

Wrapping and unwrapping multifractal fields: Application to fatigue and abrupt failure fracture surfaces

Samy Lakhal ^{1,2,3,*} Laurent Ponson ² Michael Benzaquen ^{1,3,4} and Jean-Philippe Bouchaud ^{3,4,5}

¹Chair of Econophysics and Complex Systems, *École polytechnique*, 91128 Palaiseau Cedex, France

²Institut Jean Le Rond d'Alembert, UMR CNRS 7190, Sorbonne Université, 75005 Paris, France

³LadHyX, CNRS, *École polytechnique*, Institut Polytechnique de Paris, 91120 Palaiseau, France

⁴Capital Fund Management, 23 Rue de l'Université, 75007 Paris, France

⁵Académie des Sciences, 23 Quai de Conti, 75006 Paris, France



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We develop a spectral procedure to generate multifractal random fields in any dimensions d , parameterized by their roughness exponent H , intermittency exponent λ , and multifractal range ξ . We retrieve all the salient features of the log-normal multifractal random walk introduced by E. Bacry *et al.* [*Phys. Rev. E* **64**, 026103 (2001)], namely, a quadratic scaling exponent spectrum ζ_q , and fat-tailed and spatially correlated fluctuations. Such construction procedure is then used in $d = 2$ to “unwrap” experimental fracture surfaces of an aluminum alloy obtained by abrupt and fatigue failure. Our analysis demonstrates the presence of anisotropic and filamental structures, ascribed to the dissipative nature of the damage mechanisms that underlie material failure.

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Whether in the oceans [1], sky [2–4], or mountains [5], scale invariance is everywhere around us. This feature, formalized and popularized by Mandelbrot under the concept of fractals [6–8], has reached a general audience who can recognize and appreciate its unfathomable beauty. For physicists, systems are said to be fractal when their observables display deterministic or statistical invariance under affine transformations of their space-time variables. For a fluctuating or stochastic field $h(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^d$, this property corresponds to the scale invariance of the absolute moments of its increments, which, for isotropic scaling, translates as the following variogram property:

$$V_q(\delta r) = \langle |h(\mathbf{r} + \delta \mathbf{r}) - h(\mathbf{r})|^q \rangle_{\delta r = \|\delta \mathbf{r}\|} \sim \delta r^{\zeta_q}, \quad (1)$$

where $\langle \cdot \rangle$ denotes the spatial or ensemble average, and ζ_q is a *scaling exponent spectrum*. A linear spectrum $\zeta_q = qH$ with Hurst exponent H corresponds to a monofractal scaling, usually found in linear systems. Instead, multifractals are characterized by a nonlinear spectrum $\zeta_q \neq qH$, reported in forced nonlinear systems [9,10], which may originate from a wide range of mechanisms such as self-organization [11], exotic fluctuations [12], or hierarchical interactions [13]. For the latter, a classic example may be of fluid turbulence [14–16], in which scale invariance originates from the cascading of energy from large eddies at the injection/forcing scale down to the mesoscopic dissipation scale.

Looking back, our ability to generate random monofractal objects [17–19] has allowed scientists to better understand their emergence in natural phenomena [5], and has led to many applications, for which a nonexhaustive list may include roughness characterization methods [20,21], simulations of frictions [22], deformations [23] or flows [24,25] on fractal landscapes, and even in the training of learning-based image-recognition algorithms [26–28]. The wide diversity of phenomena displaying scaling invariance makes fractal generation algorithms an essential tool for physicists, but also for graphics engineers [29,30] and artists; see, e.g., the self-similarity structures of Pollock's paintings or of traditional middle-eastern ornaments [31,32].

Yet, while synthetic multifractals too could benefit from various applications, generation methods are scarce and generally limited to one-dimensional data (see [21] and references therein). Indeed, sampling methods for multifractal random fields in $d > 1$ suffer from several limitations, related to the symmetry [33,34] of their statistics and the discreteness of their scale-invariance properties [35,36]. In this Letter, we provide a powerful, yet simple method for the generation of multifractal random fields in any dimensions (see Fig. 1 for an example in $d = 2$). Our model builds on the work of Bacry *et al.* [37], who defined a multifractal process with continuous dilation invariance properties, and that we generalize to higher dimensions by using fractional operators.

