

Enhanced signal response through synchronization competition in a ring of four bistable systems

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(Received 6 August 2024; accepted 31 December 2024; published 7 January 2025)

Recent research indicates that small-world networks can enhance weak signals through a phenomenon called topological resonance. However, the underlying mechanism of this effect remains unclear. In this study, we examine a ring of four bistable oscillators to analyze how it responds to an external subthreshold signal. Our findings reveal that the ring amplifies the signal via synchronization competition, where two desynchronized oscillators compete to synchronize with the other two, leading to increased oscillation and improved response to the subthreshold signal. Additionally, we find that synchronization competition is influenced by both the initial states of the oscillators and the characteristics of the subthreshold signal. We identify three synchronization transitions during signal amplification and propose three feedforward motifs to analyze the signal response of the ring. Finally, we apply the synchronization competition mechanism to explain the topological resonance of small-world networks.

DOI: [10.1103/PhysRevResearch.7.013024](https://doi.org/10.1103/PhysRevResearch.7.013024)

I. INTRODUCTION

The relationship between network structure and information-processing capabilities has attracted considerable interest in both biology and physics [1–4]. This research improves our understanding of signal amplification in neural networks [5–8] and contributes to the development of bio-inspired devices [9–16]. Traditionally, real networks have often been modeled as all-to-all networks or regular arrays [17–21]. When exposed to appropriate noise, the size of these networks significantly affects their response to threshold signals. This phenomenon is referred to as system-size resonance or array-enhanced stochastic resonance [22–25]. However, these models overlook two essential features of real networks: the scale-free property and the small-world property [26].

The scale-free property describes a network structure where a few nodes (hubs) have a high degree of connectivity to many other nodes (leaves) [27]. These types of networks are referred to as scale-free networks. They can amplify weak signals through their heterogeneous connectivity without relying on noise-induced stochastic resonance [28]. This amplification effect is called topological resonance. Topological resonance occurs when leaf nodes are divided into two roughly equal groups. The collective dynamics of these groups effectively lower the potential barriers of the hub nodes, enabling them to generate large oscillations in response to weak signals. As a result, a resonant signal response is observed in relation to the coupling strength [28–33]. Moreover, this mechanism allows for the simplification of scale-free

networks into a three-node motif, where similar resonant response phenomena can also be observed [28,31–33].

In contrast, the small-world property refers to networks where the average distances between nodes are relatively short [34]. Networks that exhibit this characteristic are called small-world networks. Recent research indicates that small-world networks can also enhance weak signals solely through their structure, demonstrating a topological resonance effect as well [35]. Unlike scale-free networks, small-world networks have homogeneous structures that include many loops of various lengths [36–38]. This feature is crucial for neural networks, as it facilitates the generation of self-sustained oscillations and the attainment of synchronization [39–43]. Therefore, understanding how loops respond to weak signals is crucial for gaining insights into topological resonance in small-world networks.

In this study, we examine the response of a small ring consisting of four bistable oscillators to an external subthreshold signal. Our findings show that the ring achieves signal amplification by synchronization competition, where two asynchronous oscillators compete for synchronization with two already synchronized oscillators. This competition enhances oscillatory behavior, thereby amplifying the subthreshold signal. We demonstrate that synchronization competition arises from the interplay between the initial conditions of the oscillators and the subthreshold signal. Additionally, we introduce three three-node feedforward motifs to predict the signal response observed in the ring, yielding similar outcomes. These findings enhance our understanding of the topological resonance effect in small-world networks.

II. MODEL

We consider a ring composed of four identical bistable oscillators that are coupled to their nearest neighbors and subjected to a periodic signal. This system can be described

by the following equation:

$$\dot{x}_i = x_i - x_i^3 + \frac{\lambda}{2}(x_{i-1} + x_{i+1} - 2x_i) + Af(t), \quad (1)$$

where $\lambda \geq 0$ represents the coupling strength, $f(t)$ denotes a periodic signal characterized by unit amplitude and frequency ω , and A signifies the amplitude of this periodic signal. In the absence of coupling, Eq. (1) reduces to a single bistable oscillator forced by the periodic signal $Af(t)$. When the signal amplitude is below a critical threshold, denoted as A_c , the oscillator generates small oscillations around one of its two stable fixed points, $x^* = \pm 1$, which depends on the initial condition. In contrast, when the signal amplitude exceeds this critical threshold, the oscillator switches between $x^* = \pm 1$. In both cases, the oscillation frequency aligns with that of the periodic signal. For lower-frequency signals, the critical amplitude A_c is approximated to be a constant value of $\sqrt{4/27} \approx 0.385$ [19,44].

Since the oscillation frequency corresponds to the external periodic signal, the dynamics of the oscillators can be effectively described by their amplitudes and centers, as follows [28,31]:

$$g_i = \frac{\max x_i - \min x_i}{2}, \quad (2)$$

$$c_i = \frac{\max x_i + \min x_i}{2}. \quad (3)$$

According to Eq. (2), the response of the ring to the periodic signal is defined by the maximum response of the oscillators [28],

$$G = \frac{\max_{i=1}^N g_i}{A}, \quad (4)$$

where $N = 4$. When $G > 1$, this indicates that the subthreshold signal is amplified. Additionally, the spatial synchronization of oscillators is measured by the order parameter ρ , defined as follows [28,35]:

$$\rho = \frac{|n_+ - n_-|}{N}, \quad (5)$$

where n_+ and n_- represent the number of oscillators around the fixed points $x^* = 1$ ($c_i > \theta$) and $x^* = -1$ ($c_i < \theta$), respectively, with a defined threshold of $\theta = 0.2$. When half of the oscillators are at $x^* = 1$ and the other half at $x^* = -1$ (i.e., $n_+ = n_- = N/2$), the ring is asynchronous, yielding an order parameter of $\rho = 0$. In contrast, when all oscillators are around a single fixed point, either $x^* = 1$ or $x^* = -1$, the ring is synchronous with $\rho = 1$ (i.e., $n_+ = N, n_- = 0$ or $n_+ = 0, n_- = N$). Oscillators that switch between $x^* = \pm 1$ ($-\theta \leq c_i \leq \theta$) do not contribute to n_+ and n_- . Thus, $\rho = 0$ can also occur in two other scenarios in the four-node ring: when half of the oscillators oscillate between $x^* = \pm 1$ or when all oscillators oscillate within that range.

