

Gapless nonhydrodynamic modes in relativistic kinetic theory

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We provide rigorous criteria to determine whether the nonhydrodynamic sector of a relativistic kinetic theory is gapless. These general criteria apply to both Boltzmann's equation and approximate models thereof, provided that the latter are consistent with thermodynamic principles. As an application of these criteria, we prove that the nonhydrodynamic sector is gapless whenever the scattering cross section decays to zero at large energies. Physically, this happens because, if particles with very high energy (compared to the temperature) are free streaming, we can use them to build hot nonhydrodynamic waves, which live longer than any hydrodynamic wave. Since many standard cross sections in quantum field theory vanish at high energies, the existence of these nonthermal long-lived waves is a rather general feature of relativistic systems.

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Introduction. The existence of hydrodynamics usually relies on a “separation of timescales” assumption [1–8], which can be summarized as follows: Nonconserved quantities equilibrate (through collisions) over microscopic timescales, while conserved quantities equilibrate (through transport) over macroscopic timescales [9]. To appreciate the rationale of this assumption, consider the following example. Let $\rho(t, x^j)$ be the coarse-grained energy density of a gas of photons immersed in a medium. Suppose that interactions with the medium happen through elastic scattering, so ρ is a conserved density, and we can write a continuity equation $\partial_t \rho + \partial_j F^j = 0$, with F^j the energy flux [10]. If scatterings are frequent (compared to the evolution timescale of ρ), we can average the motion of photons over a few mean free times, and treat it as a random walk. Then, F^j is uniquely determined in terms of ρ by Fick's law, $F^j = -D \partial_j \rho$ [11], resulting in a hydrodynamic equation for the density: $\partial_t \rho = D \partial_j \partial_j \rho$. If, instead, the interactions are rare, we cannot construct a differential equation in terms of ρ alone, because F^j depends on infinitely many microscopic degrees of freedom (namely, the initial direction of propagation of each individual photon). Thus, a self-contained macroscopic description of the flow does not exist.

The tendency of gases and liquids to exhibit a scale separation at small gradients is often taken for granted [12–15]. The standard argument goes as follows: Let τ_{hy} be the decay timescale of a hydrodynamic wave of interest (e.g., a sound wave), and let τ_{nonhy} be the equilibration time of the longest-living collective excitation that is not described by hydrodynamics (i.e., that is not protected from decaying by

conservation laws). If

$$\tau_{\text{nonhy}} \ll \tau_{\text{hy}}, \quad (1)$$

then all the “nonhydrodynamic excitations” decay faster than the hydrodynamic wave, which will dominate the late-time behavior. But since the damping factor of hydrodynamic waves due to viscosity scales as $\sim e^{-k^2 t}$, where k is the wave number, one can make $\tau_{\text{hy}} \sim k^{-2}$ arbitrarily large by taking $k \rightarrow 0$. Thus, at sufficiently small gradients (i.e., at sufficiently large length scales), τ_{hy} will outlive all nonhydrodynamic excitations, and (1) must hold.

Unfortunately, the above argument has a flaw [16,17]. At infinitely large length scales, a fluid cell possesses infinitely many microscopic degrees of freedom. Thus, there may be an infinite sequence of collective excitations, whose equilibration times $\{\tau_n\}_{n \in \mathbb{N}}$ diverge at large n . Then, $\tau_{\text{nonhy}} = \infty$, and (1) never occurs. For example, suppose that $\tau_n = n$. Then, if we assign to each excitation an initial amplitude $1/n^2$, the total perturbation decays as

$$\sum_{n=1}^{\infty} \frac{e^{-t/n}}{n^2} \stackrel{\text{large } t}{\approx} \int_0^{+\infty} \frac{e^{-t/x}}{x^2} dx = \frac{1}{t}, \quad (2)$$

which survives longer than any sound wave (at nonzero viscosity). This possibility was extensively studied for nonrelativistic gases [18–21]. There, it was shown that, if the collision cross section decays to zero at high energies,¹ then there is a continuous infinity of nonhydrodynamic excitations whose equilibration times are unbounded above. It was also shown that continuous superpositions of these excitations decay as $\sim e^{-t^b}$ (with $b < 1$), outliving sound waves. Whether realistic quantum field theory (QFT) interactions give rise to similar phenomena in relativistic systems is an open problem, known as the “poles or cut?” dilemma [23] (poles: $\tau_{\text{nonhy}} < \infty$;