In practice, as described in the first part of our Letter, our method consists in passing a log-correlated Gaussian field $\omega(\mathbf{r})$ through three distinct steps of (i) exponentiation, (ii) symmetrization, and (iii) fractional integration, resulting in a multifractal field $h(\mathbf{r})$ with fully controlled scaling properties. This construction allows for the prediction of the density probability of increments, at any scale within the multifractal regime ($\delta r < \xi$), and from only two scaling parameters

*Contact author: samy.lakhal@ladhyx.polytechnique.fr

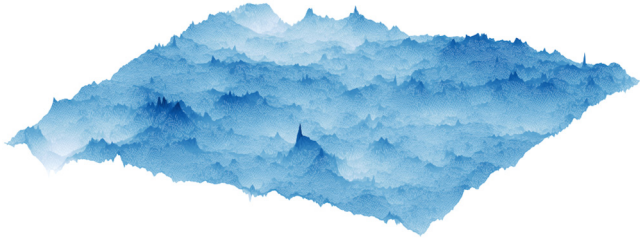


FIG. 1. Topographical representation of a synthetic multifractal field in dimension $d = 2$.

H and λ that we define later. By applying our generation method in reverse, we define a scale-free transformation which extracts—*unwraps*—the local amplitude of fluctuations $\hat{\omega}(\mathbf{r})$ also called *volatility*, and of which correlations tune the multiscaling properties of $h(\mathbf{r})$.

In the second part of our Letter, we apply this methodology to experimental fracture surfaces formed by the fatigue and abrupt failure of a metallic alloy. Both surfaces display nearly identical multifractal features H and λ . Yet, the upper range ξ of that multifractal regime varies with the failure mode. We characterize the spatial organization of strongest volatility patterns and reveal filamentary structures which constitute the fingerprint of the most fluctuating—and expectedly most dissipative—events ruling crack growth mechanisms.

Monofractal fields. Monofractal Gaussian fields can be defined from the application of fractional operators to white Gaussian noise. In dimension $d = 1$, these operators result from classic integration and derivation [38]. For $d > 1$, one may enforce isotropy and translation invariance by introducing the fractional Laplacian $(-\Delta)^\alpha$ [39] and define a monofractal field $\omega(\mathbf{r})$ ($\mathbf{r} \in \mathbb{R}^d$) from

$$\omega(\mathbf{r}) = (-\Delta)^{-\frac{H+d/2}{2}} \eta(\mathbf{r}), \quad (2)$$

where η is a d -dimensional white Gaussian noise. In Fourier space, this operation amounts to using the filter $G(\mathbf{k}) = 1/k^{H+d/2}$ ($k = \|\mathbf{k}\|$), which possesses scale and rotation invariance [40]. We may distinguish two main families of power-law correlated fields. For $H \in [-d/2, 0]$, $\omega(\mathbf{r})$ is a zero mean stationary Gaussian field with decaying power-law correlations $C_\omega(\delta\mathbf{r}) = \langle \omega(\mathbf{r} + \delta\mathbf{r})\omega(\mathbf{r}) \rangle \propto \delta r^{2H}$. For $H \in]0, 1]$, $\omega(\mathbf{r})$ is a fractional Gaussian field [41,42] of monofractal scaling $V_q(\delta\mathbf{r}) \propto \delta r^{qH}$. The particular case $H = 0$, sitting at the transition point between the two cases mentioned above, corresponds to system size and regularization-dependent logarithmic scaling.

In order to mimic natural fractals that only expand over a finite range of length scales, we introduce an upper cutoff ξ using the modified operator $(-\Delta) \rightarrow (\xi^{-2} - \Delta)$ that dampens long-range correlations. In Fourier space, this translates as $k^2 \rightarrow (2\pi/\xi)^2 + k^2$, which prevents the power-law scaling of the low-frequency regime. Simple calculations [43] provide the correlations of ω , $C_\omega(\delta\mathbf{r}) \propto \delta r^H K_H(\delta r/\xi)$, where K_H is the modified Bessel function of the second kind with parameter H . We retrieve the classic Whittle-Matérn correlations [44] which behave similarly to unregularized fields at short scales $\delta r \ll \xi$, and decay exponentially such that $C_\omega(\delta\mathbf{r}) \propto e^{-\delta r/\xi}$ for $\delta r \gg \xi$. Taking $H = 0$ yields $C_\omega(\delta\mathbf{r}) \propto -\ln(\delta r/\xi) + k$

