

Strong-field magnetohydrodynamics for neutron stars

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We present a formulation of magnetohydrodynamics which can be used to describe the evolution of strong magnetic fields in neutron star interiors. Our approach is based on viewing magnetohydrodynamics as a theory with a one-form global symmetry and developing an effective field theory for the hydrodynamic modes associated with this symmetry. In the regime where the local velocity and temperature variations can be neglected, we derive the most general constitutive relation consistent with symmetry constraints for the electric field in the presence of a strong magnetic field. This constitutive relation not only reproduces the phenomena of Ohmic decay, ambipolar diffusion, and Hall drift derived in a phenomenological model by Goldreich and Reisenegger, but also reveals terms in the evolution of the magnetic field which cannot easily be seen from such microscopic models. This formulation gives predictions for diffusion behaviors of small perturbations around a constant background magnetic field, and for the two-point correlation functions among various components of the electric and magnetic fields.

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Introduction. Hydrodynamics provides a universal description of an interacting many-body system in the large distance and long time limit by capturing the effective dynamics of conserved quantities. For an electrically conducting fluid with a magnetic field, it is necessary to incorporate electromagnetism in describing the motion of the fluid, leading to magnetohydrodynamics (MHD). MHD plays an important role in many disciplines including plasma physics, astrophysics, and cosmology. In this paper, we provide a formulation of MHD at strong magnetic fields for neutron stars, which exhibit some of the strongest (up to 10^{11} Tesla) and most dramatically variable magnetic fields in the universe (see Refs. [1–3]). We also discuss some immediate implications of this theory for magnetic diffusion. The dynamical variables of MHD for a neutral fluid include the local velocity $\mathbf{v}(x)$, local temperature $T(x)$, and magnetic field $\mathbf{B}(x)$. Here, boldface letters denote spatial vectors. The equations of motion consist of the conservation of the energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$ [4] and two of the Maxwell equations,

$$\nabla \cdot \mathbf{B} = 0, \quad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}. \quad (1)$$

The components of $T^{\mu\nu}$ and $\mathbf{E}$ are expressed in terms of the hydrodynamical variables via constitutive relations. In the standard formulation of MHD, such relations are written on phenomenological grounds and become ambiguous in the presence of a strong magnetic field. In applications to a neutron star, a frequently used approximation to illustrate some of the main physical effects is to take $\mathbf{v}(x) = 0$ [5] and $T(x) = \text{const}$. In Ref. [6], a phenomenological model for neutron stars was used to derive an expression for $\mathbf{E}$ in terms of $\mathbf{B}$ as well as fast-evolving microscopic variables such as the densities of electrons and protons. The results of Ref. [6] have been used to formulate MHD equations in a variety of contexts, both in neutron star physics [1–3, 7–18] and in astrophysics more generally [19–21]. Despite these successes, the inadequate knowledge of particle interactions inside neutron stars makes it difficult to assess the accuracy of such formulations or to improve upon them systematically using phenomenological approaches. In this paper, we provide a formulation of strong-field MHD based on effective field theory, which makes it possible to derive the most general constitutive relations consistent with the symmetries of the system, regardless of the details of microscopic interactions (such as whether weak interactions are included or not). For illustration of the formalism, we will use the static neutron approximation, and neglect possible neutron superfluidity and proton superconductivity [22, 23]. Our theory is valid in the regime where all other physical quantities relax much faster than $\mathbf{B}$. Our main result is a constitutive relation for $\mathbf{E}$ in terms of $\mathbf{B}$ given in (6), which together with (1) gives a closed equation for the evolution of $\mathbf{B}$. In this paper, we outline the derivation of this result and its consequences. A detailed exposition of the effective field theory is provided in Ref. [24]. Our formulation combines two significant recent developments. The first

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ingredient is a formulation of hydrodynamics as the low-energy effective field theory (EFT) of a general many-body system [25,26] (see Ref. [27] for a review and also Refs. [28–30]). In addition to providing a systematic framework for treating hydrodynamical fluctuations, the EFT approach also has a number of advantages at the level of equations of motion compared with the traditional approach. It provides a systematic derivation of the constitutive relations without the need of imposing any phenomenological constraints by hand, and introduces an alternative set of dynamical variables that are often more convenient than traditional ones. The second ingredient is the important observation of Ref. [31] that MHD can be formulated as an effective field theory for a system with a higher-form global symmetry [32]. See Refs. [33–40] for various developments. This makes it possible to formulate MHD based on symmetries alone.

MHD for neutron stars. The basic idea behind the formulation of MHD in Ref. [31] is that the MHD Eq. (1) can be written as the conservation equations for a two-form current $J^{\mu\nu} = -J^{\nu\mu}$:

$$\partial_\mu J^{\mu\nu} = 0, \quad J^{0i} = B^i, \quad J^{ij} = -\epsilon^{ijk} E^k, \quad (2)$$

i.e., $J^{\mu\nu} = \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Accordingly, Eq. (1) can be interpreted as resulting from an underlying one-form symmetry. This realization is not simply a renaming of Eq. (1), but provides a powerful symmetry principle for formulating MHD. In particular, the choice of dynamical variables and the structure of the constitutive relation are dictated by this symmetry principle. An effective action for a dissipative hydrodynamic system can be formulated in terms of the closed-time-path (CTP) or Schwinger-Keldysh formalism (for a review of these methods, see Ref. [27]). A key ingredient of the formulation of Refs. [25,26] is the fact that the dynamical hydrodynamic variables are the Stueckelberg fields associated with the global symmetries responsible for the conservation laws. For the one-form symmetry relevant for MHD, this is a vector field A_μ , which is a collective effective field describing the long-distance and long-time behavior (not the microscopic electromagnetic potential for $\mathbf{E}$ and $\mathbf{B}$ fields of the system). The hydrodynamic effective action S_{EFT} for MHD is then the most general action written in terms of the combination $G_{\mu\nu} \equiv b_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu$ that is invariant under various symmetries. Here, for later convenience, we have also turned on an external source $b_{\mu\nu}$ for the two-form current $J^{\mu\nu}$. The symmetries to be imposed can be separated into those universal for all hydrodynamic systems [25,26] and those specific to the system under study. A neutron star can be regarded as a neutral plasma consisting of electrons, protons, and neutrons. In this system, the energy is high enough that electrons can be treated as relativistic, but not high enough for the existence of positrons. Thus, the charge conjugation $\mathcal{C}$ is badly broken. Weak interactions also break parity $\mathcal{P}$. See Appendix B for the explicit form of the action. Here, we give the final form of the constitutive relations for $J^{\mu\nu}$ at the lowest dissipative order:

$$J^{0i} = a G_{0i} + m \delta_{ij} \epsilon_{kl n} G_{0n} H_{jkl}, \quad (3)$$

$$\begin{aligned} J^{ij} = & -2\beta_0(d \delta_{ik} \delta_{jl} + \tilde{d} \epsilon_{ijm} \epsilon_{kl n} G_{0n} G_{0m}) \partial_0 G_{kl} \\ & - m \epsilon_{lij} \partial_l G_{0k}^2 + 2p \delta_{k[i} \epsilon_{j]ln} G_{0n} \partial_0 G_{kl}, \end{aligned} \quad (4)$$

where $H_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} + \partial_\lambda b_{\mu\nu} + \partial_\nu b_{\lambda\mu}$ and $[ij] \equiv \frac{1}{2}(ij - ji)$. All coefficients $a, d, \tilde{d}, m, p$ should be understood as functions of G_{0i}^2 , and they satisfy the constraints

$$a \geq 0, \quad d \geq 0, \quad d + 2\tilde{d} G_{0i}^2 \geq 0. \quad (5)$$

We can write (3) and (4) in a more conventional form by solving for G_{0i} in terms of $J^{0i} = B^i$. Then, on setting $b_{\mu\nu} = 0$ [41], (4) can be written as the constitutive relation for the electric field:

$$\begin{aligned} \mathbf{E} = & \eta \mathbf{j} + c_a (\mathbf{j} \times \mathbf{B}) \times \mathbf{B} + c_H \mathbf{j} \times \mathbf{B} \\ & - c_H (\nabla \ln a(\mathbf{B}^2) \times \mathbf{B}) \times \mathbf{B} - c_\eta \nabla \ln a(\mathbf{B}^2) \times \mathbf{B} + \nabla f, \end{aligned} \quad (6)$$

where $\mathbf{j} \equiv \nabla \times \mathbf{B}$, and we have introduced a function f of $\mathbf{B}^2$ such that $m = f'$. The coefficients are

$$\eta = c_\eta + c_a \mathbf{B}^2, \quad c_\eta = \frac{2\beta_0 d}{a}, \quad c_a = \frac{4\beta_0 \tilde{d}}{a^3}, \quad c_H = -\frac{p}{a^2}, \quad (7)$$

where $\beta_0 = 1/T$ is the constant inverse temperature. Now all coefficients should be understood as functions of $\mathbf{B}^2$. We will see below that $a(\mathbf{B}^2)$ can be interpreted as a one-form susceptibility in the linear response theory around a constant background magnetic field $\mathbf{B}$, and must therefore be non-negative. Note that $\nabla \ln a(\mathbf{B}^2) = \frac{a'}{a} \nabla \mathbf{B}^2$, and, hence, is proportional to the gradient of the magnetic energy density. Coefficients c_η and η will be interpreted as magnetic “diffusivities.” From (5), we must have

$$\eta \geq 0, \quad c_\eta \geq 0. \quad (8)$$

Note that (6) can also be written in terms of the three independent coefficients c_η, c_a , and c_H as

$$\mathbf{E} = c_\eta \mathbf{j}_B + c_a (\mathbf{B} \cdot \mathbf{j}) \mathbf{B} + c_H \mathbf{j}_B \times \mathbf{B} + \nabla f, \quad (9)$$

$$\mathbf{j}_B = \mathbf{j} - \nabla \ln a(\mathbf{B}^2) \times \mathbf{B}. \quad (10)$$