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¹In the mathematics literature, interactions of this type are called “soft potentials” [20,22].

cut: $\tau_{\text{nonhy}} = \infty$). Several hints point toward “cut,” coming from qualitative models [24–26] and numerical experiments [23,27,28]. Here, we provide a kinetic-theory discussion that is both mathematically rigorous and physically intuitive, as it relies solely on general principles and quick estimates. We find that, in most cases, the answer is indeed “cut.” From a mathematician’s perspective, this is related to the fact that most QFT interactions happen to be “soft,” in the sense of Ref. [29] (see also Ref. [30]).

The metric signature $(-, +, +, +)$ is adopted throughout, and we work in natural units: $c = \hbar = k_B = 1$.

Intuitive argument. Consider a relativistic gas of particles with mass m , density J^0 , and temperature T . Suppose that one particle in this gas has energy much higher than the thermal energy, i.e., $p^0 \gg \sqrt{m^2 + T^2} \equiv E_{\text{th}}$. The probability that this particle travels freely for a time interval t without interacting is $\mathcal{P}_s(t) = e^{-J^0 \sigma t}$ [31,32], where σ is the mean collision cross section. We take, as a “realistic” (i.e., QFT-inspired) high-energy interaction, the cross section of $\lambda \phi^4$ scalar field theory, $\sigma \sim g/E_{\text{CM}}^2$, where g is a constant, and E_{CM} is the energy in the center of momentum frame [33]. For collisions between our high-energy particle and the rest of the gas, we have on average $E_{\text{CM}}^2 \approx 4E_{\text{th}}p^0$, giving [34]

$$\mathcal{P}_s(t) \approx \exp \left[-\frac{J^0 g t}{4E_{\text{th}} p^0} \right]. \quad (3)$$

Now, suppose that, instead of one high-energy particle, we have a population of such particles. If these are sufficiently diluted, we can treat each particle independently. Thus, working in the thermodynamic limit, imagine that there are N particles with energy p^0 ($\gg E_{\text{th}}$), plus other $N/2^3$ particles with energy $2p^0$, plus other $N/3^3$ particles with energy $3p^0$, plus other $N/4^3$ particles with energy $4p^0$, ..., and so on (up to infinity). Then, the total energy associated with hard particles that have traveled freely without interacting from time 0 to time t is a series, whose sum is finite:

$$E_{\text{nonhy}}(0) = \pi^2 N p^0 / 6, \\ E_{\text{nonhy}}(t) = \sum_{n=1}^{\infty} \frac{N p^0}{n^2} \exp \left[-\frac{J^0 g t}{4E_{\text{th}} n p^0} \right] \stackrel{\text{large } t}{\approx} \frac{4NE_{\text{th}}(p^0)^2}{J^0 g t}. \quad (4)$$

Compare this situation with the photon gas in the introduction: E_{nonhy} is the energy carried by particles that have traveled freely across the medium *without* relaxing to a hydrodynamic constitutive relation. No macroscopic law relates the large-scale flux of E_{nonhy} to large-scale densities. The amount of this “nonhydrodynamic energy” falls as $1/t$, while hydrodynamic waves decay exponentially. Therefore, free-streaming hard particles dominate the late-time transport, and $\tau_{\text{nonhy}} = \infty$. Similar conclusions hold in any gas where $\sigma \rightarrow 0$ at large E_{CM} .

