

All product eigenstates in Heisenberg models from a graphical construction

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Recently, large degeneracy based on product eigenstates has been found in spin ladders, kagome-like lattices, and motif magnetism, connected to spin liquids, anyonic phases, and quantum scars. We unify these systems by a complete classification of product eigenstates of Heisenberg XXZ Hamiltonians with Dzyaloshinskii-Moriya interaction on general graphs in the form of Kirchhoff rules for spin supercurrents. By this, we construct spin systems with extensive degree of degeneracy linked to exotic condensates which could be studied in atomic gases and quantum spin lattices.

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In recent years, spin systems were realized in a multitude of physical platforms, including ultra cold atom systems like quantum gas microscopes [1], Rydberg atoms [2,3], neutral atom arrays with optical tweezers [4,5], and others [6] as well as in qubit lattices, quantum computers [7–9] and chains of magnetic atoms on surfaces [10–12]. The eigenstates of spin models with Heisenberg exchange interactions, which is an archetypal example for strongly-correlated matter, are highly nontrivial due to their high degree of entanglement, especially of the excited states. This poses a challenge to both experimental realization and theoretical description. These models are exactly solvable only in certain limits, e.g., using the Bethe ansatz [13], Jordan-Wigner transformations [14,15], or by selecting particular types of interactions [16,17]. Contrasting this, there recently have been a number of studies on product eigenstates in quantum spin systems that are connected to quantum many-body scars [18], which evade the eigenstate thermalization hypothesis [19,20].

First examples were spin helices in 1D easy-plane quantum magnets [21–32] that have been dubbed phantom helices [25–27]. These states belong to a chiral reformulation of the Bethe ansatz at fine-tuned parameters of the one-dimensional XXZ quantum Heisenberg model where they consist of zero-energy quasi-particles. Phantom helices were experimentally observed in ultra cold atoms and generalized to higher dimensions [6]. Beyond one spatial dimension, product eigenstates can appear as ground states of Heisenberg models with carefully engineered magnetic fields manipulating the individual spins [24,33,34]. In the vicinity of a critical point with macroscopic degeneracy, these systems carry a rich phase diagram including spin-liquid and broken symmetry phases. The prod-

uct eigenstates were studied for their quantum scar properties [35] and generalized to lattices built from motifs, i.e., elementary building blocks, where they are ground states of frustrated lattices at a critical value of their exchange anisotropy [36,37], also further showing that product eigenstates exist in quasicrystalline Heisenberg magnets [38]. Product states are also a starting point for semiclassical and effective approaches to spin systems [39,40]. For example, product states have been used to elucidate the connection between classical and quantum skyrmions [41,42], in particular, studies indicate that the central spin of quantum skyrmions could decouple in the ground state [43,44].

In this manuscript, we give a procedure to completely classify product eigenstates in quantum Heisenberg models on general graphs. Our work unifies the approaches of Refs. [6,22,36,38] by supplying a graphical interpretation reminiscent of Kirchhoff's law for electrical currents, which we use to solve the generally appearing coupled trigonometric equations [24,33,34]. According to these rules, solving for product eigenstates amounts to choosing consistent flow patterns of spin supercurrent in the graph. In particular, we construct Heisenberg models with large degenerate eigenspaces containing a considerable number of product eigenstates. To this end, our work builds on Cerezo *et al.* [33], who classified product eigenstates in spin-1/2 systems with nonuniform exchange interactions and magnetic fields in order to systematically engineer models with fully factorized ground states. We find that for models such as the square lattice with a zigzag edge, the degeneracy scales exponentially in the size of the boundary of the system, and describe a suspension procedure to construct spin systems with a degeneracy of the product eigenstate subspace growing exponentially in system size. Other quantum spin models with extensive degeneracy have been recently proposed [36,45–51].