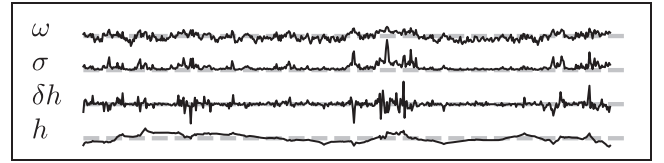


FIG. 2. Construction of a log-normal multifractal signal, illustrated in dimension $d = 1$. From top to bottom, (i) a log-correlated ($H = 0$) field ω is exponentiated into $\sigma = e^\omega$, (ii) symmetrized into $\delta h = s \cdot \sigma$ and (iii) fractionally integrated into h . Horizontal lines indicate the origins $y = 0$ of each curve.

$\delta r \ll \xi$, with $k = \gamma - \ln 2 \approx .07$ a small shift correction to the asymptotic scaling of $K_0(u)$, and $\ln(\cdot)$ the natural logarithm function. We observe the effect of ξ on the visual and statistical properties of $H = 0$ fields in the Supplemental Material [45].

Synthetic multifractal fields. We now describe the construction of multifractal fields, consisting of the three steps shown in Fig. 2. First, one builds non-Gaussian fluctuations by taking the exponential of a Gaussian field ω . The resulting field $\sigma = e^\omega$ displays log-normal statistics and amplifies the largest fluctuations of ω while tempering the lowest ones; see Fig 2. Such signals, often referred to as crackling noise, are observed in elastic models driven in disordered media, where the largest fluctuations organize in bursts called avalanches [46,47]. The correlations of σ can be directly computed from the characteristic functions of multivariate Gaussian distributions. In particular, taking $H = 0$ with $C_\omega(\delta\mathbf{r}) \approx -\lambda \ln \delta r/\xi$ for $\delta r < \xi$, one gets

$$\langle \sigma(\mathbf{r}_1) \dots \sigma(\mathbf{r}_q) \rangle = \prod_{1 \leq i < j \leq q} \|\mathbf{r}_i - \mathbf{r}_j\|^{-\lambda}, \quad (3)$$

where $\mathbf{r}_1, \dots, \mathbf{r}_q \in \mathbb{R}^d$, and $\|\mathbf{r}_i - \mathbf{r}_j\| < \xi$ for all (i, j) . Here, the *intermittency coefficient* λ and integral range ξ control the strength and the spatial extension of the bursts of σ , respectively. The resulting process is a *log-normal continuous cascade* [48], and is scale invariant since applying $\mathbf{r}_i \rightarrow \gamma \mathbf{r}_i$ multiplies Eq. (3) by $\gamma^{-\lambda q(q-1)/2}$. We will see that this quadratic scaling in q directly influences the shape of ζ_q . However, we note that the statistics of σ are skewed and do not reproduce the quasisymmetry observed in experimental data, e.g., turbulence velocity increments [49,50]. The second step, introduced by Bacry *et al.* for the log-normal multifractal random walk (MRW) [37], consists in symmetrizing the non-Gaussian fluctuations by multiplying them by a zero-mean white Gaussian noise $s(\mathbf{r})$. The resulting field $\delta h = s \cdot \sigma$ possesses symmetrical statistics, where σ defines an intermittent volatility envelope in which δh fluctuates; see Fig. 2. The third and last step consists in integrating δh . For $d = 1$, δh exactly defines the increments of the MRW and an integration retrieves a multifractal signal. For $d > 1$, we sample multifractal fields by fractionally integrating δh , into $h := (-\Delta)^{-\frac{H+d/2}{2}} \delta h$, using Eq. (2). Similarly to the construction proposed in Ref. [51], these fields display an asymptotic ($\delta r \ll \xi$) multifractal scaling of the exponent spectrum,