In summary: If the high-energy tails of the kinetic distribution are out of equilibrium, and the cross section decays to zero at large energies (as in models of QCD thermalization [24,35–37]), high-energy particles form a nonhydrodynamic excitation that equilibrates slower than sound waves, and $\tau_{\text{nonhy}} = \infty$. Note that also a small (e.g., an exponential)

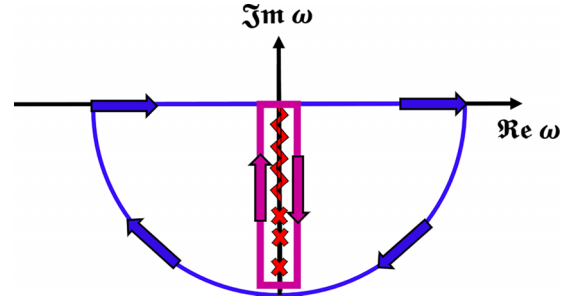


FIG. 1. To evaluate the integral $G(t) = (2\pi)^{-1} \int_{\mathbb{R}} G(\omega) e^{-i\omega t} d\omega$ for $t > 0$, we close the path for negative $\text{Im } \omega$ (blue line), since $e^{-i\omega t}$ decays exponentially there. Then, we shrink the path around the singularities (purple line). In the limit in which the vertical lines approach the imaginary axis, we obtain (5), with $z(\nu) = (2\pi i)^{-1} [G(0^+ - i\nu) - G(0^- - i\nu)]$.

nonequilibrium tail lives longer than a sound wave (see Supplemental Material [38] for two explicit examples).

The nonhydrodynamic sector is gapless. Let $G(t - t')$ be the retarded linear-response Green’s function of the shear stress component $\pi_{12}(t)$ induced by some external force $F(t')$, in the spatially homogenous limit. The singularities of its Fourier transform, $G(\omega)$, are the eigenfrequencies of the system, which (in kinetic theory) lie on the imaginary axis of the complex ω plane [39,40]. Thus, by deforming the integration path as in Fig. 1, we obtain [41]

$$G(t) = \Theta(t) \int_0^{+\infty} z(\nu) e^{-\nu t} d\nu, \quad (5)$$

where $z(\nu)$ is the spectral density of singularities at $\omega = -i\nu$. Comparing (3) with (5) as $\nu \rightarrow 0$, we see that $z(\nu)$ is proportional to the density of hard particles per unit $(p^0)^{-1} \propto \nu$ that are disturbed by F . If the force F couples with particles at all energies [this is the case of deformation rates, Eq. (48) in Ref. [39]], $z(\nu)$ is a continuous function near $\nu = 0$ [24,27] (and not a sum of Dirac deltas), meaning that $G(\omega)$ has a branch cut touching the origin. Hence, the $\lambda \phi^4$ gas has infinite nonhydrodynamic modes accumulating at $\omega = 0$, making the nonhydrodynamic sector gapless [23,28].

In the rest of this Letter, we provide rigorous theorems supporting the above arguments, for general interactions.

Formulation of the problem. We consider a relativistic gas of classical² particles of mass m (which may vanish), with kinetic distribution $f_p(x^\mu)$. The relativistic Boltzmann equation [42] has the form $p^\mu \partial_\mu f_p = \mathcal{C}[f_p]$, where $\mathcal{C}$ is some collision functional (it may be Boltzmann’s collision integral or some approximation thereof [43]). Fixed a uniform equilibrium state $f_p^{\text{eq}} = e^{\alpha + \beta_\nu p^\nu}$ (α, β_ν constant), we make the decomposition $f_p = f_p^{\text{eq}}(1 + \phi_p)$, and we linearize in ϕ_p . This results in the evolution equation $p^\mu \partial_\mu \phi_p = L\phi_p$, where L is a linear operator (whose domain of definition will be specified later). Fixed one reference frame, not necessarily at rest with respect to the gas, we restrict our attention to the homogenous

²The long-lived modes of interest are high-energy tails of the distribution. Since $f_p \ll 1$ when $p^0 \rightarrow \infty$, quantum statistics may be neglected.

problem,³ thereby dropping the term $p^j \partial_j \phi_p$. Hence, have

$$\partial_t \phi_p = (p^0)^{-1} L \phi_p \implies \phi_p(t) = e^{(p^0)^{-1} L t} \phi_p(0). \quad (6)$$

Our goal in this Letter is to discuss the properties of the operator $(p^0)^{-1} L$, which generates the homogenous evolution. We stress the importance of keeping the factor $(p^0)^{-1}$, since the spectral properties of L may be very different from those of $(p^0)^{-1} L$. For example, if we take $L = -1$, whose spectrum is just $\{-1\}$, the spectrum of $(p^0)^{-1} L = -(p^0)^{-1}$ is continuous, covering the interval $[-m^{-1}, 0]$.