We numerically find that the degeneracy of the energy eigenspace that accommodates the product eigenstates exceeds the dimension of the space spanned by the product eigenstates in accordance with earlier studies [22,23,36]. It is

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built up by additional nonproduct states that evade a simple description in terms of symmetries, and, in spin ladders, these additional states were linked to an anyonic condensate [23]. The class of models with nontrivial product eigenstates that we characterize allow for significant simplifications in Bethe ansatz approaches to XXZ quantum spin chains, including the existence of fully factorized pseudo vacua [21], homogeneous T – Q -relations for chains with nondiagonal—i.e., $U(1)$ -symmetry breaking—boundary conditions [52], and the chiral reformulation of the ordinary Bethe ansatz where the elemental excitations are defect angles in helices [25–27]. By the general theory and the derived models with extensive degeneracies, spin product eigenstates and Heisenberg models on graphs hint at a crucial role of graph topology for entanglement, phase transitions and exact solvability. The condensate corresponding to the macroscopically degenerate eigenspace can be investigated in the described experimental platforms [1–12,53–59] and could be particularly easily initialized with high fidelity in quantum computers and cold atom systems when compared to entangled states [6]. After completion of our work, we became aware of the recent Ref. [60], which touches on similar ideas.

Model and method. We investigate quantum spins-1/2 on the vertices of a connected simple undirected graph $\Lambda = (V, E)$, where V is the set of vertices and E is the set of edges. This excludes edges connecting the same pair of vertices and loops connecting a vertex to itself. The dynamics are given in terms of a Hamiltonian constructed by assigning an XXZ spin-exchange interaction term and a Dzyaloshinskii-Moriya interaction (DMI) term to each edge:

$$H = \sum_{\{i,j\} \in E} \left[J(S_i^x S_j^x + S_i^y S_j^y) + \Delta S_i^z S_j^z - \kappa_{ij} D(S_i^x S_j^y - S_i^y S_j^x) \right], \quad (1)$$

with the exchange interaction J , the anisotropy Δ and the spin-1/2 operators $S_i^{x/y/z} = \hbar \sigma_i^{x/y/z} / 2$ acting on the spin at vertex i , where $\sigma_i^{x/y/z}$ are Pauli matrices. $D \geq 0$ is the strength of the DMI and $\kappa_{ij} = -\kappa_{ji} = \pm 1$ is its sign. For $J = D = 0$, this is the quantum Ising model where an eigenbasis is given by products of eigenvectors of σ^z . In the following, we assume $J^2 + D^2 \neq 0$.

The fully polarized states $|\uparrow \dots \uparrow\rangle$ and $|\downarrow \dots \downarrow\rangle$ are product eigenstates for all Hamiltonians of the form in Eq. (1). For finding additional product eigenstates, we follow the approach in Ref. [33] and insert the general ansatz for a product state

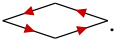
$$|\Psi(\boldsymbol{\vartheta}, \boldsymbol{\varphi})\rangle = \bigotimes_{i \in V} \left(\cos\left(\frac{\vartheta_i}{2}\right) e^{-i\frac{\varphi_i}{2}} |\uparrow\rangle_i + \sin\left(\frac{\vartheta_i}{2}\right) e^{i\frac{\varphi_i}{2}} |\downarrow\rangle_i \right) \quad (2)$$

into the eigenvalue equation for H . Here, $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_{|V|})$ are the angles relative to the hard axis z and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_{|V|})$ are the angles in the easy-plane x – y , taking values $\vartheta_i \in (0, \pi)$ and $\varphi_i \in [0, 2\pi)$. The eigenvalue problem $H|\Psi(\boldsymbol{\vartheta}, \boldsymbol{\varphi})\rangle = \varepsilon(\boldsymbol{\vartheta}, \boldsymbol{\varphi})|\Psi(\boldsymbol{\vartheta}, \boldsymbol{\varphi})\rangle$ yields a coupled set of trigonometric equations constraining the angles $\boldsymbol{\vartheta}$ and $\boldsymbol{\varphi}$, see Refs. [33,61].

