilon^{i_1 i_2 \dots i_{d+1}}$ being the fully antisymmetric tensor and $i_1, i_2, \dots, i_{d+1} \in \{0, 1, 2, \dots, d\}$, and “ d ” denotes the exterior derivative.

We illustrate it with three-dimensional (3D) NH topological insulators. The Hamiltonian takes the form of Eq. (7) with $h_0 = m_0 - 2t_0 \sum_{i=1}^3 \cos k_{r_i}$, $h_{i>0} = 2t_{s_0} \sin k_{r_i}$, and

$h'_2 = \eta$. Here we denote $(r_1, r_2, r_3) \equiv (x, y, z)$ and take $\gamma_0 = \sigma_z \otimes \tau_x$, $\gamma_1 = \sigma_x \otimes \mathbb{1}$, $\gamma_2 = \sigma_y \otimes \mathbb{1}$, and $\gamma_3 = \sigma_z \otimes \tau_y$, where σ_i and τ_i are both Pauli matrices. The Hermitian part $H_0(\mathbf{k})$ has been simulated in solid-state spin systems [94,95]. The line-gap topology can be unveiled through the winding of the dynamical field $\mathbf{g}(\mathbf{k}) = (g_0, g_1, g_2, g_3)$, with $g_i = \langle \gamma_i \rangle$, across the entire BZ. One can find that on each constant- g_0 surface—such as the $g_0 = 0$ surface shown in Fig. 2(a)— $\mathbf{g}(\mathbf{k})$ displays a hedgehog configuration, indicating a nonzero winding $W_3 = 1$ [92].

Besides the gapped cases, the proposed scheme can also be applied to topological semimetals. Here, we consider the 3D NH nodal-line semimetals [96,97]: $H(\mathbf{k}) = [m + t(\cos k_x + \cos k_y + \cos k_z)]\sigma_z + (t \sin k_z + \mathfrak{i}\eta)\sigma_x$. When $m = -2t$ and $\eta = -0.5t$, the system features two exceptional rings in the $k_z = 0$ plane [see Fig. 2(b)]. As each ring comprises a collection of EPs, the vorticity of an EP can be used to characterize these exceptional rings [96]. For a closed loop encircling one ring (two rings), our dynamical measurement can give exactly a half-integer (integer) vorticity [Figs. 2(c) and 2(d)].

Generalized Brillouin zone. The above discussion only considers periodic boundary conditions (PBCs). Here we generalize the present scheme to open boundary conditions (OBCs). As proposed in Ref. [9], we will deal with it in the GBZ. Since only the construction of 1D GBZ is well