$$\zeta_q = qH - \frac{\lambda}{2} q(q-2), \quad (4)$$

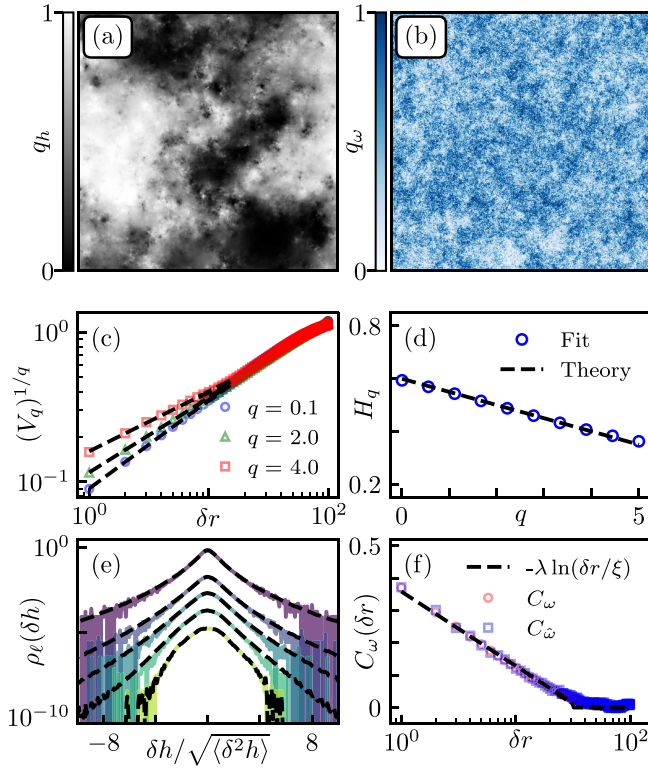


FIG. 3. Analysis of a synthetic multifractal field, with dimension $d = 2$, size 512×512 , and parameters $(H, \lambda, \xi) = (0.5, 0.1, 32)$. (a) $h(\mathbf{r})$ and (b) $\omega(\mathbf{r})$ fields in quantile distribution representation, with lowest and highest field values corresponding to (0,1) in color bars. (c) $(V_q)^{1/q}$ vs δr , with power-law fits. (d) $H_q = \zeta_q/q$ vs q , fitted from variograms, with theory from Eq. (4). (e) p.d.f.'s ρ_ℓ vs δh , for $\ell/\xi = 1/32, 1/16, 1/8, 1/4, 1/2$, and 1 (top to bottom, shifted for visibility). Theoretical predictions from Eq. (6) in black, with reference scale $L = \xi$. (f) Spatial correlations C_ω and C_h , with theoretical curve in black.

a result that we derive in the Supplemental Material [45]. When $d = 1$, taking $H = 1/2$ yields the classic log-normal MRW scaling. For $\delta r > \xi$, the correlations of ω vanish and the scaling of increments transitions towards monofractality ($\zeta_q = qH$). Note that introducing a second cutoff ξ_h in the last integration step leads to a saturation of the variograms V_q for $\delta r > \xi_h$, as observed in numerical and experimental data [9,10,52].

Intermittency characterization. We verify the effectiveness of our method by generating surfaces of size 512×512 pixels, as the one shown in Fig. 3(a) for $H = 0.5$, $\lambda = 0.1$, and $\xi = 32$. We first check its multifractal properties: in Fig. 3(c), we analyze the multifractal scaling of h by computing the power-law exponents of the rescaled variograms $(V_q)^{1/q}$ in the multifractal regime $\delta r < \xi$. We observe in Fig. 3(d) that the generalized Hurst exponents $H_q = \zeta_q/q$ follow the linear behavior $H_q = \zeta_q/q = H - \lambda(q-2)/2$ expected from Eq. (4). Note that the multi-affine to mono-affine crossover ξ is evidenced from the collapse of the rescaled variograms $(V_q)^{1/q}$ for $\delta r > \xi$.

As shown in the following and as previously reported in Refs. [53,54], the analysis of h only may be insufficient to ascertain multifractal properties. This difficulty can be

circumvented by directly studying the field ω previously introduced. For the MRW, λ and ξ can be measured from the local log-volatility field $\hat{\omega}_\epsilon = \ln |\delta_\epsilon h|$, also called *magnitude* [55–57] or log-dissipation rate in turbulence. However, for $H \neq 1/2$, that estimation may be biased by the long-range correlations of increments. An idea is then to suppress such contributions by applying a scale-free fractional differentiation [33,34] of h , hence defining

$$\hat{\omega}(\mathbf{r}) = \ln |(-\Delta)^{\frac{H+d/2}{2}} h(\mathbf{r})|. \quad (5)$$