The first line of (6) includes each of the explicitly $\mathbf{B}$ -dependent terms derived in Eq. (13) of Ref. [6], that lead to behaviors in the magnetic field evolution known as Ohmic decay, ambipolar diffusion, and Hall drift, respectively. We would, however, like to emphasize two significant ways in which this equation provides a generalization of the formalism of Ref. [6]. Firstly, the terms in the second line of (6) are not explicitly present in Ref. [6]. The physical origin of these terms has been clarified in Ref. [42] and their implications will be further elaborated in Ref. [43]. More explicitly, $a(\mathbf{B}^2)$ can be identified as the magnetic permeability of the system. Thus, the second line of (6) captures implications of $\mathbf{H} = \frac{\mathbf{B}}{a(\mathbf{B}^2)}$ being different from $\mathbf{B}$ [44]. Note that these terms only depend on the gradient of the magnetic permeability. Another way in which the above constitutive relation generalizes the one from Ref. [6] is that the coefficients η, c_a , and c_H are all functions of $\mathbf{B}$ in (6). Note also that the value of c_a , which is negative in Ref. [6], can in principle have either sign here. If negative, its absolute value is bounded by $|c_a| \mathbf{B}^2 < c_\eta$ due to (8). Here, we see the major advantages of our formulation. The constitutive relation of $\mathbf{E}$ in terms of $\mathbf{B}$ is rather complicated, and there appears to be no general principle to write it down directly.

But, in terms of the new dynamical variables A_μ (or equivalently G_{0i}), the fields $\mathbf{E}$ and $\mathbf{B}$ are given in a parametric form in (3) and (4), which can be written down systematically and straightforwardly based on symmetries and derivative expansions. We believe the parametric form (3) and (4) is likely also more convenient for nonlinear numerical simulations.

Magnetic diffusion with a strong background magnetic field. To develop intuition for the physics encoded in (3) and (4) [or equivalently, (6)], we consider the behavior of small perturbations around a constant $\mathbf{B}$ along the z direction with magnitude B_0 . Such a constant magnetic field configuration can be conveniently thought of as being due to a constant background “chemical potential” $b_{0z} = \mu$ with $B_0 = a(\mu^2)\mu$. For small perturbations around such a configuration, we can write

$$G_{\mu\nu} = 2\mu\delta_{0[\mu}\delta_{\nu]z} + f_{\mu\nu}, \quad f_{\mu\nu} = b_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (11)$$

where we have also introduced an infinitesimal external source $b_{\mu\nu}$ and we will work to linear order in both $b_{\mu\nu}$ and $f_{\mu\nu}$. Expanding (3) and (4) in small $f_{\mu\nu}$, we then find

$$J^{0z} = B_0 + \chi_{\parallel} f_{0z} + 2m\mu H_{xyz}, \quad J^{0a} = \chi_{\perp} f_{0a}, \quad (12)$$

$$J^{za} = -(\sigma_1^{\perp} \delta_{ab} + \sigma_2^{\perp} \epsilon_{ab})(H_{0zb} + \partial_z f_{0b}) + [\sigma_1^{\perp} \delta_{ab} + (\sigma_2^{\perp} - 2m\mu) \epsilon_{ab}] \partial_b f_{0z}, \quad (13)$$

$$J^{xy} = -\sigma^{\parallel}(H_{0xy} + K_{xy}) - 2m\mu \partial_z f_{0z}, \quad (14)$$

where subscripts a, b index x, y directions, and

$$\chi_{\parallel} = a + \tilde{a}\mu^2, \quad \chi_{\perp} = a, \quad \tilde{a} = 2a'(\mu^2), \quad (15)$$

$$\sigma_1^{\perp} = c_{\eta} \chi_{\perp}, \quad \sigma_2^{\perp} = B_0 c_H \chi_{\perp}, \quad \sigma^{\parallel} = \eta \chi_{\perp}, \quad (16)$$

$$K_{xy} \equiv \partial_x f_{0y} - \partial_y f_{0x}, \quad (17)$$

where all coefficients should be evaluated at $\mathbf{B}^2 = B_0^2$ through μ . As discussed in Appendix B, systems with charge conjugation symmetry have $p = 0$, and hence $\sigma_2^{\perp} = 0$. We stress that $\tilde{a}$ arises from $\mathbf{B}$ dependence of the coefficient a , and hence cannot be seen naively from the formulation of Ref. [6], where all coefficients appear to be $\mathbf{B}$ independent. The functions $\chi_{\perp}$ and $\chi_{\parallel}$ can be interpreted as “one-form susceptibilities,” defined by the Kubo formulas

$$\chi_{\parallel} = \lim_{\mathbf{k} \rightarrow 0} \lim_{\omega \rightarrow 0} G^R(B_z, B_z), \quad \chi_{\perp} = \lim_{\mathbf{k} \rightarrow 0} \lim_{\omega \rightarrow 0} G^R(B_x, B_x), \quad (18)$$

where $G^R(B_z, B_z)$ denotes the retarded correlation functions of B_z and B_z , and similarly for other components (see Appendix C for details). From thermodynamic stability we should have $\chi_{\perp} \geq 0, \chi_{\parallel} \geq 0$. In the linear response theory derived from Ref. [6], we have $\chi_{\parallel} = \chi_{\perp}$ since $\tilde{a} = 0$, but more generally, these susceptibilities should be different. In Fig. 1, we show their ratio in a holographic MHD model [34] (see also Ref. [45]). The σ 's can be interpreted as magnetic “viscosities,” defined by the Kubo formulas

$$\sigma^{\parallel} = \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{1}{i\omega} G^R(E_z, E_z), \quad (19)$$

$$\sigma_1^{\perp} = \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{1}{i\omega} G^R(E_x, E_x), \quad (20)$$

$$\sigma_2^{\perp} = \lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0} \frac{1}{i\omega} G^R(E_x, E_y), \quad (21)$$

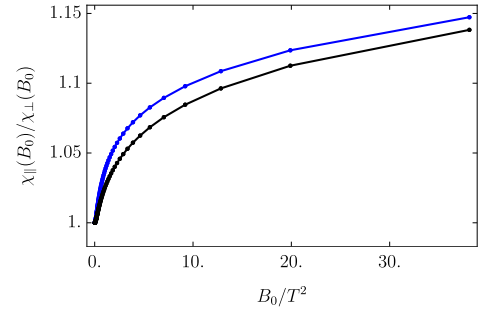


FIG. 1. Ratio of one-form susceptibilities $\chi_{\parallel}/\chi_{\perp}$ computed numerically from a holographic MHD model [34] as a function of equilibrium B_0/T^2 . Black and blue dots correspond to smaller and larger choices of the electromagnetic coupling, respectively.

where $\sigma_2^{\perp}$ can be understood as a “Hall” magnetic viscosity. We stress the differences in the order of limits between (18) and (19)–(21). Comparing (8) and (16),

$$\sigma^{\parallel} \geq 0, \quad \sigma_1^{\perp} \geq 0, \quad (22)$$

but $\sigma_2^{\perp}$ can have either sign. Note that $\sigma^{\parallel} - \sigma_1^{\perp} \propto c_a$, and thus the effect of the ambipolar term at this order is to generate anisotropy in the magnetic viscosities.