Features of the collision operator. We require L to have the following properties (taking $\phi_p \in \mathbb{C}$ for later convenience):

$$\text{Equilibrium states: } L1 = Lp^v = 0, \quad (7)$$

$$\text{Dissipation: } \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} \phi_p^* L \phi_p \leq 0, \quad (8)$$

$$\begin{aligned} \text{Onsager symmetry: } & \left[\int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} \psi_p^* L \phi_p \right]^* \\ &= \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} \phi_p^* L \psi_p, \end{aligned} \quad (9)$$

all of which are fulfilled by Boltzmann's collision integral [43] (see Supplemental Material [38] for the proof). Equation (7) expresses the requirement that all equilibrium states $f_p = e^{\tilde{\alpha} + \tilde{\beta}_v p^v}$ (with $\tilde{\alpha}$ and $\tilde{\beta}_v$ constant) must be solutions of the Boltzmann equation. In fact, taking $\tilde{\alpha} = \alpha + a$ and $\tilde{\beta}_v = \beta_v + b_v$, and linearizing in a and b_v , we obtain $\phi_p = a + b_v p^v$ which, plugged into the linearized Boltzmann equation, gives $L[a + b_v p^v] = p^\mu \partial_\mu (a + b_v p^v) = 0$. To derive (8), one expresses the entropy production in terms of the information current, $\varsigma = -\partial_\mu E^\mu \geq 0$, and imposes the second law of thermodynamics, $\varsigma \geq 0$. In kinetic theory, the information current E^μ is known [26,45], and we obtain

$$\begin{aligned} E^\mu &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu |\phi_p|^2, \\ \varsigma &= -\text{Re} \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} \phi_p^* L \phi_p. \end{aligned} \quad (10)$$

Equation (9) arises from microscopic time-reversal invariance [40,46], and allows us to drop the real part in the formula for ς . Combining (7) and (9), one finds that the perturbed particle current and stress-energy tensor,

$$\begin{aligned} \delta J^\mu &= \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu \phi_p, \\ \delta T^{\mu\nu} &= \int \frac{d^3 p}{(2\pi)^3 p^0} f_p^{\text{eq}} p^\mu p^\nu \phi_p, \end{aligned} \quad (11)$$

automatically obey the conservation laws $\partial_\mu \delta J^\mu = \partial_\mu \delta T^{\mu\nu} = 0$.

A convenient Hilbert space. To establish rigorous results, we must fix a space of functions. We choose the (separable)

Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, f_p^{\text{eq}})$, namely the space of (complex) square-integrable functions on $\mathbb{R}^3$ with respect to the measure $f_p^{\text{eq}} d^3 p$. The associated inner product is

$$(\psi_p, \phi_p) = \int \frac{d^3 p}{(2\pi)^3} f_p^{\text{eq}} \psi_p^* \phi_p, \quad (12)$$

with corresponding norm $\|\phi_p\| = \sqrt{(\phi_p, \phi_p)} = \sqrt{2E^0}$. The space $\mathcal{H}$ is the set of homogeneous states with finite free-energy density [47–49], and we use the free energy as a square norm⁴ [29]. Clearly, any physically meaningful perturbation ϕ_p should belong to $\mathcal{H}$. Indeed, all the products $p^{v_1} p^{v_2} \dots p^{v_l}$ belong to $\mathcal{H}$, meaning that all the moments $\delta \rho^{0v_1 v_2 \dots v_l} = (p^{v_1} p^{v_2} \dots p^{v_l}, \phi_p)$ are well defined inside $\mathcal{H}$, including $\delta J^0 = (1, \phi_p)$ and $\delta T^{0v} = (p^v, \phi_p)$.