This differs from other definitions [58,59], which relied on scale-dependent transformations of h . Our definition exactly reverses the construction steps illustrated in Fig. 2, by fully *unwrapping* the multifractal field h . This log-volatility field—shown in Fig. 3(b)—displays localized structures, the high values of which correspond to strongly fluctuating parts of h . The fit of its correlations in Fig. 3(f) retrieves $\lambda = 0.1$ and $\xi = 32$, perfectly matching the prescribed parameters of the initial ω . In order to assess the versatility of our method, we generate a wide variety of multifractal fields of size 512×512 with prescribed properties in the range $0 \leq \lambda \leq 0.5$ and $0 \leq H \leq 1$ for a fixed cutoff length $\xi = 30$ in [45]. The parameter values λ and H are then measured either from the volatility field $\hat{\omega}(\mathbf{r})$ or the height field $h(\mathbf{r})$. We find a good agreement with the prescribed values, especially when the $\hat{\omega}$ field is used, suggesting that the study of the spatial correlations of the volatility field is a more direct and accurate way to characterize multifractal fields.

The multifractal scaling directly implies that increments $\delta_\ell h = h(\mathbf{r} + \ell) - h(\mathbf{r})$ are linked together through a cascading rule, defined as the ratio of increments $W_{\ell/L} = \delta_\ell h / \delta_L h$, and such that $\langle W_{\ell/L}^q \rangle = (\ell/L)^{\zeta_q}$. Here, the cascading rule exclusively depends on the scale ratios, hence describing a scale-invariant cascade. As proposed by Castaing *et al.* [60], this relation can be used to link the probability density functions (p.d.f.'s) of increments $\rho_\ell(\delta h)$ through the following dilation invariance:

$$\rho_\ell(\delta h) = \int G_{\ell/L}(u) e^{-u} \rho_L(e^{-u} \delta h) du. \quad (6)$$

The kernel $G_{\ell/L}(u)$ is the Gaussian p.d.f. of $\ln W_{\ell/L}$, which depends on ζ_q and thus on H and λ only. Its analytical expression is provided in [45]. In practice, the statistics of fluctuations is computed at the scale $L = \xi$ from which the statistics at smaller scales is inferred using Eq. (6). This construction provides an alternative characterization of the multifractal behavior of h , as it captures quantitatively the ever stronger departure from Gaussianity as we go deeper into the multifractal regime. In Fig. 3(e), we compute the distributions $\rho_\ell(\delta h)$ on our synthetic surfaces. The transition from fat to Gaussian tail as ℓ increases is very well captured by Eq. (6).

Application to fracture surfaces. We now focus on the application of these concepts to experimental fracture surfaces. The topography of surfaces formed by the failure of materials are archetypes of multifractal fields [58,61–63]. Yet the physical mechanisms underlying their formation is a matter of debate [64–66]. Here, we study the fracture surface of an aluminum alloy of grade 7075 that is commonly used for aeronautic applications. We consider two different failure modes, namely, an abrupt failure resulting from