In numerical simulations, the initial values of the oscillators are randomly chosen from $x^* = \pm 1$. Figure 1 illustrates six distributions of initial values for the four-node ring: two one-point distributions (I and II) and four two-point distributions (III–VI). The one-point distributions (I and II) cause the ring to behave as a single oscillator driven by a periodic signal, as described in Eq. (1) with $\lambda = 0$. Therefore, this study will

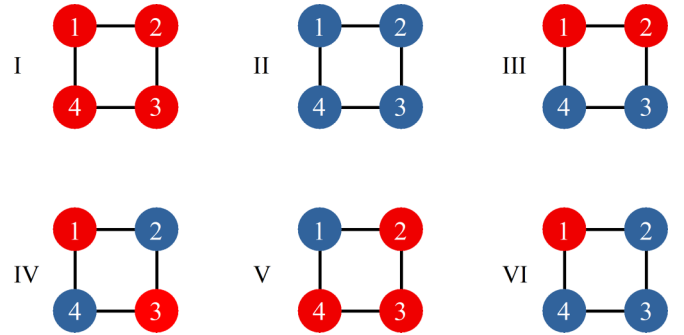


FIG. 1. Six possible distributions of initial values for four oscillators. In these distributions, red circles represent $x_i(0) = 1$, while blue circles represent $x_i(0) = -1$. Distributions I and II are one-point distributions, III and IV are uniform two-point distributions, and V and VI are nonuniform two-point distributions.

concentrate on the four two-point distributions (III–VI) to investigate the response of the ring to subthreshold signals.

III. SIMULATIONS

In the following discussions, we will set the signal amplitude A in Eq. (1) to be less than the critical amplitude A_c . Additionally, we will discard a large number of initial iterations in the calculations of G and ρ to eliminate the influence of transients.

A. Effect of initial value distributions

We consider a sinusoidal signal described by $f(t) = \sin(\omega t)$, where the frequency is set to $\omega = \pi/25$ and the amplitude is $A = 0.3$. Using uniform distribution III as the initial condition, the response of the ring to this signal continuously increases with the coupling strength until it reaches a critical threshold of $\lambda_1 = 0.146$. Beyond this point, the signal response abruptly drops to a constant value and becomes unaffected by any further changes in the coupling strength [see Fig. 2(a)]. When using distribution IV as the initial condition, the signal response is similar to that of distribution III; however, the critical coupling value decreases to $\lambda_1 = 0.073$ [see Fig. 2(b)]. Additionally, if distribution IV is employed as the initial condition, the ring produces the same signal response as that of distribution IV [see Fig. 2(c)]. Finally, with the nonuniform distribution VI, the ring exhibits a response that peaks not only at $\lambda_1 \approx 0.073$, but also within the coupling interval $(\lambda_2, \lambda_3) = (0.186, 0.197)$ [see Fig. 2(d)]. The response in this interval ($G > 1$) is significantly greater than at λ_1 ($G \approx 1$).

Figure 2 also plots the order parameter of the ring for distributions III–VI. When the coupling strength reaches the threshold λ_1 , the order parameter attains $\rho = 1$, signifying that the oscillators are synchronized. As the coupling strength increases to the values (λ_2, λ_3) , the order parameter drops to $\rho = 0$ in distribution VI. A value of $\rho = 0$ is also observed in distributions III and IV when $\lambda < \lambda_1$, but with a weaker signal response. This indicates that the desynchronization seen in distribution VI is due to large oscillations. Overall, these findings suggest that synchronization reduces signal response,

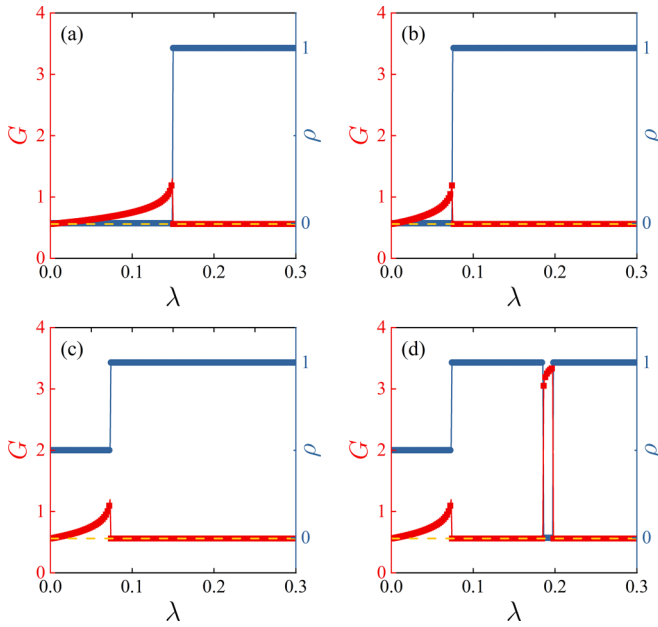


FIG. 2. The signal response G (left axis) and the spatial synchronization ρ (right axis) vary with coupling strength λ for different initial value distributions: (a) III, (b) IV, (c) V, and (d) VI. The dashed line in each panel denotes the signal response without coupling strength.

while the transition between desynchronization and synchronization enhances signal amplification.