The inner product (12) allows us to rewrite (8) and (9) as follows,

$$\begin{aligned} \varsigma &= -\left(\phi_p, \frac{1}{p^0} L \phi_p\right) \geq 0, \\ \left(\psi_p, \frac{1}{p^0} L \phi_p\right)^* &= \left(\phi_p, \frac{1}{p^0} L \psi_p\right), \end{aligned} \quad (13)$$

meaning that the operator $(p^0)^{-1} L$ is Hermitian negative-semidefinite. Furthermore, for $(p^0)^{-1} L$ to be physically meaningful, it must be densely defined in $\mathcal{H}$, since all functions in $\mathcal{H}$ are acceptable physical states.⁵ The above facts allow us to apply the Friedrichs extension (Theorem 5.1.13 of Ref. [55]), and promote $(p^0)^{-1} L$ to a self-adjoint operator.

A key theorem. Let us decompose $\mathcal{H} = \mathcal{H}_{\text{hy}} \oplus \mathcal{H}_{\text{nonhy}}$, where $\mathcal{H}_{\text{hy}}$ is the space of states that are described by hydrodynamics, and $\mathcal{H}_{\text{nonhy}}$ is the space of nonhydrodynamic excitations. Since we are working in the homogeneous limit, all hydrodynamic “waves” take the form of global equilibria $\phi_p = a + b_v p^v$, so that $\mathcal{H}_{\text{hy}} = \text{span}\{1, p^v\}$. Conversely, the nonhydrodynamic excitations are those states that are “invisible” to hydrodynamics. Since hydrodynamics is only aware

⁴Textbooks [42,43] use a different inner product, replacing our measure $f_p^{\text{eq}} d^3 p$ with the invariant measure $f_p^{\text{eq}} d^3 p / p^0$. This convention presents a serious problem: If the initial state has finite norm in $L^2(\mathbb{R}^3, f_p^{\text{eq}} / p^0)$, there is no guarantee that such norm will remain finite at later times. By contrast, the second law forces $E^0(t)$ to be nonincreasing. Thus, if $\phi_p(0) \in \mathcal{H}$, then $\phi_p(t) \in \mathcal{H}$ for all $t \geq 0$. Some readers may feel that, in relativity, the invariant measure $d^3 p / p^0$ is “more natural.” This feeling is an artifact of the homogenous limit. Thinking of the inhomogeneous case, it is clear that the norm depends on the Cauchy surface Σ upon which we define the state, and thus should *not* be Lorentz invariant [50–52]. Indeed, using $2E^0$ as a squared norm is more natural, since it generalizes to $2 \int_\Sigma E^\mu d\Sigma_\mu$ in the inhomogeneous case, and we can use Gauss’ theorem [53] to link norms on different Cauchy surfaces: $\|\phi_p\|_{\Sigma_2}^2 - \|\phi_p\|_{\Sigma_1}^2 = -\int 2\varsigma d^4 x$.

⁵For sufficiently regular cross sections, Boltzmann’s collision operator is well defined on C^∞ functions with compact support. These are dense in $\mathcal{H}$, making Boltzmann’s operator densely defined (see, e.g., Refs. [22,29,54] and references therein). In the following, we will adopt the usual physics convention of saying that “statement X holds in Hilbert space Y ” whenever X holds in a *dense linear subset* of Y .

³The main results extend by continuity to states with $\phi_p \propto e^{ikx^1}$ (k small), because the operator p^1 / p^0 is bounded (see Chap. 5, Sec. 4.3 in Ref. [44]).

