

Weak superfluidity in twisted optical potentials

Dean Johnstone¹, Shanya Mishra¹, Zhaoxuan Zhu¹, Hepeng Yao², and Laurent Sanchez-Palencia¹¹CPHT, CNRS, Ecole Polytechnique, IP Paris, F-91128 Palaiseau, France²DQMP, University of Geneva, 24 Quai Ernest-Ansermet, CH-1211 Geneva, Switzerland

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A controlled twist between separate potentials can drastically influence localization properties of quantum particles between ordered and (quasi)disordered limits. Transport properties of single-particle and correlated fermionic materials have been extensively studied in connection with twisted bilayer graphene, but their bosonic counterpart remains largely unexplored. Here, we study bosonic matter in twisted potentials. We use continuous-space quantum Monte Carlo simulations to determine the unique phase diagrams of strongly correlated ultracold bosons in twisted optical lattices. For commensurate twisting angles, spectral gaps govern the formation of insulators, separated by thin superfluid domains. These domains form weak superfluids; with low superfluid fraction at zero temperature and high sensitivity to thermal fluctuations, but may be stabilized under appropriate parameter control. In contrast, slightly changing the twisting angle to an incommensurate value destroys most spectral gaps, leaving behind a prominent Bose glass phase. Our results are directly applicable to current generation experiments that quantum simulate moiré physics.

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Introduction. In the field of condensed matter, strongly correlated materials exhibit exotic phase transitions and entanglement properties. This behavior may be enhanced in systems that have flat energy bands, such as stacked two-dimensional (2D) materials, for they exaggerate the role of interactions over kinetic energy. To that end, much attention has been devoted to the novel superconducting and insulating phases of twisted bilayer graphene [1–3], which forms flat bands at a set of magic twisting angles [4–10]. The resulting lattice has an enlarged unit cell, forming a so-called moiré superlattice pattern [11–13]. Away from magic angles, twisted models exhibit long-range quasiperiodic order, connecting to the physics of quasicrystals [14–19]. This case is, however, unlike in solid-state bilayers, where commensurability via interactions is strongly favoured during fabrication [11,20,21]. Twisted models can also be emulated in quantum simulators, such as photonic materials [22,23], cavity polaritons [24], and ultracold atoms [25–30]. These systems offer high control over interactions and the underlying geometry of potentials. Moreover, they pave the way to a new class of models, namely, twisted bosonic matter, which has, so far, been little studied compared to fermionic counterparts. Ultracold bosons in simple periodic potentials have been extensively investigated and used to demonstrate Mott-insulator (MI) to superfluid (SF) phase transitions [31,32]. If disorder is present, the Bose glass (BG), a special kind of compressible insulator, can also appear [33–36], and has been observed in both disordered [37–43] and quasiperiodic models [37,44–52]. In recent years, these procedures have also been extended to twisted optical lattices

[53–57]. Here, the twist angle can be freely tuned between commensurate and incommensurate values. So far, the role of incommensuration has been discussed in connection with ergodicity breaking in single-particle dynamics [58] and SF-MI transitions in one-dimensional interacting Bose gases [59].

For this Letter, we study the exotic phase diagrams of strongly correlated ultracold bosons within a twisted optical lattice, for both commensurate and incommensurate angles. Using a combination of quantum Monte Carlo (QMC) and exact diagonalization, we show that, in spite of the potentials having a very similar character, both cases behave completely differently. For incommensurate angles, our results confirm the presence of MI, SF, and BG phases, which we find to be closely linked to the localization of single-particle states. For commensurate angles, the BG is replaced by a series of density-waves (DWs), separated by weak SF domains: a thermally unstable SF with low SF fraction at zero temperature. These regions leave behind prominent normal fluid (NF) phases even at low temperature. We show that weak SF phases can, however, be stabilized with the parameters available in current generation experiments.