Results. Among the constraining equations there are pairs, one for each edge, that constrain the relative angles of adjacent

spins. The remaining equations can then be condensed to very few rules that characterize all product eigenstates, which we discuss in the following. For easy-axis magnetism, $\Delta^2 > J^2 + D^2$, we find that there are only the trivial product eigenstates $|\uparrow \dots \uparrow\rangle$ and $|\downarrow \dots \downarrow\rangle$ [61]. Additional interactions such as magnetic fields are required to obtain further product eigenstates in this case. For easy-plane magnetism, $\Delta^2 \leq J^2 + D^2$, we develop a graphical interpretation of the constraints for the product eigenstates. First, the polar angle of the spins is constant $\vartheta_j = \Theta$, corresponding to a canted spin configuration. Secondly, the exchange interactions constrain the azimuthal angles of adjacent spins to $\varphi_i - \varphi_j = \kappa_{ij}\delta + \sigma_{ij}\gamma$, where the DMI defines a bias angle $\delta = \arg(J + iD)$ and the anisotropy determines the anisotropy angle $\gamma = \arccos(\Delta/\sqrt{J^2 + D^2})$. Here, each edge has a binary degree of freedom with $\sigma_{ij} = -\sigma_{ji} = \pm 1$. As a consequence, the degrees of freedom of the product eigenstates are two continuous global angles Θ and Φ , fixing the orientation of a reference spin $\vartheta_1 \equiv \Theta$ and $\varphi_1 \equiv \Phi$, as well as the discrete degree of freedom of choosing signs σ_{ij} for the edges. The latter degree of freedom corresponds to choosing an orientation of the underlying graph described by assigning a direction to an edge $\{i, j\}$, that we depict by an arrow pointing towards vertex i if $\varphi_i - \varphi_j = \kappa_{ij}\delta + \gamma$ and in the other direction if $\varphi_i - \varphi_j = \kappa_{ij}\delta - \gamma$. The choice of the orientation σ has to be commensurate with two rules reminiscent of Kirchhoff's laws for electrical currents.

(i) *Vertex rule.* At each vertex i , the number of ingoing arrows equals the number of outgoing arrows. $\sum_{\{i,j\} \in E} \sigma_{ij} = 0$, depicted by, e.g., .

(ii) *Circuit rule.* The change of the azimuthal angle along a circuit $\Gamma \subset E$ in the graph must consistently be an integer multiple of 2π . $\sum_{\{i,j\} \in \Gamma} (\kappa_{ij}\delta + \sigma_{ij}\gamma) \equiv 0 \pmod{2\pi}$, depicted by, e.g., .

There are two special cases for $\Delta^2 = J^2 + D^2$. First, for $\gamma = 0$, σ is not well-defined since $-\gamma \equiv \gamma \pmod{2\pi}$. Yet, the circuit rule remains meaningful and allows for nontrivial product eigenstates only at special values of δ . By means of local rotations about the anisotropy axis, these Hamiltonians are unitarily equivalent to the XXX Hamiltonian [62] whose product eigenstates are fully polarized states along an arbitrary direction. Second, for $\gamma = \pi$, σ is not well-defined either. Still, like in the previous case, nontrivial product eigenstates exist if the circuit rule can be fulfilled. The vertex rule is not to be confused with Gauss' law in lattice gauge theories [63]. In physical terms, the vertex rule corresponds to the local conservation of the z component of the spin, i.e., compensating spin supercurrents C_{ij} at vertex i

$$\langle S_i^z \rangle = \sum_{\{i,j\} \in E} C_{ij} = \frac{\sin^2(\Theta) \sin(\gamma) \hbar^2 J}{4 \cos(\delta)} \sum_{\{i,j\} \in E} \sigma_{ij} = 0. \quad (3)$$

In accordance with this interpretation, the vertex rule can be extended to Hamiltonians including local magnetic fields $\mathbf{B}_i S_i$ such that the fields act as source and sink terms compensating the difference between ingoing and outgoing arrows [61]. Fine-tuning the DMI [64–66] and local magnetic fields can hence help to fulfill the circuit and vertex rule, respectively.

