

Atom-Atom entanglement generation via collective states of atomic rings

Nadav Antman Ron, Maor Carmi , and Rivka Bekenstein *Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel*

(Received 6 April 2024; accepted 4 October 2024; published 26 November 2024)

Subwavelength atomic lattices have been established as a promising platform for quantum applications, utilizing collective superradiant and subradiant behavior to enhance atom-light interactions. While previous work has demonstrated high-fidelity atom-photon entanglement in these systems, the generation of atom-atom entanglement, a crucial ingredient for full scalability, remains a conceptual challenge. Here, we propose methods for high-fidelity entanglement generation between atomic qubits by identifying specific atoms within a ring array as qubits. The entanglement generation is achieved by collective atomic excitation involving both subradiant and superradiant states and leveraging the novel symmetry properties of the ring array. Specifically, we utilize symmetry breaking within the ring by preparing atomic qubits in a superpositions of detunings for our entanglement protocols. A fidelity analysis of the protocols provides practical experimental parameters for state-of-the-art setups.

DOI: [10.1103/PhysRevResearch.6.L042051](https://doi.org/10.1103/PhysRevResearch.6.L042051)

Introduction. Since the early work of Dicke [1], a great deal of interest has arisen in subwavelength atomic ensembles and their coherent interaction with single photons. In the regime of subwavelength atomic spacing, coherent exchange-interactions between the dipoles leads to enhanced collective response to light [1,2]. The result is an emergence of collective eigenstates with enhanced or suppressed photon-emission rates, so-called superradiance and subradiance, respectively. Recent works have explored employing these collective effects in subwavelength arrays in quantum applications, including long-lived quantum memories [3], quantum metasurfaces that generate atom-photon entanglement [4], and atom array control over light [5]. Additional works range from analysis of single atoms interaction with the arrays [6–9] through nonlinear interactions to more complex geometries [5,9–12]. Atom-photon correlations in this systems have been recently observed in ultracold atoms experiments [13], while new potential systems are also being explored [14].

While much of this research focuses on the generation of atom-photon entanglement, for scalability a crucial need for efficient entanglement generation between atomic qubits within an array arises. Achieving this will open the door for using subwavelength atomic arrays as a scalable platform for quantum applications, which also requires efficient entanglement generation between atomic qubits. While atom-atom entanglement have been extensively demonstrated in waveguides and cavity QED systems [15–20], we aim to do so without the need of any additional structure (e.g., cavity, waveguide) but rather based solely on the atomic collective effects.

Here, we analyze novel schemes for generating atom-atom entanglement by employing solely the collective states of subwavelength ring lattices [21–24] without the need of any additional structure. We show how employing symmetry breaking within the ring lattice by varying the resonance frequency of the specific atoms (qubits) via external fields, gives rise to a new symmetry and a set of preserved quantities in our system. We demonstrate how this symmetry breaking in the system leads to a drastic spectral change, which—combined with a measurement of the scattered photon—allows the generation of high-fidelity (>0.99) entangled states between atomic qubits. We analyze how properties of the system, such as the number of atoms and the spacing between them, affects the collective response and hence the fidelity of the proposed entanglement protocols.

System: Symmetries in an atomic ring with embedded qubits. We consider a one-dimensional ring array of N two-level atoms with an interparticle distance d , interacting with the vacuum electromagnetic field. The ground ($|g_1\rangle$) and excited states ($|e\rangle$) have a dipole-allowed transition with frequency ω_0 and dipole moment $\mathbf{d}_0$. We first consider specifically the atoms at positions $\theta = 0$ and θ_{q_2} and mark them as qubit 1 (q_1) and qubit 2 (q_2), respectively, see Fig. 1(a). By employing atomic internal-energy levels (e.g., Zeeman splitting) we initialize these atoms in a superposition of $|g_2\rangle$ and $|g_1\rangle$ ground states, forming a superposition of regular and impurity atoms, as depicted in Fig. 1(b). We define the detuning of the $|g_2\rangle$ state with respect to the bare frequency ω_0 as Δ , such that $\Delta \ll \omega_0$. By initializing the initial states of the qubits (q_1, q_2) we achieve different configurations of the atomic system, see Fig. 1(c). The state of the qubits determines the specific array configuration and hence the symmetries of the system.

We mark the different array configurations by the states of the two qubits: - $|g_1g_1\rangle, |g_1g_2\rangle$ or $|g_2g_1\rangle$ and $|g_2g_2\rangle$, see Fig. 1(c). Each configuration yields a distinct set of collective states, with distinct decay rates and energy shifts (Fig. 2).

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

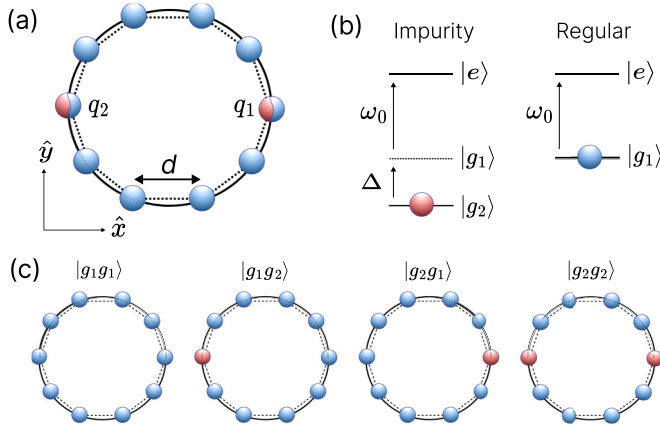


FIG. 1. Implementing atomic qubits within an atomic-ring. (a) Atomic ring array in the $\hat{x}\hat{y}$ plane with lattice constant d . The atomic qubits q_1, q_2 are the atoms at $\theta = 0, \theta = \pi$ respectively, which become impurity atoms after being detuned by external fields. (b) Energy level diagram of an impurity atom (left) and a regular atom (right). The qubits are initialized in a superposition state of the two ground states: $|g_1\rangle$ and $|g_2\rangle$, suggesting four different configurations of the ring lattice, as illustrated in (c).

In the following we employ the difference in the collective response of specific atomic states for entanglement generation between the qubits.