Model. We consider an interacting, 2D gas of ultracold bosons in an $L \times L$ box, with periodic boundary conditions. The Hamiltonian is

$$\hat{H} = \int d\mathbf{r} \, \hat{\Psi}^\dagger [\hbar^2/2M(-\nabla^2 + \tilde{g}_0 \hat{\Psi}^\dagger \hat{\Psi}) + V(\mathbf{r})] \hat{\Psi}, \quad (1)$$

where $\hat{\Psi}$ is the bosonic field operator at point $\mathbf{r} = (x, y)$, M is the atomic mass, $\tilde{g}_0$ is the dimensionless 2D contact interaction strength [60–63], and $V(\mathbf{r})$ is the optical potential. The potential is constructed via two rotated square lattices, with period a ,

$$V(\mathbf{r}) = V_1 v(R^- \mathbf{r}) + V_2 v(R^+ \mathbf{r}), \quad (2)$$

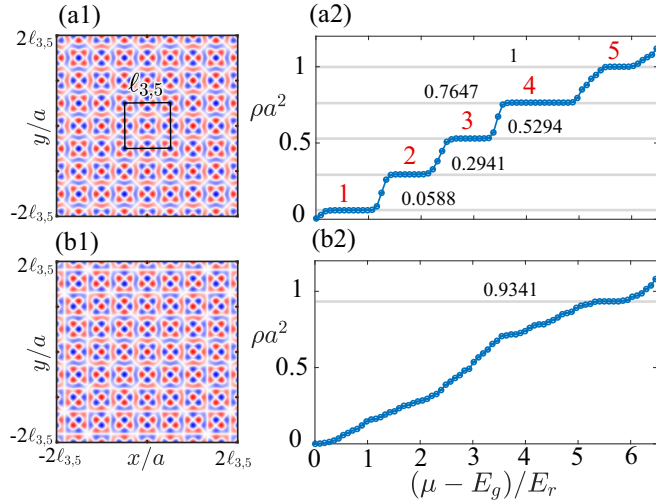


FIG. 1. (a1) Commensurate potential with $\theta_{3,5} \approx 28.07^\circ$ (black square shows the unit cell) and (b1) incommensurate potential with $\theta \approx 27.13^\circ$. (a2), (b2) QMC results for density versus shifted chemical potential at each respective angle for $V = 6E_r$, $\tilde{g}_0 = 4$, $k_B T = 0.04E_r$, and $L = 5\ell_{3,5} \approx 20.62a$. Plateaus in density are marked by grey lines, with $\rho a^2 < 1$ (DW) and $\rho a^2 = 1$ (MI). Red numbers in (a2) denote the index of the plateau p , which has $1 + 4(p - 1)$ atoms per unit cell.

where $v(\mathbf{r}) = \cos^2(2\pi x/a) + \cos^2(2\pi y/a)$, $V_{1,2}$ are the potential depths, and $R^\pm$ the rotation with angle $\pm\theta/2$. Without loss of generality, we consider cases where $V_1 = V_2 = V$ [64].

All energies will be expressed in terms of the recoil energy $E_r = \pi^2 \hbar^2 / 2Ma^2$. For almost any angle θ , $V(\mathbf{r})$ forms an incommensurate (quasiperiodic) potential. However, for a countable set of commensurate angles $\theta_{m,n}$, it takes the form of a commensurate (periodic) moiré potential. These angles are defined by two coprime integers m and n that sum to an even number [53], with $\theta_{m,n} = \cos^{-1}(\frac{2mn}{m^2+n^2})$, where the length $\ell_{m,n}$ of the square unit cell is given by $\ell_{m,n} = a\sqrt{(m^2+n^2)}/2$. We compare Fig. 1(a1) a commensurate potential for the angle $\theta_{3,5} \approx 28.07^\circ$ with Fig. 1(a2) an incommensurate potential for a small shift in angle to $\theta = 27.13^\circ$. The first is strictly periodic, with period $\ell_{3,5} \simeq 4.12a$, and exhibits an exotic, decorated unit cell. In contrast, the second is not periodic, but shows a very similar potential profile, with hardly visible distortions. These small differences, however, result in markedly distinct quantum phase diagrams.

Many-Body Phases. To calculate the exact phases, we use continuous-space QMC simulations within the grand canonical ensemble (temperature T and chemical potential μ) [65,66]. We compute the average density $\rho = N/L^2$, where N is the total number of particles, the compressibility $\kappa = \partial\rho/\partial\mu$, and the SF fraction $f_s = \Upsilon/\rho$, where Υ is the superfluid stiffness [67]. Note, working in continuous-space allows us to overcome frustration effects of lattice models that would otherwise impede convergence of QMC. Such frustration effects appear due to a mix of positive and negative tunnelings, known in incommensurate potentials [68] and also found here in commensurate cases. We plot the density ρ versus the shifted chemical potential $\mu - E_g$ (E_g is the single-particle ground-state energy) for Fig. 1(a2) commensurate