An orientation σ that satisfies the vertex rule at each vertex is called an *Euler orientation*. Graphs with Euler orientations have at least one closed path that travels each edge in the direction of the Euler orientation exactly once. Such a path can be explicitly constructed, e.g., by the Hierholzer algorithm [67,68]. To simplify terminology, we refer to an Euler orientation fulfilling the circuit rule for a Hamiltonian in Eq. (1) as a *Kirchhoff orientation* of a *Kirchhoff graph*. This should not be confused with the notion of Kirchhoff graphs in the context of reaction route graphs [69,70]. Any product state commensurate with the Kirchhoff rules has an appealing interpretation: When we unravel a closed path as above, the state looks like a helix with local pitch angles γ that are distorted by local bias angles $\pm\delta$ in presence of DMI. In this sense, the product states are a natural extension of the recently discovered phantom helices [25–27] to arbitrary graphs and dimensions. Remarkably, all product eigenstates of Eq. (1) have the same energy independent of DMI

$$\varepsilon = \hbar^2 \Delta |E|/4, \quad (4)$$

with $|E|$ the number of edges. We note that these results carry over to higher spin representations when replacing the ansatz in Eq. (2) by a product of spin-coherent states. These states are still parameterized by two angles obeying the Kirchhoff laws and populating an energetically degenerate subspace. In the following, we refer to the space of all eigenstates with the same energy as the product eigenstates as the ε -space and the subspace spanned by the product eigenstates as *product eigenspace*. An upper bound for the product eigenspace dimension (spin-1/2) is given by

$$d_{\max} = 2 + \sum_{k=1}^{|V|-1} \min \left(|\sigma|, \binom{|V|}{k} \right), \quad (5)$$

where $|\sigma|$ is the number of Kirchhoff orientations and $|V|$ the number of spins. For $\gamma = 0$ and $\gamma = \pi$, we set $|\sigma| = 1$. This bound follows from considering the projections on the $|V| + 1$ eigenspaces P_k , $k \in \{0, \dots, |V|\}$, of the total z -spin symmetry operator $\sum_{i \in V} S_i^z$. By Eq. (2), a product eigenstate $|\Psi(\Theta, \Phi)\rangle$ with $\Theta \neq 0, \pi$ gives $|V| + 1$ orthogonal eigenstates $P_k |\Psi(\Theta, \Phi)\rangle$ of the same energy. Given another nontrivial product eigenstate $|\Psi(\Theta', \Phi')\rangle$ with the same Kirchhoff orientation, the projected states differ merely by a nonzero factor, i.e., $P_k |\Psi(\Theta', \Phi')\rangle \sim P_k |\Psi(\Theta, \Phi)\rangle$. So, for any projector P_k , the image of the product eigenspace under this projector contains at most $|\sigma|$ linear independent vectors and is further bounded by the dimension of $\text{im}(P_k)$ which is $|V|$ choose k . Since the trivial product eigenstates $|\uparrow \dots \uparrow\rangle$ and $|\downarrow \dots \downarrow\rangle$ span the one-dimensional spaces $\text{im}(P_0)$ and $\text{im}(P_{|V|})$, $d_{\max}$ is at least 2. A general lower bound of $|V| + 1$ originates from the $U(1)$ symmetry if at least one nontrivial product eigenstate exists since such a state has nonzero weight in every space $\text{im}(P_k)$. For graphs with vanishing DMI and finite anisotropy angle $\gamma = \pi/(2n)$, n a positive integer, the number of Kirchhoff orientations $|\sigma|$ gives an improved lower bound on the product eigenspace degeneracy. In this case, the dimension of the product eigenspace has both an upper and a lower bound linear in $|\sigma|$. For details, see Ref. [61].

We can interpret the product eigenstates physically in terms of delocalized spin flips. For a given Kirchhoff orientation,

the span of the product eigenstates $|\Psi(\Theta, \Phi)\rangle$ is generated by states $R^{(n)} |\Psi(\pi/2, \Phi_0)\rangle$ consisting of n delocalized spin flips with $0 \leq n \leq |V|$ and a fixed Φ_0 . Here,

$$R^{(n)} = \sum_{\{i_1, \dots, i_n\} \subset V} P_{i_1} \dots P_{i_n}, \quad (6)$$