of the conserved densities, the nonhydrodynamic excitations are the states such that $\delta J^0 = \delta T^{0v} = 0$. Recalling that $\delta J^0 = (1, \phi_p)$ and $\delta T^{0v} = (p^v, \phi_p)$, we conclude that $\mathcal{H}_{\text{nonhy}} = \mathcal{H}_{\text{hy}}^\perp$. Therefore, $\mathcal{H}_{\text{nonhy}}$ is a Hilbert space in its own right. Furthermore, $\mathcal{H}_{\text{nonhy}}$ is an invariant space of $(p^0)^{-1}L$. This follows from the second equation of (13): Just take $\phi_p \in \mathcal{H}_{\text{nonhy}}$ and $\psi_p = a + b_v p^v$, so that, using (7),

$$\begin{aligned} \left(a + b_v p^v, \frac{1}{p^0} L \phi_p\right)^* &= \left(\phi_p, \frac{1}{p^0} L[a + b_v p^v]\right) \\ &= 0 \implies \frac{1}{p^0} L \phi_p \in \mathcal{H}_{\text{hy}}^\perp = \mathcal{H}_{\text{nonhy}}. \end{aligned} \quad (14)$$

Thus, we can restrict the operator $(p^0)^{-1}L$ to a self-adjoint operator on the Hilbert subspace $\mathcal{H}_{\text{nonhy}}$ of pure deviations from local thermodynamic equilibrium. Then, we have the following result, whose proof is provided in the Supplemental Material [38].

Theorem 1 (gaplessness criteria). Suppose that 1 and p^v are the only collisional invariants belonging to $\mathcal{H}$, i.e., $\text{Ker}(L) = \mathcal{H}_{\text{hy}}$. Then, the following facts are equivalent:

(a) We can construct nonhydrodynamic excitations that survive as long as we want, i.e., for any time $t > 0$,

$$\sup_{\phi_p \in \mathcal{H}_{\text{nonhy}}} \frac{||\phi_p(t)||}{||\phi_p(0)||} = 1. \quad (15)$$

(b) We can construct nonhydrodynamic states ϕ_p such that $|(p^0)^{-1}L\phi_p|$ is arbitrarily smaller than $||\phi_p||$, i.e.,

$$\inf_{\phi_p \in \mathcal{H}_{\text{nonhy}}} \frac{|(p^0)^{-1}L\phi_p|}{||\phi_p||} = 0. \quad (16)$$

(c) We can construct nonhydrodynamic states that produce negligibly small entropy, i.e.,

$$\inf_{\phi_p \in \mathcal{H}_{\text{nonhy}}} \frac{\mathcal{S}}{E^0} = 0. \quad (17)$$

(d) Working within the space $\mathcal{H}_{\text{nonhy}}$, the frequency $\omega = 0$ is an accumulation point of the spectrum of $(p^0)^{-1}L$. Namely, for any $\epsilon > 0$, there is a nonhydrodynamic frequency $\omega \neq 0$ such that $|\omega| < \epsilon$.

Statements (a)–(d) are four alternative (necessary and sufficient) criteria for τ_{nonhy} to be infinite. If any one of them is met, all four are met. Criterion (d) is the “cut” option considered in Ref. [23]. The implication (d)→(a) is a rigorous formulation of the “dehydrodynamization” mechanism discussed in Ref. [24]. Note that, since (6) is linear, the states in (15) may be rescaled by an arbitrary constant factor. Thus, if (a) holds, there are long-lived nonhydrodynamic perturbations whose amplitude is (and will forever remain) much larger than an arbitrary preassigned sound wave.

The real novelty of Theorem 1 are criteria (b) and (c). In fact, the recent literature tries to assess (d) by direct evaluation of the spectrum of $(p^0)^{-1}L$ [23,28]. This task requires numerical techniques that are by nature approximate, and sensitive to the details of L . Criteria (b) and (c) remarkably simplify the problem, since now one only needs to engineer a sequence $\{\phi_p^{(n)}\}_{n \in \mathbb{N}} \subset \mathcal{H}$ with unit norm, vanishing conserved densities, $(1, \phi_p^{(n)}) = (p^v, \phi_p^{(n)}) = 0$, and such that either $|(p^0)^{-1}L\phi_p^{(n)}|$

or $\mathcal{S}[\phi_p^{(n)}]$ converges to zero as $n \rightarrow \infty$. Below, we provide an explicit example.