The values of λ_1 differ between distributions III and IV, which can be understood by simplifying Eq. (1) to a two-node system. In distribution III, the conditions $x_1 = x_2$ and $x_3 = x_4$ hold. By substituting these into Eq. (1), we derive the following expression:

$$\dot{x}_{1,3} = x_{1,3} - x_{1,3}^3 + \frac{\lambda}{2}(x_{3,1} - x_{1,3}) + Af(t). \quad (6)$$

In distribution IV, where the equalities $x_1 = x_3$ and $x_2 = x_4$ are observed, Eq. (1) simplifies to

$$\dot{x}_{1,2} = x_{1,2} - x_{1,2}^3 + \lambda(x_{2,1} - x_{1,2}) + Af(t). \quad (7)$$

In this case, the effective coupling strength becomes λ . Thus, to achieve synchronization, Eq. (7) requires half the coupling strength compared to Eq. (6).

To elucidate the effects of the initial value distributions, Fig. 3 presents the time series of the oscillators and the subthreshold signal. In distribution III, the oscillators are evenly distributed at $x^* = \pm 1$, with oscillators 1 and 2 at $x^* = 1$ and oscillators 3 and 4 at $x^* = -1$. Before $t = T/4$, the subthreshold signal increases over time, which gradually pushes the oscillators towards $x^* = 1$. This movement aligns with the forces exerted by oscillators 1 and 2 on oscillators 3 and 4, while opposing the forces from oscillators 3 and 4 on oscillators 1 and 2. As a result, oscillators 1 and 2 remain near $x^* = 1$, while oscillators 3 and 4 cross the midline $x = 0$ and synchronize with them [see Fig. 3(a)]. Here, oscillators 1 and 2, as well as oscillators 3 and 4, exhibit identical time series because they share the same initial values and their neighboring oscillators. A similar but faster synchronization

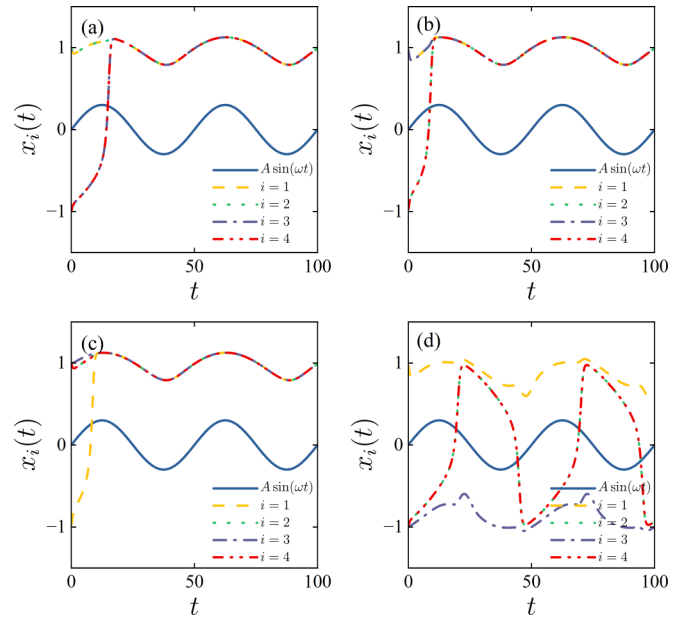


FIG. 3. The time series of four oscillators with four different initial value distributions: (a) III, (b) IV, (c) V, and (d) VI. The parameters used are $A = 0.3$, $\omega = \pi/25$, and $\lambda = 0.19$.

occurs in distribution IV [see Fig. 3(b)]. The accelerated synchronization in this case is due to a greater effective coupling strength in distribution IV [see λ in Eq. (7)], compared to distribution III [see $\lambda/2$ in Eq. (6)]. In distribution V, the oscillators are unevenly distributed at $x^* = \pm 1$, with oscillator 1 at $x^* = -1$ and oscillators 2, 3, and 4 at $x^* = 1$. Influenced by the subthreshold signal and oscillators 2 and 4, oscillator 1 quickly crosses the midline and synchronizes with them [see Fig. 3(c)]. Conversely, in distribution VI, oscillators 2, 3, and 4 are located at $x^* = -1$, while oscillator 1 is at $x^* = 1$. When $t < T/4$, oscillators 1 and 3 oscillate around $x^* = 1$ and $x^* = -1$, respectively, while oscillators 2 and 4 deviate slightly from $x^* = -1$. After $t = T/4$, the subthreshold signal begins to decline, driving all oscillators towards $x^* = -1$ [see Fig. 3(d)]. Due to the attractions exerted by oscillator 1, oscillators 2 and 4 cross the midline $x = 0$; however, oscillator 3 remains near $x^* = -1$ due to insufficient stimulation. Subsequently, influenced by the subthreshold signal and oscillator 3, oscillators 2 and 4 reverse their direction towards oscillator 3. Meanwhile, oscillator 1 stays at $x^* = 1$ due to insufficient external force. At $t = 3T/4$, the direction of the signal force reverses. This reversal, in conjunction with the attraction from oscillator 1, prompts oscillators 2 and 4 to move back towards oscillator 1. This cycle continues, with oscillators 1 and 3 competing for synchronization with oscillators 2 and 4. As a result, oscillators 2 and 4 generate large oscillations, which serve to amplify the subthreshold signal.

