

Nonlinear thermoelectric probes of anomalous electron lifetimes in topological Fermi liquids

Johannes Hofmann^{1,2,*} and Habib Rostami^{3,†}¹Department of Physics, Gothenburg University, 41296 Gothenburg, Sweden²Nordita, Stockholm University and KTH Royal Institute of Technology, 10691 Stockholm, Sweden³Department of Physics, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom

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In two-dimensional Fermi liquids (FLs), odd-parity Fermi surface deformations have anomalously slow relaxation rates that are suppressed as T^4 with temperature T , distinct from the standard FL T^2 scaling. We demonstrate here that these long-lived modes, which are often hidden in linear response, have a significant impact on nonlinear transport by establishing a direct proportionality of nonlinear thermoelectric currents to the anomalously large relaxation time. These currents exist in topological time-reversal invariant FLs, and their magnitude is characterized by topological heat capacitance terms that we refer to as the *Berry curvature capacity* and the *velocity-curvature capacity*. We quantify the effect in bismuth telluride, which is an efficient thermoelectric and a topological insulator with a hexagonal Fermi surface. The proposed field-induced thermoelectric currents are well within the range of current experiments. Our findings demonstrate the potential to explore topological and many-body effects in FLs through the nonlinear thermoelectric response, urging further experimental studies.

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Introduction. Recent advances in the fabrication of ultraclean two-dimensional (2D) materials create interaction-dominated electron gases [1–14]. As the temperature is increased to a sizable fraction of the Fermi temperature T_F in these materials, there is a crossover from a ballistic or diffusive transport regime to a hydrodynamic regime, where electron-electron interactions dominate over scattering from disorder and phonons. The change in the relaxation mechanism has a profound impact on the transport properties: For example, the characteristic relaxation time τ in Ohm's law $\sigma = ne^2\tau/m^*$ diverges in clean systems since binary collisions cannot relax the total charge current. Transport is instead set by the shear viscosity $\eta \sim \tau_{FL}$, which diverges at low temperatures with a Fermi liquid (FL) lifetime $\tau_{FL} \sim (T_F/T)^2$ [15–22]. This characteristic quadratic temperature dependence reflects a phase-space constraint where only quasiparticles close to the Fermi surface interact, and a common expectation is that this FL scale sets the temperature dependence of all relaxation times (excluding conserved modes like the current mode).

However, the simple quadratic-in-temperature FL scaling only applies to a restricted class of collective deformations that have even parity. By contrast, odd-parity deformations are predicted to have anomalously long lifetimes that diverge

as a quartic function of temperature $\tau_{AN} \sim (T_F/T)^4$. Here, odd and even parity refers to the components of the electron distribution function that are odd or even under momentum inversion, $\mathbf{p} \rightarrow -\mathbf{p}$. This odd-even anomaly had been noted previously in the literature [23–25] but has only recently been quantified, analytically at low temperatures by Ledwith *et al.* [26,27] and numerically by Hofmann and Gran [28], Nilsson *et al.* [29] for a parabolic dispersion. Generally, the odd-even effect requires time-reversal symmetry, $\varepsilon(\mathbf{p}) = \varepsilon(-\mathbf{p})$, ensuring no mixing of opposite-parity deformations. Furthermore, it is independent of the specific electron interaction since the anomalous relaxation occurs via soft small-angle scattering [26]. However, these exotic FL features are still waiting for an experimental confirmation. Authors of previous studies on nontopological materials have suggested that odd-parity dynamics primarily impacts the transverse response, introducing additional transverse diffusive modes [30] and higher-derivative corrections to shear viscosity [31], but a practical measurement scheme that probes long-lived odd-parity modes is currently lacking.

In this letter, we show that nonlinear thermoelectric transport in anisotropic and topological FLs is sensitive to electron interactions and is moreover significantly enhanced by the anomalously long-lived odd-parity features. The key element of our proposal is that breaking spatial inversion symmetry $\mathbf{r} \rightarrow -\mathbf{r}$ induces topological transport related to the Berry curvature, which in time-reversal invariant systems has odd parity, $\Omega(\mathbf{p}) = -\Omega(-\mathbf{p})$ [32,33]. The Berry curvature impacts the semiclassical quasiparticle dynamics as [32,34]

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{p}) - \frac{e}{\hbar} \mathcal{E}(t) \times \Omega(\mathbf{p}). \quad (1)$$

