

Maxwell origami tube

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Maxwell lattices are periodic frameworks characterized by a balance between the number of kinematic variables and constraints in each unit cell, attracting attention as a source of topological mechanical metamaterials. In particular, one-dimensional (1D) Maxwell lattices maintain a constant number of degrees of freedom (DOFs) as the number of unit cells increases, offering advantages in the design and control of their kinematics. Here, we construct a 1D Maxwell lattice with tunable DOFs, termed the Maxwell origami tube, by closing a triangulated origami tessellation, which is a 2D Maxwell lattice. In topological mechanics, the infinitesimal deformation modes of uniformly configured Maxwell lattices are classified into edge and bulk modes. Unlike conventional 1-DOF 1D Maxwell lattices, multi-DOF 1D Maxwell lattices exhibit a mixture of DOFs corresponding to edge and bulk modes, which we term eDOFs and bDOFs. This paper investigates how the eDOFs and bDOFs of the Maxwell origami tube depend on the crease patterns and folded states using a discrete dynamical system model. We find that ground states with zero and nonzero bDOFs can coexist within a single crease pattern. Additionally, in states with nonzero bDOF, the ratio of bDOF to total DOF decreases as the total DOF increases. In contrast, we present another origami tube with a constant DOF of 2, which represents the first overconstrained lattice to exhibit nonuniform bulk modes. This work highlights the versatility of Maxwell lattices and provides a theoretical foundation for designing novel mechanical metamaterials.

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I. INTRODUCTION

Mechanical metamaterials are the repetitive assemblies of a simple geometric structure with unconventional mechanical functionalities, which have attracted much attention over the past decade [1–5]. One representative class of mechanical metamaterials is reconfigurable metamaterials that can configure their shape smoothly on demand [6,7]. In principle, the reconfigurability comes from the kinematics of underlying periodic frameworks consisting of unit cells. Kinematics concerns the possible *configurations* of vertices connected with edges with prescribed lengths in k -dimensional Euclidean space $\mathbb{E}^k$ [8]. Hence, adding one vertex and edge to a framework induces new k variables, and one constraint, respectively.

Maxwell lattices, also called locally isostatic lattices, are periodic frameworks that have the singular topology characterized by the balancing between the number of variables and constraints in each unit cell, i.e., $k \times \#(\text{vertices in a unit cell}) = \#(\text{edges in a unit cell})$ [9–12]. The class includes many periodic frameworks that have led the metamaterial designs; for example, sheets of triangulated periodic origami patterns embedded in $\mathbb{E}^3$ [13–15] and corner-connected tiles embedded in $\mathbb{E}^2$ [16–19] are 2-dimensional

(2D) Maxwell lattices having two lattice directions. From the definition, the *degrees of freedom (DOFs)* of Maxwell lattices only come from boundary vertices; in other words, fixing the configurations of boundary unit cells rigidifies the entire lattice. In particular, 1D Maxwell lattices with only one lattice direction have a remarkable property: their DOFs remain constant to the DOFs of their single unit cell even if we increase unit cells infinitely. The invariance of DOFs under the increase of unit cells is the common property with previous metamaterials [20–25] which is desirable in predicting or controlling their kinematics. However, conventional metamaterials are based on a different design approach. They assume symmetry that the entire configuration is the repetition of a single unit cell configuration. With this conventional approach, we can realize only uniform deformation where all the unit cells deform identically. On the other hand, since Maxwell lattices do not assume uniform deformation, they serve as the underlying framework of metamaterials that can exhibit nonuniform deformations.

Topological mechanics, which has developed inspired by quantum physics [11,12], offers basic concepts in the kinematics of Maxwell lattices. Consider the *infinitesimal deformation mode* that preserves edge lengths up to first order (also known as a zero mode or floppy mode) of a uniformly configured Maxwell lattice. Based on Bloch's theorem, an infinitesimal deformation mode can be classified into an *edge mode* or a *bulk mode*. The edge and bulk modes are characterized by the localization of displacements around boundary unit cells and the distributed displacements over all unit cells, respectively. Looking at 1D Maxwell lattices, representative instances are the strip of a degree-4 single vertex origami [26,27] and the linkage imitating the Su-Schrieffer-Heeger model [27–29].

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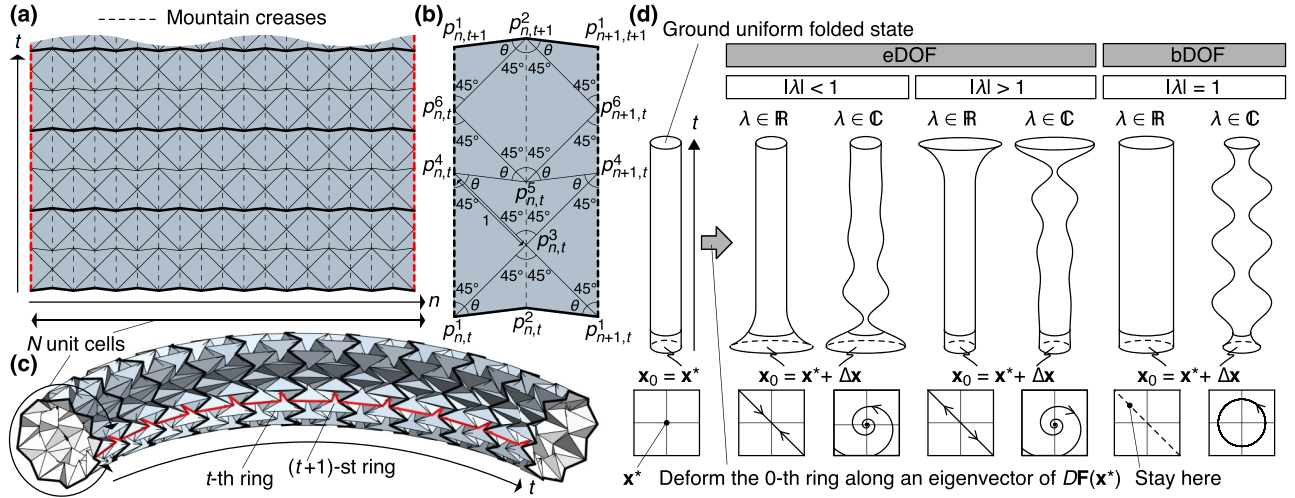


