

Second harmonic Bessel-Gauss beam shaping with elliptic axicon aberrations

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The second harmonic (SH) of an axicon generated Bessel-Gauss beam is created through the nonlinear interaction of photons with a crystal, resulting in the energy doubling of the output photons. In this work, we show experimentally that in addition to frequency doubling, the SH of Bessel-Gauss beams under asymmetric aberrations from an elliptic axicon exhibit intriguing beam formation. Particularly, the central region of the SH beam profile is composed of two central spots of various geometries surrounded by nested ellipses; one of which is the configuration of two central gamma dots with similar radius knotted by nested ellipses for a zeroth-order Bessel-Gauss pump. These SH beams consistently maintain their spatial profile throughout propagation, reminiscing the behavior of screw dislocations in wave patterns. Our numerical simulations produce beam dynamics consistent with that of experiments and further implicate the remarkable interweaving of bright spots with dark vortices. This is especially noticeable when the beams dynamically oscillate along the optical axis, resulting in the genesis of spatially polarized beams with a knotted framework. The insights gained from our study establish a novel paradigm for exploring interactions of Bessel-like beams with vortex dynamics. This, in turn, has the potential to spark innovations in optical applications, fostering new methodologies to harness and manipulate complex light structures. Our experimental findings could spur new methods of generating logical states of light and new opportunities for material processing control.

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I. INTRODUCTION

In their seminal 1974 paper, Nye and Berry made a pivotal discovery by observing that wavefronts can exhibit dislocation lines resembling those found in crystals [1]. These dislocations can take the form of loops or pairs, and their morphology was extensively analyzed, revealing intriguing characteristics such as curvature, intersection, collision, and rebound.

Subsequently, experimental studies in optics explored waves with screw dislocations [2,3]. These investigations resulted in the creation of double screw dislocations or phase singularities during the diffraction of plane waves or Gaussian beams by computer-synthesized gratings. The mathematical expressions for waves with phase singularities are exact

solutions of the Helmholtz equation, representing “cylindrical tubes” where wave intensity vanishes, and phase changes by multiples of 2π around the dislocation [4]. Notably, a superposition of Bessel beams has been proposed as an ideal method to generate such singularities. Bessel beams, initially proposed by Durnin, possess unique properties, including immunity to diffractive spreading and self-healing capabilities after encountering obstacles [5,6]. These beams exhibit a well-defined bright central spot surrounded by nested circular rings, and their intensity distribution remains invariant along the propagation direction. However, realizing ideal Bessel beams in the laboratory requires an infinite amount of energy.

Nevertheless, it is possible to obtain approximations of ideal Bessel beams within a limited spatial range. Such approximations offer practical solutions for various applications such as sensing [7], ablation [8,9], manufacturing [10,11], such as display glass precise machining [12,13], quantum optics [14,15], free space optical communication [16,17], beyond diffraction limited imaging [11,18] to name a few, bridging the gap between theoretical concepts and experimental realizations.

One such light field is termed Bessel-Gauss (BG) beams [19]. These beams combine the advantageous properties of

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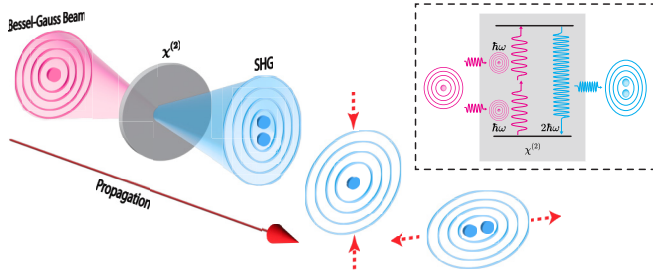


FIG. 1. Generation and behavior of knotted bright dots with a zeroth order Bessel-Gauss pump for a frequency doubling BBO crystal. The knotted dots surrounded by dark ellipses demonstrate collision, rotation, and perpendicular rebound relative to their initial alignment. The inset shows the SH process where two photons are combined through a $\chi^{(2)}$ crystal to give one photon at half of the wavelength and twice the amount of energy.

ideal Bessel beams with an additional Gaussian factor that modulates the light amplitude away from the origin. Experimental realizations of BG beams have been achieved using various techniques, such as conical lenses like axicons [20], reflective axicons [21], holographic masks [22], and spatial light modulators (SLMs) [23]. Arrays of BG beams are also generated by different methods and for different purposes [24–26]. BG beams not only maintain the self-healing property inherent in ideal Bessel beams [27] but also exhibit relatively diffraction-free propagation. As a result, these beams have found diverse applications in fields such as optical communication [28,29], encryption [30], optical trapping [31], and beyond [32]. In particular, researchers have explored the potential of nonlinear wave mixing to superpose BG beams and generate intriguing structures, such as screw dislocations, through processes like second harmonic (SH) generation [2–4]. Among the various approaches for SH generation with laboratory-generated Bessel beams, particular interest has been drawn to Bessel-like distributions [33,34]. By tuning phase-matching conditions in the crystal, the SH excited by Bessel-like distributions can be transformed into a high-brightness axial frequency-doubled beam [34].