B. Effects of signal frequency

Next, we keep the amplitude at $A = 0.3$ to explore the influence of signal frequency on the response of the ring. The initial condition is set according to distribution VI. For subthreshold signals at both high and low frequencies, the

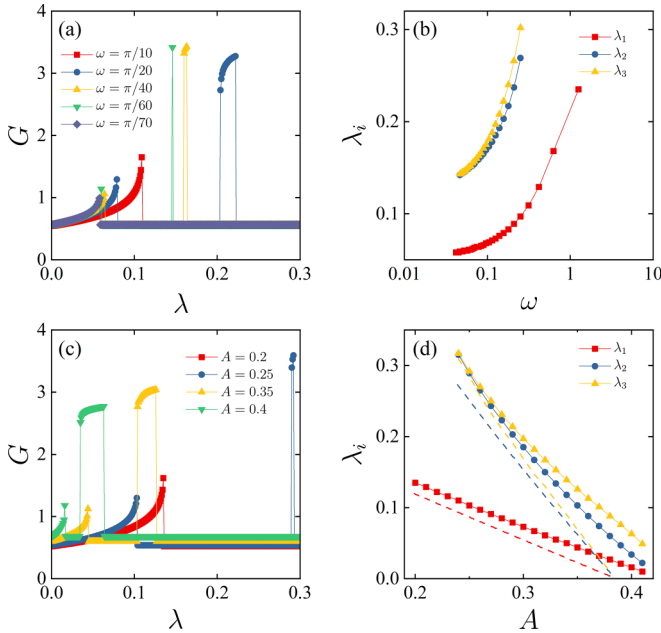


FIG. 4. The left panels show the signal response as a function of the coupling strength for sinusoidal signals with different (a) frequencies and (c) amplitudes. The right panels illustrate the correlations between the critical couplings λ_1 , λ_2 , and λ_3 with respect to the (b) signal frequency and (d) amplitude. Dashed lines represent the results of Eqs. (14), (18), and (23). In (a) and (b), the parameter $A = 0.3$ is used, while in (c) and (d), the parameter $\omega = \pi/25$ is employed. The initial values of distribution VI are taken into consideration in all panels.

response peaks at λ_1 [see Fig. 4(a)]. However, when the signal frequency falls within the intermediate range, the response not only peaks at λ_1 , but also shows considerable amplification within the coupling interval (λ_2, λ_3) . This behavior can be explained as follows: at high frequencies, the rapid oscillation of the subthreshold signal causes oscillators 2 and 4 to return to their initial positions before they cross the midline, resulting in small oscillations around $x^* = -1$. As a result, these oscillators influence oscillator 1 to synchronize around $x^* = -1$. In contrast, the gradual variations in the signal at low frequencies permit oscillators 2 and 4 to approach oscillator 1, leading to synchronization around $x^* = 1$. In the intermediate-frequency range, however, the signal enables oscillators 2 and 4 to cross the midline while limiting their duration in that region, which prevents synchronization.

From this discussion, it is clear that oscillators require stronger coupling at higher signal frequencies to improve their interaction and counteract the effects of the rapid switching of the subthreshold signal. As a result, the threshold λ_1 steadily increases as the signal frequency rises. Likewise, oscillators 2 and 4 also need greater coupling strengths to cross the midline, leading to an increase of λ_2 and λ_3 as the signal frequency increases [see Fig. 4(b)].

C. Effect of signal amplitude

After that, we fix the frequency at $\omega = \pi/25$ to investigate how variations in signal amplitude affect the response of the

ring. The initial condition is set to distribution VI. At low signal amplitudes, the response shows a peak at λ_1 . As the amplitude increases, the response not only peaks at λ_1 but also enhances in the coupling interval (λ_2, λ_3) [see Fig. 4(c)]. At lower amplitudes, the subthreshold signal does not provide enough driving force, requiring a larger coupling strength for the oscillators to cross the midline. However, as the amplitude approaches the critical value A_c , less coupling strength is needed. Consequently, the critical coupling strengths λ_i (where $i = 1, 2, 3$) decrease as the amplitude A approaches A_c [see Fig. 4(d)].

D. Effect of signal waveform

Finally, we demonstrate that the signal waveform is a relevant factor affecting signal amplification. This is illustrated using an elliptic function signal, which is defined as follows [32,45]:

$$f(t) = N(m) \text{sn}(2K\omega t/\pi) \text{dn}(2K\omega t/\pi), \quad (8)$$

where $\text{sn}(\cdot)$ and $\text{dn}(\cdot)$ are Jacobian elliptic functions, and $K \equiv K(m)$ is the complete elliptic integral of the first kind. The normalization function $N(m) = (0.43932 + 0.69796\{1 + \exp[(m - 0.3727)/0.26886]\}^{-1})^{-1}$ is used to ensure that the elliptic signal maintains a unit amplitude and a frequency of ω for any waveform. The parameter m serves to modulate the signal waveform, taking values within the interval $[0, 1]$. At $m = 0$, the signal corresponds to a simple sinusoidal wave $f(t) = \sin(\omega t)$. As m increases, the waveform begins to flatten. When m surpasses 0.5, the waveform develops an inward curvature, forming an M shape. This inward curvature becomes more pronounced with further increases in m [see Fig. 5(a)]; ultimately, the signal disappears at $m = 1$.

Additionally, the impulse transmitted by the signal per unit period is given by [45]

$$I = I(m) = \frac{1}{T} \int_0^{T/2} f(t) dt = \frac{N(m)}{2K(m)}, \quad (9)$$

where $T = 2\pi/\omega$. The impulse I has a single maximum at $m \approx 0.72$ [see Fig. 5(b)].

Figure 5(c) depicts the signal response as a function of the waveform parameter m and the coupling strength λ , with the signal amplitude set at $A = 0.3$ and the frequency at $\omega = \pi/25$. As m increases from 0, the critical coupling strengths λ_2 and λ_3 initially decline but then rise again once m exceeds 0.6. The two curves intersect at $m \approx 0.8$, beyond which the signal amplification effect disappears. The initial decline in the values of λ_2 and λ_3 is due to the continuous increase of the signal impulse I acting on the oscillators until $m \approx 0.72$. However, when m surpasses 0.5, the signal transforms into an M-shaped waveform that resembles a high-frequency signal, requiring a higher coupling strength for signal amplification, as shown in Fig. 4(b). Thus, this M-shaped waveform reduces the influence of the enhanced signal impulse starting from $m = 0.5$. At $m \approx 0.6$, the effects of the increased signal impulse and the M-shaped waveform reach a balance. After this point, λ_2 and λ_3 begin to rise. For values of m greater than 0.72, there is a sharp decline in the signal impulse. This decline, combined with the effect of the M-shaped waveform, causes