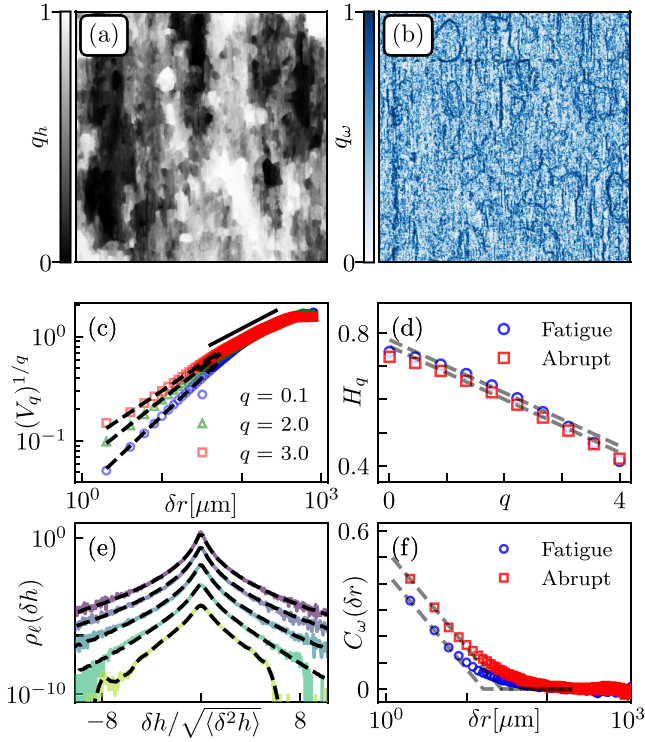


FIG. 4. Multifractal analysis of a fractured metallic alloy, obtained from abrupt failure (unless indicated otherwise). (a) Height map $h(\mathbf{r})$, captured over a surface $2 \times 2 \text{ mm}^2$ with $2 \mu\text{m}/\text{pixel}$. (b) $\hat{\omega}(\mathbf{r})$, computed from Eq. (5). Both (a) and (b) are displayed in quantile distribution values (see Fig. 3). (c) $(V_q)^{1/q}$ vs δr , with power-law fits. The black line of equation $y \propto \delta r^{H_{\text{mono}}}$ ($H_{\text{mono}} = 0.45$) indicates the monofractal antipersistent regime observed for $\delta r > \xi$. (d) $H_q = \zeta_q/q$ vs q , fitted from variograms analysis. The theory from Eq. (4), in black dotted lines, uses the values from Table I. (e) ρ_ℓ vs δh , with $\ell/\xi = 1/16, 1/8, 1/4, 1/2$, and 1. Theoretical prediction from Eq. (6) in black, with reference scale $L = \xi$. (f) Correlations C_ω , with fitted logarithmic asymptote values reported in Table I.

monotonically increasing load and fatigue failure resulting from cycling loading. For both cases, a prenotched compact tension specimen of size $75 \times 72 \times 30 \text{ mm}^3$ as specified by the norm ASTM E1820 [67] is used for the tests. Abrupt failure takes place when the load reaches the toughness $K_c \simeq 32 \text{ MPa m}^{1/2}$ of the material while fatigue crack growth is carried using a constant load amplitude $\Delta K \simeq 12 \text{ MPa m}^{1/2}$ with a load ratio $R = 0.1$. In the following, both types of fracture surfaces are studied separately.

The methodology previously developed on synthetic fields is employed in Fig. 4 for abrupt failure, while its application to fatigue failure is detailed in [45]. Both types of fracture

TABLE I. Multifractal parameters for the surfaces of identical alloys. H is fitted from V_2 , and (λ, ξ) from C_ω . Error intervals are computed from analysis of submaps.

Failure mode	H	λ	ξ
Abrupt	$0.63 \pm .05$	$0.15 \pm .02$	$30 \pm 3 \mu\text{m}$
Fatigue	$0.60 \pm .07$	$0.15 \pm .03$	$14 \pm 2 \mu\text{m}$

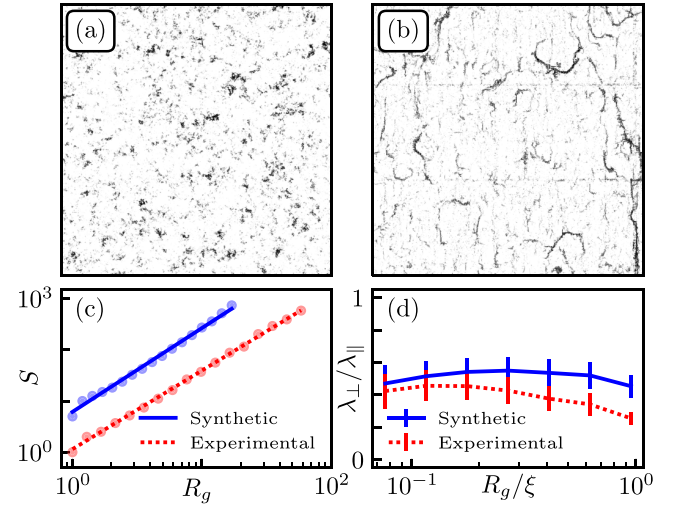


FIG. 5. Cluster analysis of ω . (a) and (b) $\hat{\omega}$, computed from artificial and abrupt failure surfaces, thresholded at 10% of their largest values. Larger clusters are displayed in darker colors. (c) S vs R_g , fitted by power-law exponents $D = 1.65 \pm .03$ and $1.53 \pm .02$ for synthetic and experimental clusters, respectively. (d) $\lambda_{||}/\lambda_{\perp}$ vs R_g/ξ , with error bars corresponding to standard deviations.

surface display the salient features of the MRW: linearly decaying exponents $H_q = H - \lambda(q - 2)/2$, dilation invariance of increments p.d.f. $\rho_\ell(\delta h)$, and log-correlated volatility $\hat{\omega}$. The multifractal parameters, fitted from V_2 for H and C_ω for (λ, ξ) , are reported in Table I. H and λ remain nearly identical, suggesting that the robustness of the multifractal properties of fracture surfaces with respect to the material's type evidenced in Ref. [58] extends to the type of failure. In contrast, the upper bound ξ of the multifractal domain differs by a factor 2 between brutal and fatigue failure. This observation is in line with the interpretation that ξ provides a characteristic size of the damage processes taking place at the tip of the growing crack. Indeed, under fatigue, crack growth resistance is lower, implying a lower process zone size [68], hence leading to $\xi_{\text{fatigue}} < \xi_{\text{abrupt}}$.