After tracing out the electromagnetic degrees of freedom in the Born-Markov approximation, the interaction of the atoms is described by the effective non-Hermitian Hamiltonian [3]:

$$H_{eff} = \hbar \sum_{i=1}^N \omega_i \hat{\sigma}_i^\dagger \hat{\sigma}_i + \hbar \sum_{i,j} \left(J_{ij} - i \frac{\Gamma_{ij}}{2} \right) \hat{\sigma}_i^\dagger \hat{\sigma}_j. \quad (1)$$

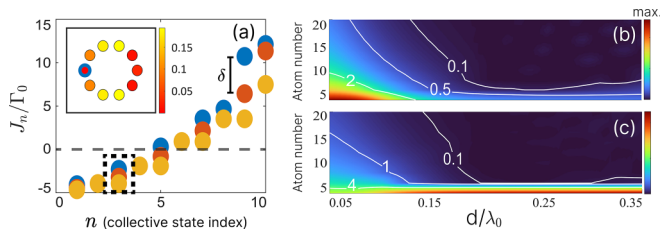


FIG. 2. Distinct collective resonance shift for the different qubits' state. (a) Calculated collective frequency shift for the different lattice configurations, corresponding to $|g_1g_1\rangle$ (yellow) $|g_1g_2\rangle + |g_2g_1\rangle$ (red) and $|g_2g_2\rangle$ (blue) qubits' state. The x-axis represents the collective states for the different configurations sorted according to the energy shift. It is clear that the collective resonance shifts depend on the state of the qubits. The square dashed line marks the states for which we conduct our analysis with (third energy state). The energy separation δ results from the symmetry breaking and is further analyzed in Appendix D. The calculation is for a ring of $N = 10$ atoms, and spacing of $d = 0.1\lambda_0$, and detuning of $\Delta = 10\Gamma_0$. Inset: the atomic excitation distribution of the third $|g_1g_2\rangle + |g_2g_1\rangle$ collective state, where the excitation is shared between all the atoms, excluding the impurities. (b) and (c) Relative resonance shift of the third energetic collective state $|g_1g_2\rangle + |g_2g_1\rangle$ with respect to $|g_1g_1\rangle$ (relative to Γ_0) ($|J_{g1g2} - J_{g1g1}|/\Gamma_0$) (b) and to $|g_2g_2\rangle$ ($|J_{g1g2} - J_{g2g2}|/\Gamma_0$) (c) as function of atom number and atomic spacing. Contours mark lines of constant collective resonance shift.

Here, different qubit states result in different effective Hamiltonians by adding additional energy terms that are proportional to Δ (see Appendix A for more details). The index j labels the atom at angle $\theta_j = 2\pi/N(j-1)$ with bare frequency ω_j . Γ_{ij} and J_{ij} are the coherent and dissipative interactions between the i -th and j -th atoms, and are given by $J_{ij} - i \frac{\Gamma_{ij}}{2} = -\frac{\omega_j^2}{\hbar\epsilon_0 c^2} \mathbf{d}_i^* \cdot \mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega_j) \cdot \mathbf{d}_j$ for $i \neq j$ [25,26], where $\mathbf{G}(\mathbf{r}_i, \mathbf{r}_j, \omega_0)$ is the dyadic Green's function in homogeneous media (see Appendix B and [27]) and $\mathbf{d}_j$ is the dipole moment of the j 'th atom, which is assumed to be identical for all atoms. The expressions of J_{ij} and Γ_{ij} represent the most general form of the coherent and dissipative interactions while assuming the atoms' detuning for each atom is small compared to the variation scale of the Green tensor, allowing us to replace ω_j with ω_0 [25]. $\Gamma_{jj} = \Gamma_{0,j}$ is the spontaneous emission rate of the j 'th atom. We set $J_{jj} = 0$ as we include the Lamb shift in ω_j .

The ring structure ($|q_1, q_2\rangle = |g_1g_1\rangle$) possesses an N -fold rotational symmetry ($\mathcal{U}_{\frac{2\pi}{N}}$), leading to the conservation of orbital angular momentum (OAM) around the $\hat{z}$ axis [21,23] and the good quantum number m . Hence, the collective states corresponding to the single excitation manifold have a Bloch form, i.e., delocalized states with well-defined OAM of value m that takes the form

$$|m\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N e^{i\theta_i \cdot m} |e_i\rangle. \quad (2)$$

The periodicity of the eigenstates under a 2π rotation implement the periodic boundary conditions of the system and determine that $m = 0, \pm 1, \pm 2, \dots, [\pm \frac{(N-1)}{2}]$. $|e_i\rangle$ describes the state for which the i 'th atom is excited. When detuning specific atoms within the atomic ring, this symmetry breaks, resulting in a new set of collective states with distinct energies and undefined OAM in the spectrum. Here, we use the spectral difference between the collective states to distinguish between the different configurations. In Fig. 2, we display the collective energy shift of each configuration.

When detuning the qubits to either the $|g_1g_2\rangle$ or $|g_2g_1\rangle$ or the $|g_2g_2\rangle$ configurations, the rotational symmetry described above is broken. Hence, we further leverage an additional symmetry that exhibit in atomic ring to achieve high-fidelity entanglement protocol. For this aim we introduce a new axis of symmetry, which we choose as the $\hat{x}$ axis that is in common to all three configurations, introducing the x-parity symmetry: $\hat{\mathcal{P}}_x : \theta_i \rightarrow -\theta_i$. We now introduce the states that respect the x-parity symmetry and span them over the OAM basis given by Eq. (2). We define the symmetric and antisymmetric state for every m that fulfils $0 < m < [\frac{(N-1)}{2}]$:

$$|\alpha_+^m\rangle = \frac{1}{\sqrt{2}}(|m\rangle + |-m\rangle), \quad (3)$$

$$|\alpha_-^m\rangle = \frac{1}{\sqrt{2}}(|m\rangle - |-m\rangle), \quad (4)$$

where $\hat{\mathcal{P}}_x |\alpha_+^m\rangle = |\alpha_+^m\rangle$ and $\hat{\mathcal{P}}_x |\alpha_-^m\rangle = -|\alpha_-^m\rangle$, with eigenvalues ± 1 . These states, along with the states with $m = 0, [\frac{(N-1)}{2}]$ in Eq. (2), complement a full x-parity basis for our system. These antisymmetric states exhibit the same energy

shift and decay rate as the m 'th OAM state in Eq. (2) in both the $|g_1g_2\rangle + |g_2g_1\rangle$ and the $|g_2g_2\rangle$ structure with $\theta_{q_2} = \pi$.

In the following we will employ both the rotational and parity symmetry breaking for generating entanglement between the atomic qubits. The symmetry breaking gives rise to new sets of collective states for each structure, with drastic spectral differences between them that enables distinguishability.