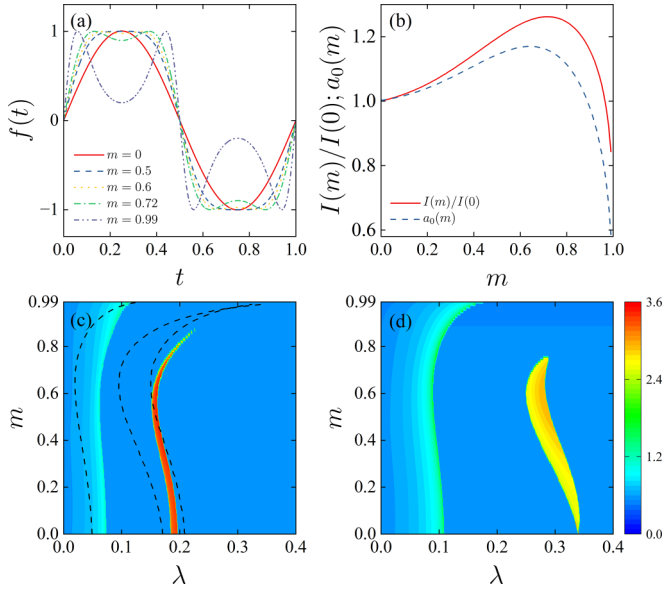


FIG. 5. (a) The waveform of the elliptic function signal $f(t)$ for $m = 0, 0.5, 0.6, 0.72$, and 0.99 . (b) The normalized impulse $I(m)/I(0)$ and the first Fourier coefficient $a_0(m)$ as functions of m . (c), (d) The signal responses as a function of the coupling strength and the signal waveform for frequencies $\omega = \pi/25$ and $\omega = \pi/10$, respectively. The dashed lines in (c) represent the results from Eqs. (14), (18), and (23) using the first Fourier coefficient $a_0(m)$. The parameter A is fixed at 0.3 , and the initial value of distribution VI is taken into account.

the signal amplification to disappear at $m > 0.8$. Figure 5(d) further illustrates a scenario where the signal frequency is set to $\omega = \pi/10$. As shown in Fig. 4(a), this signal cannot be amplified when $m = 0$. However, for suitable values of $m \neq 0$, the signal can undergo significant amplification within the coupling interval (λ_2, λ_3) . These findings highlight the importance of the signal waveform in its amplification.

IV. ANALYSIS

We now conduct a quantitative analysis of the four-node ring with an initial distribution value of distribution VI and a sinusoidal signal for simplicity. During the signal amplification process, the ring experiences three distinct synchronization transitions [see Fig. 2(d)].

The first transition occurs at λ_1 , where oscillators 2, 3, and 4 oscillate around $x^* = 1$ with similar, small behaviors, while oscillator 1 crosses the midline and synchronizes with them. Since oscillator 1 undergoes a significant dynamic change, we can assume that oscillators 2 and 4 have a considerable effect on it, whereas oscillator 1 has a weaker effect on them. Meanwhile, the interactions among oscillators 2, 3, and 4 can also be regarded as weak. Consequently, the first synchronization transition can be characterized as a feedforward motif, as illustrated in Fig. 6(a). The second transition takes place at λ_2 , where oscillators 1 and 3 oscillate with small amplitudes around $x^* = 1$ and $x^* = -1$, respectively. These oscillators compete for synchronization with oscillators 2 and 4, resulting in large oscillations in the latter. Since oscillators 2 and 4 experience significant dynamic changes while oscillators 1

and 3 remain relatively stable, we can conclude that oscillators 1 and 3 exert a strong influence on oscillators 2 and 4, while the reverse influence is negligible. Thus, the second synchronization transition can also be approximated using the feedforward motif depicted in Fig. 6(b). The third transition happens at λ_3 . In this scenario, oscillators 2 and 4 exhibit large oscillations between $x^* = 1$ and $x^* = -1$, while oscillators 1 and 3 generate small oscillations near $x^* = 1$ and $x^* = -1$, respectively. Driven by the subthreshold signal and the large oscillations of oscillators 2 and 4, oscillator 1 moves towards $x^* = 1$ and synchronizes with oscillators 2, 3, and 4. Similarly, it can be inferred that oscillator 1 does not significantly affect the behavior of oscillators 2 and 4, allowing us to neglect its impact on them. Therefore, the third synchronization transition can be represented by the feedforward motif shown in Fig. 6(c).

A. The first synchronization transition

The feedforward motif presented in Fig. 6(a) can be formulated as follows:

$$\begin{aligned} \dot{x}_1 &= (1 - \lambda)x_1 - x_1^3 + \frac{\lambda}{2}(x_2 + x_4) + A \sin(\omega t), \\ \dot{x}_2 &= x_2 - x_2^3 + A \sin(\omega t), \\ \dot{x}_4 &= x_4 - x_4^3 + A \sin(\omega t). \end{aligned} \quad (10)$$

The signal response of this motif (10), as depicted in Fig. 6(a), exhibits a peak at $\lambda = 0.072$, followed by a sudden drop to a constant value. Consequently, motif (10) effectively captures the first synchronization transition observed in Figs. 2(d) and 3(d).