Application: Massless scalar particles. Let us apply Theorem 1 to a system of massless classical particles with $\lambda\varphi^4$ interaction (working in the equilibrium rest frame, so $\beta^\mu = \beta\delta_0^\mu$). In this model, the spectrum of Boltzmann’s collision operator L is known analytically, and $\text{Ker}(L) = \mathcal{H}_{\text{hy}}$ [34]. We consider the sequence

$$\phi_p^{(n)} = L_n^{(5)}(\beta p^0) p^2 p^3, \quad (18)$$

where $L_n^{(2\ell+1)}$ are Laguerre polynomials. These states are eigenvectors of L , with eigenvalues

$$\chi_{n2} \propto -\frac{n+1}{n+3}. \quad (19)$$

Showing that $(1, \phi_p^{(n)}) = (p^v, \phi_p^{(n)}) = 0$ is straightforward. Furthermore, in the Supplemental Material [38] we verify that

$$\frac{|(p^0)^{-1}L\phi_p^{(n)}|}{||\phi_p^{(n)}||} \leq \frac{\text{const}}{\sqrt{n+3}} \xrightarrow{n \rightarrow +\infty} 0. \quad (20)$$

Thus, criterion (b) is met. Applying Theorem 1, we conclude from (a) that the gas possesses nonhydrodynamic excitations that survive longer than sound waves. We also conclude from (d) that the nonhydrodynamic sector is gapless, in agreement with Eq. (4), and with numerical studies [23,28].

High-energy criterion. For Boltzmann’s equation, $\text{Ker}(L)$ always equals $\mathcal{H}_{\text{hy}}$, and the entropy production is [42]

$$\begin{aligned} \mathcal{S} &= \frac{1}{8} \int \frac{d^3 p}{(2\pi)^3 p^0} \frac{d^3 p'}{(2\pi)^3 p'^0} \frac{d^3 q}{(2\pi)^3 q^0} \frac{d^3 q'}{(2\pi)^3 q'^0} \sigma(s, \theta) \\ &\quad \times \delta^4(p + p' - q - q') f_p^{\text{eq}} f_{p'}^{\text{eq}} |\phi_p + \phi_{p'} - \phi_q - \phi_{q'}|^2, \end{aligned} \quad (21)$$

where $\sigma(s, \theta)$ is the differential cross section, $s = E_{\text{CM}}^2$ is the Mandelstam variable, and θ is the scattering angle in the center of momentum frame. Now, suppose that there are two constants $A \geq 0$ and $0 < a \leq 1$ such that $\sigma(s, \theta) \leq A/s^a$. Then, if we consider the sequence of perturbations

$$\phi_p^{(n)} = e^{\frac{\beta}{2}(1-n^{-1})p^0} \frac{2p^2 p^3}{(p^2)^2 + (p^3)^2}, \quad (22)$$

which are orthogonal to 1 and p^v , we find that (see Supplemental Material [38])

$$\frac{\mathcal{S}[\phi_p^{(n)}]}{E^0[\phi_p^{(n)}]} \leq \frac{\text{const}}{n^a} \xrightarrow{n \rightarrow +\infty} 0. \quad (23)$$

Hence, condition (c) of Theorem 1 is fulfilled. Considering that $\mathcal{S}$ is linear in σ , we have the following rigorous result.

Theorem 2 (high-energy criterion). Consider a classical gas of relativistic particles, governed by Boltzmann’s equation, with nonvanishing cross section. Suppose that there are three constants $A \geq 0$, $B \geq 0$, and $0 < a < 1$ such that

$$\sigma(s, \theta) \leq \frac{A}{s^a} + \frac{B}{s}, \quad (24)$$

for all θ and s . Then, the assumptions of Theorem 1 are met, and facts (a)–(d) occur. This means that, if σ decays for $s \rightarrow +\infty$ as a power law s^{-a} (or faster), we can construct

nonhydrodynamic excitations that survive as long as we want. The second term in (24) is there to remind us that $\sigma(s, \theta)$ is allowed to diverge at $s = 0$ (which occurs only for massless particles), but it should not grow faster than s^{-1} (for our proof to work). Note that Theorem 2 only provides sufficient conditions for (a)–(d) to hold. In the Supplemental Material [38], we generalize Theorem 2 to all soft interactions, following the definition of Ref. [29].