The equations of motion following from (12)–(14) are a set of coupled diffusion equations, which give the following dispersion relations (see Appendix C for details):

$$\omega = -i \frac{\sigma_1^{\perp}}{\chi_{\perp}} k_z^2 - \frac{i}{2} \left(\frac{\sigma^{\parallel}}{\chi_{\perp}} + \frac{\sigma_1^{\perp}}{\chi_{\parallel}} \right) k_{\perp}^2 \mp \frac{i}{2} \sqrt{\left(\frac{\sigma^{\parallel}}{\chi_{\perp}} - \frac{\sigma_1^{\perp}}{\chi_{\parallel}} \right)^2 k_{\perp}^4 - 4 \left(\frac{\sigma_2^{\perp}}{\chi_{\perp}} \right)^2 \left(k_z^2 + \frac{\chi_{\perp}}{\chi_{\parallel}} k_{\perp}^2 \right) k_z^2}, \quad (23)$$

where $k_{\perp}^2 = k_x^2 + k_y^2$. The special case of (23) with $\chi_{\parallel} = \chi_{\perp}$ was derived from the formulation of Ref. [6] in Ref. [46], but the general case is a different prediction from our formalism. Note also the special cases

$$k_{\perp} = 0: \quad \omega = -\frac{i}{\chi_{\perp}} (\sigma_1^{\perp} \pm i\sigma_2^{\perp}) k_z^2, \quad (24)$$

$$k_z = 0: \quad \omega = -i \frac{\sigma^{\parallel}}{\chi_{\perp}} k_{\perp}^2 \quad \text{or} \quad \omega = -i \frac{\sigma_1^{\perp}}{\chi_{\parallel}} k_{\perp}^2. \quad (25)$$

The above dispersion relations lead to interesting patterns in the diffusion of magnetic field lines. First, consider $\sigma_2^{\perp} = 0$ (note that this approximation is valid in certain regions of the neutron star such as the core [8]), in which case (23) becomes

$$\omega = -\frac{i}{\chi_{\perp}} (\sigma_1^{\perp} k_z^2 + \sigma^{\parallel} k_{\perp}^2) \quad \text{or} \quad \omega = -i\sigma_1^{\perp} \left(\frac{k_z^2}{\chi_{\perp}} + \frac{k_{\perp}^2}{\chi_{\parallel}} \right). \quad (26)$$

Both modes diffuse anisotropically between z and some direction in the x - y plane. When either $\sigma_1^{\perp} = \sigma^{\parallel}$ (i.e., $c_a = 0$) or $\chi_{\perp} = \chi_{\parallel}$, one of the modes becomes isotropic. See Fig. 2 for an example of the qualitatively different evolutions with $\chi_{\perp} = \chi_{\parallel}$, corresponding to the formalism of Ref. [6], and the generic case of $\chi_{\perp} \neq \chi_{\parallel}$ [47].

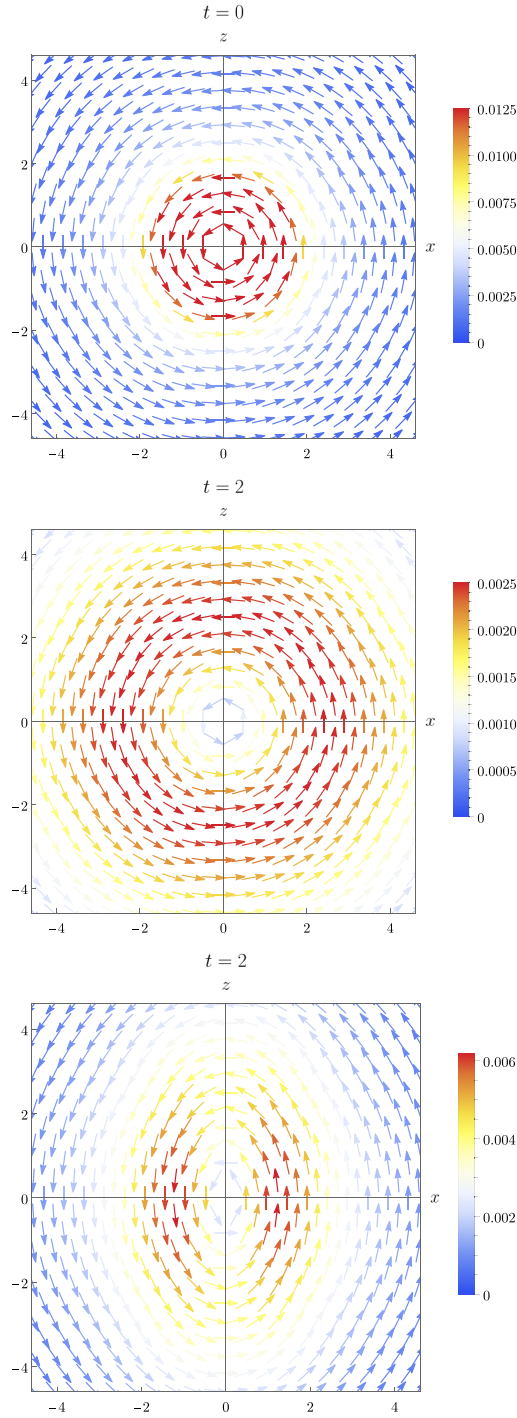


FIG. 2. Evolution of the initial magnetic field configuration $\mathbf{B}(0, \mathbf{x}) = \frac{\sqrt{\pi}r}{(2\pi)^2} e^{-\frac{r^2}{2}} [I_0(\frac{r^2}{2}) - I_1(\frac{r^2}{2})] \hat{\phi}$, where r, ϕ are the polar coordinates in the x - z plane, and $I_n(r)$ is the modified Bessel function of the first kind. We take $\sigma_1^\perp = 1$. The top figure shows the initial condition. The middle figure shows the configuration at $t = 2$ with $\chi_\parallel = \chi_\perp = 1$, while the bottom figure shows the configuration at $t = 2$ with $\chi_\parallel = 1$ and $\chi_\perp = 8$. The color bars indicate the magnitude of $\mathbf{B}$.

A simple example illustrating the consequences of the full dispersion relation (23) is given in Fig. 3, where the initial $\mathbf{B}$ points in the y direction and is concentrated around the y axis. The B_y component diffuses anisotropically, and in addition B_x

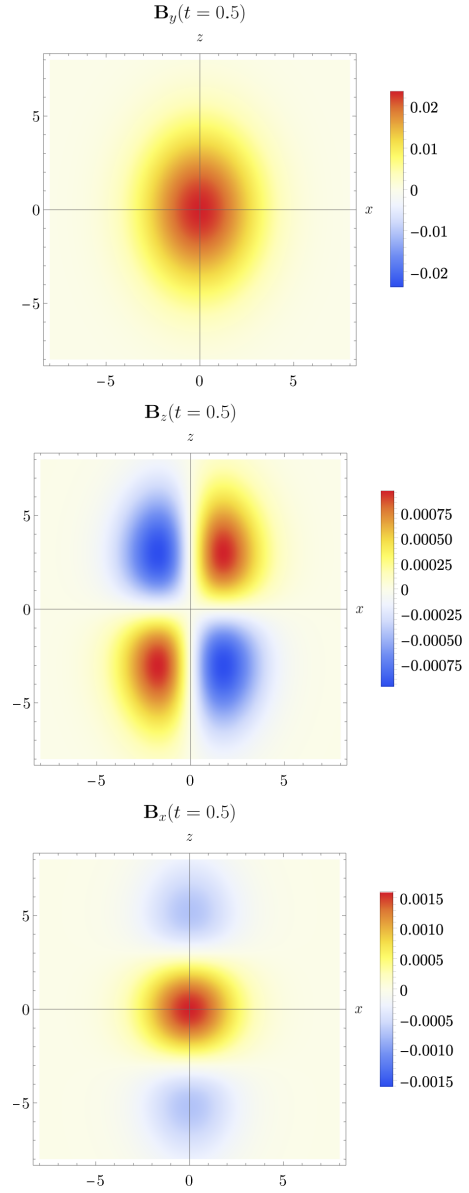


FIG. 3. Different components of $\mathbf{B}$ at $t = 0.5$, evolved from the initial configuration $\mathbf{B}(0, \mathbf{x}) = \frac{1}{\pi\sigma^2} e^{-\frac{x^2+z^2}{\sigma^2}} \hat{y}$. All planes orthogonal to the y direction are identical. The parameter values are $\sigma_1^\perp = 4$, $\sigma_\parallel = 2$, $\sigma_2^\perp = -1$, $\chi_\perp = 1$, and $\chi_\parallel = 8$.

and B_z components are also generated and subsequently diffuse. Similar patterns were previously observed in nonlinear simulations of neutron stars in Ref. [46] and of magnetized filamentary clouds in Ref. [48]. We see here that these effects may be understood as simple consequences of the dispersion relations (23). Note that if we had set $\sigma_2^\perp = 0$ for the initial condition in Fig. 3, we would still see anisotropic diffusion of B_y , but the B_x and B_z components would remain zero for all times.