FIG. 1. (a) Crease pattern of the waterbomb tessellation and (b) its (n, t) th unit cell ($t \in \mathbb{Z}_{\geq 0}$, $n \in \{0, \dots, N-1\}$) where n and t refer to the horizontal and vertical direction, respectively. The unit cell is mirror symmetric to the central creases. The mountain creases are indicated by dashed lines. To form a tubular shape, we must fold the crease pattern in a way that the red boundary creases coincide. (c) Folded state of the waterbomb tube. The indices t of the rings along the longitude serve as the “time” variable of the discrete dynamical system. (d) Schematics of the infinitesimal deformation from a ground uniform folded state represented by a fixed point $\mathbf{x}^*$. The deformation is along the vector $\Delta \mathbf{x}$ which is an element of the space spanned by some eigenvectors of $D\mathbf{F}(\mathbf{x}^*)$. The bottom diagrams illustrate typical phase diagrams of $\Delta \mathbf{x}$ by $D\mathbf{F}(\mathbf{x}^*)$. In the diagrams for edge modes, the diagrams exhibit the convergence or the divergence to the fixed point while the diagrams corresponding to bulk modes neither converge nor diverge (see Supplemental Material [30] for an actual phase diagram example).

Since both are 1-DOF mechanisms, they must have a single edge mode or bulk mode at each uniform configuration, and previous works have mainly focused on their edge modes. On the other hand, 1D Maxwell lattices having multiple DOFs have been little investigated. Unlike 1-DOF lattices, in multi-DOF lattices, some of the total DOF is spanned by edge modes while bulk modes span the others. Here, we name the DOF corresponding to the edge (resp. bulk) modes the *eDOF* (resp. *bDOF*). Clarifying how many of the total DOF are eDOF/bDOF and how it depends on ground uniform configurations are crucial problems in capturing the complex nature of multi-DOF 1D Maxwell lattices.

Here, we construct a 1D Maxwell lattice with tunable DOFs by stitching the boundaries of a sheet of triangular origami tessellation that is a 2D Maxwell lattice. From the construction, the DOFs that come from the stitched boundaries vanish, and the resulting tubular origami tessellation becomes a 1D Maxwell lattice having a single lattice direction along its longitude. Furthermore, we can increase the DOFs of the tubular origami by increasing the unit cells of the original origami tessellation in the hoop direction. We name the 1D Maxwell lattice constructed in this way the *Maxwell origami tube*. Specifically, we focus on the waterbomb tube obtained from the waterbomb tessellation [Figs. 1(a)–1(c)]. Although the waterbomb tube or the waterbomb tessellation is instances of the most studied origami pattern [31–37], no work considers the general kinematics of the waterbomb tube without any symmetry assumptions. In this paper, we capture the complex nature of the waterbomb tube as a multi-DOF 1D Maxwell lattice using a discrete dynamical system model [37–42]. Here, the proposed dynamical system model does not refer to the dynamics of the structure but is entirely grounded in geometry. Instead of addressing the state evolution over time, our model formalizes the geometric and kinematic couplings of unit cells

in a 1D lattice, describing their configuration changes with each cell repetition.

This paper is organized as follows. First, we define a multidimensional dynamical system representing the coupling folding motion of adjacent rings. Then, we define the eDOF and bDOF at a uniform folded state based on the stability analysis of a fixed point of the dynamical system. We numerically calculate the eDOFs/bDOFs for different ground folded states in the variation of crease pattern. Focusing on the bulk mode, we visualize finite deformation modes of the waterbomb tube tangent to a bulk mode. Furthermore, we show the 2-DOF “extruded” waterbomb tube as the first overconstrained origami structure with a nonuniform bulk mode, which is in contrast to the extensively studied overconstrained quadrilateral origami [43–47] represented by the Miura-Ori or the corner-connected quadrilaterals [16,19,48] which have a single uniform bulk mode.

II. MATHEMATICAL FORMULATION

A. Dynamical system model

We obtain the waterbomb tube by closing the boundaries of the waterbomb tessellation. The waterbomb tessellation is a 2D Maxwell lattice whose unit cell is indexed by a pair of the two non-negative integers $(n, t) \in \mathbb{Z}_{\geq 0}^2$ and satisfies $3 \times \#[\text{vertices in the } (n, t)\text{th cell}] - \#[\text{edges in the } (n, t)\text{th cell}] = 18 - 18 = 0$ [Figs. 1(a) and 1(b)]. In this paper, we vary the metric of the unit cell by changing the sector angle $\theta \in (0, 135^\circ)$ between creases. Then, we prescribe the number of unit cells along the n -direction by $N \in \mathbb{Z}_{\geq 2}$ and stitch the boundaries of the $(0, t)$ th and $(N-1, t)$ th unit cells to form a tubular shape [Fig. 1(c)]. As a result, the N unit cells represented by the pairs $[(n, t)]_{n \in \{0, \dots, N-1\}}$ form the t th ring indexed

along the longitudinal direction of the tube. The obtained waterbomb tube is a 1D Maxwell lattice regarding the ring as a unit cell, where $3 \times \#(\text{vertices in the } t\text{th ring}) - \#(\text{edges in the } t\text{th ring})$ is equal to $18N - 18N = 0$. Here, for a given crease pattern specified by θ and N , we consider the rigid-origami kinematics of the waterbomb tube, i.e., the isometric deformation. Let x_t write for the folded state of the t th ring; then, we can represent the folded state of the waterbomb tube as the sequence $(x_t)_{t \in \mathbb{Z}_{\geq 0}}$. Note that the folded state x_t is an element of the *moduli space* $\mathcal{M}$ defined as the quotient space of the *configuration space* $\mathcal{C}$ by the special Euclidean group $\text{SE}(3)$:

$$\mathcal{M} := \mathcal{C}/\text{SE}(3),$$

$$\mathcal{C} := \left\{ \left((\mathbf{p}_{n,t}^i)_{i \in \{1, \dots, 6\}}, (\mathbf{p}_{n,t+1}^i)_{i \in \{1, 2\}} \right)_{n \in \{0, \dots, N-1\}} \in \mathbb{R}^{24N} \mid \right. \\ \|\mathbf{p}_{n,t}^1 - \mathbf{p}_{n,t}^2\| = \|\mathbf{p}_{n+1,t}^1 - \mathbf{p}_{n,t}^2\| = \frac{1}{\sin \theta + \cos \theta}, \\ \|\mathbf{p}_{n,t}^1 - \mathbf{p}_{n,t}^3\| = \|\mathbf{p}_{n+1,t}^1 - \mathbf{p}_{n,t}^3\| = 1, \\ \|\mathbf{p}_{n,t}^2 - \mathbf{p}_{n,t}^3\| = \|\mathbf{p}_{n,t}^5 - \mathbf{p}_{n,t}^3\| = \frac{\sqrt{2} \sin \theta}{\sin \theta + \cos \theta}, \\ \|\mathbf{p}_{n,t}^4 - \mathbf{p}_{n,t}^1\| = \|\mathbf{p}_{n,t}^5 - \mathbf{p}_{n,t+1}^2\| = \sqrt{2}, \\ \|\mathbf{p}_{n,t}^4 - \mathbf{p}_{n,t}^5\| = \|\mathbf{p}_{n+1,t}^4 - \mathbf{p}_{n,t}^5\| = \frac{1}{\sin \theta + \cos \theta}, \\ \|\mathbf{p}_{n,t}^4 - \mathbf{p}_{n,t}^3\| = \|\mathbf{p}_{n+1,t}^4 - \mathbf{p}_{n,t}^3\| = 1, \\ \|\mathbf{p}_{n,t}^6 - \mathbf{p}_{n,t}^5\| = \|\mathbf{p}_{n+1,t}^6 - \mathbf{p}_{n,t}^5\| = 1, \\ \|\mathbf{p}_{n,t}^6 - \mathbf{p}_{n,t+1}^2\| = \|\mathbf{p}_{n,t+1}^2 - \mathbf{p}_{n,t}^5\| = 1, \\ \|\mathbf{p}_{n,t}^4 - \mathbf{p}_{n,t}^6\| = \|\mathbf{p}_{n,t+1}^1 - \mathbf{p}_{n,t}^6\| = \frac{\sqrt{2} \sin \theta}{\sin \theta + \cos \theta}, \\ \|\mathbf{p}_{n,t+1}^1 - \mathbf{p}_{n,t+1}^2\| = \|\mathbf{p}_{n+1,t+1}^1 - \mathbf{p}_{n,t+1}^2\| = \frac{1}{\sin \theta + \cos \theta}, \\ \left. n \equiv 0, \dots, N-1 \pmod{N} \right\}. \quad (1)$$

In $\mathcal{M}$, we identify the configurations in $\mathcal{C}$ mapped to each other by a rigid transformation. The dimension of $\mathcal{M}$ that gives the DOF of a single ring, which is equal to the DOF of the entire tube, is $24N - 20N - \dim[\text{SE}(3)] = 4N - 6$; therefore, we can tune the DOF of the waterbomb tube by changing N .

There is a relationship between x_t and x_{t+1} , which we formulate as a discrete dynamical system. Since the waterbomb tube is a 1D Maxwell lattice, determining x_t generically constrains x_{t+1} on a finite number of points in $\mathcal{M}$; hence, the adjacent folded state is generally not unique under only the edge-length preservation that is the intrinsic constraint. Here, we give the extrinsic constraint called the *mountain-valley assignment* which constrains the folding direction by prescribing the sign of the *fold angle* defined by the exterior dihedral angle of creases. Specifically, we assign the mountain labels to the vertical creases in the crease pattern, with which we can uniquely select one from the possible folded states if they exist. Here, the *discrete dynamical system* of the waterbomb

tube is defined as follows:

$$F : \mathcal{M} \rightarrow \mathcal{M}, \quad x_t \mapsto x_{t+1}. \quad (2)$$

Using F , if it exists, the entire folded state is uniquely determined by the initial value x_0 and represented by its *orbit* $[F^t(x_0)]_{t \in \mathbb{Z}_{\geq 0}}$. In addition, the continuous deformation or the folding motion is achieved by a change of x_0 . Note that there might not be the $(t+1)$ st folded state compatible with a given t th folded state at some t , in which we cannot further increase the number of rings. For the later analysis, we introduce the coordinate system on the moduli space $\varphi : \mathcal{M} \rightarrow \mathbb{R}^{4N-6}$ and parameterize F as follows:

$$\mathbf{F} := \varphi \circ F \circ \varphi^{-1},$$

$$\mathbf{F} : \varphi(\mathcal{M}) \rightarrow \varphi(\mathcal{M}), \quad \mathbf{x}_t \mapsto \mathbf{x}_{t+1}, \quad (3)$$

where $\mathbf{x}_t := \varphi(x_t)$ represents the parameterized t th folded state (see Supplemental Material [30] for the definition of the parameterization). With the help of the topology of the crease pattern, we can derive the function $\mathbf{F}$ by the construction based on taking the intersection of spheres repeatedly [37,41,49] (see Supplemental Material [30] for the derivation of $\mathbf{F}$).