Here, we investigate the second-harmonic generation for BG beams realized using both physical axicons and virtual axicons encoded as holograms on SLMs, following the methods described in Rosales-Guzmán and Forbes [23]. We first present a numerical model for a BG beam generated with a tilted axicon (also known as elliptic axicon) placed before a nonlinear crystal. Then, we compare our numerical results to the experimental observations and discuss the profound influence of the tilted axicon's angle on the spatial distribution of generated SH signal. We demonstrate the production of a SH beam with unique spatial profiles at the center surrounded by nondiffractive dark rings, signature of phase dislocations, similar to those of the pump beam [35], as illustrated in Fig. 1. Owing to the breaking of radial symmetry by the elliptic axicon, dark rings in these beams become slightly elliptic but intensity undulations at the center of the SH beam do not bear any resemblance to those at the pump beam. More revealing is in the SH phase where the collapse of the perturbed inner dark ring spawning vortex pairs, i.e., the formation of vortices

and antivortices is shown, akin to the instability of spatial dark solitons [36].

Importantly, our findings reveal a fascinating twist in the behavior of the SH light. Complementing previous discussions by Berry and Nye, where light intensity vanishes at phase singularities, we observe that bright elliptic central spots are manipulated by a quadrupole of vortex-antivortex pairs in dark regions where the vortex spacing contracts or expands with the propagation distance. These bright modes can be split into two distinct central spots of similar radius. This distinct dynamic is reminiscent of a Gaussian beam embedded with multiple vortice in free space propagation [37] and vortex quadrupoles in Bose-Einstein condensation (BEC) [38], as suggested in Refs. [39,40] that one may study the interaction process in BEC with the GrossPitaevskii equation, which shares similar form to the two-dimensional (2D) Schrödinger wave equation. Even more spectacular is the uncovering of vortex quadrupoles with high topological charges during the breaking up of multiple dark ring in our simulations.

Interestingly, these vortex beams retain the self-healing properties of BG beams, exhibiting rotation and adopting springlike oscillations as they propagate through space. The superpositionlike nature of BG beams allows the second-harmonic light to form linked and knotted structures, enabling the central spots to intersect, collide, rotate, and rebound in novel and remarkable ways. This discovery represents a significant advancement in our understanding of BG beams and opens up exciting possibilities for their applications in various fields.

II. NUMERICAL MODEL

We present a numerical model to describe the upconversion of BG beams generated by a tilted axicon through a three-wave mixing process as illustrated in Fig. 2.

We consider a time harmonic electric field in the form of $E(r, \phi) \exp(-i\omega t)$ where $E(r, \phi)$ assumes a Gaussian beam profile with an amplitude $\mathcal{E}$ and a beam waist w_0 expressed in the polar coordinates (r, ϕ) :

$$E(r, \phi) = \mathcal{E} e^{-\left(\frac{r}{w_0}\right)^2} \quad (1)$$

is incident on an elliptic axicon, i.e., an axicon tilting at an angle φ ($0 \leq \varphi < \frac{\pi}{2}$) with respect to the plane normal to the propagation axis. The elliptic axicon imposes a phase shift $\Phi'_{ax}(x, y)$:

$$\begin{aligned} \Phi'_{ax}(x, y) &= \Phi_{ax}(x, y \sec(\varphi)) \\ &= C_a \sqrt{x^2 + (y \sec(\varphi))^2}, \end{aligned} \quad (2)$$

where $C_a = -kD/4f_{ax}$ is a constant associated with the physical axicon parameters: D is the aperture, f_{ax} is the focal length and k is the wave number. The modulated Gaussian input now becomes

$$\begin{aligned} E_{ax}(x, y, \varphi) &= \mathcal{E} \exp\left(-\frac{x^2 + y^2}{w_0^2}\right) \\ &\times \exp(iC_a \sqrt{x^2 + (y \sec(\varphi))^2}) \end{aligned} \quad (3)$$