and Fig. 1(b2) incommensurate potentials. We fix $V = 6E_r$, and consider a strongly interacting gas with $\tilde{g}_0 = 4$. Roughly speaking, ultracold atom experiments can reach temperatures of about 10 nK [69–71]. For ^{87}Rb atoms and a typical $a = 350$ nm, it gives $k_B T \approx 0.04E_r$, which we use in the QMC calculations for Fig. 1. Lower ratios of $k_B T/E_r$ may be reached at the same temperature using different atomic species, e.g., ^{40}K , for which $k_B T \approx 0.02E_r$, or ^7Li , for which $k_B T \approx 0.0036E_r$.

First, for the commensurate case with $V = 6E_r$ in Fig. 1(a2), five plateaus in density (red numbers) are observed. These correspond to incompressible, insulating phases with $\kappa = 0$, in which we consistently find $f_s = 0$ (not shown). If the dimensionless density ρa^2 is an integer, we have the standard MI phase. Here, we also find noninteger ρa^2 plateaus, indicating a DW phase. This phase is similar in character to those identified in both superlattice [72,73] and quasiperiodic [49,74] tight-binding models. Furthermore, a single DW-like phase was also observed in experiments and mean-field calculations at $\theta_{1,22} \approx 5.21^\circ$ in Ref. [56]. The increments in density are directly related to the geometry of the moiré potential. First, the lowest DW plateau ($p = 1$) fills the lowest potential minima [blue points at the four corners of the unit cell in Fig. 1(a1)], giving one atom per unit cell. The next DW plateau ($p = 2$) fills the second four lowest minima (located near the midpoints of the unit cell edges), giving now $1 + 4 = 5$ atoms per unit cell. This pattern continues for all insulating phases with $\rho a^2 \leq 1$. The density for the p th plateau is then $\rho_p = [1 + 4(p - 1)]/\ell_{m,n}^2$, corresponding to the black numbers in Figs 1(a2), 1(a3), and the total number of plateaus is $\mathcal{M}_{m,n} = 1 + (\ell_{m,n}^2/a^2 - 1)/4$. For the commensurate potential, we, therefore, have $\ell_{3,5}^2 = 17a^2$ and $\mathcal{M}_{3,5} = 5$, consistent with the results of Fig. 1(a2).

Figure 1(b2) shows the counterpart of Fig. 1(a2) for the incommensurate potential. In spite of the strong similarity between the potentials in Figs. 1(a1) and 1(b1), most plateaus are now absent. Comparatively speaking, commensurate angles form a singular, countable set immersed in a continuous range of incommensurate angles, making the appearance of vastly different physical properties for such a small change in θ surprising. To explain this, we note, however, that any incommensurate potential can be approximated by a commensurate one with very large m and n . This implies that $\mathcal{M}_{m,n}$ diverges, so that each plateau becomes vanishingly narrow. The spectral gaps then vanish, and the compressibility is finite. Moreover, the moiré period increases, which weakens phase coherence and induces localization in the incommensurate limit. This forms a BG, with $\kappa > 0$ and $f_s = 0$. Note, a DW with $\rho a^2 \approx 0.93$ survives due to a small single-particle gap in the quasiperiodic potential.

Weak Superfluids. In our calculations, we find $f_s = 0$ for both potentials, consistent with the onset of DW, MI, and BG phases. However, $f_s = 0$ is also found for the commensurate potential in compressible domains, while a SF is to be expected due to the underlying periodicity of the system. The absence of finite f_s may be attributed to the moiré-enhanced lattice period and the finite temperature. In other words, tunneling is vanishingly small, and the system is thus unable to stabilise superfluidity, even against very weak thermal fluctuations.