Entanglement protocols based on rotational symmetry breaking. We aim to measure scattered photons in different states conditioned on their interaction with specific atomic configuration of the array in order to achieve entanglement between the qubits. To this end we utilize the spectral profile of the emitted electric field. Under the Markovian approximation, the field scattered by the atoms is given by the generalized input-output equation [25,26]:

$$\hat{E}(\mathbf{r}, t) = \hat{E}_{\text{input}}(\mathbf{r}, t) + \mu_0\omega_0^2 \sum_{j=1}^N \mathbf{G}(\mathbf{r}_j, \mathbf{r}, \omega_0) \cdot \mathbf{d}_0 \hat{\sigma}_j(t). \quad (5)$$

Here, $\hat{E}_{\text{input}}(\mathbf{r})$ is the input field which we consider as a weak coherent field, and the second term is the field scattered from the atomic collective state. By Fourier transforming the field scattered by the atoms over the angular component θ and over time, the field m 'th term in the Fourier expansion can be written as [see Appendix C for the full derivation]:

$$\hat{E}(m, z, \rho, \omega) = \sqrt{N}\mu_0\omega_0^2 \mathbf{d}_0 \cdot \mathbf{G}(m, z, \rho, \omega_0) \hat{b}_m(\omega). \quad (6)$$

Here, $\hat{b}_m(\omega)$ is the Fourier transform of $\hat{b}_m(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{im\theta_j} \hat{\sigma}_j(t)$. From the Eqs. (5) and (6) we conclude that due to the rotational symmetry of the ring configuration, an atomic collective state of the ring configuration with a given OAM of value m couples only to modes of the electromagnetic field with the same OAM [28,29]. In addition, the spectral profile of the excited collective state determines the spectral distribution of the emitted electric field. As a consequence, the m -th collective state centered at $\omega_0 + J_m$ (where J_m is the associated energy shift of the collective state with spectral width Γ_m) leads to the generation of photons with corresponding central frequency and width (see Fig. 3). We hereby study the spectrum for different atom number (N) inter-atomic distance (d) and detuning Δ . We identify specific parameters that will result in the optimal energies and decay rates that will enable the generation of Bell state between the impurities, e.g., $|q_1, q_2\rangle = \frac{1}{\sqrt{2}}(|g_2g_1\rangle + |g_1g_2\rangle)$. For this purpose the optimal states will exhibit the best spectral separation with minimal decay rates (i.e., subradiant states) along with having practical lifetimes so that we can access these states in finite time within a protocol.

We analyze the state marked by the square dash line in Fig. 2(a), i.e., the third energy state. This state, relative to the others, features a good energy separation along with a narrow bandwidth response (decay rate). Figures 2(b) and 2(c) shows the relative energy shift of the collective state between the different configurations ($J_{g_2g_1+g_1g_2} - J_{g_1g_1}$ and $J_{g_2g_1+g_1g_2} - J_{g_2g_2}$), as a function of the number of atoms (N) and interatomic distance (d). The results displayed in Fig. 2 consider detuning parameters of: $\Delta/\Gamma_0 = 5$ and 10, although in order to achieve sufficient spectral separation that will lead to high fidelity

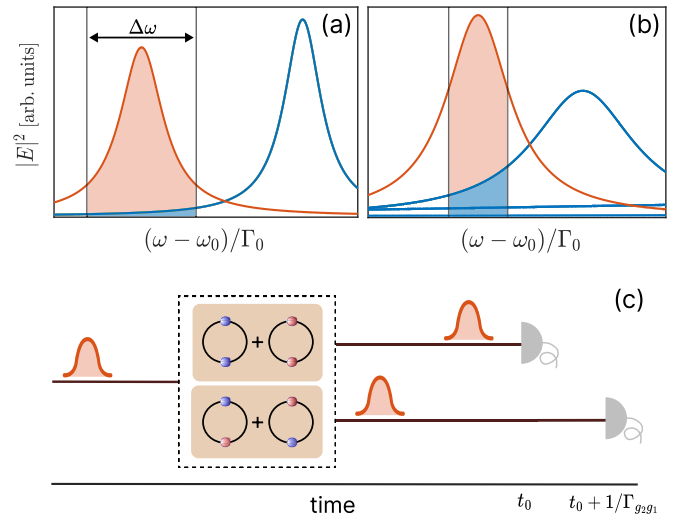


FIG. 3. Entanglement protocols based on rotational symmetry breaking. (a), (b) Emission spectrum for the rotational breaking protocol, for $|g_1g_1\rangle$ and $|g_2g_2\rangle$ error states (blue), and $|g_1g_2\rangle + |g_2g_1\rangle$ target state (red). Calculated for (a) a ring of $N = 10$ atoms and spacing of $d/\lambda_0 = 0.165$, showing fidelity of $\mathcal{F} = 0.95$ and (b) a ring of $N = 8$ atoms and spacing of $d/\lambda_0 = 0.165$, showing fidelity of $\mathcal{F} = 0.81$. $\Delta\omega$ is the incoming-photon bandwidth at the relevant frequency range considered in the fidelity calculation. (c) Schematic illustration of the entanglement protocol in which a photon tuned to the collective resonance $J_{g_1g_2+g_2g_1}$ with bandwidth $\Delta\omega$ is launched to interact with the atomic ring. The lower panel describes the case where the system was initialized in $|g_1g_2\rangle + |g_2g_1\rangle$ followed by a delayed photon. The upper panel describes the case where the system was initialized in $|g_1g_1\rangle + |g_2g_2\rangle$ and the photon will pass through without any delay.

(>0.99), one can work with smaller detuning, down to even $\Delta/\Gamma_0 = 1$.

Next we detail the steps of our entanglement scheme for generating a Bell state between the qubits by utilizing the rotational symmetry breaking.

First, we initialize the system in the state:

$$|\psi\rangle \propto \frac{1}{2}(|g_2g_2\rangle + |g_2g_1\rangle + |g_1g_2\rangle + |g_1g_1\rangle) \otimes \prod_{i \neq q_{1,2}} |g_1\rangle_i, \quad (7)$$

where both of the qubits are in the superposition state $\frac{1}{\sqrt{2}}(|g_2\rangle + |g_1\rangle)$ and the remaining atoms are in the ground state. Second, we apply a laser tuned to the collective resonance $J_{g_1g_2+g_2g_1}$ with bandwidth $\Delta\omega$ [vertical black lines in Figs. 3(a) and 3(b)] for a duration longer than the decay rate of the collective state ($1/\Gamma_{g_1g_2+g_2g_1}$), which is our target state marked as $|\psi\rangle_{\text{target}}$. This results in the excitation of an atomic collective state in the form of $e^{-i(\omega_0 + J_{g_1g_2+g_2g_1} - i\frac{\Gamma_{g_1g_2+g_2g_1}}{2})t} |\psi\rangle_{\text{target}}$ if the qubit state is $|q_1, q_2\rangle \propto |g_1g_2\rangle + |g_2g_1\rangle$. But, if the qubits are in the $|g_2g_2\rangle$ or $|g_1g_1\rangle$ states, the photon is detuned from the collective transition and will pass through without any delay. Finally, by measuring a delayed scattered photon the atoms have been prepared in the Bell state: $\frac{1}{\sqrt{2}}(|g_2g_1\rangle + |g_1g_2\rangle)$. Following a similar logic if the photon is measured in an undelayed