where P_i acts on a product eigenstate as in Eq. (2) by $\vartheta_i \mapsto \pi - \vartheta_i$ and $\varphi_i \mapsto \varphi_i - \pi$ and hence flips the expectation value of the i th spin, i.e., $\langle S_i \rangle \mapsto -\langle S_i \rangle$. We set $R^{(0)} \equiv 1$ and note that $R^{(n)} = 0$ for $n > |V|$ since the sum is empty. The spin-flip states can be alternatively expressed in terms of a linear combination of derivatives of $|\Psi(\Theta, \Phi)\rangle$ with respect to Θ and Φ which are also eigenstates of H , given by Eq. (4),

$$\partial_{\Theta} |\Psi(\Theta, \Phi)\rangle = \frac{i}{2} \sum_{i \in V} P_i |\Psi(\Theta, \Phi)\rangle, \quad (7)$$

$$\partial_{\Phi} |\Psi(\Theta, \Phi)\rangle = \frac{1}{2} \sum_{i \in V} e^{-\frac{i}{\hbar} \pi S_i^z} |\Psi(\Theta, \Phi)\rangle, \quad (8)$$

and at $\Theta = \pi/2$, $\Phi = \Phi_0$ the derivatives can be conversely expressed in terms of the spin-flip states. The delocalized spin-flip states then span the product eigenspace because the eigenspace generated by all product eigenstates and the eigenspace generated by the derivatives at the angles $\Theta = \pi/2$ and $\Phi = \Phi_0$ are the same. This is because on the one hand every product state has a converging Taylor expansion in Θ and Φ [see Eq. (2) with $\vartheta_1 = \Theta$, $\varphi_1 = \Phi$ and the other angles according to the Kirchhoff orientation], and on the other hand, every derivative can be expressed as the limit of a differential quotient of product eigenstates.

Applications. We next discuss well-known systems without DMI and their product eigenstates in our graphical framework. In 1D, spin helices are well-studied product eigenstates. For vanishing DMI, a periodic spin chain with N sites has two Kirchhoff orientations if the periodic closure condition $\gamma N \equiv 0 \pmod{2\pi}$ [24,25] is fulfilled, corresponding

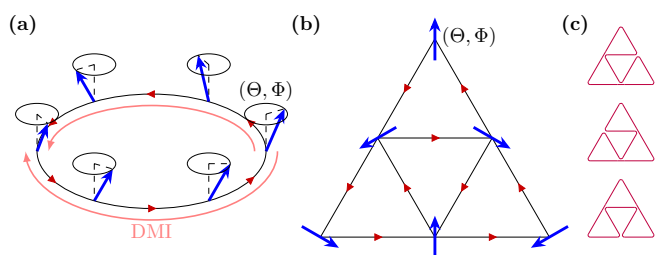


FIG. 1. Graphical solution to the product eigenstate problem, shown by a Heisenberg model on two graphs with spins at the vertices (blue arrows depict spin expectation values $\langle S \rangle = (\langle S^x \rangle, \langle S^y \rangle, \langle S^z \rangle)^T$) and edges denote the exchange interaction and the DMI, see Eq. (1). The change of the azimuthal angle along an edge can take the values $\kappa_{ij}\delta + \sigma_{ij}\gamma$. The sign σ_{ij} is indicated by the direction of the red arrows. (a) Spin chain with anisotropy and bias angles $\gamma = \pi/3$ and $\delta = \pi/6$. The signs of the DMI κ_{ij} are indicated by rose arrows giving bias angles of $+\delta$ (top) and $-\delta$ (bottom). The reference angles are $\Theta = \pi/3$ and $\Phi = \pi/6$. (b) Triangular lattice with $\gamma = 2\pi/3$ and $D = 0$ with a product eigenstate and the corresponding Kirchhoff orientation (red arrows) for reference angles $\Theta = \Phi = \pi/2$ (top view). (c) Euler circuit for the triangular lattice in (b).