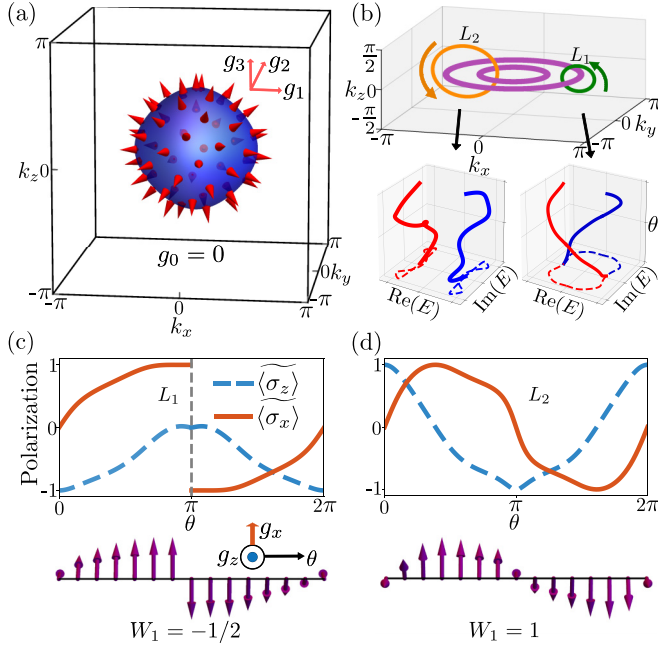


FIG. 2. Measuring the NH topology of 3D models. (a) Characterization of the 3D line-gap topology. The dynamical field $\mathbf{g}(\mathbf{k})$ (arrows) exhibits a hedgehog configuration on the $g_0 = 0$ surface. Here $m_0 = 4t_0$, $t_{so} = 0.5t_0$, and $\eta = 0.2t_0$. (b)–(d) Characterization of the NH nodal-line semimetal, which features two exceptional rings in the BZ, as highlighted in magenta in (b). Along the loop L_1 (L_2), the vorticity of the enclosed EP (EPs) takes a half-integer (an integer) value, which is revealed by the winding of the dynamical field $\mathbf{g}(\mathbf{k}) = (g_z, g_x)$ constructed from $\langle \sigma_{z,x}(\mathbf{k}) \rangle$ in (c) [(d)]. The evolution of complex eigenvalues is parametrized by $\theta \in [0, 2\pi)$.

established [9,14], we will restrict our discussion to 1D cases here.

In 1D GBZ theory [9], the non-Bloch Hamiltonian $H(k_G)$, corresponding to $H(k)$ in Eq. (1), is obtained by the replacement $e^{ik} \rightarrow \beta \equiv e^{ik_G}$, where k_G is complex. The auxiliary Q matrix, which is topologically equivalent to $H(k_G)$, can be accordingly introduced as in Eq. (2) [92]. For an initial state $\rho_0(x) = |\psi_0(x)\rangle\langle\psi_0(x)|$ in real space, we introduce a projection operator $P_{k_G} \equiv |\beta^R(k_G)\rangle\langle\beta^L(k_G)|$ to project the time evolution $\rho(x, t) = e^{-iHt}\rho_0(x)e^{iH^\dagger t}$ onto the GBZ: $\rho(k_G, t) = P_{k_G}\rho(x, t)P_{k_G}^\dagger$ [89], where $|\beta^R(k_G)\rangle = L^{-1/2} \sum_i e^{ik_G x_i} |x_i\rangle$ and $|\beta^L(k_G)\rangle = L^{-1/2} \sum_i e^{ik_G^* x_i} |x_i\rangle$, with L the lattice length, which satisfy $\langle\beta^L(k_G)|\beta^R(k'_G)\rangle = \delta_{k_G, k'_G}$ and $\sum_{k_G} |\beta^R(k_G)\rangle\langle\beta^L(k_G)| = \mathbb{1}$ [98]. This gives the normalized LTSTs $\langle\sigma_{x,y}(k_G)\rangle$, as defined in Eq. (3), with the replacement $e^{-iHt}\rho_0(k)e^{iH^\dagger t} \rightarrow \rho(k_G, t)$.

The dynamical measurement in the GBZ is divided into two cases: (i) When the OBC spectrum is complex, the procedure follows exactly as outlined in Eqs. (4) and (5). An illustrative example is provided in Figs. 3(a) and 3(b), where we examine the 1D NH SSH model under OBCs using the same parameters as those in Fig. 1(a). (ii) In contrast, when the spectrum is entirely real, the situation becomes analogous to the characterization of Hermitian systems [99]. Rather than