The first term $\mathbf{v}(\mathbf{p}) = \partial_{\mathbf{p}}\varepsilon(\mathbf{p})$ is the band velocity, and the second term is the anomalous velocity driven by the

*Contact author: johannes.hofmann@physics.gu.se

†Contact author: hr745@bath.ac.uk

homogeneous electric field $\mathcal{E}(t)$ of light with negligible magnetic component. In time-reversal symmetric systems, the Berry curvature plays a crucial role in nonlinear photocurrent generation [35–47]. Specifically, nonlinear Hall [40,48–51] and injection [37–39,52] photocurrents depend on the product of a relaxation time scale and an average over the Berry curvature. For example, the nonlinear Hall current, measured recently in 2D metals [53–55], is determined by the Berry curvature dipole $\mathbf{D} = \langle \partial_{\mathbf{p}} \Omega \rangle = \sum_{\mathbf{p}} (\partial_{\mathbf{p}} \Omega) f_0$, which is an average over the Fermi-Dirac distribution function f_0 . Like Ohm's law, however, this charge current is relaxed by momentum-nonconserving processes like impurity scattering and does not probe FL effects.

In the following, we explore the nonlinear thermoelectric response of topological FLs. This response is highly sensitive to electron interactions and depends on the topological heat capacitance expressed in terms of topological Berry curvature moments that we term the *Berry curvature capacity*:

$$C_{\Omega} = \frac{\partial \langle \Omega^2 \rangle}{\partial T} = \frac{1}{T} \sum_{\mathbf{p}} (\varepsilon - \mu) \Omega^2 (-\partial_{\varepsilon} f_0), \quad (2)$$

and the *velocity-curvature capacity*:

$$C_{v_a} = \frac{\partial \langle v_a \Omega \rangle}{\partial T} = \frac{1}{T} \sum_{\mathbf{p}} v_a = (\varepsilon - \mu) \Omega (-\partial_{\varepsilon} f_0). \quad (3)$$

Both of these quantities have integrands with even parity and are thus finite in the presence of time-reversal symmetry. Intuitively, the form of these topological heat capacities is inferred by comparing with the conventional thermoelectric current $\mathbf{j}^{\nabla T} = \chi^{\nabla T} \nabla T$, where the thermoelectric response function $\chi^{\nabla T} \sim C_e \sim \partial_T \langle v^2 \rangle$ is proportional to the conventional electronic heat capacity.¹ Incorporating the anomalous velocity into this definition naturally leads to the topological heat capacities characterized by Eqs. (2) and (3) with a respective quadratic and linear dependence on the electric field.

The topological capacities emerge in the nonlinear thermoelectric currents as sketched in Fig. 1. We propose a second-order rectified (i.e., static) field-induced thermoelectric current that is proportional to C_{Ω} . Specifically, in the case of a circularly polarized field, this current is transverse to the direction of the temperature gradient [Fig. 1(a)]. This nonlinear (in the electric fields) thermoelectric current takes the form [56]:

$$\mathbf{j}^0 = T \frac{e^3}{4\hbar^2} (\text{Im}[\tau_{\text{AN}}(\omega)] [\mathcal{E}^T \times \mathcal{F}(\omega)] + \text{Re}[\tau_{\text{AN}}(\omega)] \times \{ [\mathcal{E}^T \times \mathcal{E}(\omega)]_z [\mathcal{E}^*(\omega) \times \hat{\mathbf{z}}] + \text{c.c.} \}) C_{\Omega}, \quad (4)$$

where $\tau_{\text{AN}}(\omega) = \tau_{\text{AN}}/(1 + i\omega\tau_{\text{AN}})$, $\mathcal{F}(\omega) = i\mathcal{E}(\omega) \times \mathcal{E}^*(\omega)$ is a real-valued circular-light induced field, and we introduce the effective thermal electric field $\mathcal{E}^T = -\nabla T/T$ [57–59]. The first and second terms in Eq. (4) correspond to circular and linear polarized light responses, respectively. Crucially, for Fermi surfaces with hexagonal symmetry or higher, the

¹To obtain the electronic heat capacity in correct units, consider $m_e^* C_e$, $m_e^* (e\mathcal{E}/\hbar)^2 C_{\Omega}$, and $m_e^* (e\mathcal{E}/\hbar) C_{v_a}$, where m_e^* is the electronic effective mass.