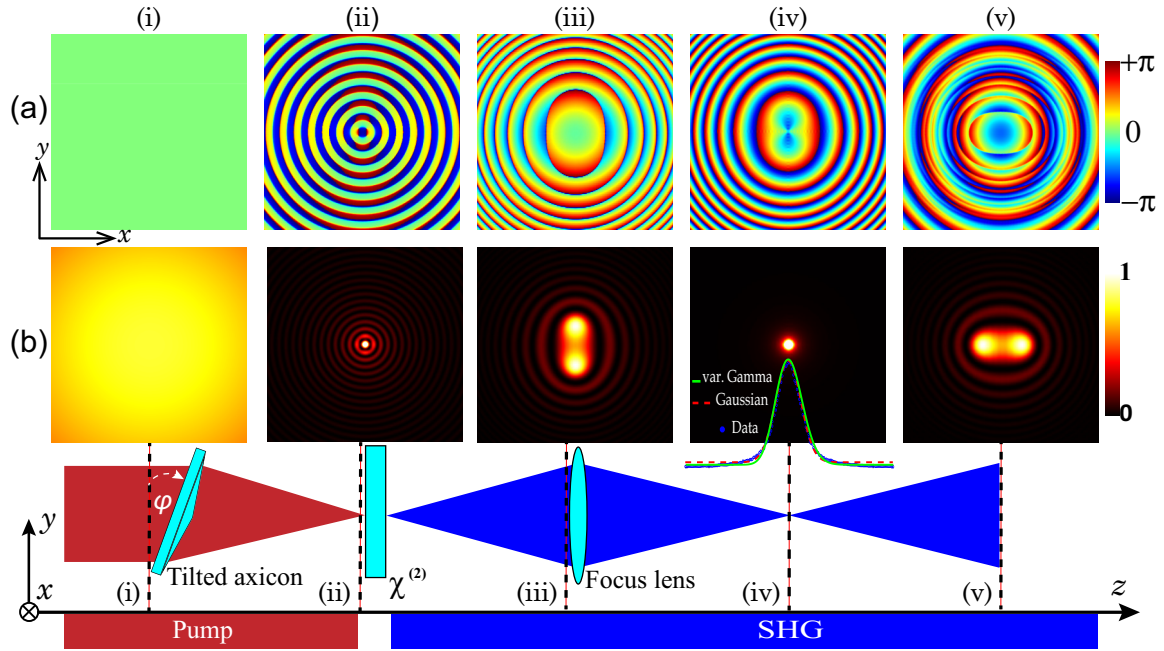


FIG. 2. Schematic of the numerical model showcasing simulation results for the second harmonic generation process. (a) Depicts the phases of the intensity profile shown in (b). (i) to (v) highlight the intensity and phase profiles at different positions along the propagation axis (z). A Gaussian pump (i) is focused by a tilted axicon, with a tilt angle $\phi = 4^\circ$, to form a slightly asymmetric zeroth-order Bessel-Gauss (BG) beam that is projected onto the $\chi^{(2)}$ material (ii). The second harmonic response is examined in the far field (iii), and at different positions after passing through a focusing lens [(iv) and (v)]. Each image is normalized for clarity. A line plot, fitted with both a Gaussian and a variant (var.) of Gamma distribution (see Supplemental Material [63], A22), is included in b(iv). A video demonstrates the behavior of the generated second harmonic near the Fourier plane.

The angular spectrum method is used to calculate the Fresnel integral and propagate the field E_{ax} in Eq. (3) to obtain E_f at the axicon's focus [41,42]. The beam profile E_f shown in Fig. 2(ii) is incident on the nonlinear crystal $\chi^{(2)}$. We note here that unlike the ideal zeroth-order BG beam, E_f lacks radial symmetry, thus showing the slight ellipticity in its intensity and phase profiles. However, the characteristic rings of a BG beam are maintained under this asymmetry and manifest as phase dislocations, reminiscent to those of (2+1) dimensional dark soliton stripes. In the Supplemental Material (SM) Appendix A, we provide an alternate method to describe the effect of the tilted axicon. With ideal phase matching and the absence of pump depletion, the generated SH field immediately after the $\chi^{(2)}$ crystal simplifies to

$$E_{SH} = C_{SH} E_f^2. \quad (4)$$

Here, C_{SH} is a constant related to the upconversion efficiency; it is proportional to the effective nonlinear optical coefficient d_{eff} and the square of the crystal thickness L : $C_{SH} \propto d_{\text{eff}} L^2$. The SH signal's far-field intensity and phase profile is shown in Fig. 2(iii). At this location, the rings of phase location is further squeezed by the asymmetry with more conspicuous ellipticity, particularly for the inner rings. In fact, the innermost ring becomes unstable and leads to the spawning of vortex-antivortex pairs under the asymmetric perturbations, a similar process as the breaking up of dark soliton stripes [36]. Most remarkable is the vortex quadrupole [37] formed by the vortex pairs at the upper and lower end of the ellipse that encloses the bright region with a dumbbell-like mode distribution.

To account for a lens in the path, we multiply the SH field by a phase term:

$$\exp(i\phi_l) = \exp[iC_l(x^2 + y^2)] \quad (5)$$

for a thin lens with a phase of ϕ_l comprising a constant $C_l = -k/(2f)$ determined by the focal length f , and the wave number k . After the beam propagates through the thin lens, the dumbbell-like mode collides at the focus as shown in Fig. 2(iv) and rebounds at a 90° angle of rotation along with the return of more vortex-antivortex pairs in Fig. 2(v). The phase profile of the dumbbell-like mode [Fig. 2a(iv)] suggests the vortex quadrupole annihilation at the collide plane while the dumbbell-like mode is merged as a single dot that can be best fitted with a variant of a Gamma distribution as indicated by the line plot in Fig. 2b(iv), leading us to refer each bright spot of the dumbbell-like mode as Gamma-dot (GD) (more details in SM, Appendixes B & C).