By utilizing the linearization method, we can obtain approximate solutions for x_2 and x_4 as follows:

$$x_{2,4}(t) = x_{2,4}(0) + \frac{2A}{4 + \omega^2} \sin(\omega t) - \frac{\omega A}{4 + \omega^2} \cos(\omega t), \quad (11)$$

where $x_{2,4}(0) = -1$. Consequently, Eq. (10) can be reformulated as

$$\dot{x}_1 = (1 - \lambda)x_1 - x_1^3 - \lambda + AB \sin(\omega t + \phi_1), \quad (12)$$

where $B = \sqrt{(2\lambda + 4 + \omega^2)^2 + \omega^2 \lambda^2} / (4 + \omega^2)$ and ϕ_1 represents a phase shift. Equation (12) has three fixed points, $x_1^* = -1$ and $x_1^* = (1 \pm \sqrt{1 - 4\lambda})/2$, when $A = 0$. For $A \neq 0$ and small λ , Eq. (12) exhibits small oscillations around $x_1^* = (1 + \sqrt{1 - 4\lambda})/2 \approx 1$ described by the following equation:

$$\begin{aligned} x_1(t) &= \frac{1 + \sqrt{1 - 4\lambda}}{2} + \frac{1}{\sqrt{\left(\frac{1}{2} - 2\lambda + \frac{3\sqrt{1 - 4\lambda}}{2}\right)^2 + \omega^2}} \\ &\quad \times AB \sin(\omega t + \phi_2), \end{aligned} \quad (13)$$

where ϕ_2 denotes another phase shift.

As the coupling strength exceeds λ_1 , the signal described in Eq. (12) becomes a suprathreshold. For a very low signal frequency ω , we can determine the critical coupling strength λ_1 using the following equation [19]:

$$\left(\frac{\lambda + AB}{2}\right)^2 = \left(\frac{1 - \lambda}{3}\right)^3. \quad (14)$$

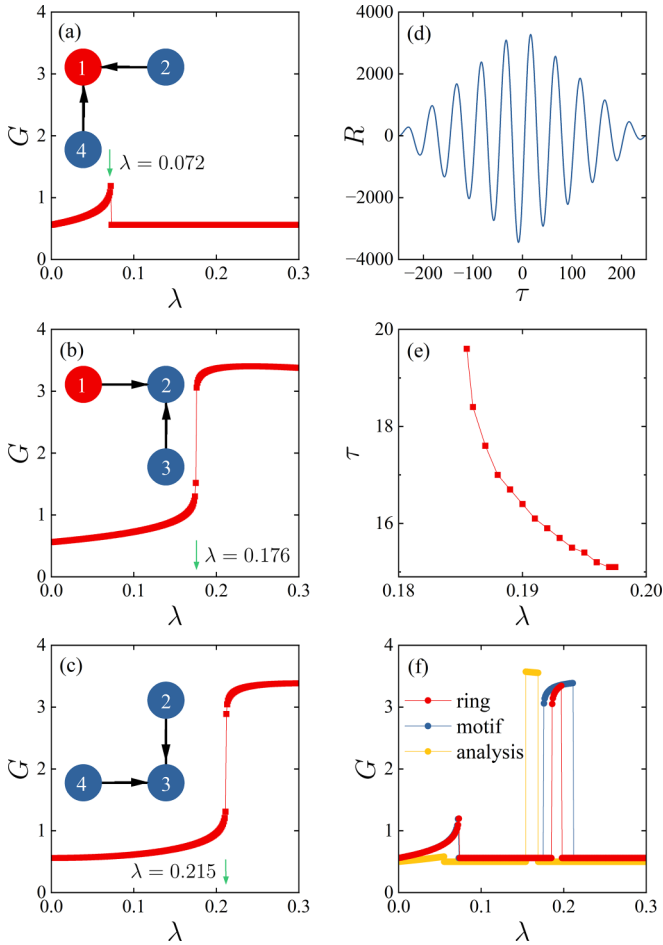


FIG. 6. The left panels illustrate the signal response as a function of the coupling strength for subthreshold signals with varying (a) frequencies and (c) amplitudes. The right panels display the correlations between the critical couplings λ_1 , λ_2 , and λ_3 and the (b) signal frequency and (d) amplitude, where dashed lines denote the results of Eqs. (14), (18), and (23). The parameters $A = 0.3$, $\omega = \pi/25$, and distribution VI are considered.

By substituting $A = 0.3$ and $\omega = \pi/25$ into Eq. (14), we find that the analytical threshold is $\lambda'_1 = 0.054$. Thus, when $\lambda > \lambda_1$, x_1 can cross the midline $x = 0$ and synchronize with x_2 and x_4 . As a result, motif (10) behaves as a single oscillator, as described in Eq. (11).

B. The second synchronization transition

The dynamics of the feedforward motif, seen in Fig. 6(b), can be described by the following equations:

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1^3 + A \sin(\omega t), \\ \dot{x}_2 &= (1 - \lambda)x_2 - x_2^3 + \frac{\lambda}{2}(x_1 + x_3) + A \sin(\omega t), \\ \dot{x}_3 &= x_3 - x_3^3 + A \sin(\omega t).\end{aligned}\quad (15)$$

Equation (15) resembles the three-node motif derived from scale-free networks [28]. The signal response, shown in Fig. 6(b), indicates a discontinuous rise at $\lambda = 0.176$, which is close to the critical coupling strength $\lambda_2 = 0.186$ seen in

Fig. 2(d). Thus, motif (15) may help explain the synchronization competition mechanism in the ring.

From Eq. (11), we can express the solutions for x_1 and x_3 in Eq. (15) as follows:

$$x_{1,3}(t) = x_{1,3}(0) + \frac{2A}{4 + \omega^2} \sin(\omega t) - \frac{\omega A}{4 + \omega^2} \cos(\omega t), \quad (16)$$

where $x_1(0) = 1$ and $x_3(0) = -1$. By substituting these into Eq. (15), we derive

$$\dot{x}_2 = (1 - \lambda)x_2 - x_2^3 + AB \sin(\omega t + \phi_1). \quad (17)$$

In Eq. (14), the signal $AB \sin(\omega t + \phi_1)$ becomes a suprathreshold at λ_2 , enabling it to drive x_2 to generate large oscillations between $x^* = \pm 1$. According to Eq. (14), the critical value of λ_2 satisfies

$$\left(\frac{AB}{2}\right)^2 = \left(\frac{1 - \lambda}{3}\right)^3. \quad (18)$$

For the parameters $A = 0.3$ and $\omega = \pi/25$, the analytical threshold is determined to be $\lambda'_2 = 0.153$.