Equation (23) has the form $\omega = -iD(\theta)k^2$, where $k^2 = k_z^2 + k_\perp^2$ and θ is the angle between the background $\mathbf{B}$ and $\mathbf{k}$. In general, for $\sigma_2^\perp \neq 0$, $D(\theta)$ is complex, with its real part being the “diffusion constant” and the imaginary part describing oscillatory propagation. Such propagating

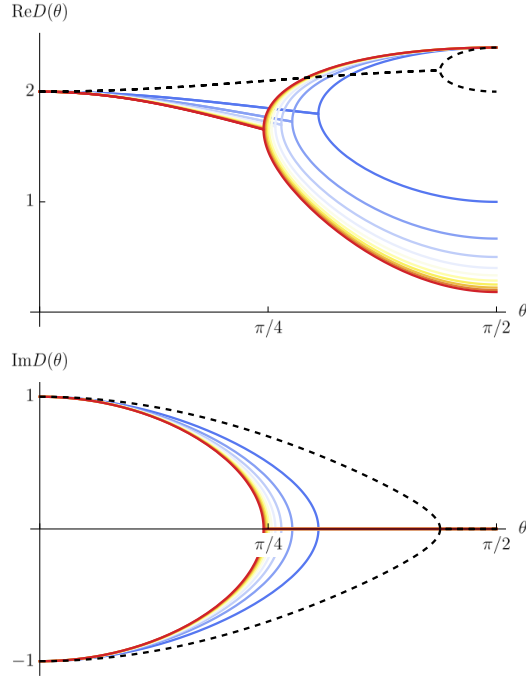


FIG. 4. Real and imaginary parts of $D(\theta) = i\omega/k^2$ for the following choice of parameters: $\beta_0 = \mu = p = a = d = 1$ and $\tilde{d} = 0.1$, where $k_\perp = k \sin \theta$ and $k_z = k \cos \theta$, for $\theta \in [0, \pi/2]$. The black dashed line represents $\tilde{a} = 0$ (corresponding to the theory in Ref. [6]) and the colored lines represent $\tilde{a} = 1, 2, \dots, 10$, going from blue to red with other parameters fixed.

behavior was considered in Ref. [6] as a mechanism for the transport of magnetic energy from the inner to the outer crust in neutron stars. $D(\theta)$ at $\theta = 0$ and $\theta = \frac{\pi}{2}$ can be read from (24) and (25), while for general θ , it is plotted in Fig. 4. There is a critical value θ_c beyond which $\text{Im}D(\theta)$ for both modes vanishes, and the real parts become different. We also notice that as $\chi_\parallel - \chi_\perp$ is increased (with other parameters fixed), θ_c decreases.

The effective action formulation allows us to understand the precise way in which the sources $b_{\mu\nu}$ are coupled to the magnetohydrodynamic variables, and hence enables us to obtain the retarded two-point functions among components of **E** and **B**. The full expressions for generic **k** are complicated, and are presented in Appendix C. As examples, we show the special cases of all *nonzero* correlation functions with $k_z \neq 0$ and $k_x = k_y = 0$ in Appendix A.

Discussion. We have developed a formulation for MHD applicable to neutron stars, in the presence of strong magnetic fields and in the regime where flow velocities and temperature variations can be neglected. This theory leads to the most general constitutive relation for the electric field in terms of the magnetic field for this setup, which in particular reproduces the phenomena of Ohmic decay, ambipolar diffusion, and Hall drift derived in previous formulations such as Ref. [6]. Moreover, the constitutive relation includes terms which capture implications of a nontrivial magnetic permeability. As a simple application of our formulation, we examined the behavior of small perturbations on top of a strong constant magnetic field, and uncovered different magnetic diffusion patterns. We

also found the two-point correlation functions among electric and magnetic field components.

It will be of great interest to apply our formalism to more realistic problems concerning neutron stars and other astrophysical systems with strong magnetic fields, using full nonlinear evolution based on (6). In particular, it would be interesting to see if the different terms in the second line of (6) lead to qualitatively new phenomena which were not accessible with earlier simulations such as those in Refs. [7–9]. The MHD formulation here can be generalized to include flow velocities, going beyond the static neutron approximation, as well as temperature variations. It would also be interesting to include the effects of superfluidity or superconductivity, extending the discussion of Refs. [22,23].

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Appendix A: Correlation functions. In this Appendix, we write the correlation functions involving the **B** and **E** fields for the special case of $\omega \neq 0$, $k_z \neq 0$, and $k_x = k_y = 0$. The complete set of all correlation functions is derived and shown in Appendixes B and C. The *nonzero* correlation functions among the **B** components are given by

$$G^R(B_x, B_x) = \frac{k_z^2 \chi_\perp (k_z^2 \xi^2 - i\sigma_\perp^\perp \chi_\perp \omega)}{k_z^4 \xi^2 - 2ik_z^2 \sigma_\perp^\perp \chi_\perp \omega - \chi_\perp^2 \omega^2}, \quad (\text{A1})$$

$$G^R(B_x, B_y) = \frac{-ik_z^2 \chi_\perp \sigma_\perp^\perp \chi_\perp \omega}{k_z^4 \xi^2 - 2ik_z^2 \sigma_\perp^\perp \chi_\perp \omega - \chi_\perp^2 \omega^2}, \quad (\text{A2})$$

where $\xi = \sqrt{\sigma_\perp^{\perp 2} + \sigma_\perp^{\perp 2}}$. The mixed nonzero correlation functions among **E** and **B** field components are

$$G^R(B_x, E_x) = \frac{\omega}{k_z} G^R(B_x, B_y), \quad (\text{A3})$$

$$G^R(B_x, E_y) = -\frac{\omega}{k_z} G^R(B_x, B_x), \quad (\text{A4})$$

and among the **E** field components are

$$G^R(E_z, E_z) = -\frac{4k_z^2 m^2 \mu^2}{\chi_\parallel} + i\sigma^\parallel \omega, \quad (\text{A5})$$

$$G^R(E_x, E_x) = \frac{\omega^2}{k_z^2} G^R(B_x, B_x), \quad (\text{A6})$$

$$G^R(E_x, E_y) = \frac{\omega^2}{k_z^2} G^R(B_x, B_y). \quad (\text{A7})$$

Equations (19)–(21) arise by taking $k_z \rightarrow 0$ in the above expressions, while the definition of $\chi_\perp$ in (18) can be seen by taking $\omega \rightarrow 0$ with k_z finite. Note that while from (6), the coefficient m plays no role in the equation of motion of the magnetic field at leading order in derivatives, it does appear at

leading order in the correlation functions among electric field components, as in (A5).

Appendix B: Details of the effective CTP action for MHD. In this Appendix, we outline the steps leading to Eqs. (3)–(5) in the Letter. See Ref. [24] for technical details, other generalizations, and a more extensive discussion.

The hydrodynamical variables for MHD in the static neutron approximation are given by two vector fields, $A_{1\mu}$ and $A_{2\mu}$, as on a closed time path all quantities come in two copies. The one-form symmetry implies that the action S_{EFT} can only depend on the dynamical variables $A_{s\mu}$ (s labels the copies) through the combinations

$$G_{s\mu\nu} \equiv b_{s\mu\nu} + \partial_\mu A_{sv} - \partial_\nu A_{s\mu}, \quad s = 1, 2, \quad (\text{B1})$$

where $b_{\mu\nu}$ denotes the external source for the two-form current $J^{\mu\nu}$. By definition, $G_{s\mu\nu}$ and thus S_{EFT} are invariant under the transformation $A_{s\mu} \rightarrow A_{s\mu} + \partial_\mu \alpha_s$, for arbitrary functions α_s . We can use the freedom to set $A_{s0} = 0$. $S_{\text{EFT}}[G_{1\mu\nu}, G_{2\mu\nu}]$ is then the most general action invariant under a number of symmetries.

To describe the symmetries to be imposed, it is convenient to introduce the Keldysh basis of symmetric (“ r ”) and anti-symmetric (“ a ”) combinations of all fields: $\varphi_r \equiv (\varphi_1 + \varphi_2)/2$ and $\varphi_a \equiv \varphi_1 - \varphi_2$. The symmetries that the MHD action S_{EFT} should satisfy include [25,26]

$$\text{Diagonal shift: } A_{ri}(t, \mathbf{x}) \rightarrow A_{ri}(t, \mathbf{x}) + \lambda_i(\mathbf{x}), \quad (\text{B2})$$

and the following constraints from unitarity:

$$S_{\text{EFT}}[G_{r\mu\nu}, G_{a\mu\nu} = 0] = 0, \quad (\text{B3})$$

$$S_{\text{EFT}}^*[G_{r\mu\nu}, G_{a\mu\nu}] = -S_{\text{EFT}}[G_{r\mu\nu}, -G_{a\mu\nu}], \quad (\text{B4})$$

$$\text{Im } S_{\text{EFT}} \geq 0. \quad (\text{B5})$$

We assume that the underlying microscopic system has an antiunitary discrete symmetry Θ involving time reversal $\mathcal{T}$. The macroscopic implications of Θ and the fact that the system is in local equilibrium are realized by a $\mathbb{Z}_2$ dynamical Kubo Martin Schwinger(KMS) symmetry

$$\begin{aligned} G_{a\mu\nu}(x) &\rightarrow \Theta G_{a\mu\nu}(x) - i\beta_0 \partial_0 \Theta G_{r\mu\nu}(x), \\ G_{r\mu\nu}(x) &\rightarrow \Theta G_{r\mu\nu}(x). \end{aligned} \quad (\text{B6})$$

Here, β_0 is the inverse temperature. Finally, we need to impose on S_{EFT} any other discrete symmetry respected by the underlying system. Depending on what discrete symmetries are present, we obtain different classes of actions, which are fully classified in Ref. [24].