III. EXPERIMENTAL SETUP

The Bessel-Gauss beams are created with two methods: a physical axicon and a virtual axicon rendered on a SLM. The virtual axicon pattern is created by encoding both an azimuthal variation (with modulus 2π) and a blazed grating, ensuring the separation of the primary order from others [23]. This is mathematically described by

$$\Phi_{SLM} = \text{mod} [k\alpha(n-1)r + \ell\phi + 2\pi(G_x x + G_y y), 2\pi]. \quad (6)$$

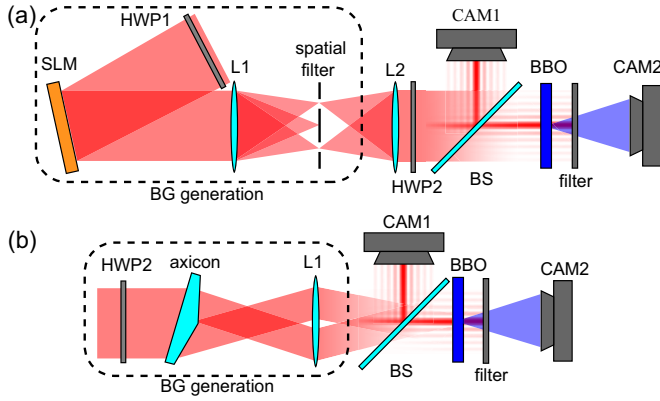


FIG. 3. Illustration of the experimental configuration for structured beam generation and characterization with a BBO crystal. (a) An expanded Gaussian laser beam is directed at the SLM. The structured output from the SLM is collected by lens L1. The fundamental beam's polarization is modulated by a half-wave plate (HWP2) before being focused onto the BBO crystal via lens L2. A spatial filter located in the Fourier plane of L1 out all orders except for the primary one from the generated beam. (b) An alternative configuration employs an axicon instead of the SLM and lens L1. The beam incident on the BBO is captured with a scientific CMOS (sCMOS, CAM1) camera. The resulting second harmonic is filtered and analyzed with an Electron Multiplying CCD (EMCCD, CAM2).

Here, mod represents the modulus function. The parameters G_x and G_y define grating frequencies in the x and y directions, n is the refractive index of the virtual axicon, and α represents the axicon angle or its base angle. The experimental setup is illustrated in Fig. 3. The laser source is a mode-locked regenerative Ti:sapphire with a central wavelength of 808-nm and a 38-fs pulse. The spatially filtered laser output beam is linearly polarized and then expanded to cover the SLM, a Santelec SLM-200 phase-only reflective liquid crystal on silicon with a resolution of 1080 x 1920 with 8- μm pixel size. For SH generation, a 500- μm thick β -Barium Borate (BBO) nonlinear crystal is mounted on a stage allowing axial translations and angular adjustments for optimal phase matching. The desired phase mask or holographic grating is computer-generated and displayed on the SLM. A custom-made aperture serves as a spatial filter, ensuring only the selected order passes through in the Fourier plane of the lens L1. The resultant order postaperture is then focused on the BBO crystal by the lens L2. An sCMOS camera (CAM1) captures the beam's shape and intensity before it interacts with the BBO crystal. The phase-matching conditions are optimized by adding a half-wave plate (HWP2) placed in front of the BBO. For comparative purposes, the system composed of the SLM and lens L1 is occasionally swapped with an axicon, generating a zeroth-order BG beam [as illustrated in Fig. 3(b)]. Both the camera CAM1 and the BBO crystal are strategically placed at equidistant positions from the beam splitter. This confirms that the desired fundamental beam profile is incident on the crystal. The resulting SH signal, captured in the far field, is imaged with an EMCCD camera (CAM2). We utilize a pair of interference filters ($405 \pm 5\text{-nm}$) to ensure that only the SH signal is captured by the camera CAM2.

IV. RESULTS AND DISCUSSION

The electric field of an ideal Bessel beam is mathematically described by

$$E(r, \phi, z) = \mathcal{E} \times J_\ell(k_r r) e^{ik_z z} e^{i\ell\phi}, \quad (7)$$

where $J_\ell(k_r r)$ are the Bessel functions of the first kind with order ℓ , k_r and k_z are the radial and longitudinal components of the wave vector $\vec{k}$, respectively, obeying the relation $|k|^2 = k_r^2 + k_z^2$, and $\mathcal{E}$ is the amplitude. This beam has an infinite extent and its intensity profile is a bright central spot ($\ell = 0$) surrounded by nested rings. Each bright ring carries approximately the same power as the central spot. The beam's intensity profile is also propagation invariant. This combination prohibits an exact implementation in experiments. There are, however, a number of ways to generate good approximations. Here, we use the BG approximation where an additional Gaussian factor with a width of w'_0 [43] is introduced to decrease the amplitude away from the origin, as follows [23]:

$$E(r, \phi, z = 0) = \mathcal{E} \times e^{-\left(\frac{r}{w'_0}\right)^2} J_\ell(k_r r) e^{i\ell\phi}. \quad (8)$$

