

Decoupled sound and amplitude modes in trapped dipolar supersolids

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We theoretically investigate elementary excitations of dipolar quantum gases across the superfluid to supersolid phase transition in a toroidal trap. We show how decoupled first sound, second sound, and Higgs modes emerge by following their origin from superfluid modes across the transition. The structure of these excitations reveals the interplay between crystal and superfluid oscillations. Our results unify previous notions of coupled Goldstone and Higgs modes in harmonic traps, allowing us to establish a correspondence between excitations of trapped and infinitely extended supersolids. We propose protocols for selectively probing these sound and amplitude modes, accessible to state-of-the-art experiments.

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Spontaneous symmetry breaking permeates our understanding of nature and is central to concepts ranging from the emergence of elementary particles to phase transitions in condensed matter [1]. These phase transitions are often signaled by a characteristic spectrum of Goldstone and Higgs collective excitation modes across criticality. In superfluids at zero temperature, the Goldstone mode emerges as superfluid sound from breaking the $U(1)$ symmetry [2,3]. The coupling to another (nonsuperfluid) component can then lead to a second sound mode. This concept historically originated from a hydrodynamic two-fluid model describing the out-of-phase oscillation between thermal and superfluid components of helium [4–6], which finally enabled its experimental observation by surface temperature modulations [7,8].

In a supersolid [9–11], the coexistence of superfluidity and a crystal structure results in a nonsuperfluid component even at zero temperature. Corresponding theoretical investigations of sound in helium have a long history [12], yet in experiments to date the supersolid phase of helium remains elusive [13]. Supersolids were first realized in the context of ultracold gases, using BECs with spin-orbit coupling [14–16] or in multimode optical cavities [17], as well as dipolar quantum gases [18–20]. In the latter system, indications of a novel Goldstone mode associated with translational symmetry breaking were observed [21,22]. Understanding the interplay between crystal and superfluid components was crucial for proving the superfluidity of trapped supersolids [21–24].

The connection of this Goldstone mode to the concept of second sound and more generally the development of a consistent picture describing all elementary excitations has proven challenging, both experimentally and theoretically [18,21,23–28], in particular due to coupling to center-of-mass (COM) modes in harmonically confined systems. In addition, hydrodynamic theories identifying two sound modes are unable to capture gapped modes such as the Higgs mode [29–32], which is yet to be observed experimentally in dipolar supersolids. To circumvent COM coupling, toroidal potentials [33–37] are a better choice: The continuous rotational symmetry and periodic boundary conditions along the azimuthal direction link experimentally achievable finite-size supersolids with those in the bulk addressed by mean-field [32,38] and quantum Monte Carlo methods [39,40].

In this work, we demonstrate how first sound, second sound, and Higgs amplitude modes emerge in dipolar supersolids by calculating the elementary excitation spectrum in a toroidal trap across criticality. Using group theory, we classify the spatial symmetries of the eigenmodes and assign them a quasimomentum $\tilde{q}$. This classification establishes a direct correspondence between the excitations of finite-sized systems and infinite systems, where the description using band structures becomes meaningful. We show that the second (first) sound is an out-of-phase (in-phase) oscillation between superfluid and crystal component, establishing an analogy to sound in finite temperature superfluids. In the zero-temperature supersolid, both sound modes emerge from the superfluid sound branch near the roton momentum q_{rot} , which defines the crystal periodicity. At the critical point, q_{rot} becomes the $\tilde{q} = 0$ point of the emergent Brillouin zone, at which the superfluid sound mode splits into the $\tilde{q} = 0$ instance of the second sound branch—the zero-energy Goldstone mode—and a separate isolated Higgs amplitude mode. The analysis of these modes allows us to design experimental protocols for probing single elementary excitations of the supersolid and, more

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generally, to define selection rules of the supersolid excitation spectrum.