The dynamical system of the waterbomb tube has the *equivariance* that comes from the symmetries of the crease pattern along the ring direction. The equivariance is a parameter-free property of the underlying system F . Because of the symmetry of the crease pattern, the configuration space $\mathcal{C}$ has N symmetries $g^j : \mathcal{C} \rightarrow \mathcal{C}$ defined by the shift of the index corresponding to the ring direction by $j \in \{0, \dots, N-1\}$:

$$g^j : \left((\mathbf{p}_{n,t}^i)_{i \in \{1, \dots, 6\}}, (\mathbf{p}_{n,t+1}^i)_{i \in \{1, 2\}} \right)_{n \in \{0, \dots, N-1\}} \\ \mapsto \left((\mathbf{p}_{n+j,t}^i)_{i \in \{1, \dots, 6\}}, (\mathbf{p}_{n+j,t+1}^i)_{i \in \{1, 2\}} \right)_{n \in \{0, \dots, N-1\}}. \quad (4)$$

Thus, the moduli space $\mathcal{M}$ also has symmetries. Let $\underline{p}$ be the canonical surjection of $\mathcal{C}$ onto $\mathcal{M}$, and $\bar{p}$ be the injection which maps an $x \in \mathcal{M}$ to one representative element in its inverse image $\underline{p}^{-1}(x)$. Then, the symmetry on $\mathcal{M}$ is represented by $G^j := \underline{p} \circ g^j \circ \bar{p}$, and the equivariance $G^j \circ F = F \circ G^j$ holds; i.e., if $(x_t)_{t \in \mathbb{Z}_{\geq 0}}$ is an orbit of F , then $[G^j(x_t)]_{t \in \mathbb{Z}_{\geq 0}}$ is also an orbit. Remarkably, because G^j -invariance is F -invariant, the restriction of F to a G^j -invariant subspace of $\mathcal{M}$ defines a reduced dynamical system. For example, the N -fold-symmetric folded state of a ring, in which the folded states of the N unit cells in the same ring are identical, is G^j -invariant for arbitrary j ; thus, we can reduce F to a subspace of N -fold-symmetric folded states. As studied in the previous research [37], the dimension of this subspace is 2, and the N -fold-symmetric waterbomb tube has 2-DOF. The equivariance is an important property used in extracting a specific DOF of the waterbomb tube by the extrusion latter.

B. Linear stability analysis

We define the eDOF and the bDOF of the waterbomb tube at a uniform folded state using the linear stability analysis in the parameterized dynamical system $\mathbf{F}$. (Again, note that “stability” here is not used in the mechanical sense. It refers to the propagation of geometric configurations through the repetition of unit cells.) In the discrete dynamical system

model, a uniform folded state that corresponds to a uniform configuration is represented by the orbit of a *fixed point* $\mathbf{x}^* \in \varphi(\mathcal{M})$ satisfying $\mathbf{F}(\mathbf{x}^*) = \mathbf{x}^*$. Consider deforming the 0th ring at $\mathbf{x}_0 = \mathbf{x}^*$ along a vector $\Delta\mathbf{x}$. To understand the “time” evolution of $\mathbf{x}^* + \Delta\mathbf{x}$, the linear stability analysis approximates the orbit $[\mathbf{F}'(\mathbf{x}^* + \Delta\mathbf{x})]_{t \in \mathbb{Z}_{>0}}$ by $\{\mathbf{x}^* + [\mathbf{DF}(\mathbf{x}^*)]^t \Delta\mathbf{x}\}_{t \in \mathbb{Z}_{>0}}$, where $\mathbf{DF}(\mathbf{x}^*)$ is the Jacobian matrix of $\mathbf{F}$ at $\mathbf{x}^*$ which defines the linear mapping from the tangent space of $\varphi(\mathcal{M})$ at $\mathbf{x}^*$ to itself. Here, if we take an eigenvector of $\mathbf{DF}(\mathbf{x}^*)$ as $\Delta\mathbf{x}$, in the first-order approximation, $[\mathbf{DF}(\mathbf{x}^*)]^t(\Delta\mathbf{x})$ that represents the deformation from $\mathbf{x}^*$ grows exponentially with exponent given by the eigenvalue corresponding to $\Delta\mathbf{x}$ [Fig. 1(d)]. Hence, if the magnitude of the eigenvalue is less than 1 or more than 1, the deformations of successive rings decrease or increase, respectively. We can identify these cases by reversing the “time” direction of $\mathbf{F}$. Intuitively, a kinematic limit of deformation that comes from the isometric constraints exists, and it forces the deformation to localize at the boundary rings when the magnitude is not equal to 1; thus, it corresponds to an edge mode. Here, if the eigenvalue has magnitude 1, the deformations of successive rings neither converge nor diverge and stay in the kinematically allowable range; hence, it corresponds to a bulk mode. Unlike edge modes that inherently result in nonuniform deformation, bulk modes can exhibit uniform and nonuniform deformations. The uniform bulk mode corresponds to the eigenvalue +1, where the deformation remains constant with respect to t . On the other hand, the nonuniform bulk mode corresponds to a conjugate pair of complex eigenvalues with magnitude 1 where the deformation oscillates [37,41]. From the above, we define the *edge/bulk degree of freedom (eDOF/bDOF)* of the waterbomb tube at a uniform folded state by the number of *multipliers* of the corresponding fixed point $\mathbf{x}^*$, i.e., the eigenvalues of $\mathbf{DF}(\mathbf{x}^*)$, with magnitude unequal/equal to 1. Note that there exists the $\mathbf{F}$ -invariant submanifold of $\varphi(\mathcal{M})$ called the stable/unstable manifold or the center manifold which is tangent to the subspace spanned by the eigenvectors corresponding to edge modes or bulk modes, respectively. The orbit within these submanifolds represents the finite deformation modes tangent to a bulk mode or an edge mode; however, the asymptotic behaviors of orbits in the submanifolds and the linear approximated orbits in their tangent space are not always the same. Specifically, although the behavior on the stable/unstable manifold is the same as the linear approximation, in general, an orbit on the center manifold might exhibit behavior different from the linear approximation, including the convergence or divergence [50]. Thus, the “real” bDOF might be smaller than the bDOF obtained from the first-order approximation.