As discussed in the main text, in neutron stars, the charge conjugation symmetry $\mathcal{C}$ is badly broken, while parity $\mathcal{P}$ may be broken due to weak interactions. At the derivative order of the EFT that we work with, parity breaking does not yield any difference in either the dispersion relations or the two-point functions (see Ref. [24] for details). Below, we will first write down the theory with parity conservation and then comment on the parity violating case. The microscopic system respects time reversal $\mathcal{T}$. With $\mathcal{P}$ conserved, we can take Θ to be either $\mathcal{T}$ or $\mathcal{PT}$, which give equivalent actions. The identification in Eq. (2) in the Letter determines how each discrete symmetry

acts on $J^{\mu\nu}$, and since the source $b_{\mu\nu}$ should transform in the same way as $J^{\mu\nu}$, we conclude that

$$\mathcal{P}: (G_{0i}, G_{ij}) \rightarrow (G_{0i}, -G_{ij}), \quad (\text{B7})$$

$$\mathcal{T}: (G_{0i}, G_{ij}) \rightarrow (-G_{0i}, G_{ij}), \quad (\text{B8})$$

$$\mathcal{C}: (G_{0i}, G_{ij}) \rightarrow (-G_{0i}, -G_{ij}). \quad (\text{B9})$$

Our proposal of MHD for neutron stars is then given by the most general action invariant under (B2)–(B6) with $\Theta = \mathcal{T}$ and with invariance under $\mathcal{P}$. With $S_{\text{EFT}} = \int d^4x \mathcal{L}_{\text{EFT}}$, we write

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_0 + \mathcal{L}_{\mathcal{P}, \mathcal{T}}, \quad (\text{B10})$$

where $\mathcal{L}_0$ contains terms invariant under all the $\mathcal{C}, \mathcal{P}, \mathcal{T}$ symmetries, while $\mathcal{L}_{\mathcal{P}, \mathcal{T}}$ includes those violating $\mathcal{C}$. To first derivative order, $\mathcal{L}_0$ has the form

$$\begin{aligned} \mathcal{L}_0 = & a G_{r0i} G_{a0i} + i(c \delta_{ij} + \tilde{c} G_{r0i} G_{r0j}) G_{a0i} \tilde{G}_{a0j} + i(d \delta_{ik} \delta_{jl} \\ & + \tilde{d} \epsilon_{ijm} \epsilon_{klm} G_{r0m} G_{r0n}) G_{aij} \tilde{G}_{akl}, \end{aligned} \quad (\text{B11})$$

where $\tilde{G}_{a\mu\nu} = G_{a\mu\nu} + i\beta_0 \partial_0 G_{r\mu\nu}$, and $\mathcal{L}_{\mathcal{P}, \mathcal{T}}$ is written as

$$\begin{aligned} \mathcal{L}_{\mathcal{P}, \mathcal{T}} = & \epsilon_{ijk} G_{r0k} (h \partial_0 G_{r0j} G_{a0i} + p \partial_0 G_{rjl} G_{ail}) \\ & + \epsilon_{ijk} \left(\frac{f}{6} H_{aijk} + f' G_{r0k} H_{rlij} G_{a0l} \right), \end{aligned} \quad (\text{B12})$$

where $H_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu}$. Various coefficients $a, c, \tilde{c}, d, \tilde{d}, h, p, f$ are functions of $g \equiv G_{r0i}^2$. Note that invariance under (B2) implies that G_{rij} must enter S_{EFT} through $\partial_0 G_{rij}$ or H_{rij} , and note that

$$\partial_0 G_{rij} = H_{r0ij} + \partial_i G_{r0j} - \partial_j G_{r0i}. \quad (\text{B13})$$

We will see that the system is dominated by diffusion at leading order, which means that when doing derivative counting we should assign weight 2 to ∂_0 , weight 1 to ∂_i , and weight 0 to G_{r0i} . Under this assignment, G_{rij} , G_{a0i} , G_{aij} have respective weights of $-1, 2, 1$. The terms proportional to $c, \tilde{c}, h$ then have weights 3 and are higher order than the other terms, which have weights 2. We will suppress them below. When parity is broken, additional first-derivative terms exist, but they all have higher weights.

Equations of motion following from S_{EFT} are

$$\partial_\mu J_r^{\mu\nu} = 0, \quad \partial_\mu J_a^{\mu\nu} = 0, \quad (\text{B14})$$

$$J_r^{\mu\nu} \equiv \frac{\delta S_{\text{EFT}}}{\delta b_{a\mu\nu}}, \quad J_a^{\mu\nu} \equiv \frac{\delta S_{\text{EFT}}}{\delta b_{r\mu\nu}}. \quad (\text{B15})$$

With $b_{a\mu\nu} = 0$, the second equation of (B14) can be solved by setting $G_{a\mu\nu} = 0$, in which case $J_a^{\mu\nu}$ are identically zero. With $G_{\mu\nu} \equiv G_{r\mu\nu}$ and $J^{\mu\nu} \equiv J_r^{\mu\nu}$, the equations of motion then reduce to Eq. (2) in the Letter with $J^{\mu\nu}$ (to weight 1) given by (3) and (4). Note that m appearing in (3) and (4) is equal to f' . The first lines of (3) and (4) come from $\mathcal{L}_0$ while the second lines come from $\mathcal{L}_{\mathcal{P}, \mathcal{T}}$. Equations (3) and (4) along with (2) give our formulation of MHD equations for neutron stars to first-derivative order. Lastly, Eq. (B5) requires that

$$c \geq 0, \quad c + \tilde{c} G_{r0i}^2 \geq 0, \quad d \geq 0, \quad d + 2\tilde{d} G_{r0i}^2 \geq 0. \quad (\text{B16})$$

Appendix C: Derivation of dispersion relations and linear response theory. Here, we study the linear responses of the system (in the presence of a constant magnetic field) to an infinitesimal external source. From the responses of various components of electric and magnetic fields we can extract physical interpretations for various parameters introduced in Eqs. (15)–(17) in the Letter.

Recall that

$$J^{\mu\nu} = \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad (C1)$$

and let a_μ be the vector potential for $F_{\mu\nu}$. We then have

$$\frac{1}{2}\int d^4x J^{\mu\nu}b_{\mu\nu} = \int d^4x j^\mu a_\mu, \quad (C2)$$

$$j^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu b_{\rho\sigma} = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}. \quad (C3)$$

j^μ thus corresponds to an external current for a_μ , and

$$j^0 = H_{123}, \quad j^i = -\frac{1}{2}\epsilon^{ijk}H_{0jk}. \quad (C4)$$

Given the expansions (12)–(14), the linearized equations of motion (2) can be written as

$$\tilde{K} = -\frac{\chi_\parallel}{\chi_\perp}K - \frac{2m\mu}{\chi_\perp}\partial_z H_{xyz}, \quad (C5)$$

$$\chi_\parallel\partial_0 f_{0z} + \sigma_1^\perp(\partial_z \tilde{K} - \partial_z^2 f_{0z}) + \sigma_2^\perp\partial_z K_{xy} \\ = -\sigma_1^\perp\partial_a H_{0za} - \sigma_2^\perp\epsilon_{ab}\partial_a H_{0zb} - 2m\mu\partial_0 H_{xyz}, \quad (C6)$$

$$\chi_\perp\partial_0 f_{0a} + \sigma^\parallel\epsilon_{ab}\partial_b K_{xy} - \sigma_1^\perp(\partial_z^2 f_{0a} - \partial_a K) \\ - \sigma_2^\perp\epsilon_{ab}(\partial_z^2 f_{0b} - \partial_b K) \\ = -\sigma^\parallel\epsilon_{ab}\partial_b H_{0xy} + \sigma_1^\perp\partial_z H_{0za} + \sigma_2^\perp\epsilon_{ab}\partial_z H_{0zb}, \quad (C7)$$

where we have introduced

$$K \equiv \partial_z f_{0z}, \quad \tilde{K} \equiv \partial_a f_{0a}, \quad (C8)$$

which can be further written as

$$(\partial_0 + \mathcal{D})\begin{matrix} K \\ K_{xy} \end{matrix} = \begin{matrix} s \\ s_{xy} \end{matrix}, \quad \mathcal{D} = \begin{matrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{matrix}, \quad (C9)$$

where

$$D_{11} = -\sigma_1^\perp\left[\frac{1}{\chi_\perp}\partial_z^2 + \frac{1}{\chi_\parallel}\partial_a^2\right] = \sigma_1^\perp\left[\frac{1}{\chi_\perp}k_z^2 + \frac{1}{\chi_\parallel}k_a^2\right], \quad (C10)$$

$$D_{12} = \frac{\sigma_2^\perp}{\chi_\parallel}\partial_z^2 = -\frac{\sigma_2^\perp}{\chi_\parallel}k_z^2, \quad (C11)$$

$$D_{21} = -\frac{\sigma_2^\perp\chi_\parallel}{\chi_\perp}\left(\frac{\partial_z^2}{\chi_\perp} + \frac{\partial_a^2}{\chi_\parallel}\right) = \frac{\sigma_2^\perp\chi_\parallel}{\chi_\perp}\left(\frac{k_z^2}{\chi_\perp} + \frac{k_a^2}{\chi_\parallel}\right), \quad (C12)$$

$$D_{22} = -\frac{1}{\chi_\perp}[\sigma_1^\perp\partial_z^2 + \sigma^\parallel\partial_b^2] = \frac{1}{\chi_\perp}[\sigma_1^\perp k_z^2 + \sigma^\parallel k_b^2], \quad (C13)$$

and

$$s = -\frac{\sigma_1^\perp}{\chi_\parallel}\partial_z\partial_a H_{0za} - \frac{\sigma_2^\perp}{\chi_\parallel}\epsilon_{ab}\partial_a\partial_z H_{0zb} \\ - \frac{2m\mu}{\chi_\parallel}\left(\partial_0\partial_z H_{xyz} - \frac{\sigma_1^\perp}{\chi_\perp}\partial_z^3 H_{xyz}\right), \quad (C14)$$

$$s_{xy} = \frac{\sigma^\parallel}{\chi_\perp}\partial_a^2 H_{0xy} + \frac{\sigma_1^\perp}{\chi_\perp}\epsilon_{ab}\partial_a\partial_z H_{0zb} \\ - \frac{\sigma_2^\perp}{\chi_\perp}\partial_z\partial_a H_{0za} + \frac{2m\mu\sigma_2^\perp}{\chi_\perp^2}\partial_z^3 H_{xyz}. \quad (C15)$$