We numerically calculate the ground state (GS) of our atomic dipolar system using the extended Gross-Pitaevskii (eGPE) framework [41–45], which includes quantum fluctuations to leading order in a local density approximation [43,46–49]. The contact and dipolar two-body interaction strengths of the N atom system are controlled by the s -wave scattering length a_s and dipolar length $a_{dd} = \mu_0 \mu_d^2 M / 12\pi \hbar^2$, respectively. Here μ_d denotes the magnetic moment and M the atomic mass. We introduce a toroidal trap geometry by setting the external trapping potential $V_t = M\omega_r^2[(\rho - \rho_0)^2 + \lambda^2 z^2]/2$, where ρ_0 denotes the radius of the torus, ω_r the radial trap frequency, and λ the aspect ratio between transversal and radial confinement [34]. We perform the 3D GS search using imaginary time evolution [50] and conjugate gradient [41,51–53] techniques. On top of the GS, we study the elementary excitations within the Bogoliubov-de Gennes (BdG) formalism described in detail, e. g., in Refs. [23,41]. It is based on a linear expansion in the small amplitude parameter α of the wavefunction $\psi(\mathbf{r}, t) = \{\psi_0(\mathbf{r}) + \alpha[u_i(\mathbf{r})e^{-i\omega t} + v_i(\mathbf{r})e^{i\omega t}]\}e^{-i\mu t/\hbar}$, with ψ_0 the real valued GS wavefunction at a chemical potential μ and i labeling the i th excitation. With this ansatz, we arrive at a system of linear BdG equations. Solving them numerically, we obtain the Bogoliubov amplitudes u_i , v_i as well as the linear combination $f_i = u_i + v_i$ which relate to the density fluctuations of the GS density $n(t) = \psi_0^2 + 2\alpha \delta n_i \cos(\omega_i t)$, where $\delta n_i = \psi_0 f_i$. Due to the finite system size, the spectrum of elementary excitations is discrete.

For a direct comparison to the experimental platform of strongly dipolar ^{162}Dy atoms in optical dipole traps, we choose parameters yielding solutions with $n_d = 7$ droplets [34]. Scaling relations allow for a direct transfer of the results to systems with other dipolar strengths, atom numbers, and trapping strengths [26,54,55,59]. We assume a torus radius $\rho_0 = 1 \mu\text{m}$, radial trap frequency $\omega_r/2\pi = 1 \text{ kHz}$, aspect ratio $\lambda = 1.7$, particle number $N = 5e3$, and dipolar length $a_{dd} = 130a_0$. We calculate the GS for s -wave scattering lengths between $a_s = 59a_0$ and $a_s = 67a_0$, corresponding to relative dipolar strengths $\epsilon_{dd} = a_{dd}/a_s \in [1.95, 2.20]$. As a function of ϵ_{dd} we find the GS to be either superfluid (SF) ($\epsilon_{dd} \lesssim 2.09$) or supersolid (SS). Upon further increasing ϵ_{dd} , the SS eventually develops into a state of n_d isolated droplets.

As shown in the inset of Fig. 1(c), we classify the GSs based on the SF fraction. It is unity in the SF and continuously drops in the SS toward higher ϵ_{dd} . The spectrum of the low energy modes across the SF to SS phase transition is shown in Fig. 1(b), which constitutes one of the main findings of this work. In the SF, each mode is twice degenerate due to the cylindric symmetry [41], and deep in the SF phase the energy of the modes increases monotonically with the circulation number m , which we extract from the number of nodes in the corresponding eigenfunction. For a sufficiently large torus radius we can interpret this as a quantization of the one-dimensional momentum scale $q_m = 2\pi m/\Theta$, where Θ is the circumference of toroidal density distribution [56]. In the long-wavelength limit, we expect a phononic dispersion relation $\omega_m = c_s q_m$, with the SF speed of sound c_s . Due to the influence of the dipolar interactions (for the given system size), only the lowest mode ($q = q_1$) falls into this linear