III. ANALYSIS

A. Ground uniform folded states

Our analysis considers the two kinds of uniform folded states of the waterbomb tube as a ground state, which we call the *cylindrical state* and *toroidal state*. In general, the positions of adjacent unit cells in a uniform configuration placed in $\mathbb{E}^3$ are related by a screw transformation which is the composition of rotation and translation about a screw axis

[14,15,51]. Both the cylindrical and toroidal states are the degenerate case where the screw transformation becomes the pure translation, and the rotation, respectively. Generally, each crease pattern of the waterbomb tube has a finite number of cylindrical/toroidal states rather than the parameter families. This is because they can be characterized as special instances in the 1-parameter families of folded states of the waterbomb tessellation with necessary symmetries, where the boundaries coincide to form a tubular shape [Figs. 2(a) and 2(b)]. Note that the toroidal state is closed in the poloidal direction, but not closed in the toroidal direction. Based on the necessary symmetries, we numerically calculated these folded states by solving the 2D equation instead of solving the $(4N - 6)\text{D}$ equation $\mathbf{F}(\mathbf{x}^*) = \mathbf{x}^*$. For the cylindrical state, it is represented as a fixed point of the reduced 2D dynamical system for the N -fold-symmetric waterbomb tube [37,41]. On the other hand, we computed the toroidal state using another 2D dynamical system defined by interchanging the roles of the n -direction and t -direction, where the t -direction and n -direction become the ring direction and longitudinal direction (“time” direction), respectively (see Supplemental Material [30] for the calculation of the cylindrical and toroidal folded states). Through the computation, we observed that the existence and the number of cylindrical/toroidal states change depending on the crease pattern parameters [Fig. 2(c)]. Specifically, each crease pattern has at most 3 cylindrical states having different radii and N toroidal states. Note that if a single toroidal state exists, there are N toroidal states that are congruent but have different “bending directions” due to the symmetry of the crease pattern in the ring direction or the equivariance induced by the symmetry; however, we will use only one of the N toroidal states as a representative state in calculating their eDOF/bDOF.

B. Bulk DOF at ground states

We numerically calculated the multipliers of the fixed points of $\mathbf{F}$ corresponding to cylindrical/toroidal states in crease patterns under various values of θ and N to know how the eDOF/bDOF depends on the ground state and the crease pattern. First, we visualized multipliers under the fixed N and varied θ in the range $(0, 45^\circ)$, in which there are at most 2 cylindrical states with smaller or larger radius and N toroidal states [Fig. 3(a)]. The result shows that the bDOF changes depending on the ground folded states even in the identical crease pattern. Specifically, the bDOF tends to be nonzero at the cylindrical state with a smaller radius or the toroidal state; on the other hand, a cylindrical state with a larger radius has zero bDOF. Note that the diagrams in Fig. 3(a) are mirror symmetric with respect to the horizontal line $\log |\lambda| = 0$, which implies that each eigenvalue of $\mathbf{DF}(\mathbf{x}^*)$ must coexist with its reciprocal. This property would come from the time-reversal symmetry of the system; i.e., the system behaves in the same way even if we flip the direction of the “time” t . In particular, $\det \mathbf{DF}(\mathbf{x}^*) = 1$ holds. This is the first-order necessary condition for the dynamical system to be conservative, where there cannot exist a stable (resp. unstable) fixed point all of whose multipliers have magnitude less (resp. more) than 1.

Focusing on the smaller cylindrical states with nonzero bDOF, regardless of the crease pattern parameters, their