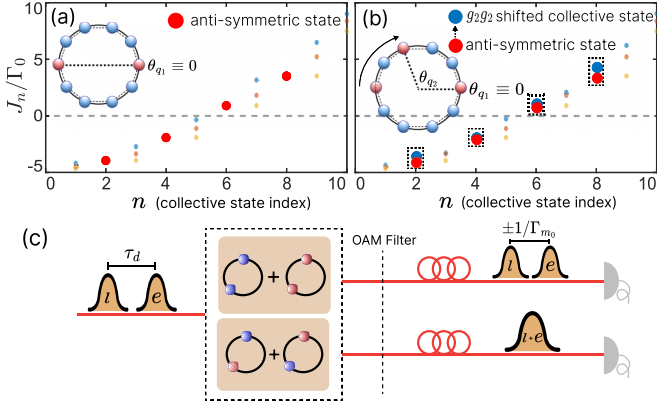


FIG. 4. Entanglement protocol based on breaking x-Parity symmetry. (a), (b) Calculated collective frequency shift for different qubit state showing distinguishable spectra: $|g_1g_1\rangle$ (faded blue), and $|g_1g_2\rangle + |g_2g_1\rangle$ (faded red), and $|g_2g_2\rangle$ (faded yellow). The red dots refer to the common $|\alpha_m^m\rangle$ asymmetric state. We specifically consider an atomic ring of $N = 10$ atoms, spacing of $0.1\lambda_0$, and detuning of $\Delta = 5\Gamma_0$. (a) Detuning the atom at $\theta_{q_2} = \pi$. (b) When detuning the atom at $\theta_{q_2} \neq \pi$, x-parity symmetry breaking is induced in the $|g_2g_2\rangle$ configuration, and the asymmetric state exhibits an energy shift (blue). (c) Schematic illustration of the entanglement protocol in which an earlier and a later photonic state $|\alpha_{m_0}^-\rangle$ is launched to interact with the atomic ring, where an OAM filter transmits only photons with OAM that matches the initial photon anti-symmetric state, followed by a delay line that will combine the parts conditioned on the qubits being in the state $|g_1g_2\rangle + |g_2g_1\rangle$ (lower panel).

time, the system is projected to the atomic entangled state: $\frac{1}{\sqrt{2}}(|g_1g_1\rangle + |g_2g_2\rangle)$. The fidelity of this procedure is displayed in Fig. 5(b) for various system parameters.

Entanglement protocols based on parity symmetry breaking. As explained above, in order to generate entangled states, additionally to the N -fold rotational symmetry, we utilize the

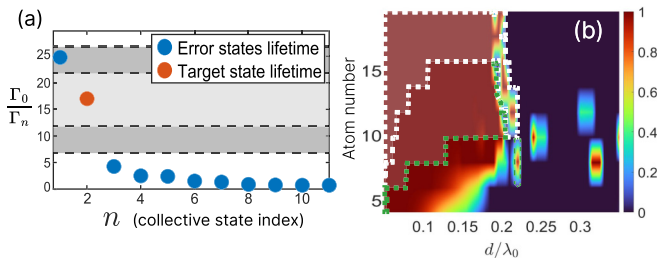


FIG. 5. Fidelity of the entanglement protocols. (a) Lifetime of the collective states that contribute to the error in Eq. (8) for the parity symmetry breaking scheme. States of the $|g_2g_2\rangle$ lattice configuration (blue dots) and the target state $|g_2g_1\rangle + |g_1g_2\rangle$ configuration (red dot) displayed in decrease order. For a ring of $N = 10$ atoms, spacing of $d/\lambda_0 = 0.27$ and detuning $\Delta/\Gamma_0 = 5$. Light and dark grey area mark the regimes $\frac{\Gamma_0}{\Gamma_i} \pm 5$ and $\frac{\Gamma_0}{\Gamma_i} \pm 10$, respectively. (b) Calculated fidelity for the rotational symmetry breaking entanglement protocol, displayed as function of atomic distance and atom number for laser bandwidth of $\Delta\omega = 100$ KHz. The graphs shows fidelity of above 0.9 for certain system parameters. The dashed white (green) line area mark the states with access times that are longer than $\frac{10^2}{\Gamma_0}$ ($\frac{10^1}{\Gamma_0}$), respectively.

parity symmetry that exists in the system. In the following, we will break the x-parity symmetry only for the $|g_2g_2\rangle$ configuration by setting the relative angle between the qubits such that $\theta_{q_2} \neq \pi$ [Fig. 4(b)]. For this case, a state with defined x parity will be a collective state for the $|g_1g_2\rangle + |g_2g_1\rangle$ and the $|g_1g_1\rangle$ configurations with a common energy shift and decay rate. Whereas for the $|g_2g_2\rangle$ structure, due to the symmetry breaking the collective state is not an x-parity state, but rather has distinct energy [see blue dots in Fig. 4(b)]. Likewise, as described above with regards to breaking of rotational symmetry, we will employ this spectral separation to distinguish between the $|g_1g_1\rangle$, $|g_1g_2\rangle + |g_2g_1\rangle$ states and the $|g_2g_2\rangle$ [see Fig. 4(b) red and blue dots respectively]. As we will describe below, this allows us, to generate entanglement between the atomic qubits for specific incoming photon. In the following, we describe the protocol in detail.

The qubits q_1 and q_2 are first initialized each in a superposition state: $|q_1\rangle \otimes |q_2\rangle_{\text{init}} \propto |g_1g_1\rangle + |g_2g_2\rangle + |g_2g_1\rangle + |g_1g_2\rangle$. A photon is prepared in the antisymmetric state associated with m_0 OAM in a superposition of an earlier and later pulse: $|\phi_{ph}\rangle = \frac{1}{\sqrt{2}}(|l\rangle + |e\rangle)$ [30]. The photon is centered around the frequency $\omega_0 + J_{m_0}$ with bandwidth $\Delta\omega$. We assure the later and earlier parts are separated by a time difference: $\tau_d \gg \frac{1}{\Gamma_{m_0}}$. The entanglement protocol will be based on manipulating the time difference between the earlier and later pulse, where a delay line will combine them conditioned on the qubits being in the state $|g_1g_2\rangle + |g_2g_1\rangle$.