In momentum space, we then have

$$\begin{matrix} K \\ K_{xy} \end{matrix} = \frac{1}{-i\omega + \mathcal{D}}\begin{matrix} s \\ s_{xy} \end{matrix} = \frac{1}{\Lambda}\begin{matrix} -i\omega + D_{22} & -D_{12} \\ -D_{21} & -i\omega + D_{11} \end{matrix}\begin{matrix} s \\ s_{xy} \end{matrix}, \quad (C16)$$

where $\Lambda = \det(-i\omega + \mathcal{D})$. It will also be useful to introduce in some of the expressions below the quantity

$$\Gamma = \chi_\parallel\chi_\perp^2\Lambda. \quad (C17)$$

Let us first set the external sources to zero, so that all $H_{\mu\nu\sigma}$ and hence s, s_{xy} are zero. We must then have $\det\Lambda = 0$, which leads to the dispersion relations (23) in the Letter.

In general, the s, s_{xy} are nonzero, and we can solve for f_{0i} in terms of these sources by noting that

$$f_{0z} = -\frac{i}{k_z}K, \quad (C18)$$

$$f_{0a} = \frac{i}{k_\perp^2}\left(k_a\frac{\chi_\parallel}{\chi_\perp}K + \epsilon_{ab}k_b K_{xy}\right) - \frac{2m\mu}{\chi_\perp k_\perp^2}k_x k_z H_{xyz}, \quad (C19)$$

which gives

$$f_{0z} = -\frac{i}{k_z\Lambda}[(-i\omega + D_{22})s - D_{12}s_{xy}], \quad (C20)$$

$$f_{0a} = \frac{i}{k_\perp^2\Lambda}\left\{k_a\frac{\chi_\parallel}{\chi_\perp}[(-i\omega + D_{22})s - D_{12}s_{xy}] + \epsilon_{ab}k_b[-D_{21}s \right. \\ \left. + (-i\omega + D_{11})s_{xy}]\right\} - \frac{2m\mu}{\chi_\perp k_\perp^2}k_x k_z H_{xyz}. \quad (C21)$$

These expressions can be substituted back into Eqs. (11)–(14) in the Letter to find expressions for $J^{\mu\nu}$ in terms of the external sources $b_{\mu\nu}$, which in turn can be used to find the retarded correlation functions

$$G^R(J^{\mu\nu}, J^{\lambda\rho}; t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1) \\ = i\theta(t_2 - t_1)\langle[J^{\mu\nu}(t_2, \mathbf{x}_2), J^{\lambda\rho}(t_1, \mathbf{x}_1)]\rangle = \frac{\delta J^{\mu\nu}(t_2)}{\delta b_{\lambda\rho}(t_1)}. \quad (C22)$$

(See, for instance, Ref. [25] for details on why the last two expressions above are equal.) From the invariance of the above expression under $\mathcal{PT}$ symmetry, the retarded functions should satisfy

$$G^R(J^{\mu\nu}, J^{\lambda\rho}) = G^R(J^{\lambda\rho}, J^{\mu\nu})_{\mu \rightarrow -\mu}. \quad (C23)$$

From $J^{\mu\nu}$ being Hermitian, we have

$$G^R(J^{\mu\nu}, J^{\lambda\rho}; k^\sigma)^* = G^R(J^{\mu\nu}, J^{\lambda\rho}; -k^\sigma) \\ = \pm G^R(J^{\mu\nu}, J^{\lambda\rho}; -\omega, \mathbf{k}), \quad (C24)$$

where in the last expression we have also used parity symmetry. Further constraints on the correlation functions are given by Ward identities

$$k_\mu G(J^{\mu\nu}, J^{\lambda\rho}; k^\sigma) = 0, \quad (C25)$$

and, moreover, there is a rotational symmetry in the x - y plane.

The expressions for the retarded correlation functions with general k_i , ω , and μ are complicated, so let us first consider some special limits, which in particular will justify the interpretations of various coefficients given in Eqs. (18) and (19)–(21) in the Letter.

1. Case 1: $\mu = 0$

In this case, the system becomes rotationally symmetric and we have $\sigma_1^\perp = \sigma^\parallel = \sigma$, $\chi_\perp = \chi_\parallel = \chi$, and

$$\Gamma = \chi(\sigma k^2 - i\chi\omega)^2. \quad (\text{C26})$$

The correlation functions are given by

$$\begin{aligned} G(B_i, B_j) &= \frac{(k^2\delta_{ij} - k_i k_j)\sigma\chi}{k^2\sigma - i\chi\omega}, & G(B_i, E_j) \\ &= \frac{-\epsilon_{ijm}k_m\sigma\chi\omega}{k^2\sigma - i\chi\omega}, \end{aligned} \quad (\text{C27})$$

$$\begin{aligned} G(E_i, E_j) &= \frac{\sigma\omega(ik_i k_j\sigma + \delta_{ij}\chi\omega)}{k^2\sigma - i\chi\omega} \\ &= i\sigma\omega\delta_{ij} + i\sigma^2\omega \frac{k_i k_j - k^2\delta_{ij}}{k^2\sigma - i\chi\omega}. \end{aligned} \quad (\text{C28})$$

Any correlation functions that are not written explicitly can be derived using the symmetries listed around (C23)–(C25). Note further that the expression for $G(B_i, B_j)$ can be determined from rotational symmetry and $\nabla \cdot \mathbf{B} = 0$, and similarly with $G(B_i, E_j)$ and $G(E_i, E_j)$.

By rotational symmetry we can align momentum to be along the z direction and the above expressions can be written more explicitly as

$$\begin{aligned} G^R(B_z, B_i) &= 0, & G^R(B_a, B_b) &= \frac{\sigma\chi k_z^2 \delta_{ab}}{k_z^2\sigma - i\chi\omega}, \\ G^R(E_z, E_z) &= i\sigma\omega, & G^R(E_z, E_a) &= 0, \end{aligned} \quad (\text{C29})$$

The mixed correlation functions among electric and magnetic fields are

$$G^R(B_z, E_a) = G^R(B_x, E_z) = 0, \quad G^R(B_x, E_x) = \frac{\omega}{k_z} G^R(B_x, B_y), \quad G^R(B_x, E_y) = -\frac{\omega}{k_z} G^R(B_x, B_x). \quad (\text{C37})$$

The correlation functions among electric field components are

$$G^R(E_z, E_z) = -\frac{4k_z^2 m^2 \mu^2}{\chi_\parallel} + i\sigma^\parallel \omega, \quad G^R(E_z, E_a) = 0, \quad (\text{C38})$$

$$G^R(E_x, E_x) = \frac{\omega^2}{k_z^2} G^R(B_x, B_x), \quad G^R(E_x, E_y) = \frac{\omega^2}{k_z^2} G^R(B_x, B_y). \quad (\text{C39})$$

Now, taking $k_z \rightarrow 0$, the only nonvanishing pieces are

$$G^R(E_z, E_z) = i\sigma^\parallel \omega, \quad G^R(E_x, E_x) = i\sigma_1^\perp \omega, \quad G^R(E_x, E_y) = i\sigma_2^\perp \omega. \quad (\text{C40})$$

This leads to Eqs. (19)–(21) in the Letter, and we now interpret $\sigma^\parallel$, $\sigma_1^\perp$, and $\sigma_2^\perp$, respectively, as the longitudinal, transverse, and Hall magnetic viscosities.

With $\omega \rightarrow 0$ (and k_z finite) we have

$$G^R(B_x, B_x) = \chi_\perp, \quad G^R(E_z, E_z) = -\frac{4k_z^2 m^2 \mu^2}{\chi_\parallel}. \quad (\text{C41})$$

This gives the second equation in (18), and we thus should interpret $\chi_\perp$ as transverse one-form susceptibility.