This BG maintains its profile within a finite distance z_{max} , also known as the depth of focus (DoF). For a BG generated by either an axicon or a holographic mask, z_{max} is determined by the (virtual) axicon angle α , the incident beam radius w_0 in Eq. (1), and the deflection angle $\theta = \arcsin(n \sin \alpha) - \alpha$ between the refracted ray and the z -axis. Specifically, $z_{\text{max}} = w_0(\cot \theta - \tan \alpha)$ [44].

In this section, we aim to explore the evolution of the SH intensity and profile shape along propagation direction under various axicon tilt angles and pumps with high order BG beams. Additionally, we investigate the effect of the focusing lens' (L_2) tilt angle combined with the SLM's shallow angle in the case of a virtual axicon.

SH generation with a tilted axicon. Here, we consider the zeroth-order of the Bessel-Gauss beam generated by an axicon. Figure 4 depicts the dependence of the SH signal intensity distribution on the axicon's tilt angle φ . We present the numerical results in Figs. 4(a) and 4(b) and the experimental observations in Fig. 4(c). In the numerical model, the value of φ is varied from 0° , 4° , 6° , and 8° . Figures 4(a) and 4(b) show, respectively, the SH phases and their corresponding intensity profiles.

In the experiment, we mount the axicon (130° apex angle, $\varnothing = 25.4\text{-mm}$, and $\alpha = 25^\circ$) onto a high-precision rotation stage (Thorlabs, PR01), to change φ to 0° , 2° , 4° , and 6° . The generated SH for each φ are shown in Fig. 4(c). Experimentally, when the axicon is tilted beyond 6° the fundamental beam starts to deviate too far from the nonlinear crystal axis for the SH generation.

At normal incident $\varphi = 0$, the corresponding SH signal has an intense central dot surrounded by circular rings as demonstrated earlier by Belyi *et al.* [34,45]. As φ is increased, the intensity distribution changes gradually from a single bright GD surrounded by circular rings to two separate GDs encased by dark ellipses.

We observe that the two central spots are similarly bounded by dark ellipses both experimentally for $2^\circ \leq \varphi < 6^\circ$ and numerically for $4^\circ \leq \varphi < 8^\circ$. When the axicon is tilted even