First, the earlier pulse encounters the array. For interaction with the $|g_1g_1\rangle$ and $|g_1g_2\rangle + |g_2g_1\rangle$ structure, parity is conserved and the photon couples only to $|\alpha_{m_0}^-\rangle$. However, while interacting with the $|g_2g_2\rangle$ structure due to the x-parity symmetry breaking the photon cannot couple to $|\alpha_{m_0}^-\rangle$ and will pass our system. After a time t , such that $\tau_d > t \gg \frac{1}{\Gamma_{m_0}}$, ensuring that the earlier part has scattered from the array, the qubits' state is flipped such that the state of the system is given by $(|g_2g_2\rangle + |g_1g_2\rangle + |g_2g_1\rangle) \otimes (|e\rangle)$. The later part $|l\rangle$ then interacts with the structure, where again interaction with the $|g_2g_2\rangle$ will not contribute to the final state as we do not collect it. After scattering of the later pulse the system state is $\frac{1}{2}(|g_1g_2\rangle + |g_2g_1\rangle) \otimes (|e\rangle + |l\rangle)$. This is followed by a measurement of a photon in the $|e\rangle + |l\rangle$ basis (by implementing a proper delay line [30]) which generates the desired atomic Bell state $\frac{1}{\sqrt{2}}(|g_1g_2\rangle + |g_2g_1\rangle)$, in Fig. 4(c).

In the case of an error, either the earlier or the later part couple to unwanted collective states that overlap with their spectrum in the $|g_2g_2\rangle$ configuration. This in turn, enables the emergence of new OAMs - $m \neq \pm m_0$ of the atomic collective states, which will directly affect the orbital angular momentum of the scattered photon. This part of the photon will not be collected by filtering out its distinct OAM [31] leading to further reduction in the error of the protocol.

Fidelity analysis for practical implementation systems. We finally estimate the fidelity of the entanglement protocols described above for practical implementation systems. The system of a ring atomic lattice can be realized with arrays of trapped neutral atoms [32,33] with single-atom control capabilities [34]. Recently, sub-wavelength ultracold atomic arrays have implemented quantum coherence of reflection off a mirror from Rb atomic array [13] with subwavelength

spacing of $d/\lambda_0 = 0.68$. Other atomic species such as Yb can offer sub-wavelength spacing down to $d/\lambda_0 = 0.27$ [35], which will correspond to eliminating major error channels as analyzed below. For example, an alkaline-earth atom system has been proposed for achieving a ratio of $d/\lambda_0 = 0.08$ [36]. In atom-array systems with spacing larger than the wavelength, single-atom control and generation of entanglement have been realized in recent years [37–40]. Whereas some of the error mechanisms such as atomic motion and imperfect array filling have been previously analyzed [4,13], we focus here on error mechanisms that are unique to our suggested protocols. The specific entanglement protocols suggested above rely on symmetry breaking, which results in a distinguishable photon spectrum for the different atomic structures. Therefore, the main error channel for these protocols is estimated by the probability of an incoming photon to couple to undesired states:

$$\epsilon = \frac{\sum_i \int_{\Delta\omega} |\hat{E}_{\text{error}}^i(\omega)|^2 d\omega}{\int_{\Delta\omega} |\hat{E}_{\text{target}}(\omega)|^2 d\omega}. \quad (8)$$

We define the error (ϵ) as the ratio between the probability of the photon with a certain bandwidth ($\Delta\omega$) to couple to the array prepared in an undesired state versus the probability of the photon to couple to the array prepared in a desired state for the specific protocol. For each protocol, $\hat{E}_{\text{error}}^i(\omega)$ [blue area in Figs. 3(a) and 3(b)] is the photonic spectral profile contributed by the undesired collective state, where we sum over all the possible undesired collective states, whereas $\hat{E}_{\text{target}}(\omega)$ [red area in Figs. 3(a) and 3(b)] is the photonic spectral profile contributed by the desired collective state. The fidelity is given by $\mathcal{F} = 1 - \epsilon$. It is important to note that the incoming-photon bandwidth has a direct effect on the error and that a smaller bandwidth generates a narrower response around J_m and hence minimizes the error.

In order to minimize the error of the protocols, the photons are measured within a defined time frame, where smaller time frames will lead to smaller error.

To estimate the fidelity in this scenario, we consider only excitations with lifetimes bounded by the spontaneous emission time $1/\Gamma_0$. In Fig. 5(a) we display the lifetime of the target state (red dot) and the lifetimes of the other states that contribute to the error (blue dots) for the parity symmetry breaking scheme. The light and dark gray areas contain the error states with a lifetime within $\frac{1}{\Gamma_i} \pm \frac{\alpha}{\Gamma_0}$ for $\alpha = 5$ and $\alpha = 10$, respectively. For example, if the experimental measurements are performed in the time frame that corresponds to $\alpha = 5$, there are no error states affecting the measurement outcome, which means that this error channel becomes negligible. By contrast, for $\alpha = 10$ one error state will contribute, resulting in a significant increase in error: $\epsilon = 0.82$.

For the rotational symmetry breaking, we analyze the fidelity for the third-most energetic state [see Fig. 2(a) dashed-line] in the $|g_1g_2\rangle + |g_2g_1\rangle$ configuration as the target state $|\psi\rangle_{\text{target}}$ as it suggests the optimal spectral distinguishability, and consider all the possible collective states of $|g_2g_2\rangle$ and $|g_1g_1\rangle$ as error channels. ϵ will then be the probability of coupling to $|g_1g_1\rangle$ or $|g_2g_2\rangle$ versus the probability of coupling to the desired state $|\psi\rangle_{\text{target}}$ [see Figs. 3(a) and 3(b)]. In that case, $\hat{E}_{\text{target}}(\omega)$ will be the spectral profile of the emitted

electric field related to the state $|\psi\rangle_{\text{target}}$ calculated, according to Eq. (8) and $\hat{E}_{\text{error}}(\omega)$ will be the sum over all the electric field spectral profile of the $|g_2g_2\rangle$, $|g_1g_1\rangle$ energy states.

Figure 5(b) displays the fidelity, where high fidelity ($\mathcal{F} > 0.99$) is achieved for small spacings and for subradiant states, which offer good distinguishability but also have the disadvantage of long access times. The left upper corner in Fig. 5(b) includes strong subradiant states that can be challenging to access in experiments.