3. Case 3: $\mu \neq 0$ and momentum along transverse direction

$$\begin{aligned} G^R(E_a, E_b) &= i\sigma\omega\delta_{ab} - i\sigma^2\omega \frac{k_z^2 \delta_{ab}}{k_z^2\sigma - i\chi\omega} \\ &= i\sigma\omega\delta_{ab} - i\frac{\sigma\omega}{\chi} G^R(B_a, B_b). \end{aligned} \quad (\text{C30})$$

We can then interpret σ as a magnetic “viscosity” as defined by the Kubo formula

$$\sigma = \lim_{\omega \rightarrow 0} \lim_{k_z \rightarrow 0} \frac{1}{i\omega} G^R(E_z, E_z) \quad (\text{C31})$$

and χ as the one-form susceptibility defined by

$$\chi = \lim_{k_z \rightarrow 0} \lim_{\omega \rightarrow 0} G^R(B_z, B_z). \quad (\text{C32})$$

We note that the order of limits is important in (C31) and (C32).

2. Case 2: $\mu \neq 0$ and momentum along the z direction

In this case we have

$$\begin{aligned} \Gamma &= k_z^4 \xi^2 - 2ik_z^2 \sigma_1^\perp \chi_\perp \omega - \chi_\perp^2 \omega^2 \\ &= (k_z^2 \sigma_2^\perp)^2 + (k_z^2 \sigma_1^\perp - i\chi_\perp \omega)^2, \end{aligned} \quad (\text{C33})$$

where

$$\xi = \sqrt{\sigma_1^{\perp 2} + \sigma_2^{\perp 2}}. \quad (\text{C34})$$

The correlation functions of the magnetic fields are

$$G^R(B_z, B_a) = 0, \quad G^R(B_x, B_x) = \frac{k_z^2 \chi_\perp (k_z^2 \xi^2 - i\sigma_1^\perp \chi_\perp \omega)}{k_z^4 \xi^2 - 2ik_z^2 \sigma_1^\perp \chi_\perp \omega - \chi_\perp^2 \omega^2}, \quad (\text{C35})$$

$$G^R(B_x, B_y) = \frac{-ik_z^2 \chi_\perp \sigma_2^\perp \chi_\perp \omega}{k_z^4 \xi^2 - 2ik_z^2 \sigma_1^\perp \chi_\perp \omega - \chi_\perp^2 \omega^2}. \quad (\text{C36})$$

For the spatial momentum along $k_\perp$ we have

$$\Gamma = \chi_\perp(\sigma_1^\perp k_\perp^2 - i\chi_\parallel\omega)(\sigma^\parallel k_\perp^2 - i\chi_\perp\omega). \quad (\text{C42})$$

The correlation functions among magnetic field components are

$$G^R(B_z, B_z) = \frac{\chi_\parallel \sigma_1^\perp k_\perp^2}{k_\perp^2 \sigma_1^\perp - i\chi_\parallel\omega}, \quad G^R(B_x, B_x) = \sigma^\parallel \chi_\perp \frac{k_y^2}{k_\perp^2 \sigma^\parallel - i\chi_\perp\omega}, \quad (\text{C43})$$

$$G^R(B_z, B_x) = 0, \quad G^R(B_x, B_y) = -\chi_\perp \sigma^\parallel \frac{k_x k_y}{k_\perp^2 \sigma^\parallel - i\chi_\perp\omega}. \quad (\text{C44})$$

The mixed correlation functions among electric and magnetic fields are

$$G^R(B_z, E_x) = \frac{-2im\mu k_x \sigma_1^\perp k_\perp^2 + \chi_\parallel\omega(-k_y \sigma_1^\perp - k_x \sigma_2^\perp)}{\sigma_1^\perp k_\perp^2 - i\chi_\parallel\omega}, \quad (\text{C45})$$

$$G^R(B_z, E_z) = G^R(B_x, E_x) = G^R(B_x, E_y) = 0, \quad G^R(B_x, E_z) = \frac{\omega}{k_y} G^R(B_x, B_x). \quad (\text{C46})$$

The correlation functions among electric field components are

$$G^R(E_z, E_z) = \frac{\sigma^\parallel \chi_\perp \omega^2}{\sigma^\parallel k_\perp^2 - i\chi_\perp\omega} = \frac{\omega^2}{k_y^2} G^R(B_x, B_x), \quad G^R(E_z, E_x) = 0, \quad (\text{C47})$$

$$G^R(E_x, E_x) = i\omega \frac{k_x^2(\sigma_1^\perp{}^2 + (2m\mu - \sigma_2^\perp)^2) - i\omega \sigma_1^\perp \chi_\parallel}{\sigma_1^\perp k_\perp^2 - i\chi_\parallel\omega}, \quad (\text{C48})$$

$$G^R(E_x, E_y) = i\omega \frac{2m\mu \sigma_1^\perp k_\perp^2 + k_x k_y (\sigma_1^\perp{}^2 + (2m\mu - \sigma_2^\perp)^2) - i\omega \sigma_2^\perp \chi_\parallel}{\sigma_1^\perp k_\perp^2 - i\omega \chi_\parallel}. \quad (\text{C49})$$

We can use rotational symmetry to take $k_\perp = k_y$. We then find that

$$G^R(B_z, E_x) = -\frac{\chi_\parallel \sigma_1^\perp \omega k_y}{\sigma_1^\perp k_y^2 - i\chi_\parallel\omega}, \quad G^R(B_z, E_y) = k_y \frac{-2im\mu \sigma_1^\perp k_y^2 - \chi_\parallel \sigma_2^\perp \omega}{\sigma_1^\perp k_y^2 - i\chi_\parallel\omega}, \quad (\text{C50})$$

$$G^R(E_y, E_y) = i\sigma_1^\perp \omega + \frac{k_y^2(2m\mu - \sigma_2^\perp)^2}{\sigma_1^\perp k_y^2 - i\chi_\parallel\omega}, \quad G^R(E_x, E_x) = \frac{\omega^2 \sigma_1^\perp \chi_\parallel}{\sigma_1^\perp k_y^2 - i\chi_\parallel\omega}, \quad (\text{C51})$$

$$G^R(E_x, E_y) = i\sigma_2^\perp \omega + i\sigma_1^\perp k_y^2 \omega \frac{2m\mu + \sigma_2^\perp}{\sigma_1^\perp k_y^2 - i\omega \chi_\parallel}. \quad (\text{C52})$$

A curious thing is that even in the direction parallel to $k_\perp = k_y$, $G^R(E_y, E_y)$ still has a diffusion pole.

With $k_\perp \rightarrow 0$, we again find (C40). For $\omega \rightarrow 0$, we have

$$G^R(B_z, E_x) = -2im\mu k_x, \quad G^R(B_z, B_z) = \chi_\parallel, \quad (\text{C53})$$

which gives the first equation in (18), and implies that we should interpret $\chi_\parallel$ as the longitudinal one-form susceptibility.

4. Case 4: $\mu \neq 0$ and general momentum

For the correlation functions among the magnetic field components in the case of general k and ω , it is useful to define the quantity

$$A = k_z^2 \xi^2 + \sigma_1^\perp \sigma^\parallel k_\perp^2. \quad (\text{C54})$$

Note that we can write

$$\begin{aligned} \Gamma &= \chi_\parallel k_z^2 (A - i\sigma_1^\perp \chi_\perp \omega) + \chi_\perp k_\perp^2 (A - i\sigma^\parallel \chi_\parallel \omega) - i\chi_\perp \sigma_1^\perp \omega (\chi_\parallel k_z^2 + \chi_\perp k_\perp^2) - \chi_\perp^2 \chi_\parallel \omega^2 \\ &= (\chi_\parallel k_z^2 + \chi_\perp k_\perp^2) (A - i\sigma_1^\perp \chi_\perp \omega) - i\chi_\parallel \chi_\perp \omega (\sigma_1^\perp k_z^2 + \sigma^\parallel k_\perp^2) - \chi_\perp^2 \chi_\parallel \omega^2. \end{aligned} \quad (\text{C55})$$

We introduce the following combinations of the B_a and E_a :

$$B_1 = \frac{B_a k_a}{k_\perp}, \quad B_2 = \frac{\epsilon_{ab} B_a k_b}{k_\perp}, \quad E_1 = \frac{E_a k_a}{k_\perp}, \quad E_2 = \frac{\epsilon_{ab} E_a k_b}{k_\perp}. \quad (\text{C56})$$