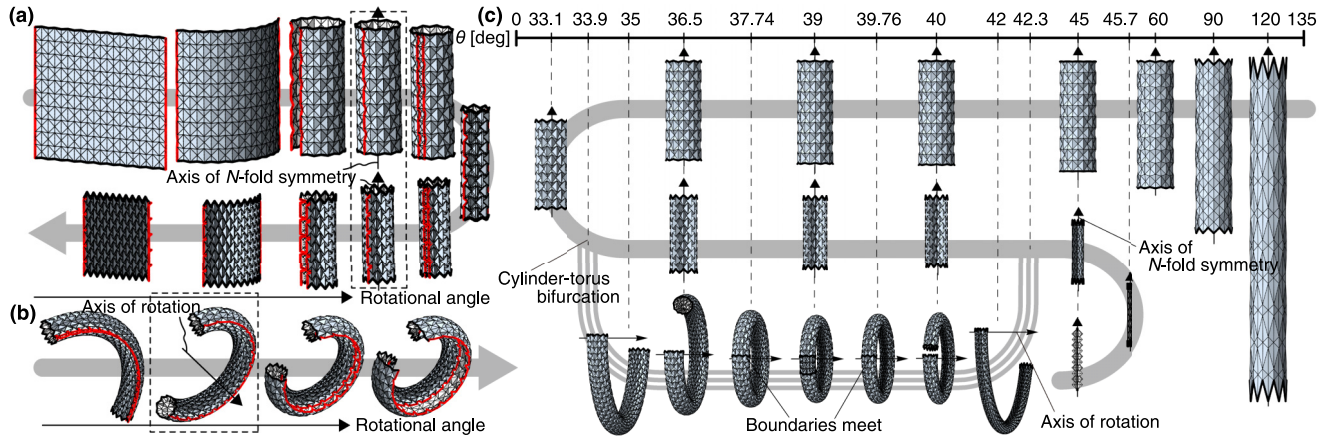


FIG. 2. (a), (b) The 1-DOF symmetric folding motions of the waterbomb tessellation in $(\theta, N) = (38^\circ, 10)$. They pass through the uniform folded state of the waterbomb tube when the red boundaries of the sheet meet, highlighted by the dashed boxes. (a) Cylindrical state. Assuming that the folded states of all the unit cells are identical, generic triangulated origami tessellations have 2-DOF that specifies a screw transformation relating the positions of the (n, t) th and $(n+1, t)$ th cells [14,15,51]; however, to form a tubular shape, we remove the translation about the screw axis and vary only the rotation angle. For each rotation angle, two uniform folded states exist; hence, the two N -fold-symmetric cylindrical states with different radii exist. (b) Toroidal state. We assume that the folded states of the unit cells indexed by $[(n, t)]_{n \in [0, \dots, N-1]}$ are identical and vary the rotation angle of the screw transformation relating the positions of (n, t) th and $(n, t+1)$ th unit cells while removing the translation. In addition, the (N, t) th folded state is required to come back to the $(0, t)$ th folded state; i.e., the folded states along the t -direction must be n -periodic. (c) The bifurcation of the cylindrical and toroidal states when we fix N to 10 and vary θ . There are critical values of θ around 33.1° , 33.9° , 42.3° , and 45.7° where some branches meet, followed by the appearance or the disappearance of the branches. The branch for the toroidal state is shown as multiple lines because there are N different branches for the toroidal state from the symmetry of the crease pattern in the ring direction. The rotation angle relating the position of the t th and $(t+1)$ st rings in the toroidal state changes depending on θ ; in particular, there are two toroidal states that can also be closed in the toroidal direction to form a closed polyhedron, which is possible if and only if the rotation angle is $2\pi/m$ ($m \in \mathbb{Z}_{>0}$).

multipliers necessarily include a conjugate pair of complex multipliers with multiplicity 1 whose magnitude is equal to 1. The 2-bDOF induced by them corresponds to the N -fold-symmetric undulation reported in the previous research [34,37,41,52]. In the finite deformation, by transitioning the orbits corresponding to a family of quasiperiodic solutions, we can tune the “amplitude” and “phase” of the N -fold-symmetric mode independently [37,41].

The bDOF of the smaller cylindrical state changes and non-rotationally symmetric modes appear or disappear at some critical values of θ , which falls into two cases: type α and type β [Fig. 3(b)]. Type α occurs when 4 multipliers meet at $+1$. Before or after type α singularity, the 4 multipliers bifurcate into a reciprocal pair of real multipliers with multiplicity 2 or a conjugate pair of complex multipliers with multiplicity 2 on the unit circle, respectively; therefore, the bDOF of the cylindrical state changes by ± 4 . The finite deformation mode corresponding to the 4-bDOF induced through type α singularity is the “sine” mode (Fig. 3(c); see Supplemental Material [30] for the optimization method to visualize the finite deformation modes). Unlike the N -fold-symmetric mode, this mode has a degree of freedom to choose a direction of deformation; therefore, 4-bDOF corresponds to the amplitude and phase of the two base modes of the sine mode. Type β singularity occurs when 4 complex multipliers and their conjugates meet at a conjugate pair on the unit circle. Before or after type β singularity, the 4 multipliers bifurcate into a reciprocal pair of complex multipliers with multiplicity 2 outside/inside the unit circle, or 2 complex multipliers with multiplicity 2 on the unit circle, respectively; therefore, the

bDOF of the cylindrical state changes by ± 8 . This bifurcation generates two kinds of directional modes: “diamond” mode and “twist” mode, each corresponding to 4 of the 8-bDOF. We observed that as N becomes larger, the “cross section” along the ring direction becomes complex.

Type α singularity occurs when the fixed points corresponding to the cylindrical and the toroidal states intersect [see also Fig. 2(c)]. The multipliers of the generated toroidal state are similar to the cylindrical state; thus, the toroidal state has a 2-bDOF bulk mode inherent from the N -fold-symmetric mode of the cylindrical state, where the radius oscillates along the toroidal direction. However, the 4 multipliers that were at $+1$ bifurcate differently from the cylindrical state. Specifically, 2 of the 4 multipliers stay at $+1$ while the other 2 multipliers bifurcate into a reciprocal pair on the real axis. Thus, the bDOF of the toroidal state is the bDOF of the cylindrical state before the bifurcation plus 2. However, note that, generically, the multipliers $+1$ with multiplicity 2 cannot be extended to the finite deformation mode from the dynamical systems theory [53]; therefore, the real bDOF of the toroidal state would be 2 rather than 4. Unlike type α , type β does not accompany the branching/merging of uniform folded states. This is because, in general, the creation/destruction of fixed points occurs only when some multipliers meet at $+1$ [50].

Next, we investigate how the bDOF of the smaller cylindrical state changes when we fix θ and increase N [Fig. 3(d)]. In contrast to the total DOF that increases linearly with the number of unit cells in a ring N , the bDOF grows not linearly. Specifically, there is a finite range of N in which bDOF stays