To solve Eq. (17) when the coupling strength exceeds λ'_2 , we consider a low signal frequency, treating the signal as a constant value of $\pm AB$. Then, we estimate the solution for Eq. (17) by finding the real roots of the cubic equation $(1 - \lambda)x_2 - x_2^3 \pm AB = 0$, leading to the following solution:

$$\begin{aligned}x_2(t) &= 2\sqrt{\frac{1 - \lambda}{3}} \cosh \left[\frac{1}{3} \operatorname{arcosh} \left(\sqrt{\frac{27A^2B^2}{4(1 - \lambda)^3}} \right) \right] \\ &\times \sin(\omega t + \phi_3),\end{aligned}\quad (19)$$

where ϕ_3 represents some phase shift.

C. The third synchronization transition

The equations that govern the motif depicted in Fig. 6(c) can be expressed as

$$\begin{aligned}\dot{x}_3 &= (1 - \lambda)x_3 - x_3^3 + \frac{\lambda}{2}(x_2 + x_4) + A \sin(\omega t), \\ x_2 &= x_4 = \sin(\omega t - \psi),\end{aligned}\quad (20)$$

where oscillators 2 and 4 are simplified by a sinusoidal signal with unit amplitude and an initial phase denoted as ψ . This simplification is based on two factors: these oscillators alternate between $x^* = 1$ and $x^* = -1$ within the coupling interval (λ_2, λ_3) , and their time series show a time lag τ (phase difference ψ) relative to the subthreshold signal [see Fig. 3(d)]. Thus, Eq. (20) can be rewritten as

$$\dot{x}_3 = (1 - \lambda)x_3 - x_3^3 + C \sin(\omega t + \phi_4), \quad (21)$$

where $C = \sqrt{\lambda^2 + A^2 + 2\lambda A \cos(\psi)}$ and ϕ_4 denotes some phase shift.

To quantify the phase difference ψ between $x_2(t)$ and the subthreshold signal $A \sin(\omega t)$, we employ the cross-correlation coefficient defined as

$$R(\tau) = \int_{-\infty}^{\infty} x_2(t) A \sin[\omega(t - \tau)] dt. \quad (22)$$

A high value of R indicates a strong similarity between the two signals at a specific time lag τ , whereas a low value

of R suggests a weak linear relationship between them. For instance, Fig. 6(d) displays the cross-correlation coefficient at $\lambda = 0.19$, where the maximum positive value occurs at $\tau \approx 16.5$. This means that a time lag of $\tau \approx 16.5$ applied to $\sin(\omega t)$ closely approximates the dynamics of $x_2(t)$. In addition, Fig. 6(e) calculates the time lag that corresponds to the maximum positive cross-correlation coefficient between $x_2(t)$ and $A \sin(\omega t)$ in the coupling interval (λ_2, λ_3) . As the coupling strength approaches λ_3 , the time lag decreases to $\tau \approx 15$, allowing us to estimate the initial phase in Eq. (20) as $\psi = \omega\tau \approx 3\pi/5$. Substituting this into Eq. (21) shows that the signal response of oscillator 3 suddenly increases at $\lambda = 0.215$, as seen in Fig. 6(c). This indicates that oscillator 3 will cross the midline after $\lambda_3 = 0.197$. Thus, Eq. (20) effectively predicts the critical coupling strength for the third synchronization transition observed in the ring.

In a similar manner, the critical coupling strength for Eq. (21) when crossing the midline is determined by the following relationship:

$$\left(\frac{B}{2}\right)^2 = \left(\frac{1-\lambda}{3}\right)^3. \quad (23)$$

For the parameters $A = 0.3$, $\omega = \pi/25$, and $\phi = 3\pi/5$, the analytical threshold is calculated to be $\lambda'_3 = 0.164$. Figure 4(d) shows the analytical thresholds λ'_i ($i = 1, 2$, and 3) obtained from Eqs. (14), (18), and (23). Although these results are slightly underestimated in comparison to the numerical data, they effectively demonstrate that the critical coupling strength λ_i decreases as the amplitude A increases.

When the coupling strength exceeds λ_3 , four oscillators achieve synchronization around $x^* = 1$, behaving as described in Eq. (11).

Based on the solutions presented in Eqs. (11), (13), and (19), the analytical signal response for the three feedforward motifs can be derived as follows:

$$G = \begin{cases} \frac{B}{\sqrt{\left(\frac{1}{2} - 2\lambda + \frac{3\sqrt{1-4\lambda}}{2}\right)^2 + \omega^2}}, & \lambda < \lambda'_1, \\ \frac{1}{\sqrt{4+\omega^2}}, & \lambda'_1 \leq \lambda < \lambda'_2, \\ \frac{2}{A} \sqrt{\frac{1-\lambda}{3}} \cosh\left(\frac{1}{3} \operatorname{arccosh} \sqrt{\frac{27A^2B^2}{4(1-\lambda)^3}}\right), & \lambda'_2 \leq \lambda < \lambda'_3, \\ \frac{1}{\sqrt{4+\omega^2}}, & \lambda'_3 \leq \lambda. \end{cases} \quad (24)$$

Figure 6(f) shows the predictions derived from Eq. (24), which qualitatively reflect the observed double peaks in the signal responses of both the ring and the motifs. However, some discrepancies are evident between the predictions and the numerical data. These discrepancies can be attributed to several factors: the simplification of the ring structure into three-node feedforward motifs with unidirectional couplings, the oversimplified dynamics of the two driving oscillators within these motifs, and the assumption of a low signal frequency.