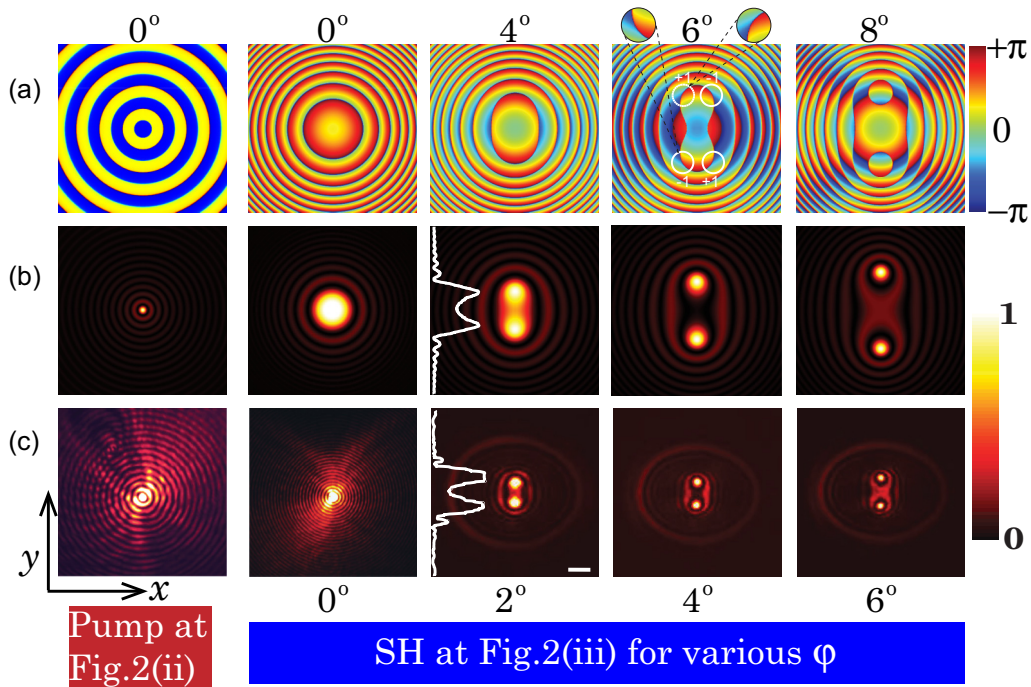


FIG. 4. Dependence of the second harmonic (SH) intensity profile on the tilt angle (φ) of an axicon, at the plane (iii) referenced in Fig. 2. (a), (b) Numerical results: From left to right, the panels show the intensity and the corresponding phase profile of the fundamental zeroth-order Bessel-Gauss (BG) beam, alongside the SH generated at axicon tilt angles of $\varphi = 0^\circ, 4^\circ, 6^\circ$, and 8° . On the phase profile for $\varphi = 6^\circ$ in (a), the vortex quadrupole with positive and negative unity topological charges are marked with circles in white lines and zoomed as visual guides. Similar quadrupole formation can be found from the phase profile for $\varphi = 8^\circ$. (c) Experimental results: From left to right, the panels display the intensity profile of the fundamental zeroth-order BG beam and the generated SH for axicon tilt angles of $\varphi = 0^\circ, 2^\circ, 4^\circ$, and 6° . All intensity profiles are normalized for comparison. A line plot is included in the third column for both (b) and (c), highlighting specific profile characteristics. In the same column, a scale bar, located at the lower right corner, represents a length of 0.5 mm.

further (experimentally: $\varphi \geq 6^\circ$, numerically: $\varphi \geq 8^\circ$), the resulting SH beam has each bright dot surrounded by its first Bessel ring merging its counterpart around the other bright dot to give an “8” like contour in the intensity distribution. This peculiar shape is in turn bounded by dark ellipses as shown in Figs. 4(b) and 4(c). The slight variation of φ is hardly noticeable for the pump beam but it is very significant for the SH intensity distribution. As shown in the Fig. 4(c), the alignment of the GDs knotted by ellipsoidal rings is along the y – axis (x – axis). When the axicon is tilted around y – axis (x – axis) (see Fig. S5), the focus in the (x, y) plane is slightly elongated along y – axis (x – axis) at the surface of the BBO. That ellipticity and the split between the Gamma dots increase as the axicon’s titled angle (φ) increases [46]. The dumbbell-like mode’s orientation after the NL crystal is directly related to the axis (x or y) around which the axicon is rotated (SM, Appendix A4, FIG. S5). The experiment yields similar results when it is repeated using a 1-mm thick BBO crystal. Under these conditions, crystal thickness does not significantly impact SH intensity distribution provided spatial walkoff is negligible and pump beam variation within the crystal is minimal.

SH generation excited by high-order BG beam. The use of an SLM to generate Bessel beams is of great advantage because the control of the encoded holograms described by Eq. (6) enables implementing a variety of axicon designs for the creation of any desired BG beam. Here, we limit our study to BG beams with topological charge $\ell = \{0, 3, 5\}$

(corresponding to BG₀, BG₃, and BG₅ beams) as shown in Fig. 5. In the case of the virtual axicon, the astigmatism aberration is generated by the sum of a shallow reflection angle (φ_s) of the SLM and a slight tilt angle (φ_l) of the focusing lens placed before the BBO. The lens’s tilt angle is negligible compared to the SLM shallow reflection angle but necessary for the formation of the dumbbell-like SH light.

The experimental SH profile generated by the virtual axicon designs are presented in Fig. 5(c). Their counterparts generated by numerical simulations are presented in [Figs. 5(a) and 5(b)]. For these sets of results, the tilt angle φ is fixed at 10° .

It is imperative to notice the consistency among these results. Particularly, the SH intensity distribution generated from the zeroth order BG beam manifests as two bright dots centered in dark Bessel-rings which in turn are cocooned in pinched Bessel-rings in the form of dark ellipses as shown in the first column of Fig. 5(c). Within these dark rings and ellipses, phase circulations enforced by dislocations and vortices are concealed. We can further observe that the topology of the input beams is conserved through the frequency doubling process. Each of the SH light possess the same topological charge as the incident beam as evidenced in the results displayed in the middle and right column of Fig. 5.

The following remarks can be added: (i) For an asymmetric fundamental beam carrying a topological charge $\ell \neq 0$, the orbital angular momentum (OAM) is conserved as $2\ell\hbar = \ell\hbar + \ell\hbar$ where the left-hand side of the equality corresponds