Application: Screened gauge theories. Most scattering differential cross sections in QED [56] and QCD [57] decay as $1/s$ at high energies [see, e.g., the (ultrarelativistic) electron-electron and the gluon-gluon scatterings]:

$$\begin{aligned}\sigma_{ee \rightarrow ee} &\propto \frac{(3 + \cos^2 \theta)^2}{s \sin^4 \theta}, \\ \sigma_{gg \rightarrow gg} &\propto \frac{1}{s} \left[3 - \frac{\sin^2 \theta}{4} + \frac{4 + 12 \cos^2 \theta}{\sin^4 \theta} \right].\end{aligned}\quad (25)$$

The factor $(\sin \theta)^{-4}$ makes the operator L small-angle divergent, reflecting the long-range nature of the interaction, which makes Boltzmann’s assumption of local collisions invalid. However, if we correct the cross sections (25) with medium-dependent effects (e.g., Debye screening [58]) the small-angle behavior is regularized [57,59]. With certain regularizations, e.g., $(\sin \theta)^{-4} \rightarrow (\epsilon^2 + \sin^2 \theta)^{-2}$ (with $\epsilon = \text{const}$), Theorem 2 applies.

Application: Yukawa theory. Ultrarelativistic fermions f interacting through either Yukawa coupling $\phi \bar{f} f$ or $\phi \bar{f} \gamma^5 f$ have a scattering cross section $\sigma_{ff \rightarrow ff} \sim 1/s$ [60,61] (the dependence on the angle disappears at high energies). Theorem 2 applies.

Final remarks. Interactions that decay to zero at high energy are the most common in relativistic physics. Nearly all “textbook” (two-body) cross sections decay as s^{-1} , as one would expect from naive dimensional analysis [33]. Indeed, $\sigma \propto s^{-1}$ is the universal scaling law of those theories that “forget” their reference mass scales in the ultrarelativistic limit. In those theories that “remember” some mass scale M

at high energy, unitarity requires that $\sigma \leq \pi M^{-2} \ln^2(s)$ [62], which allows for mild growth of σ with s [63]. Hence, some diluted gases with a nonhydrodynamic gap are likely to exist. However, they seem to be the exception, not the rule.

Let us provide two arguments suggesting that the nonhydrodynamic sector of QCD plasmas is probably gapless.

(1) All relaxation-type models of QCD matter adopt an energy-dependent relaxation time $\tau \propto (p^0)^a$, with $0 < a < 1$ [24,35,37]. If this approximation is valid, at least qualitatively, we can use it to learn about the scaling law of the cross sections. In fact, for a high-energy particle interacting with a thermal bath, $s \propto p^0$, and $\tau \propto \sigma^{-1}$. We conclude that $\sigma \propto 1/s^a$, consistently with (24). Hence, in a corresponding “full-fledged” Boltzmann equation, Theorem 2 applies, and the nonhydrodynamic sector is gapless.

(2) According to Refs. [64,65], particles with $p^0 \gg T$ traveling through a QCD plasma lose energy following the equation $\dot{p}^0 = -2C\sqrt{p^0}$, whose solution is $p^0(t) = [\sqrt{p^0(0)} - Ct]^2 \Theta[\sqrt{p^0(0)} - Ct]$. Suppose that there is a diluted population of high-energy particles whose number (per unit energy) falls as $(p^0)^{-3}$ at $t = 0$. Then, at late times, the nonthermal energy carried by these particles decays slower than exponentially,

$$E_{\text{nonth}}(t) \propto \int_{(Ct)^2}^{\infty} \frac{(\sqrt{p^0} - Ct)^2}{(p^0)^3} dp^0 = \frac{1}{6(Ct)^2}, \quad (26)$$

proving the absence of a gap.

Although our analysis was formally carried out within the linear theory, the discussed phenomena are not artifacts of linearization. In fact, in the Supplemental Material [38], we show that, for each linear state considered here, there is a family of distribution functions that behave under the nonlinear theory in a way that is indistinguishable from the linear theory prediction.

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