For the x-parity symmetry breaking, we analyze the error for different atom numbers N and for the anti-symmetric state which exhibits a stronger subradiant behavior, taking into account only states within the regime $\frac{1}{\Gamma_i} \pm \frac{\alpha}{\Gamma_0}$. For $\alpha = 3$, we find that $\epsilon = 0$ for all $d/\lambda_0 < 0.3, 0.37, 0.42$ for $N = 10, 20, 40$, respectively. For larger spacing d , the radiant behavior of the lattice becomes dominant, leading to increased error. For $\alpha = 6$, the error increases and more atomic distances become impractical.

In conclusion, we analyze practical protocols for generation of atom-atom entanglement, based on the unique symmetries of a ring atomic array. The principles of this work can be generalized to other geometries and symmetries. Our analysis, along with theory and state-of-the-art experimental efforts lay the ground for scalable quantum information processing with collective resonance of subwavelength atomic arrays.

Acknowledgments. The authors would like to thank I. Cirac for meaningful discussions. This research was funded by Grant No. 2021775 from the United States-Israel Binational Science Foundation (BSF) and by Grant No. 2207972. From the United States National Science Foundation (NSF) and ISF Grant No. 2402/22. This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant Agreement No. 101117845).

Appendix A: Effective hamiltonian. Here we provide the expressions for the possible effective Hamiltonians in our system relative to the qubits state - $|g_1g_1\rangle$, $|g_2g_1\rangle + |g_1g_2\rangle$, and $|g_2g_2\rangle$, respectively,

$$\hat{H}_{\text{eff}}^{g_1g_1} = \omega_0 \sum_{i=1}^N \hat{\sigma}_i^\dagger \hat{\sigma}_i + \sum_{i,j} \left(J_{ij} - i \frac{\Gamma_{ij}}{2} \right) \hat{\sigma}_i^\dagger \hat{\sigma}_j, \quad (A1)$$

$$\hat{H}_{\text{eff}}^{g_2g_1} = \omega_0 \sum_{i=1}^N \hat{\sigma}_i^\dagger \hat{\sigma}_i + \Delta \hat{\sigma}_{q_1}^\dagger \hat{\sigma}_{q_1} + \sum_{i,j} \left(J_{ij} - i \frac{\Gamma_{ij}}{2} \right) \hat{\sigma}_i^\dagger \hat{\sigma}_j, \quad (A2)$$

$$\begin{aligned} \hat{H}_{\text{eff}}^{g_2g_2} = & \omega_0 \sum_{i=1}^N \hat{\sigma}_i^\dagger \hat{\sigma}_i + \Delta \hat{\sigma}_{q_1}^\dagger \hat{\sigma}_{q_1} + \Delta \hat{\sigma}_{q_2}^\dagger \hat{\sigma}_{q_2} \\ & + \sum_{i,j} \left(J_{ij} - i \frac{\Gamma_{ij}}{2} \right) \hat{\sigma}_i^\dagger \hat{\sigma}_j, \end{aligned} \quad (A3)$$

where $\hat{H}_{\text{eff}}^{g_2g_1}$ describes the state where q_1 is detuned, and the state with q_2 detuned will have the same Hamiltonian.

Appendix B: Green's function. The dyadic Green's function $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega_0)$ in homogeneous media space used in the paper is

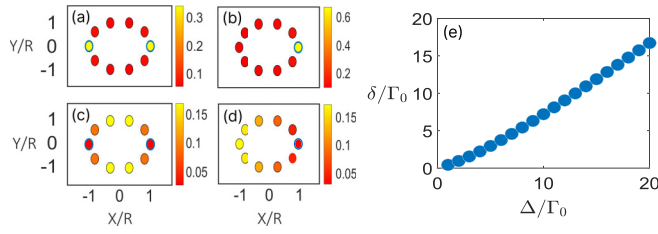


FIG. 6. Calculated atomic excitation distribution of the unique collective states. (a), (b) For these states the excitation is localized on the detuned impurities and not shared with the rest of the atoms. (c), (d) States for which the excitation is shared between all the regular atoms excluding the impurities. (a), (c) is calculated for qubits' state $|g_2g_2\rangle$ and (b), (d) for qubits' state $|g_1g_2\rangle + |g_2g_1\rangle$. All states are for $N = 10$ atoms, and spacing $d/\lambda_0 = 0.1$ and atomic detuning $\Delta/\Gamma_0 = 10$. (e) The energy separation $[\delta]$ as in Fig. 2(a) as a function of the g_2 ground state detuning Δ for ring of $N = 10$ atoms, and spacing $d/\lambda_0 = 0.1$.

given by[27]:

$$\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega_0) = \frac{e^{ik_0r}}{4\pi k_0^2 r^3} \times \left[(k_0^2 r^2 + ik_0r - 1)\mathcal{I} + (-k_0^2 r^2 - 3ik_0r + 3) \frac{\mathbf{r}'' \otimes \mathbf{r}''}{r^2} \right], \quad (\text{B1})$$

where $r = |\mathbf{r} - \mathbf{r}'|$ and $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$ and $k_0 = \frac{2\pi}{\lambda_0} n_0 = \frac{\omega_0}{c} n_0$ is the wave number corresponding to the atomic transition energy and n_0 is the refractive index of the medium. $\mathcal{I}$ is the identity matrix.

Appendix C: Emitted electric field. The expression for the electric field at any location and time is given by

$$\hat{E}(\mathbf{r}, t) = \hat{E}_{\text{input}}(\mathbf{r}, t) + \hat{E}_{\text{atoms}}(\mathbf{r}, t), \quad (\text{C1})$$

where $\hat{E}_{\text{input}}(\mathbf{r}, t)$ is known input field and $\hat{E}_{\text{atoms}}(\mathbf{r}, t)$ is the field scattered by the atoms and given by

$$\hat{E}_{\text{atoms}}(\mathbf{r}, t) = \mu_0 \omega_0^2 \sum_{j=1}^N \mathbf{G}(\mathbf{r}_j, \mathbf{r}, \omega_0) \cdot \mathbf{d}_j \hat{\sigma}_j(t), \quad (\text{C2})$$

where we assume identical polarization for all the atoms such that $\mathbf{d}_j = \mathbf{d}_0$. We then span the scattered field by the complex exponential modes and rewrite it as

$$\hat{E}_{\text{atoms}}(\mathbf{r}, t) = \sum_m \hat{E}(m, z, \rho, t) e^{-im\theta}, \quad (\text{C3})$$

where θ is the azimuthal angle in the x, y plane and $\hat{E}(m, z, \rho, t)$ will be defined as the m 'th element in the angular expansion of the electric field. Therefore, we can write

$$\hat{E}(m, z, \rho, t) = \int_0^{2\pi} d\theta \hat{E}_{\text{atoms}}(\mathbf{r}, t) e^{im\theta}. \quad (\text{C4})$$