The retarded correlation functions among magnetic field components are then given by

$$G^R(B^z, B^z) = \frac{\chi_\parallel \chi_\perp}{\Gamma} k_\perp^2 (A - i\sigma_1^\perp \chi_\perp \omega), \quad (\text{C57})$$

$$G^R(B^z, B_1) = -\frac{\chi_\parallel \chi_\perp}{\Gamma} k_z k_\perp (A - i\sigma_1^\perp \chi_\perp \omega), \quad (\text{C58})$$

$$G^R(B^z, B_2) = -i \frac{\chi_{\parallel} \chi_{\perp}^2 \sigma_2^{\perp}}{\Gamma} k_z k_{\perp} \omega, \quad (C59)$$

$$G^R(B_1, B_1) = \frac{\chi_{\perp} \chi_{\parallel}}{\Gamma} k_z^2 (A - i \sigma_1^{\perp} \chi_{\perp} \omega), \quad (C60)$$

$$G^R(B_1, B_2) = i \frac{\chi_{\perp}^2 \chi_{\parallel} \sigma_2^{\perp}}{\Gamma} \omega k_z^2, \quad (C61)$$

$$G^R(B_2, B_2) = \frac{\chi_{\perp}}{\Gamma} [\chi_{\perp} k_{\perp}^2 (A - i \chi_{\parallel} \sigma^{\parallel} \omega) + \chi_{\parallel} k_z^2 (A - i \chi_{\perp} \sigma_1^{\perp} \omega)]. \quad (C62)$$

Among the electric field components we have

$$G^R(E^z, E^z) = i \omega \sigma^{\parallel} - \frac{4m^2 \mu^2}{\chi_{\parallel}} k_z^2 + \frac{4m^2 \mu^2 \chi_{\perp}}{\chi_{\parallel}} \frac{k_z^2 k_{\perp}^2}{\Gamma} (A - i \sigma_1^{\perp} \chi_{\perp} \omega) \quad (C63)$$

$$- i \frac{\omega k_{\perp}^2 \sigma^{\parallel}}{\Gamma} [k_z^2 (\sigma_1^{\perp} \sigma^{\parallel} \chi_{\parallel} + 4m\mu \sigma_2^{\perp} \chi_{\perp}) + \sigma^{\parallel} \chi_{\perp} (\sigma_1^{\perp} k_{\perp}^2 - i \chi_{\parallel} \omega)], \quad (C64)$$

$$G^R(E^z, E_1) = \frac{k_{\perp} k_z}{\Gamma} (k_z^2 [i \omega (\xi^2 \sigma^{\parallel} \chi_{\parallel} + 8m^2 \mu^2 \sigma_1^{\perp} \chi_{\perp}) - 4m^2 \mu^2 A] \\ + \omega \chi_{\perp} \{ ik_{\perp}^2 [\sigma_1^{\perp 2} + (2m\mu - \sigma_2^{\perp})^2] \sigma^{\parallel} + \omega \sigma_1^{\perp} \sigma^{\parallel} \chi_{\parallel} + 2m\mu \omega \chi_{\perp} (2m\mu - \sigma_2^{\perp}) \}), \quad (C65)$$

$$G^R(E^z, E_2) = \frac{k_{\perp} k_z \omega \chi_{\perp}}{\Gamma} [-(\sigma_2^{\perp} \sigma^{\parallel} \chi_{\parallel} + 2m\mu \sigma_1^{\perp} \chi_{\perp}) \omega - 2im\mu A], \quad (C66)$$

$$G^R(E_1, E_1) = i \omega \sigma_1^{\perp} - i \frac{k_{\perp}^2 k_z^2 \omega}{\Gamma} [-(\sigma_2^{\perp})^2 \sigma^{\parallel} \chi_{\parallel} + \sigma_1^{\perp} \chi_{\perp} (\xi^2 - 8m^2 \mu^2)] - \frac{4m^2 \mu^2 k_{\perp}^2 k_z^2 A}{\Gamma} \\ - \frac{\omega}{\Gamma} [ik_z^4 \sigma_1^{\perp} \xi^2 \chi_{\parallel} + k_z^2 \omega (\sigma_1^{\perp 2} - \sigma_2^{\perp 2}) \chi_{\parallel} \chi_{\perp} - ik_{\perp}^2 \chi_{\perp} (2m\mu - \sigma_2^{\perp})^2 (k_{\perp}^2 \sigma^{\parallel} - i \omega \chi_{\perp})], \quad (C67)$$

$$G^R(E_1, E_2) = -i \omega \sigma_2^{\perp} - \frac{i \omega}{\Gamma} (-k_z^4 \sigma_2^{\perp} \xi^2 \chi_{\parallel} + k_{\perp}^2 \sigma_1^{\perp} \chi_{\perp} (2m\mu - \sigma_2^{\perp}) (k_{\perp}^2 \sigma^{\parallel} - i \omega \chi_{\perp}) \\ + k_z^2 \{ 2i \omega \sigma_1^{\perp} \sigma_2^{\perp} \chi_{\parallel} \chi_{\perp} + k_{\perp}^2 [-\sigma_1^{\perp} \sigma_2^{\perp} \sigma^{\parallel} \chi_{\parallel} + (2m\mu - \sigma_2^{\perp}) \xi^2 \chi_{\perp}] \}), \quad (C68)$$

$$G^R(E_2, E_2) = i \omega \sigma_1^{\perp} - \frac{i \omega}{\Gamma} \{ k_z^4 \sigma_1^{\perp} \xi^2 \chi_{\parallel} + k_{\perp}^2 \sigma_1^{\perp} \chi_{\perp} (k_{\perp}^2 \sigma^{\parallel} - i \omega \chi_{\perp}) \\ + k_z^2 [-i \omega (\sigma_1^{\perp 2} - \sigma_2^{\perp 2}) \chi_{\parallel} \chi_{\perp} + k_{\perp}^2 \sigma_1^{\perp} (\sigma_1^{\perp} \sigma^{\parallel} \chi_{\parallel} + \xi^2 \chi_{\perp})] \}. \quad (C69)$$

The mixed retarded correlation functions between the electric and magnetic field components are

$$G^R(B^z, E^z) = \frac{k_{\perp} k_z \chi_{\perp}}{\Gamma} [-\omega \sigma_2^{\perp} \sigma^{\parallel} \chi_{\parallel} - 2im\mu (A - i \sigma_1^{\perp} \chi_{\perp} \omega)], \quad (C70)$$

$$G^R(B^z, E_1) = -\frac{ik_{\perp} \chi_{\perp}}{\Gamma} \{ ik_{\perp}^2 [-(\sigma_2^{\perp} \sigma^{\parallel} \chi_{\parallel} + 2m\mu \sigma_1^{\perp} \chi_{\perp}) \omega - 2m\mu ik_z^2 \xi^2] + 2m\mu k_{\perp}^4 \sigma_1^{\perp} \sigma^{\parallel} - \omega^2 \sigma_2^{\perp} \chi_{\parallel} \chi_{\perp} \}, \quad (C71)$$

$$G^R(B^z, E_2) = \frac{k_{\perp} \omega \chi_{\parallel} \chi_{\perp}}{\Gamma} (i \omega \sigma_1^{\perp} \chi_{\perp} - k_{\perp}^2 \sigma_1^{\perp} \sigma^{\parallel} - k_z^2 \xi^2), \quad (C72)$$

$$G^R(B_1, E^z) = \frac{k_{\perp} k_z^2 \chi_{\perp}}{\Gamma} [2im\mu (k_{\perp}^2 \sigma_1^{\perp} \sigma^{\parallel} + k_z^2 \xi^2 - i \sigma_1^{\perp} \chi_{\perp} \omega) + \sigma_2^{\perp} \sigma^{\parallel} \chi_{\parallel} \omega], \quad (C73)$$

$$G^R(B_1, E_1) = \frac{ik_z \chi_{\perp}}{\Gamma} \{ ik_{\perp}^2 [-(\sigma_2^{\perp} \sigma^{\parallel} \chi_{\parallel} + 2m\mu \sigma_1^{\perp} \chi_{\perp}) \omega - 2m\mu ik_z^2 \xi^2] + 2m\mu k_{\perp}^4 \sigma_1^{\perp} \sigma^{\parallel} - \omega^2 \sigma_2^{\perp} \chi_{\parallel} \chi_{\perp} \}, \quad (C74)$$

$$G^R(B_1, E_2) = \frac{k_z \omega \chi_{\parallel} \chi_{\perp}}{\Gamma} (k_{\perp}^2 \sigma_1^{\perp} \sigma^{\parallel} + k_z^2 \xi^2 - i \omega \sigma_1^{\perp} \chi_{\perp}), \quad (C75)$$

$$G^R(B_2, E^z) = \frac{k_{\perp} \omega \chi_{\perp}}{\Gamma} [\sigma^{\parallel} \chi_{\perp} (k_{\perp}^2 \sigma_1^{\perp} - i \omega \chi_{\parallel}) + k_z^2 (\sigma_1^{\perp} \sigma^{\parallel} \chi_{\parallel} + 2m\mu \sigma_2^{\perp} \chi_{\perp})], \quad (C76)$$

$$G^R(B_2, E_1) = -\frac{k_z \omega \chi_{\perp}}{\Gamma} [k_{\perp}^2 \chi_{\perp} (\xi^2 - 2m\mu \sigma_2^{\perp}) + k_z^2 \xi^2 \chi_{\parallel} - i \omega \sigma_1^{\perp} \chi_{\parallel} \chi_{\perp}], \quad (C77)$$

$$G^R(B_2, E_2) = -\frac{ik_z \omega^2 \sigma_2^{\perp} \chi_{\parallel} \chi_{\perp}^2}{\Gamma}. \quad (C78)$$

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