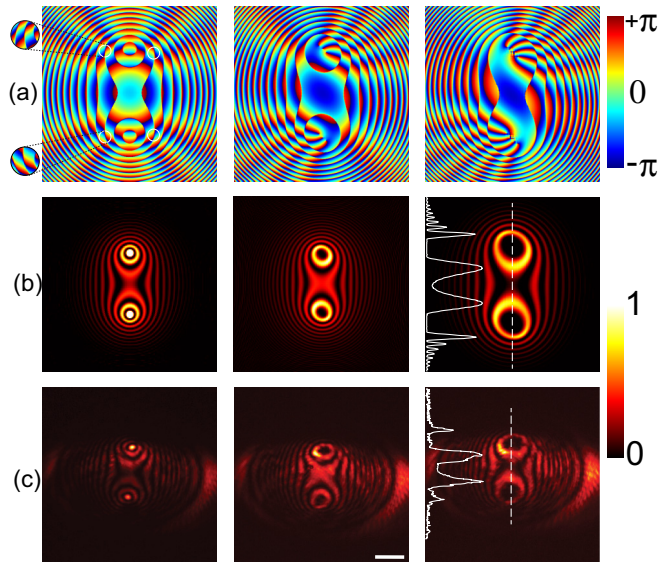


FIG. 5. Numerically and experimentally generated second harmonic (SH) signals with Bessel-Gauss (BG) beams carrying topological charges $\ell = 0, 3, 5$, at the location of Fig. 2(iii). (a) Phase profiles of numerically simulated SH signals for incident BG_0 , BG_3 , and BG_5 beams. (b) Intensity profiles corresponding to the phase profiles in (a), with the axicon tilt angle φ fixed at 10° . (c) Intensity distributions of experimentally generated SH light excited with the aforementioned beams, captured with the camera CAM1. An intensity plot for both the numerically and experimentally generated SH BG_5 beam is included. A 0.5-mm scale bar is provided in the second column for reference. Notice that more quadrupoles are formed than cases with smaller φ in Fig. 4 and even more fascinating is the doubling in topological charges. The location for one of the quadrupoles is marked by circles in white lines and zoomed for more clarity.

to the two pump photons combined through the SHG process to give the resulting SH light with the coupled vortices. Moreover, these vortices can merge to form a high-order vortex at a certain propagation distance without violating the charge conservation. (ii) For the zeroth order BG input, the resulting GD is surrounded by the first-order Bessel ring and then encased by dark ellipses as shown in Fig. 5. (iii) For higher order BG pump beam the SH beam produced is comparable to wavefront dislocations as described by Nye and Berry [1,4].

Further discussions on the control of the dumbbell-like mode structures can be found in SM, Appendix C3 and C4.

Dumbbell-like mode behavior at Fourier plane. Here, we investigate the intensity distribution of the generated SH at the Fourier plane (FP). In Figs. 6(a) and 6(b), we present the SH phase and intensity profile generated by a zeroth-order BG beam (BG_0). The virtual axicon used, is progressively tilted with $\varphi = 0^\circ, 4^\circ, 6^\circ, 8^\circ, 10^\circ$, and 15° . We observe that for $\varphi < 6^\circ$, the SH intensity pattern at the FP is a Gaussian-like intensity profile.

When $\varphi \geq 6^\circ$, the SH intensity profiles show a cross with arms progressively lengthen as φ increases. This observation confirms the increase of the separation between the GD that form the dumbbell-like mode shown in Fig. 4. In Figs. 6(c) and 6(d), we display the intensity distribution and phase profile at the FP, of the SH beams generated from higher order BG beam ($BG_{\ell \neq 0}$) with φ fixed at 10° . The topological charge of the incident beams is varied from $\ell = 1$ to 5. The SH intensity generated by a BG_ℓ beam with $\ell = 0, 1$, or 2 has a similar number of arms to that of the zeroth order beam but the structure is tilted along the antidiagonal line. For $\ell = 3$, the different arms of the SH light start to coalesce and we additionally show that the direction of the tilt depends on the