D. The synchronization transitions for elliptic function signal

We can also estimate the three critical coupling strengths λ'_i in relation to the parameter m by using three feedforward motifs. The Fourier expansion of the elliptic function is given

by [45]

$$f(t) = \sum_{n=0}^{\infty} a_n(m) \sin[(2n+1)\omega t], \quad (25)$$

where the coefficient $a_n(m)$ takes the form

$$a_n(m) = \frac{\pi^2 N(m)(n + \frac{1}{2})}{\sqrt{m} K^2(m)} \operatorname{sech}\left[\frac{(n + \frac{1}{2})\pi K(1-m)}{K(m)}\right]. \quad (26)$$

For simplicity, we will only consider the first harmonic of the Fourier expansion,

$$f(t) \approx a_0(m) \sin(\omega t). \quad (27)$$

The first coefficient $a_0(m)$ reaches its maximum normalized value $a_0(m)/a_0(m=0)$ at $m \approx 0.64$ [see Fig. 5(b)]. By substituting the amplitude $a_0(m)$ into Eqs. (14), (18), and (23), we can derive the critical coupling strengths λ'_i , which initially decrease and then increase with rising m [see Fig. 5(c)].

V. DISCUSSION AND CONCLUSIONS

In summary, we have examined the dynamics of a four-bistable-oscillator ring subjected to a subthreshold signal. The ring amplifies the signal through synchronization competition within a specific coupling range. We have analyzed the effects of the initial conditions of the oscillators, as well as the amplitude, frequency, and waveform of the subthreshold

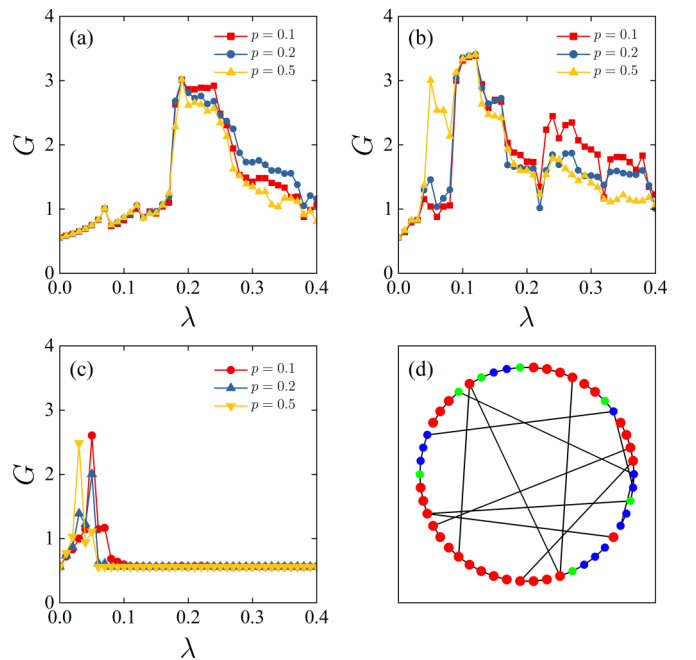


FIG. 7. The signal response of small-world networks to a sinusoidal signal with an amplitude $A = 0.3$ and a frequency $\omega = \pi/25$: (a) $N = 50$ and $\langle k \rangle = 2$, (b) $N = 100$ and $\langle k \rangle = 2$, (c) $N = 100$ and $\langle k \rangle = 4$. (d) A typical distribution of the oscillators corresponding to (a) during signal amplification. In this distribution, red circles represent the oscillators oscillating around $x^* = 1$, blue circles represent the oscillators oscillating around $x^* = -1$, and green circles represent oscillators switching between $x^* = \pm 1$. The lines represent the links between oscillators.

signal, on the phenomenon of synchronization competition. To simplify the signal amplification process, we have used three three-node feedforward motifs, which produce an analytical response similar to that of the ring.

This phenomenon is also seen in larger rings of oscillators, where the impact of the initial value distribution on signal amplification decreases. However, this is not the case in a minimal ring composed of three oscillators, in which synchronization is readily achieved due to its fully coupled structure. Furthermore, this phenomenon may elucidate the topological resonance effect observed in small-world networks. As shown in Figs. 7(a)–7(c), small-world networks may enhance their response to a subthreshold signal at intermediate-coupling strengths. However, as the average degree of the network, denoted as $\langle k \rangle$, increases, the signal response decreases [see Figs. 7(b) and 7(c)]. This decrease occurs because a higher average degree improves network synchronization, which destroys the signal amplification effect. Moreover, Fig. 7(d) illustrates a typical distribution of oscillators during signal amplification at $\lambda = 0.2$ shown in Fig. 7(a). The distribution reveals that the oscillators can be classified into three distinct groups. The first and second groups oscillate at low amplitudes, centered around $x^* = 1$ and $x^* = -1$, while the third group lies between them. The competition for synchronization between the first two groups leads to amplified oscillations in the third group, thereby enhancing the subthreshold signal. This indicates that topological resonance in small-world networks arises from the competition for synchronization. The shortcuts in these networks serve a dual purpose. They

can connect oscillators within the same groups, facilitating synchronization but disrupting amplification, or link groups at $x^* = 1$ and $x^* = -1$, creating a competing synchronization structure that enhances signal amplification. The role of these shortcuts depends on the initial value distribution of the oscillators.

Furthermore, our proposed feedforward motifs are similar to those observed in scale-free networks. Therefore, we conclude that the topological resonances seen in small-world and scale-free networks are fundamentally similar, driven by the mechanism of synchronization competition. It is important to note that the synchronization competition discussed here is different from the one analyzed in Ref. [46]. In that study, synchronization competition occurs due to a mixture of positive and negative couplings within brain networks. Oscillators with negative coupling tend to repel each other, forming clusters around $x^* = 1$ and $x^* = -1$, where they exhibit small oscillations. In contrast, oscillators with positive coupling strive to synchronize with these negatively coupled oscillators, while also being repelled by them. This interaction causes the positively coupled oscillators to oscillate between $x^* = \pm 1$, resulting in a significant response to weak signals. We hope that our findings will contribute to the development of simple and efficient artificial devices.

ACKNOWLEDGMENT

X.L. was supported by the NFSC of China under Grant No. 12175087.

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