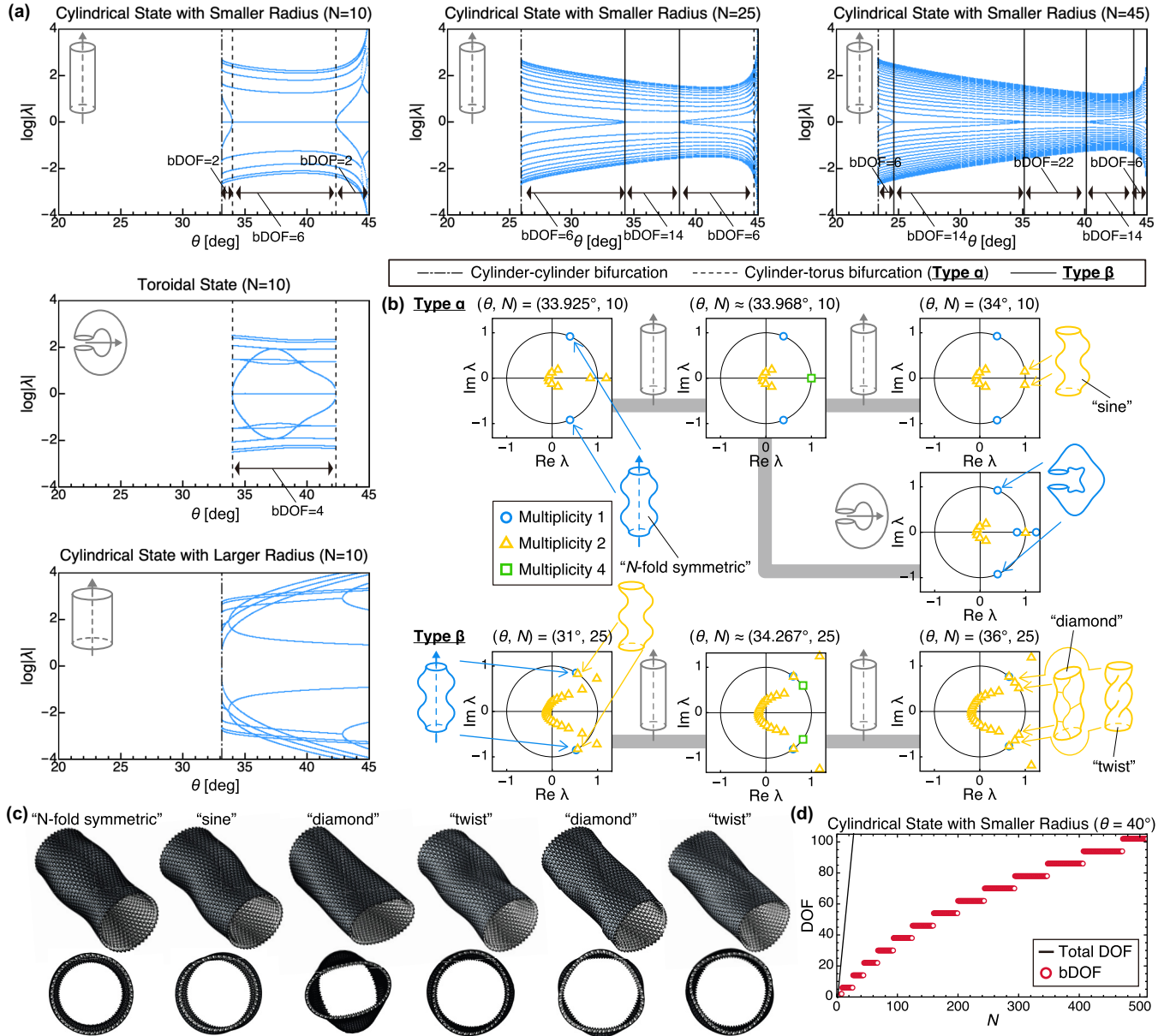


FIG. 3. (a) Plots of $(\theta, \log|\lambda|)$ where λ is multipliers of the fixed points corresponding to the larger, smaller cylindrical or the toroidal state (left: $N = 10$, middle: $N = 25$, right: $N = 45$). The bDOF corresponds to the horizontal line $\log|\lambda| = 0$. Note that multipliers having the same value or conjugate complex values are plotted at the same point. For the smaller cylindrical state and the toroidal state, their bDOF counted including the multiplicity is shown. The vertical lines indicate the three kinds of critical values of θ where fixed points bifurcate or the bDOF of the smaller cylindrical state changes. (b) Plots of $(\text{Re } \lambda, \text{Im } \lambda)$ around the two types of critical θ (top: type α , bottom: type β). The multiplicity of multipliers is indicated using different markers. For each multiplier on the unit circle, schematics of its corresponding bulk mode are attached. (c) Finite deformation modes tangent to the bulk modes around the smaller cylindrical state in $(\theta, N) = (37.5^\circ, 45)$. The bottom figures show their cross sections. From the plot in (a) for $N = 45$, the smaller cylindrical state in this crease pattern has 22-bDOF [2/22: N -symmetric mode, 4/22: sine mode, $2 \times (2 \times 4)/22$: two pairs of diamond and twist mode with different cross sections]. (d) Change of the bDOF of the smaller cylindrical state compared to the total DOF when θ is fixed to 40° and N is varied.

constant, and the width of intervals becomes longer and longer as we increase N . Between consecutive intervals, the bDOF jumps by 4 or 8 reflecting the nature of type α or type β singularity. Furthermore, we observed that the bDOF jumps by 4 just once between the first and second intervals; after that, the increment of the bDOF stays constant at 8. Therefore, the bDOF of the smaller cylindrical state can take only values

from $\{2, 6\} \cup 6 + 8\mathbb{Z}_{>0}$, and the ratio of the bDOF within the total DOF becomes smaller and smaller as we increase N .

C. Extruded Maxwell origami tube

Here, by modifying the topology of the crease pattern, we can make another tubular origami tessellation, which