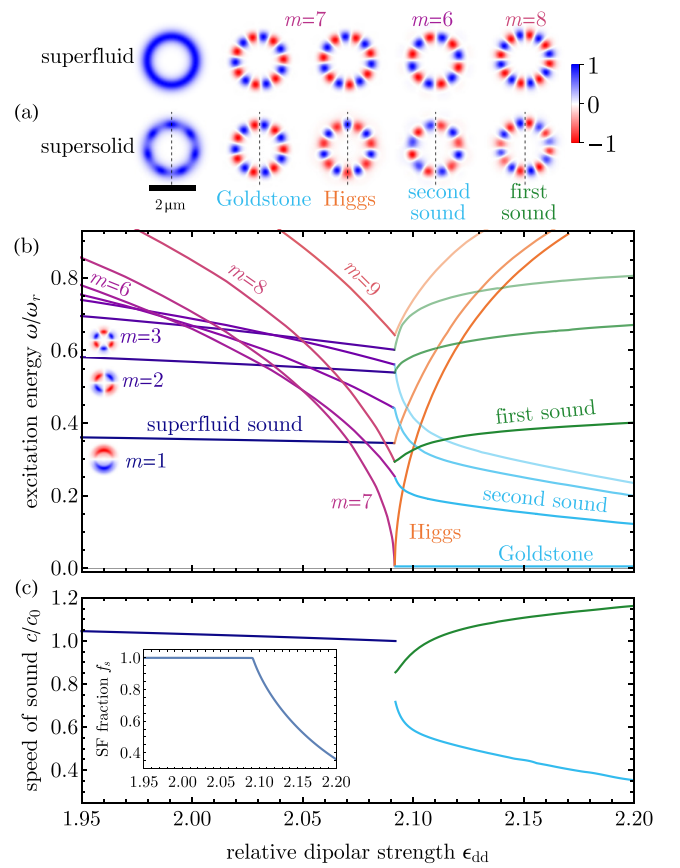


FIG. 1. *Elementary excitations across the superfluid-to-supersolid phase transition.* (a) GS density $n(x, y, z = 0)$ and Bogoliubov mode patterns $f_i(x, y, z = 0)$ in arbitrary units for a selection of modes in the superfluid (top row) and supersolid (bottom row) regime ($\epsilon_{dd} = 2.090, 2.093$, respectively). Atoms flow in the direction of the gradient of f_i , where red indicates a depletion and blue an increase of atomic density. (b) Energies (in units of the radial trap frequency ω_r) of the BdG excitations as a function of ϵ_{dd} . We label the modes by their circulation quantum number m and show the mode pattern on the superfluid side for the three lowest m . We identify a branch of superfluid sound modes (dark blue) and roton modes (purple) on the superfluid side of the transition as well as the first sound branch (green), second sound branch including the Goldstone mode (light blue), and Higgs modes (orange) on the supersolid side. A lighter color corresponds to a higher quasimomentum. (c) Speed of sound associated with the first sound as well as the second sound branches of (b) across the transition in units of the sound velocity at the phase transition $c_0 \simeq 2.2 \text{ mm/s}$ [colors as in (b)]. The inset shows the superfluid fraction, numerically calculated based on the nonclassical rotational inertia.

regime for the lowest ϵ_{dd} shown here. Upon increasing ϵ_{dd} , a number of states with higher m decrease in energy, and can be identified as the angular roton modes [26,57–59].

At the quantum critical point ($\epsilon_{dd}^{\text{crit}} \approx 2.09$), the two $m = 7$ modes (degenerate before the transition) touch zero excitation energy and trigger the phase transition from SF to SS, splitting into what we identify as a Higgs and a Goldstone mode [60]. The nature of these modes can be best understood by considering the BdG mode patterns $f_m(x, y)$ displayed in

Fig. 1(a). As both modes emerge from the $m = 7$ roton mode, the density fluctuation patterns share the periodicity of the GS. The Goldstone mode pattern is phase shifted by half a period with respect to the density, while the Higgs pattern is in phase. The two modes correspond to a pure phase and amplitude mode, with striking clarity compared to harmonic 1D systems where phase and amplitude oscillations are always coupled. Due to the periodic boundary conditions, the Higgs and the Goldstone modes are gapless at criticality.