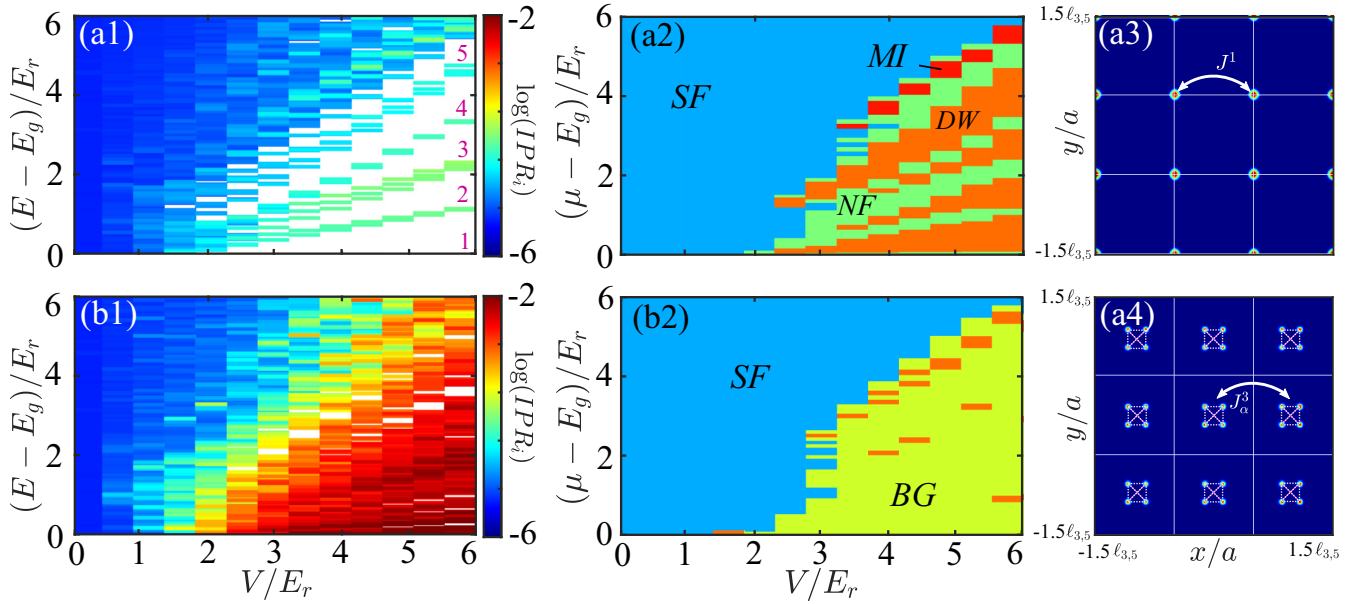


FIG. 2. (a1), (b1) IPR_i of single-particle states for $L = 15\ell_{3,5} \approx 61.85a$, at (a1) commensurate angle with $\theta_{3,5} \approx 28.07^\circ$ and (b1) incommensurate angle with $\theta \approx 27.13^\circ$. (a2), (b2) QMC phase diagrams for the same potentials when $\tilde{g}_0 = 4$, $k_B T = 0.01E_r$, and $L = 5\ell_{3,5} \approx 20.62a$. (a3), (a4) Examples of single-particle states below gap (a3) 1 and (a4) 3 are shown for the commensurate potential, which can be mapped to tight-binding models. J^1 and J^β_α are couplings between states in nearest moiré unit cells (white cells).

To check this, we perform QMC calculations at lower temperature $k_B T = 0.01E_r$, and compare with single-particle properties. Figures 2(a1) and 2(b1) show the single-particle spectra versus energy and potential depth for both commensurate and incommensurate potentials, with $L = 15\ell_{3,5} \approx 61.85a$. The spectra are colored according to the inverse participation ratio $\text{IPR}_i = \int d\mathbf{r} |\psi_i(\mathbf{r})|^4$ of each state i , with blue (red) regions mapping to extended (localized and/or critical) states. Starting with the incommensurate case in Fig. 2(b1), the low-energy states are primarily localized above a critical potential depth (here $V \approx 2.5E_r$), as expected for a quasiperiodic system. The onset of single-particle localization is both qualitatively and quantitatively consistent with the BG phase for the corresponding many-body phase diagram in Fig. 2(b2), while extended states map onto the SF phase. Narrow DW regions, consistent with single-particle gaps, are also visible.

Let us now turn to the commensurate case. The single-particle spectrum in Fig. 2(a1) displays five prominent gaps (labeled 1 to 5), which quantitatively overlap the five insulating plateaus in the many-body phase diagram of Fig. 2(a2) (1 to 4 are DWs, 5 is the MI). This quantitative mapping is to be expected, owing to 2D fermionization of strongly repelling bosons. Here, the single-particle states in different bands occupy separated regions of the plane, so that energy is minimized by filling each with up to one boson, hence mimicking local Pauli exclusion [51,75]. The spatial separation of single-particle states is illustrated in Figs. 2(a3) and 2(a4), which show the density distribution of eigenstates in the first and third band, respectively. In the first band, the density is concentrated at the nodes of the white, moiré unit cells, while in the third band, it is concentrated around four points near the center of each unit cell. Similar separation occurs with the other bands.