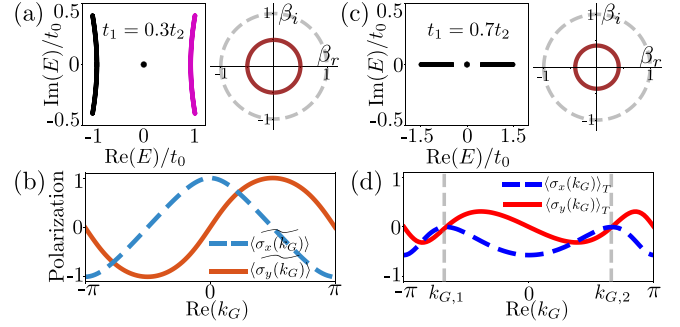


FIG. 3. Measuring non-Bloch topological invariants. (a) The OBC spectrum (left) and GBZ (right) of the NH SSH model with $t_1 = 0.3t_2$. Here β_r (β_i) denotes the real (imaginary) part of β . The dashed unit circle represents the BZ. (b) The measurement of $\langle\sigma_{x,y}(k_G)\rangle$ with $\rho_0 = |x_0\rangle\langle x_0| \otimes (\mathbb{1} - \sigma_z)/2$ [101] yields the winding number $W_1 = 1$, which characterizes the zero modes in (a). Here, x_0 represents the lattice center site. (c) The real-energy spectrum (left) and GBZ (right) for $t_1 = 0.7t_2$. (d) Time-averaged spin textures $\langle\sigma_{x,y}(k_G)\rangle_T$ with $\rho_0 = |x_0\rangle\langle x_0| \otimes (\mathbb{1} - \sigma_x)/2$. The slopes of $\langle\sigma_y(k_G)\rangle_T$ at $k_{G,1}$ and $k_{G,2}$ determine $C_0 = 1$, characterizing the zero modes in (c).

using LTSTs, we introduce the *time-averaged* spin textures:

$$\langle\sigma_i(k_G)\rangle_T = \lim_{T \rightarrow \infty} \int_0^T \frac{1}{\mathcal{N}_{k_G}} \text{Tr}[\rho(k_G, t)\sigma_i] dt, \quad i = x, y, \quad (9)$$

where $\mathcal{N}_{k_G} \equiv \text{Tr}[\rho(k_G, t)]$. We prove that when choosing the initial spin state to be $(\mathbb{1} - \sigma_x)/2$ [101], the open-boundary topology is characterized by a zero-dimensional topological invariant [92]

$$C_0 = \frac{1}{2} \sum_j \text{sgn}[\partial_{k_G} \langle\sigma_y(k_{G,j})\rangle_T], \quad (10)$$

where $k_{G,j}$ denote the paired points at which $\langle\sigma_y(k_{G,j})\rangle_T = 0$ and $|\langle\sigma_x(k_{G,j})\rangle_T|$ take local minima. An example is given in Figs. 3(c) and 3(d), and more details can be found in [92].

Before proceeding, several important points should be emphasized: (i) As described above, a well-chosen initial state can significantly simplify the measurement, with the selection highly depending on the systems dimensionality and symmetry. A summary can be found in [92]. (ii) Although we assume the initial state to be completely polarized for simplicity, the proposed scheme is also applicable in the case of a shallow quench, where the initial state is not fully polarized but remains within the same trivial regime as the fully polarized state. This robustness has been both theoretically analyzed and experimentally demonstrated in the dynamical characterization of Hermitian systems [102,103]. (iii) For OBCs, the extension to higher-dimensional GBZ is straightforward. Once the GBZ is determined through a theoretical framework, such as the theory established in Refs. [53,54], one can project the measured spin textures onto the corresponding GBZ, thereby enabling the characterization of the non-Bloch topology.

Experimental realization. Based on optical Raman lattices [104–108], we propose a highly feasible experimental setup to realize and detect 1D NH topological phases in ultracold

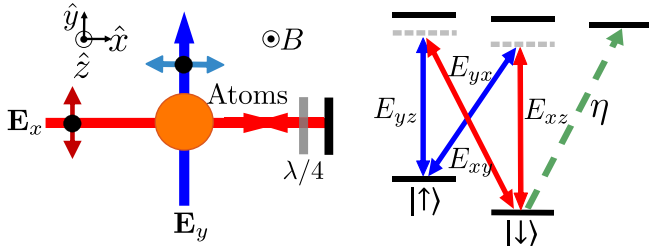


FIG. 4. Experimental setup and laser couplings. A standing wave E_x (red) and a plane wave E_y (blue) form a double- Λ -type configuration (right panel) to generate both lattice and Raman potentials. A near-resonant beam (green) is added to induce a spin-dependent atom loss.