Now, we substitute the scattered field expression in Eq. (C2) into Eq. (C3):

$$\hat{E}(m, z, \rho, t) = \mu_0 \omega_0^2 \mathbf{d}_0 \cdot \sum_{j=1}^N \int_0^{2\pi} d\theta \mathbf{G}(\mathbf{r}_j, \mathbf{r}, \omega_0) e^{im\theta} \hat{\sigma}_j(t). \quad (\text{C5})$$

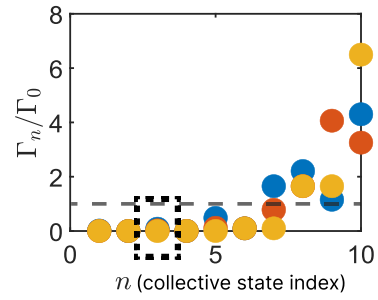


FIG. 7. Calculated collective decay rate for the different lattice configurations, corresponding to $|g_1g_1\rangle$ (yellow) $|g_1g_2\rangle + |g_2g_1\rangle$ (red), and $|g_2g_2\rangle$ (blue) qubits' state. The x axis represents the collective states for the different configurations sorted according to the energy shift in Fig. 2(a) in the paper. The square dashed line marks the states for which we conduct our analysis with (third energy state). The calculation is for a ring of $N = 10$ atoms, and spacing of $d = 0.1\lambda_0$, and detuning of $\Delta = 10\Gamma_0$.

We now define the coordinates relative to the atomic ring location - $z = 0, \rho = R$, where R is the ring radius. We work in homogeneous medium, such that we can write the Green's function as: $\mathbf{G}(\mathbf{r} - \mathbf{r}_j, \omega_0) = \mathbf{G}(\theta - \theta_j, z, \rho - R, \omega_0)$. We then define a new variable $\theta' = \theta - \theta_j$ which yields

$$\hat{E}(m, z, \rho, t) = \mu_0 \omega_0^2 \mathbf{d}_0 \sum_{j=1}^N e^{im\theta_j} \hat{\sigma}_j(t) \times \int_{-\theta_j}^{2\pi - \theta_j} d\theta' \mathbf{G}(\theta', z, \rho, \omega_0) e^{im\theta'}. \quad (\text{C6})$$

The integral is independent of θ_j and therefore we take it outside of the sum:

$$\hat{E}(m, z, \rho, t) = \sqrt{N} \mu_0 \omega_0^2 \mathbf{d}_0 \cdot \mathbf{G}(m, z, \rho, \omega_0) \hat{b}_m(t) \quad (\text{C7})$$

with $\mathbf{G}(m, z, \rho, \omega_0) = \int_0^{2\pi} d\theta \mathbf{G}(\theta, z, \rho, \omega_0) e^{im\theta}$ and $\hat{b}_m(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{im\theta_j} \hat{\sigma}_j(t)$.

Appendix D: Localized collective states. The existence of impurity atoms break the rotational symmetry resulting in an emergence of unique spectrally separated collective states for the $|g_1g_2\rangle$ and $|g_2g_2\rangle$ configurations. In Fig. 2(a) in the paper we marked the spectral gap that separates them by δ . In Fig. 6(e) we display the dependency of the spectral separation on the detuning Δ . In Fig. 6 we show the spatial profile of those separated state from Fig. 2(a) in the paper. Each one of those states describe the case where only the impurity is excited and the excitation is not shared between all the atoms in the structure as can be clearly seen in Eqs. 6(a) and 6(b). As can be easily seen in Fig. 6(e), as Δ is increased the impurities become more spectrally separated from the regular atoms and the excitation is not shared with the regular atoms. This leads to the emergence of localized collective states focused on the impurities which are spectrally separated.

Appendix E: Decay rates of the different configurations. We analyze the collective states resonance shift and decay rate. Figure 2(a) in the paper displays the different collective energy shifts of each configurations, whereas here, in Fig. 7, we display the analysis of the collective decay rates. Each

atomic ring configuration or impurities' state results in a different collective response of our system. We find a dominant change in the collective decay rate of the different configura-

tion (in the order of Γ_0), meaning a significant modification of the collective spontaneous emission for specific collective states.