In the SS regime, the GS has developed a periodic density modulation. Accordingly, we can classify the modes by a quasimomentum $\tilde{q}$ folded back into the first Brillouin zone, and identify the Higgs and Goldstone modes with $\tilde{q}_0 = 0$. The SS has three distinct excitation branches, which we identify based on their mode pattern, their shift in energy as a function of the superfluid fraction and, as discussed below, the structure factor: The Higgs branch, with higher-lying $\tilde{q} > 0$ modes in addition to the $\tilde{q}_0$ mode, and the first and second sound branches [61], as indicated in Fig. 1. The periodic boundary conditions allow us to identify these three branches predicted for infinitely extended SSs [38,62] in finite and experimentally feasible systems. The Higgs branch consists primarily of amplitude oscillations and rises quickly in energy. The two sound modes show a weaker dependence on ϵ_{dd} , with the first sound branch rising in energy while the second sound branch is decreasing, consistent with a smaller SF fraction and compressibility of the crystal structure [30,31].

Based on the circulation number m and the structure factor (to be discussed below), we can attribute the quasimomentum $\tilde{q}_m = 2\pi m/\Theta$, with $m = 1, 2, 3$, to the respective modes. This assignment is supported by employing group theory. The SF GS has the symmetry of the point group $C_{\infty,v}$ describing its invariance under rotation and reflection. At the SF-SS transition this symmetry is broken, $C_{\infty,v} \rightarrow C_{7,v}$ [63], where $C_{7,v}$ contains the irreducible representations A_1, A_2, E_1, E_2, E_3 , with symmetries of $A_{1,2}$ corresponding to an $m = 7$ angular roton mode, and of E_j to one with $m = j$ compatible with quasimomentum $\tilde{q}_j$. The allowed quasimomenta, purely based on symmetry arguments, are thus $\tilde{q}_m$ with $m = 0, 1, 2, 3$. This analysis directly generalizes to an arbitrary droplet number n_d through subduction of $C_{\infty,v}$ to $C_{n_d,v}$ [63–65].

As an example, the mode pattern of the $m = 6$ and $m = 8$ modes before and after the transition are displayed in Fig. 1(a). In the SS these modes give rise to the second and first sound at quasimomentum $\tilde{q}_1$ and show a reflection antisymmetry described by E_1 . We note that, in compliance with our observations, the dispersion relation of the two sound modes is expected to be linear for low momenta, before flattening off close to the edge of the Brillouin zone. The Higgs mode energy is a convex function of the quasimomentum. Albeit similar excitation spectra were previously reported for finite harmonically trapped systems [18,20,21,23,24], their interpretation in terms of different sound branches was hindered by the absence of well defined quasimomenta and Brillouin zones due to the coupling of linear and COM motion. In contrast, in the toroidal supersolid investigated here, the Higgs modes branch keeps its character even relatively far away from the transition point, as it is not coupled to modes with a different quasimomentum. This allows us to estimate the effective mass of a Higgs quasiparticle excitation [63].

Using the energies of the respective $\tilde{q}_1$ modes, we estimate the speed of sound $c_i = \omega_i/\tilde{q}_1$, where $i = 1, 2$ corresponds to first and second sound, respectively [31]. The results are shown in Fig. 1(c) and agree with a recent description in terms of a hydrodynamic model [30,31,63]. For our parameters, the $m = 8$ modes drop below the $m = 1$ modes in the SF, causing them to evolve into a first sound and excited Higgs mode in the SS. This reordering is consistent with an avoided crossing of Higgs and first sound branches described in an infinite system at the same density [38]. The resulting discontinuity between the SF and first sound at the phase transition point together with the continuous SF fraction, is consistent with the expectation of a second-order phase transition at intermediate densities in the corresponding bulk system.