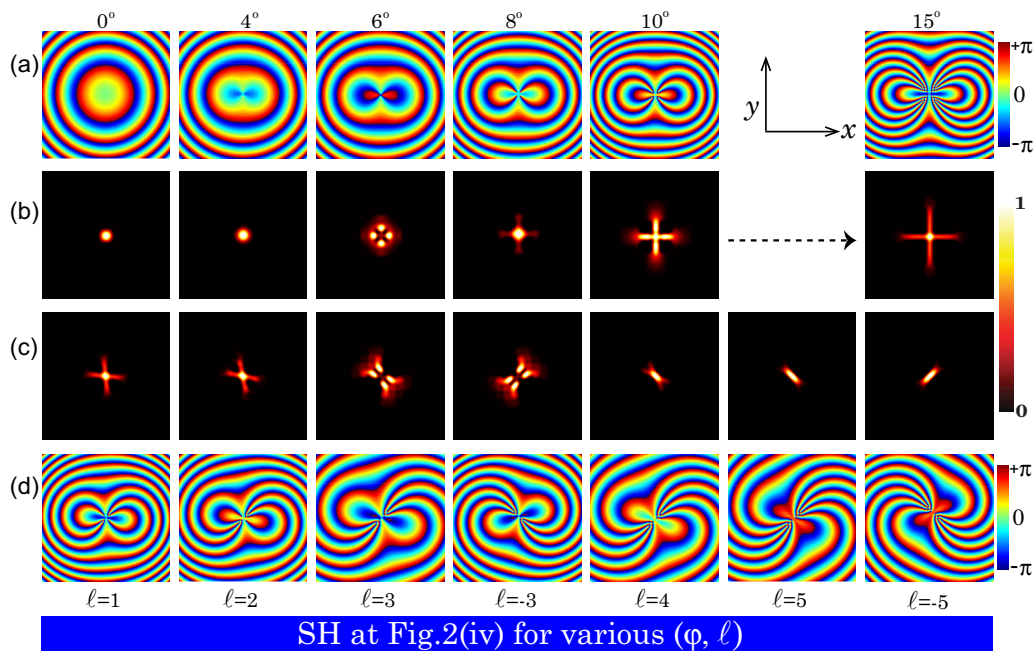


FIG. 6. Numerical simulation of the second harmonic (SH) intensity and its corresponding phase profile at the Fourier plane, as referred to in Fig. 2(iv). (a), (b) SH intensity and phase profile produced by a zeroth-order Bessel-Gauss (BG) beam of the first kind, for various axicon tilt angles ($\varphi \in [0^\circ, 15^\circ]$). (c), (d) SH intensity and phase profiles at the Fourier plane generated by high-order BG_ℓ beams, where $\ell \in [-5, 5]$, with an axicon tilt angle of $\varphi = 10^\circ$.

sign of ℓ . As ℓ increases, the arms of the cross have now combined into a single line focus, which shrinks and shortens, as shown in Fig. 6(c). This line pattern is due to the astigmatic transformation at the Fourier plane [47]. Our experimental observations and numerical simulations reveal that the SH beam with the dumbbell-like mode (generated by a BG pump with $\ell = 0$) enclosed by vortex quadruples [37] are due to the astigmatic distortion induced by the tilting of the axicon. Similarly, a $BG_{\ell \neq 0}$ beam can create, through the SHG process, a vortex-vortex pair that rotates at the FP [48] and lines up in the far-field along the axis orthogonal to the axis around which, the axicon is tilted away. However, the process for the dumbbell-like mode to reach the far-field position is fundamentally different from that of the vortex-vortex pair since the vortex-antivortex pair in a vortex quadrupole allows only rectilinear motion in contrast to the rotation of a vortex-vortex pair.

V. CONCLUSION

In summary, we have observed the second harmonic signal from zeroth-order and high-order Bessel-Gauss beams produced by an axicon and by a virtual axicon through an SLM. By producing an astigmatism aberration from slightly tilting an axicon, or the focusing lens of a virtual axicon, we have demonstrated that the intensity distribution of the generated SH with a zeroth-order BG pump is composed of bright spots of various geometries surrounded by concentric dark rings. This is also demonstrated by varying the refractive indices of a virtual axicon. Further, the shape of the bright spots and associated dark rings depends on the tilt angle or the index of refraction (φ, φ', n). We further demonstrate that the SH intensity profile is transformed from one with circularly symmetric dark rings and one bright spot for $\varphi = 0^\circ (n = 1.5)$ into an asymmetric one with dark ellipses and two bright spots of similar radius linked by an “8” shaped-contour in the intensity profile for $\varphi = 6^\circ (n = 2.5)$. The most peculiar feature is the formation of vortex quadrupoles with vortex-antivortex pairs under cover of darkness that can manipulate the motions of the bright spots.

The SH light generated with BG_ℓ beams ($\ell \neq 0$), shows angular momentum conservation with the formation of higher

order vortices. For these beams, the “8” shaped-contour in the intensity profile encloses the donut intensity distribution of vortices. Remarkably, we uncover that these GDs ($\ell = 0$) or donut modes ($\ell \neq 0$) collide, rebound, and flip as they propagate exhibiting the behavior of screw dislocations although the motion of a vortex-antivortex pair is fundamentally different from that of a vortex-vortex pair. Our experiments have demonstrated the potential of the nonlinear optical process as a versatile testbed for exploring the dynamics of vortices and their interactions with non-diffracting beams. Further studies of our results, specifically the GD configuration, could be used to describe stationary configurations of electrons in free space where Schrödinger’s equation reduces to the Helmholtz equation. Additionally, one can also note that as the GDs collide and flip direction, the two become indistinguishable. For this reason, our experimental findings could spur new methods of generating logical states of light and could broadly impact future quantum technologies, such as the preparation of newly demonstrated Gottesman-Kitaev-Preskill qubits [49]. The high-order GDs’ beam can be used to generate higher-dimensional quantum entangled photons pairs [50]. It can also be applied in 3D microscopy: for example, the GD beam generated by a BG_0 beam can be used in a confocal Raman microscopy for an enhanced in-depth scan of two different phases simultaneously.

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The authors declare no conflicts of interest.

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- [63] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevResearch.7.013012> for model conception is based on the spatial distribution of the beam formed by a tilted axicon and its Fourier transform and inverse. The result is supported by numerical results. Some of the experimental results are obtained by displaying into a spatial light modulator the beam’s phase generated by our model, which includes Refs. [51–62].