We now would like to focus on the deviations to the synthetic multifractal fields. First, we do observe height correlations at scales $\delta r > \xi$, but with a different exponent $H_{\text{mono}} \simeq 0.45$. This regime, characterized by Gaussian fluctuations [65,69,70] and described in the Supplemental Material [45], is reminiscent of the long-range elastic interactions that prevail along the crack front at scales $\delta r > \xi$ [71,72]. Dominated by elastic restoring forces, it gives rise to antipersistent ($H < 1/2$) mono-affine surfaces, a behavior that is captured by linear elastic fracture mechanics [73–75].

Another striking difference is the very different spatial organization of the most intermittent clusters identified from the field $\hat{\omega}$ in Fig. 5, a property that we investigate in more detail below. The 10% largest values of $\hat{\omega}$ organize in filamentary structures across fracture surfaces, whereas they look more compact for the synthetic fields shown in Fig. 5(a). The variations of the surface S of these clusters with their gyration radius R_g are shown in Fig. 5(c). The scaling relation $S = R_g^D$ [58] provides the fractal dimension D of clusters, which is found to be slightly smaller for fracture surfaces than

for the synthetic field that displays more compact patterns. We then go one step further in Fig. 5(d) by comparing the clusters' aspect ratio, defined as the ratio $\lambda_{\perp}/\lambda_{\parallel} < 1$ of the two eigenvalues of their inertia tensor. The lower aspect ratio of experimental clusters reveals their filamentary structure, especially for large R_g . Such patterns, reminiscent of the dissipative structures observed in fluid turbulence, constitute the fingerprint of the mechanisms underlying crack propagation and fracture surfaces formation. Such unique geometrical features put strong constraints on the competing models of crack growth previously proposed [76,77] and currently in development.

Conclusion and discussion. We now summarize the main steps of our construction and discuss our results and their implications in the context of fracture. We first generated monofractal Gaussian fields of Hurst exponent H and fractal range ξ , from which we built log-normal cascades by exponentiation in the case $H = 0$. This constitutive brick was then used to compute symmetric non-Gaussian fluctuations δh whose fractional integration generated log-normal multifractal fields. Our method produces isotropic statistics, but can be generalized to anisotropic Family-Vicsek scaling [78], allowing one to capture the two-dimensional properties of experimental fracture surfaces [70]; see the Supplemental Material [45].

We believe an important contribution is the idea of applying our methodology “backwards” such as to *unwrap* experimental fields and identify the organization of their intermittent features. Applied to the topographic map of a fractured metallic alloy, we highlighted some fundamental differences with the standard properties of synthetic fields, such as the topology of the largest intermittent bursts [see Fig. 5(d)]. This

last property echoes the patterns observed in turbulence, for which the cascade of kinetic energy up to the Kolmogorov scale leads to the emergence of coherent dissipative structures [79,80]. It suggests that the failure of metallic alloys may result from the *cooperative* evolution of the plastic cavities observed at the crack tip vicinity [81], their coalescence culminating in the formation of clifflike filamentary structures observed on the fracture surfaces. We expect the size of these structures to be related to the dissipation scale of the failure processes, and therefore to the crack growth of materials.

This last conjecture is supported by the comparison between the surface of the same metallic alloy failed in fatigue at low load level and abruptly at a higher load. Both displayed nearly identical multifractal properties, yet different values of the fractographic lengths ξ , thus revealing different dissipative length scales. A next step could consist in modeling the collective growth and coalescence of damage cavities, through discrete [82] or coagulation [83,84] based models, or from the description of damage organization in disordered materials [85]. Finally, our study implies that the topological description of intermittent structures is a promising way to investigate the emergence of multiscaling regimes in various systems, beyond material failure and turbulence.

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