Our approach allows us to develop a microscopic and intuitive understanding of the character of these elementary excitations (Fig. 2). To relate our findings to quantities which can be directly measured in experiments [59], we define the angular spectrum of the structure factor $\tilde{S}(k_\phi, \omega) = \int d\phi e^{ik_\phi\phi} \tilde{S}(\phi, \omega)$, shown in Figs. 2(a)–2(c), where $\tilde{S}(\phi, \omega)$ is readily obtained from a Fourier transform of the density fluctuation patterns $\delta n_i = f_i \psi_0$ [25] in cylinder coordinates at $z = 0$ along the torus $r \simeq \rho_0$ [63]. A mode with an m fold angular symmetry corresponding to an excitation at momentum q_m shows up as a peak in $\tilde{S}(k_\phi, \omega)$ at $k_\phi = 2q_m/q_1 = 2m$. In the SF regime [Figs. 2(a) and 2(b)], $\tilde{S}$ reproduces a discretized version of the known Bogoliubov dispersion relation, which is linear at small k_ϕ and quadratic toward larger k_ϕ . Near the phase transition, a roton minimum develops. On the SS side [Fig. 2(c)], the SF branch splits up into the first sound, second sound, and Higgs branches.

We find that for the first sound branch, the first mode shows peaks at $k_\phi = 2, 14, 16$, the second mode at $k_\phi = 4, 14, 18$, and the third mode at $k_\phi = 6, 8, 14$. The peak at $k_\phi = 14$ corresponds to a modulation at the spatial frequency of the crystal structure, and the other peaks can be mapped to a quasimomentum of $|\tilde{q}_m|$ with $m = 1, 2, 3$. This assignment is consistent with an m -fold rotational symmetry of the excitation pattern. In contrast, for the second sound branch, the peak at $k_\phi = 14$ is absent close to the phase transition for all modes. Instead, the main peaks are at 12 ($m = 1$), 10 ($m = 2$), and (6,8) for $m = 3$. While the character of these modes is strikingly different from the first sound they can be mapped to the same symmetry and quasimomentum. The structure of the Higgs modes similarly maps to a quasimomentum of $|\tilde{q}_m|$ with $m = 0, 1, 2$ [66].

It is instructive to study the linearized time evolution shown for the first mode on each branch, respectively, in Figs. 2(d)–2(f). In Fig. 2(d), the amplitude character of the Higgs mode becomes apparent. Atoms flow from the SF background into the droplets and vice versa. In contrast, the second (first) sound modes are an out-of-phase (in-phase) oscillation between crystal compression and SF flow [Figs. 2(e) and 2(f)]. As the crystal compresses toward one node on the torus, the SF density flows toward the opposite (same) direction, reducing (increasing) the density where the droplet spacing is smaller. Due to the dipolar repulsion of the droplets, the compression of the crystal is associated with an energy cost, which is reduced (increased) by the out-of-phase (in-phase) SF flow, providing an intuitive picture of the second sound