- [1] R. H. Dicke, Coherence in spontaneous radiation processes, *Phys. Rev.* **93**, 99 (1954).
- [2] D. Porras and J. I. Cirac, Collective generation of quantum states of light by entangled atoms, *Phys. Rev. A* **78**, 053816 (2008).
- [3] A. Asenjo-Garcia, M. Moreno-Cardoner, A. Albrecht, H. J. Kimble, and D. E. Chang, Exponential improvement in photon storage fidelities using subradiance and "selective radiance" in atomic arrays, *Phys. Rev. X* **7**, 031024 (2017).
- [4] R. Bekenstein, I. Pikovski, H. Pichler, E. Shahmoon, S. F. Yelin, and M. D. Lukin, Quantum metasurfaces with atom arrays, *Nat. Phys.* **16**, 676 (2020).
- [5] O. Rubies-Bigorda, V. Walther, T. L. Patti, and S. F. Yelin, Photon control and coherent interactions via lattice dark states in atomic arrays, *Phys. Rev. Res.* **4**, 013110 (2022).
- [6] T. L. Patti, D. S. Wild, E. Shahmoon, M. D. Lukin, and S. F. Yelin, Controlling interactions between quantum emitters using atom arrays, *Phys. Rev. Lett.* **126**, 223602 (2021).
- [7] D. Castells-Graells, D. Malz, C. C. Rusconi, and J. I. Cirac, Atomic waveguide QED with atomic dimers, *Phys. Rev. A* **104**, 063707 (2021).
- [8] S. J. Masson and A. Asenjo-Garcia, Atomic waveguide quantum electrodynamics, *Phys. Rev. Res.* **2**, 043213 (2020).
- [9] D. Fernández-Fernández and A. González-Tudela, Tunable directional emission and collective dissipation with quantum metasurfaces, *Phys. Rev. Lett.* **128**, 113601 (2022).
- [10] P.-O. Guimond, A. Grankin, D. V. Vasilyev, B. Vermersch, and P. Zoller, Subradiant bell states in distant atomic arrays, *Phys. Rev. Lett.* **122**, 093601 (2019).
- [11] S. J. Masson and A. Asenjo-Garcia, Universality of Dicke superradiance in arrays of quantum emitters, *Nat. Commun.* **13**, 2285 (2022).
- [12] F. Shah, T. L. Patti, O. Rubies-Bigorda, and S. F. Yelin, Quantum computing with subwavelength atomic arrays, *Phys. Rev. A* **109**, 012613 (2024).
- [13] K. Srakaew, P. Weckesser, S. Hollerith, D. Wei, D. Adler, I. Bloch, and J. Zeiher, A subwavelength atomic array switched by a single Rydberg atom, *Nat. Phys.* **19**, 714 (2023).
- [14] C. M. Lange, E. Daggett, V. Walther, L. Huang, and J. D. Hood, Superradiant and subradiant states in lifetime-limited organic molecules through laser-induced tuning, *Nat. Phys.*, **20**, 836 (2024).
- [15] S. Welte, B. Hacker, S. Daiss, S. Ritter, and G. Rempe, Cavity carving of atomic bell states, *Phys. Rev. Lett.* **118**, 210503 (2017).
- [16] J. M. Raimond, M. Brune, and S. Haroche, Manipulating quantum entanglement with atoms and photons in a cavity, *Rev. Mod. Phys.* **73**, 565 (2001).
- [17] J. Lee, J. Park, S. M. Lee, H.-W. Lee, and A. H. Khosa, Scalable cavity-QED-based scheme of generating entanglement of atoms and of cavity fields, *Phys. Rev. A* **77**, 032327 (2008).
- [18] F. Haas, J. Volz, R. Gehr, J. Reichel, and J. Estève, Entangled states of more than 40 atoms in an optical fiber cavity, *Science* **344**, 180 (2014).
- [19] X.-L. Yin and J.-Q. Liao, Generation of two-giantatom entanglement in waveguide-QED systems, *Phys. Rev. A* **108**, 023728 (2023).
- [20] I. M. Mirza and J. C. Schotland, Multiqubit entanglement in bidirectional-chiral-waveguide QED, *Phys. Rev. A* **94**, 012302 (2016).
- [21] V. Scheil, R. Holzinger, M. Moreno-Cardoner, and H. Ritsch, Optical properties of concentric nanorings of quantum emitters, *Nanomaterials* **13**, 851 (2023).
- [22] B. Olmos and I. Lesanovsky, Collective photon emission from symmetric states created with Rydberg atoms on a ring lattice, *Phys. Rev. A* **82**, 063404 (2010).
- [23] M. Moreno-Cardoner, D. Plankensteiner, L. Ostermann, D. E. Chang, and H. Ritsch, Subradiance-enhanced excitation transfer between dipole-coupled nanorings of quantum emitters, *Phys. Rev. A* **100**, 023806 (2019).
- [24] M. Moreno-Cardoner, M. Moreno-Cardoner, R. Holzinger, and H. Ritsch, Efficient nano-photon antennas based on dark states in quantum emitter rings, *Opt. Express* **30**, 10779 (2022).
- [25] H. T. Dung, L. Knöll, and D.-G. Welsch, Resonant dipole-dipole interaction in the presence of dispersing and absorbing surroundings, *Phys. Rev. A* **66**, 063810 (2002).
- [26] T. Gruner and D.-G. Welsch, Green-function approach to the radiation-field quantization for homogeneous and inhomogeneous kramers-kronig dielectrics, *Phys. Rev. A* **53**, 1818 (1996).
- [27] L. Novotny and B. Hecht, *Principles of Nano-Optics* (Cambridge University Press, Cambridge, England, UK, 2012).
- [28] L. Allen, M. J. Padgett, and M. Babiker, in *Progress in Optics*, Vol. 39 (Elsevier, Waltham, MA, USA, 1999), pp. 291–372.
- [29] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Orbital angular momentum of light and the transformation of laguerre-gaussian laser modes, *Phys. Rev. A* **45**, 8185 (1992).
- [30] H. K. C. Beukers, M. Pasini, H. Choi, D. Englund, R. Hanson, and J. Borregaard, Tutorial: Remote entanglement protocols for stationary qubits with photonic interfaces, *PRX Quantum* **5**, 010202 (2024).
- [31] D.-X. Chen, P. Zhang, R.-F. Liu, H.-R. Li, H. Gao, and F.-L. Li, Orbital angular momentum filter of photon based on spin-orbital angular momentum coupling, *Phys. Lett. A* **379**, 2530 (2015).
- [32] D. Barredo, S. de Léséleuc, V. Lienhard, T. Lahaye, and A. Browaeys, An atom-by-atom assembler of defect-free arbitrary two-dimensional atomic arrays, *Science* **354**, 1021 (2016).
- [33] M. Endres, H. Bernien, A. Keesling, H. Levine, E. R. Anschuetz, A. Krajenbrink, C. Senko, V. Vuletic, M. Greiner, and M. D. Lukin, Atom-by-atom assembly of defect-free one-dimensional cold atom arrays, *Science* **354**, 1024 (2016).
- [34] C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schauß, T. Fukuhara, I. Bloch, and S. Kuhr, Single-spin ad-

- dressings in an atomic Mott insulator, *Nature (London)* **471**, 319 (2011).
- [35] T. O. Höhn, E. Staub, G. Brochier, N. Darkwah Oppong, and M. Aidelsburger, State-dependent potentials for the $^1S_0$ and $^3P_0$ clock states of neutral ytterbium atoms, *Phys. Rev. A* **108**, 053325 (2023).
- [36] B. Olmos, D. Yu, Y. Singh, F. Schreck, K. Bongs, and I. Lesanovsky, Long-range interacting many-body systems with alkaline-earth-metal atoms, *Phys. Rev. Lett.* **110**, 143602 (2013).
- [37] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, Probing many-body dynamics on a 51-atom quantum simulator, *Nature (London)* **551**, 579 (2017).
- [38] H. Levine, A. Keesling, A. Omran, H. Bernien, S. Schwartz, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, High-fidelity control and entanglement of rydberg-atom qubits, *Phys. Rev. Lett.* **121**, 123603 (2018).
- [39] A. L. Shaw, R. Finkelstein, R. B.-S. Tsai, P. Scholl, T. H. Yoon, J. Choi, and M. Endres, Multi-ensemble metrology by programming local rotations with atom movements, *Nat. Phys.* **20**, 195 (2024).
- [40] S. J. Evered, D. Bluvstein, M. Kalinowski, S. Ebadi, T. Manovitz, H. Zhou, S. H. Li, A. A. Geim, T. T. Wang, N. Maskara, H. Levine, G. Semeghini, M. Greiner, V. Vuletić, and M. D. Lukin, High-fidelity parallel entangling gates on a neutral-atom quantum computer, *Nature (London)* **622**, 268 (2023).