Due to periodicity and Bloch's theorem, all states are extended, with the underlying density profiles in Figs. 2(a3) and 2(a4) implying that different bands with index β correspond to different tight-binding geometries [76], see also [77]. For band $\beta = 1$, the picture is straightforward since there is one site per unit cell, see Fig. 2(a3). This yields a standard square lattice with period $\ell_{3,5}$, which has a dispersion relation of

$$\varepsilon(\mathbf{k}) = \epsilon^1 - 2[J_x^1 \cos(k_x \ell_{3,5}) + J_y^1 \cos(k_y \ell_{3,5})], \quad (3)$$

where $J_x^1 = J_y^1 = J^1$ owing to four-fold rotation symmetry. Here J^1 is the tunneling rate, ϵ^1 is the on-site energy, and $\mathbf{k} = (k_x, k_y)$ spans the first Brillouin zone $[-\pi/\ell_{3,5} \cdots \pi/\ell_{3,5}]$. For $V \gtrsim 3E_r$, we find that the tight-binding prediction of Eq. (3) is in excellent agreement with the continuous space band structure.

For the other bands, we have a different scenario in which each moiré cell contains four sites, see, for instance, Fig. 2(a4) corresponding to band 3. Intracell couplings dominate over intercell couplings, so that each band splits into four subbands α ($= a, b, c$, and d) with width much smaller than the total band width. The subbands a and d are well separated in energy, and can thus be described by a standard tight-binding model, yielding dispersion relations similar to Eq. (3) with $J_{a,d;x}^\beta = J_{a,d;y}^\beta = J_{a,d}^\beta$. In contrast, the subbands b and c are quasidegenerate and the four-fold rotation symmetry is broken due to intercell coupling, i.e. $J_{b,c;x}^\beta \neq J_{b,c;y}^\beta$ [78].

By fitting these dispersion relations to the continuous space band structure, we can then extract all relevant tunneling energies. For $V = 6E_r$, we find that the largest tunneling in each band, $J_{\max}^\beta = \max_\alpha (|J_\alpha^\beta|)$ ranges from $J_{\max}^1 \sim 10^{-11}E_r$ to $J_{\max}^5 \sim 10^{-3}E_r$ [76].

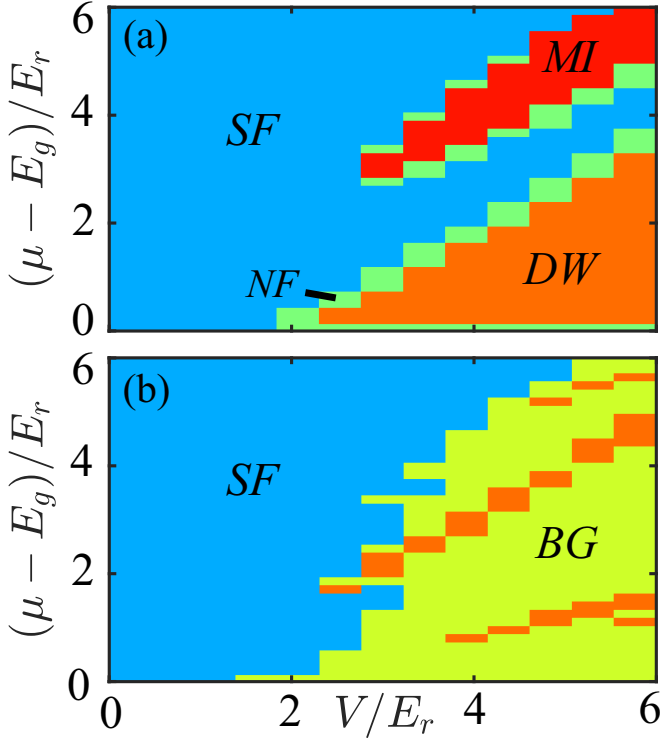


FIG. 3. QMC phase diagrams with $\tilde{g}_0 = 4$, $k_B T = 0.01 E_r$, and $L = 10\ell_{3,1} \approx 22.36a$, for the (a) commensurate angle $\theta_{3,1} \approx 36.87^\circ$ and (b) incommensurate angle $\theta \approx 37.82^\circ$.