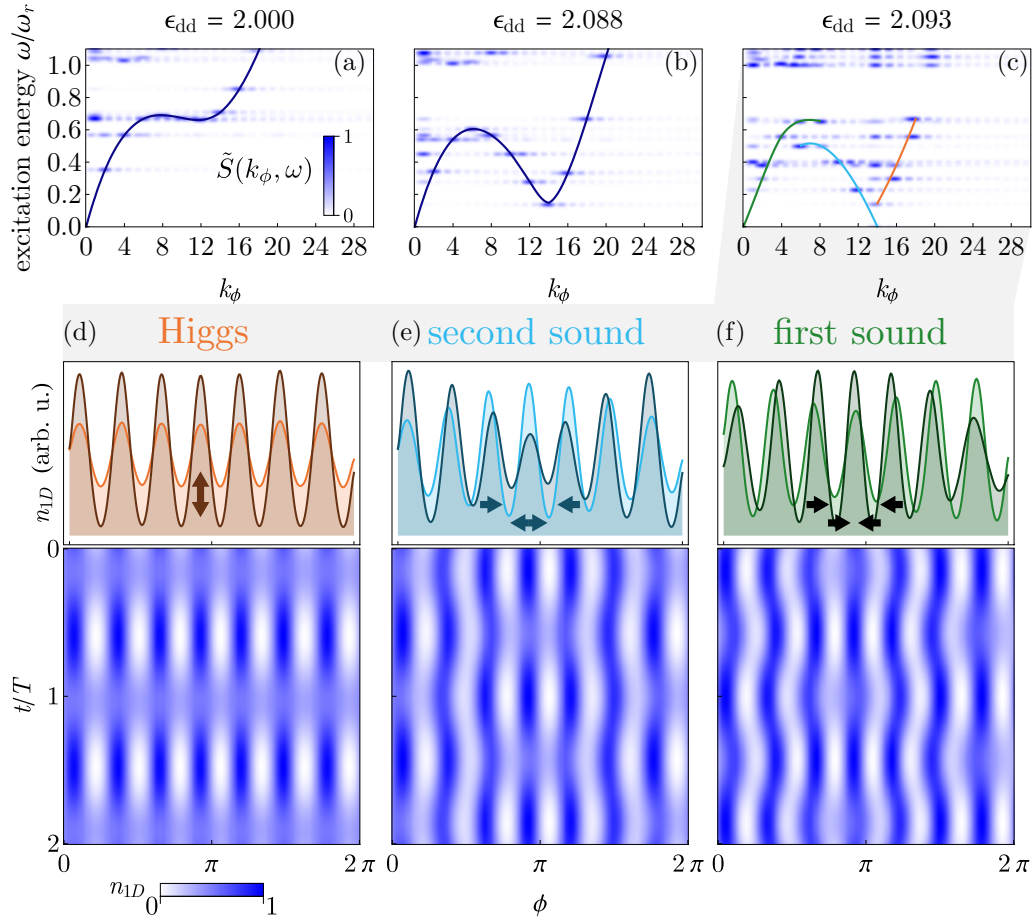


FIG. 2. *Spatial and temporal character of sound and amplitude modes.* (a)–(c) Angular spectrum of the dynamic structure factor $\tilde{S}(k_\phi, \omega)$ across the phase transition. We normalize $\tilde{S}$ at each ω_i for visual clarity. Lines through the most prominent peaks of $\tilde{S}$ are guides to the eye. (d)–(f) The upper panel shows cuts of the density along the torus $n_{1D}(\phi, t)$ at $t/T = 0, 1/2$ during the time evolution, where $T = 2\pi/\omega$. The lower panels show the full density evolution $n_{1D}(\phi, t)$ for two periods T of the Higgs mode, second sound, and first sound at $\tilde{q}_1$ from (c). The Higgs mode is an amplitude oscillation mediating a periodic change in the superfluid fraction through superfluid flow into and out of the droplets. The second (first) sound is characterized by an out-of-phase (in-phase) oscillation between the crystal structure and the superfluid background.

being slower than the first sound. This zero temperature mode structure is reminiscent of the finite temperature in- and out-of-phase oscillations between SF and normal fluid flow in a (nondipolar) SF within the two-fluid model [5,6] and microscopically explains the density (phase) character of the first (second) sound. This *a posteriori* motivates our interpretation of the corresponding modes in terms of first and second sound.

Understanding the spatial and temporal character of these elementary excitations as developed in this work helps devise simple and experimentally feasible schemes to selectively excite individual modes (Fig. 3). We employ principal component analysis (PCA) [25,67] to assess our excitation scheme. PCA is a model-free method giving access to the principal components (PCs) and their weights $w(t)$, quantifying their contribution to the spatial variance of the time evolution. We define the selectivity s as the amount of variance described by the time evolution for a given PC. The overlap between the strongest PC and the corresponding BdG mode is above 95% in all cases discussed here.