atoms. As sketched in Fig. 4, a standing-wave beam E_x (green) with x, z polarization and a plane wave E_y (orange) with y, z polarization are applied to form a double- Λ -type configuration, which generates a spin-independent lattice $V(x)$ and two Raman coupling potentials $\mathcal{M}_{x,y}(x)$ simultaneously. A $\lambda/4$ wave plate is placed before the mirror to induce an additional phase shift to the z component light, which results in $E_x = \hat{y}E_{xy} \cos(k_0x) + \hat{z}E_{xz} \sin(k_0x)$. Here $k_0 = 2\pi/\lambda$, with λ being the wavelength. Thus, the Raman potential $\mathcal{M}_x$ ($\mathcal{M}_y$), generated by the components E_{xy} and E_{yz} (E_{xz} and E_{yx}), takes the form $\mathcal{M}_x(x) = M_{0x} \cos(k_0x)$ [$\mathcal{M}_y(x) = M_{0y} \sin(k_0x)$], with $M_{0x} \propto E_{xy}E_{yz}/\Delta$ ($M_{0y} \propto E_{xz}E_{yx}/\Delta$), where Δ denotes the one-photon detuning. The lattice potential reads $V(x) = V_0 \cos^2(k_0x)$, with $V_0 \propto (E_{xz}^2 - E_{xy}^2)/\Delta$. In addition, a single near-resonant beam is added to induce a spin-selective atom loss η [79,82]. The total Hamiltonian then reads [92]

$$H = \left[\frac{k_x^2}{2m} + V_0 \cos^2(k_0x) \right] \otimes \mathbb{1} + M_{0x} \cos(k_0x) \sigma_x + M_{0y} \sin(k_0x) \sigma_y + m_z \sigma_z - i\eta |\downarrow\rangle\langle\downarrow|, \quad (11)$$

where m is the mass of an atom and m_z measures the two-photon detuning. Note that the Raman $\mathcal{M}_x$ ($\mathcal{M}_y$) and lattice potentials satisfy a relative antisymmetry (symmetry) with respect to each lattice site, ensuring that $\mathcal{M}_x$ ($\mathcal{M}_y$) only induces spin-flip hopping (on-site spin flipping). The Bloch Hamiltonian in the tight-binding limit then takes $H(q) = (m_z - 2t_0 \cos qa)\sigma_z + (m_y +$

$2t_{so} \sin qa)\sigma_y + i\eta/2(\sigma_z - \mathbb{1})$, which describes a NH SSH-like model that hosts a point-gap topological phase [92]. Here a is the lattice constant, q is the quasimomentum, and t_0 (t_{so}) represents the spin-conserved (spin-flip) hopping. The model parameters can be tuned independently.

The proposed dynamical measurement scheme can be implemented within this cold-atom system. Numerical results based on the Hamiltonian (11) are detailed in the Supplemental Material [92], where the initial state is a wave packet localized at the center of the lattice, and measurements are projected onto the BZ (GBZ) to characterize the topology under PBCs (OBCs). It is crucial to note that while the LTSTs are defined in the infinite-time limit [Eq. (3)], the characterization can be accurate enough after a finite time $\tau^* \gg 1/\eta$. Importantly, the timescale τ^* can often be significantly shorter than the lifetime of the cold-atom system. Taking ^{40}K atoms as an example, one can choose $|\uparrow\rangle = |F=7/2, m_F=7/2\rangle$, $|\downarrow\rangle = |9/2, 9/2\rangle$, and set $\lambda = 768 \text{ nm}$. The timescale for obtaining $\langle\sigma_{y,z}\rangle$ is $\tau^* = 120/E_r$ with $E_r = k_0^2/(2m)$ [92], while the lifetime for ^{40}K atoms is about $\tau \sim 1300/E_r$ [105]. The requirement $\tau^* \ll \tau$ is indeed satisfied, ensuring that there is ample time available for implementing the dynamical measurement. A detailed illustration based on ^{40}K atoms is provided in [92], while the proposed scheme is also suitable for other alkali-metal atoms, such as ^{87}Rb bosons, as well as alkaline-earth (-like) atoms, such as ^{173}Yb and ^{87}Sr .

Conclusion. We have proposed a generic and unified scheme to directly measure line-gap winding numbers, point-gap braiding degrees, and non-Bloch topological invariants in odd-dimensional systems. The experimental setup we have designed not only demonstrates the practicality of detecting 1D NH topological phases but also highlights the accessibility of these measurements with existing technologies. Despite only odd-dimensional systems being considered, the proposed scheme can certainly be generalized to NH systems in even dimensions, which will be our next study.

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