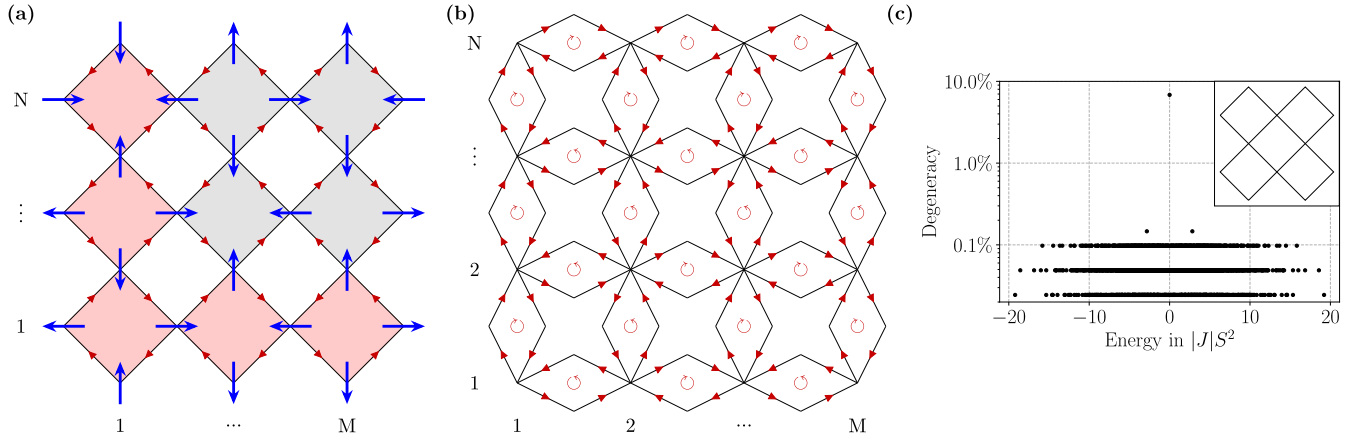


FIG. 2. (a) XX model ($\Delta = D = 0$) on a square lattice with a zigzag edge and its product eigenstates. Fixing $\Theta = \pi/2$, the spins align row- and columnwise in a Néel pattern determined by the boundary spins for the N rows and M columns resulting in 2^{N+M-1} Kirchhoff orientations. (b) The replacement of edges by “diamond” squares allows for a number of Kirchhoff orientations that grows exponentially in the number of diamonds with a lower bound of $2^{2NM+N+M}$ Kirchhoff orientations. (c) Degeneracy of the eigenspaces of the XX model on a square lattice, see inset, as a fraction of the total Hilbert space dimension. The eigenspace that accommodates the product eigenstates has energy zero commensurate with Eq. (4), and is two orders of magnitude more degenerate than typical eigenspaces. ($S = \hbar/2$).

to two helices with opposite helicity. In absence of DMI, inverting the direction of arrows in a Kirchhoff orientation generally yields another permissible orientation. Numerically, we find that the product eigenstates in the circular chain generate a degeneracy linear in the chain length N [61]. However, the product eigenstates do not span the entire ε -space for all choices of permissible γ and N . The triangular lattice depicted in Fig. 1(b) also allows for two different choices of Kirchhoff orientations independent of the system size whose product eigenstates are part of the ground space [36]. More generally for graphs made up of triangular motifs, the choice of a Kirchhoff orientation is equivalent to a three-coloring of the graph which has been used to study product eigenstates in such systems [36]. For the kagome lattice, the abundance of three-colorings yields an extensively degenerate ground space, i.e., for this family of systems, the degeneracy grows exponentially in the number of spins. We proceed with constructing novel lattices with macroscopic degeneracy. We first detail the square lattice with zigzag boundary conditions, see Fig. 2(a), where the number of Kirchhoff orientations increases exponentially with the size of the system’s boundary. To that end, we consider the XX model, i.e., $\Delta = D = 0$ in Eq. (1), corresponding to $\gamma = \pi/2$. We find that due to the zigzag boundary, the permissible Kirchhoff orientations form checkerboard patterns of vortices, cf. Fig. 2(a) where arrows go either clockwise or counterclockwise around the colored plaquettes. Specifying the M vortices at the bottom and the $N - 1$ vortices at the left boundary [rose in Fig. 2(a)] fully determines the inner ones (gray and white). Hence, the number of Kirchhoff orientations is 2^{N+M-1} , i.e., grows exponentially with the size of the boundary [61]. On the level of the individual spins, fixing the spin in the lower left corner to point along the x axis, the degrees of freedom are choosing the leftmost spins pointing either in the $(+x)$ or $(-x)$ direction and the bottom spins pointing either in the $(+y)$ or $(-y)$ direction. The remaining spins anti-align with their neighbors along each column and row of spins, see Fig. 2(a). We find numerically that the dimension of the space spanned