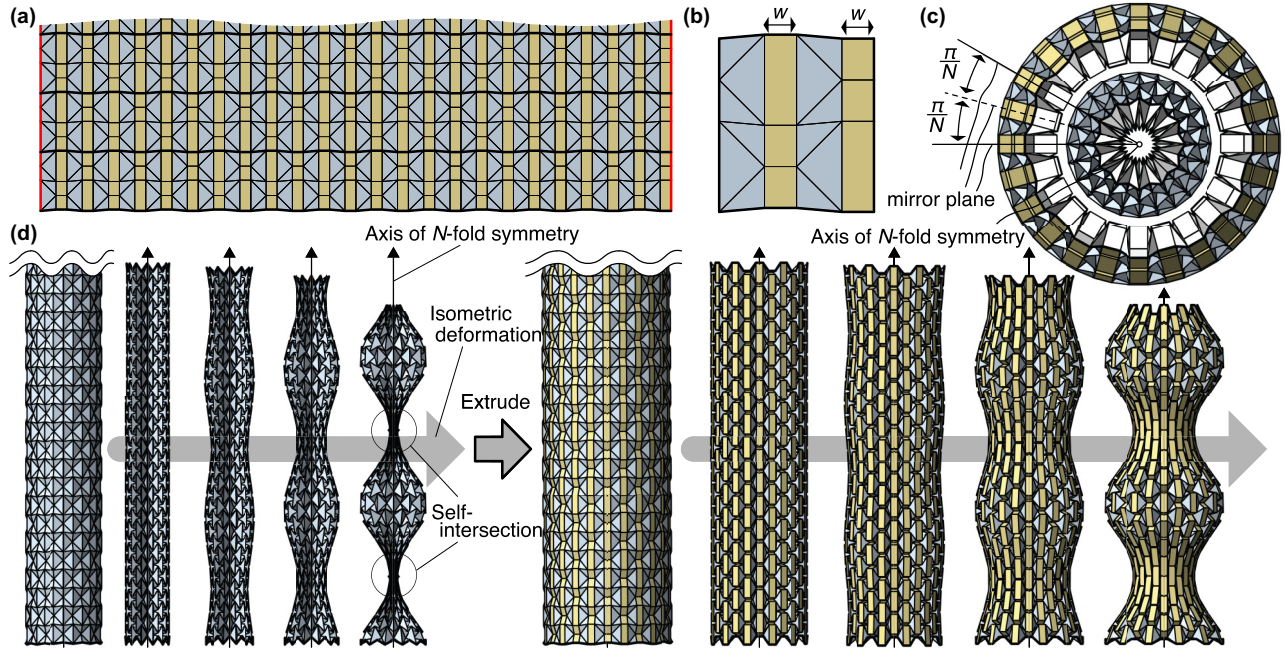


FIG. 4. (a) Crease pattern and (b) unit cell of the extruded waterbomb tube. The inserted rectangular facets are shown in yellow, whose width is specified by w . (c) N -fold-symmetric folded states before/after the extrusion viewed along the axis of N -fold symmetry. The entire folded state is the rotational copies of its $1/N$ part between the two mirror planes shown as the solid lines, which is mirror symmetric with respect to the plane shown as the dashed line; then, all the vertices in the $1/N$ part of the non-extruded state are on either of the three mirror planes. We can extrude the folded state by moving each vertex in the non-extruded state by the vectors perpendicular to the planes. (d) N -fold-symmetric folded states including the two cylindrical states before/after the extrusion in $(\theta, N, w) = (40^\circ, 12, 0.5)$. The 1-bDOF isometric deformation that changes the “amplitude” in the N -fold-symmetric mode is visualized. The radius of the tube becomes larger after the extrusion; hence, we can remove the self-intersections in slim parts of an original folded state by extruding with w larger than a necessary value.

we call the *extruded waterbomb tube*, whose DOF stays constant even if we increase N . Specifically, we copy the vertical creases and insert rectangular facets between duplicated creases [Figs. 4(a) and 4(b)]. This operation is one instance of the edge-extruding technique [54]. There are some variations of extruded origami patterns [55,56], and the (nonclosed) extruded waterbomb tessellation approximating various surfaces has been already studied [57]; however, how the extrusion changes the kinematics when being applied to tubular origami tessellations was not investigated. In the extruded waterbomb tube, $3 \times \#(\text{vertices in the } t\text{th ring}) - \#(\text{edges in the } t\text{th ring})$, in which we must count diagonals of the rectangular facets to maintain them flat, becomes $3 \times 12N - 42N = -6N < 0$. Hence, the extruded waterbomb tube is no longer a Maxwell lattice and becomes an overconstrained framework that generically has no continuous motions; nevertheless, it has 2-DOF. This is because the waterbomb tube can be extruded if and only if the folded state is N -fold symmetric [Fig. 4(c)]. Thus, since the N -fold-symmetric waterbomb tube has 2-DOF dominated by the reduced 2D discrete dynamical system, the extruded waterbomb tube has 2-DOF independent of N . After the extrusion, the larger (resp. smaller) cylindrical state has 2-eDOF (resp. 0-eDOF) and 0-bDOF (resp. 2-bDOF). In particular, around the smaller cylindrical state, the extruded waterbomb tube exhibits the 2-DOF nonuniform bulk mode (Fig. 4(d); see

Supplemental Material [30] for the deformation of the paper model of the extruded waterbomb tube), which is in contrast to the conventional overconstrained origami structures exhibiting uniform bulk modes.

IV. DISCUSSION

Focusing on the nature of multi-DOF 1D Maxwell lattices, this paper constructed the Maxwell origami tube with tunable DOFs by closing the periodic origami sheet and explored their eDOF and bDOF using the discrete dynamical system model. Also, the first example of overconstrained origami with nonuniform folding is constructed by extruding the Maxwell origami tube. We hope the concept and result of the present work serve as the theoretical basis for designing mechanical metamaterials based on Maxwell lattices or their variants. The analysis implies the feasibility of metamaterials that can switch between different topological characteristics through reconfiguration; for example, the larger (resp. smaller) cylindrical states with zero (resp. nonzero) bDOF would be the stiff (resp. soft) structure. The physical meaning of the nonlinearity of the bDOF with respect to the total DOF in the Maxwell origami tube is an open problem that could be connected with other fields like soft-matter physics. Further physical simulations and experiments are needed to verify the properties of the Maxwell origami tube including the

switchability of topological characteristics. Another future direction is investigating the relationship between the deformation modes of the Maxwell origami tube and the elastic wave propagating through a cylindrical shape. They look quite similar; however, their connection remains unclear. From the theoretical side, the present work only considers the local kinematics around the uniform folded states. Exploring global kinematics would lead to exotic nonlinear phenomena.

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