To observe a distinct SF phase within band β , we typically require that $k_B T \lesssim J_{\max}^\beta$ [79]. For $V = 6E_r$, it means $k_B T \lesssim 0.001E_r$, i.e., for ^{87}Rb atoms $T \lesssim 0.25\text{ nK}$, quite unrealistic for present-day experiments. For QMC simulations and typical experiments with $k_B T \approx 0.04E_r$, thermal broadening strongly depletes the SF domains, leaving behind a thermal NF phase, which explains the absence of observable superfluidity in the calculations of Fig. 1. This effect can, however, be somewhat mitigated by considering smaller potential depths, in which the effective couplings J_α^β then become larger. For instance, when $V = 3E_r$, the largest tunnellings culminate to $J_{\max}^\beta \approx 0.0134E_r$, within band 2. QMC calculations for varying temperatures confirm that a sizable SF fraction appears roughly for $k_B T \sim J_{\max}^\beta$ [76]. The phase diagrams in Fig. 2 are plotted for $k_B T = 0.01E_r$, slightly below $J_{\max}^\beta$ and, consistently, a small SF domain appears for $V = 3E_r$ and $(\mu - E_g) \sim 1.5E_r$ in Fig. 2(a2). Due to the small values of J_α^β , however, SF domains between insulators are very narrow and only present around $V = 3E_r$, with thermal fluctuations leading to the prevalence of the NF.

To further enhance the superfluidity between the insulating plateaus without further reducing T or V , we may exploit another controllable parameter of ultracold atoms, namely, the twist angle θ . In Fig. 3(a), we plot another phase diagram for the same parameters as in Fig. 2(a2), but with

$\theta = \theta_{3,1} \approx 36.87^\circ$. At this commensurate angle, the size of the unit cell is now $\ell_{3,1} \approx 2.24a$, giving $\mathcal{M}_{3,1} = 2$. We consistently find a single DW lobe (with $\rho a^2 = 0.2$) below the MI (with $\rho a^2 = 1$). The smaller unit cell enhances tunneling rates, which now culminate to $J_{\max}^2 = 0.042E_r$ for $V = 6E_r$. We then find a prominent SF phase, leaving behind narrow NF domains around the edges of DW and MI lobes. In between these insulators, we find a low SF fraction, which varies from $f_s \approx 0.2$ at $V = 3E_r$ to $f_s \approx 0.1$ at $V = 6E_r$. The SF phase transition is consistent with a Berezinski-Kosterlitz-Thouless (BKT) mechanism, with a critical temperature in the range $T_c \approx 0.025\text{--}0.03E_r/k_B$. For the temperature $T = 0.01E_r/k_B$ and the size $L \approx 22.36a$ used here, thermal fluctuations and finite-size effects are strongly suppressed [76]. Our results are thus representative of quantum phase diagrams. Note, weak superfluids are reminiscent of the recently observed SF-II in bilayer Bose systems [56].

Consistently with the results above, a drastic change in behavior is observed for a small angular variation to the incommensurate $\theta \approx 37.82^\circ$ in Fig. 3(b). Large DW and MI lobes vanish, as well as the intermediate SF, leaving behind a BG that is qualitatively similar to that observed for $\theta \approx 27.13^\circ$ in Fig. 2(b2).

Conclusion. We showed that bosonic matter in twisted potentials with controlled twist angle can undergo a variety of exotic phase transitions. Incommensurate angles induce quasiperiodic potentials, which support BG phases over large ranges of potential depths and chemical potentials. In contrast, commensurate angles create moiré potentials, which generates a family of DW lobes. Such insulating domains map onto single-particle gaps, which separate bands and narrow subbands. The narrow subbands support weak superfluids, characterized by low SF fraction at zero temperature and high sensitivity to thermal fluctuations. Sizeable SF domains can, however, be stabilized by controlling the potential depth and, more importantly, the twist angle. Our results are directly applicable to current generation ultracold atom experiments, by using configurations similar to Refs. [45,46,52], where phases can be identified using standard matter-wave interferometry and transport measurements. They may also impact our understanding of fermionic systems in strong-pairing regimes. Finally, our work may be extended to other bosonic quantum simulation platforms using photonic materials or cavity polaritons, where drive-dissipation processes may induce novel effects.

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