by product eigenstates is consistent with $(MN + 1)2^{N+M}$. In the special case of $N = 1$ or $M = 1$, the degeneracy scales exponentially with the system’s size, and is hence an extensive quantity. Such diamond chains have been studied among other reasons for their degeneracy in the context of exactly solvable systems [71,72] and spin ladders [73]. We find that systems with an extensively degenerate eigenspace of product eigenstates can be engineered by a suspension procedure similar to earlier ideas of building graphs from smaller motifs [36,38,74,75]. For a given graph, we replace each edge of the graph by a subgraph in a way that respects the Kirchhoff rules, e.g., for graphs with all circuits of even length like the square lattice, we substitute the edges by a single diamond square oriented clockwise or anticlockwise, see Fig. 2(b). The construction respects the vertex and the circuit rule for $\gamma = \pi/2$ and yields extensive degeneracy, exponential in the number of independently oriented diamonds. This results in a lower bound of $2^{2NM+N+M}$ in the case of the square lattice with N and M the number of rows and columns of the square lattice before the suspension, respectively. A variety of systems with an extensive number of Kirchhoff orientations can be constructed by removing arbitrary vertices and corresponding edges from the square lattice before suspending the diamonds.

Degeneracy & intrinsic structure of ε -spaces. In general, the degeneracy of the product eigenspace exceeds the number of possible Kirchhoff orientations. We analyze the ε -space for the square lattice and the diamond chain. We numerically confirm for $(N, M) = (1, 1)$, $(1, 2)$, $(1, 3)$, and $(2, 2)$ that the ε -space is the most degenerate eigenspace [61]. Typically the eigenspaces are one or two orders of magnitude more degenerate than all other eigenspaces, making up a large share of the total Hilbert space dimension, see Fig. 2(c). Additional degeneracy stems from states with nonvanishing entanglement entropy. In special cases in 1D, these have been associated with anyonic condensates [22] or spin liquid phases [36].

Discussion. We developed the complete classification scheme of product eigenstates of Heisenberg Hamiltonians

with XXZ exchange interactions and DMI by Kirchhoff rules for the underlying graph. We thereby give a concrete example for a fundamental property connecting graph topology, degeneracy, and entanglement. Some graphs feature associated Kirchhoff orientations in abundance. They possess large degenerate eigenspaces with zero-energy delocalized spin-flip excitations and additional nonproduct states associated to anyonic statistics in special systems. The degeneracy is reduced when moving away from the critical value of γ [61]. The singular nature of the degeneracy of the eigenspace accommodating the product eigenstates is therefore not explained by the natural symmetries of the model. The ε -space that appears as a condensate may be especially accessible by optically driving the system. There, the extensive degeneracy compensates the reduced transition rates caused by the energy difference which results in a macroscopic observable. In conjunction with a recent prediction of one-dimensional anyon condensates [23], our work suggests exotic characteristics of the related condensates in higher-dimensional systems, and experimental investigations could unveil exotic quasiparticles which are not accessible by analytical techniques for higher-dimensional systems. The considered graphs in general describe quantum spin systems of any dimension. As such, our study could be a step towards a partial solution of XXZ Heisenberg models

in more than one dimension for a critical value of γ , since we have shown that product eigenstates can be interpreted as spin helices along a one-dimensional Euler circuit. At special anisotropies, these paths may allow the application of ideas from well-established one-dimensional techniques like Jordan-Wigner transforms [14,15,76] and bosonization approaches [77] to higher-dimensional models.

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