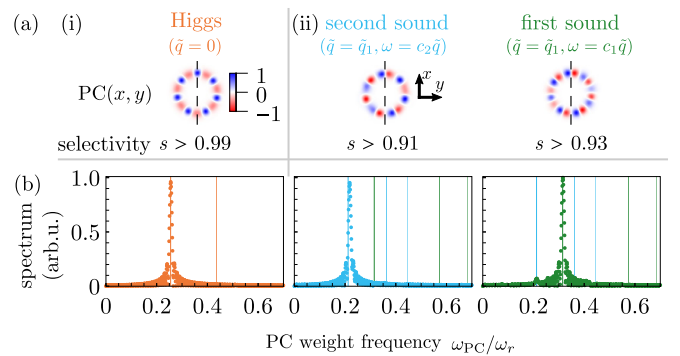


FIG. 3. *Spectroscopy of supersolids.* (a) Dominant PCs after applying an excitation scheme (i), (ii) (see main text) to the GS at $\epsilon_{dd} \simeq 2.1$. (b) Fourier spectra $|\mathcal{F}_w(\omega_{PC})|^2$ of the dominant PC weights $w(t)$. Vertical lines in the spectra indicate BdG energies [Fig. 1(b)]. All modes are individually excited with $>90\%$ selectivity. All PCs show above 95% overlap with the density fluctuation pattern obtained from the BdG analysis [Fig. 1(a)] and oscillate at frequencies corresponding to the BdG energies.

The Higgs mode [Fig. 3(a)(i)] can be selectively excited by modulating the s -wave scattering length, which can be done conveniently in state-of-the-art experiments by modulating a magnetic offset in the vicinity of a Feshbach resonance. This scheme directly couples to the (symmetric) A_1 symmetry and allows us to excite the $\tilde{q} = 0$ Higgs mode with a selectivity of $>99\%$.

The Goldstone mode corresponds to a rotation of the system at zero energy in a cylindrically symmetric trap. Any breaking of the cylindrical symmetry elevates this mode to a finite frequency and enables imparting angular momentum into the system. This can be achieved by deforming and rotating the trapping potential or by using magnetostirring [68], where the magnetic field is tilted and rotated. For more details, see [63].

The two sound modes at $\tilde{q}_1$ [Fig. 3(a)(ii)] share the symmetry E_1 . These modes therefore couple to a dipole excitation pattern and we modulate the trap with $\Delta V \propto x \sin(\omega t)$ along a symmetry axis x of the SS with the frequency ω matching the energy of the respective modes. This scheme selectively excites the resonantly driven sound with only a few percent weight in the other sound modes. This selectivity can be further increased by optimizing duration, amplitude, and other details of the modulation scheme.

Importantly, this scheme cannot couple to the modes at $\tilde{q}_2$ as their symmetry E_2 has no overlap with the symmetry E_1 , while coupling to the E_3 mode is possible in principle, but suppressed due to the frequency detuning. Similarly, a quadrupole (hexapole) excitation scheme can be used to selectively excite the $\tilde{q}_2$ ($\tilde{q}_3$) sound modes with no coupling to modes at $\tilde{q}_{1,3}$ (small coupling to modes at $\tilde{q}_1$) [63]. Based on the symmetry considerations, one can generalize these observations to a set of selection rules, establishing a complete toolbox for the spectroscopy of supersolids. The absence of linear and COM motion coupling ensures that a mode retains its character over time and a range of interaction strengths and is the basis of such spectroscopy.

In conclusion, we have shown the origin of second sound and Higgs amplitude modes in dipolar supersolids by employing a toroidal trap, establishing a correspondence between

elementary excitations of finite and bulk supersolid systems. Strikingly, toroidal supersolids in the low momentum limit realize an isolated and massive Higgs excitation that should propagate as a massive quasiparticle in the self-generated periodic structure of the supersolid. Scattering of the Higgs excitation at an obstacle, such as a weak link on the torus and its dispersion in larger tori, is the starting point for studies of massive quasiparticles in supersolids. Our work establishes an analogy between second sound in supersolids at zero temperature and second sound in superfluids at finite temperature, which is an out-of-phase oscillation between superfluid and nonsuperfluid components. We demonstrate how knowledge of the spatial and temporal character of these modes enables their selective excitation and detection. We therefore set the stage for the experimental observation of second sound and Higgs amplitude modes in zero-temperature supersolids, analogous to the example of second sound in helium [5], where the interpretation of second sound as an entropy wave enabled its observation. Finally, we believe that future studies considering finite temperature supersolids [69,70] in toroidal traps could provide insight into the origin of further sound modes arising through the presence of two nonsuperfluid (crystal and thermal) components.

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