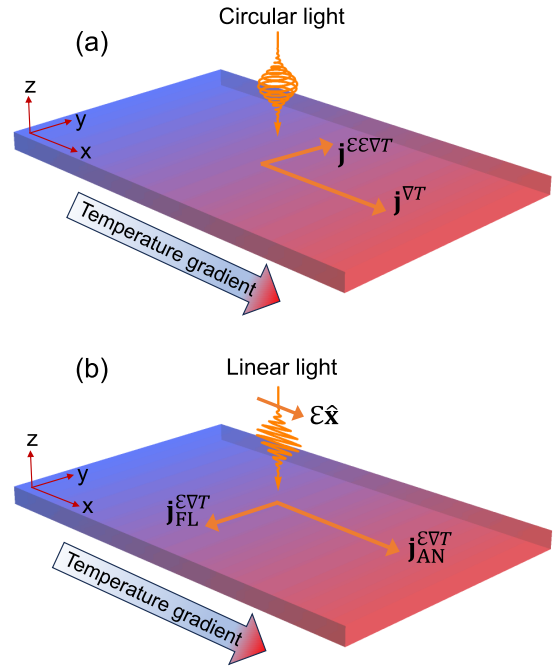


FIG. 1. Illustration of field-induced thermoelectric currents. (a) Nonlinear thermoelectric current $\mathbf{j}^{\mathcal{E} \nabla T}$ due to circular light, Eq. (4), characterized by the anomalous relaxation time τ_{AN} and Berry curvature capacity C_{Ω} , and conventional thermoelectric current $\mathbf{j}^{\nabla T}$. (b) Nonlinear thermoelectric current induced by linear polarized light, characterized by the velocity-curvature capacity C_v . $\mathbf{j}_{\text{AN}}^{\mathcal{E} \nabla T}$ and $\mathbf{j}_{\text{FL}}^{\mathcal{E} \nabla T}$ are the orthogonal contributions proportional to τ_{AN} and τ_{FL} , Eq. (5). We assume a temperature gradient and light polarization along the x direction, normal to the mirror plane of the system ($x \rightarrow -x$).

anomalous velocity does not couple to a conserved current and has an enhanced odd-parity lifetime τ_{AN} , which in turn leads to a large increase in the magnitude of the current in Eq. (4) [56].

In addition, there is a dynamical field-driven thermoelectric current that is proportional to C_{v_a} [Fig. 1(b)]:

$$\mathbf{j}^{\omega} = -T \frac{e^2}{\hbar} \{ \tau_{\text{FL}} (\mathbf{C}_v \cdot \mathcal{E}^T) [\mathcal{E}(\omega) \times \hat{\mathbf{z}}] - \tau_{\text{AN}}(\omega) [\mathcal{E}^T \times \mathcal{E}(\omega)]_z \mathbf{C}_v \}, \quad (5)$$

which couples to both odd- and even-parity relaxation times τ_{AN} and τ_{FL} . The first term is a light-induced thermoelectric Hall current that is directed perpendicular to the polarization of the light field. The second term is proportional to the anomalous damping rate τ_{AN} and thus enhanced in clean systems at low temperatures.

Model. In the following, we derive the nonlinear currents in Eqs. (4) and (5) using Boltzmann kinetic theory, where we consider transport in a single band. A general deformation of the quasiparticle distribution induces a nonzero expectation value of the charge current [60,61]:

$$\mathbf{j}(t, \mathbf{r}) = -e \sum_{\mathbf{p}} \dot{\mathbf{r}} f(t, \mathbf{r}, \mathbf{p}). \quad (6)$$

By time-reversal symmetry, the velocity $\dot{\mathbf{r}}$ in Eq. (1) is odd under momentum inversion, and hence, the current response is solely determined by odd-parity deformations. The time evolution of the quasielectron distribution follows from the FL kinetic equation [62]:

$$\left[\frac{\partial}{\partial t} + \dot{\mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}} \right] f(t, \mathbf{r}, \mathbf{p}) = \hat{\mathcal{J}}\{\delta f\}, \quad (7)$$

with $\dot{\mathbf{p}} = -e\mathcal{E}(t)$. The left-hand side is the streaming term that describes the free phase-space evolution in the presence of a Berry curvature, and the right-hand side is the collision integral that accounts for two-particle scattering. The collision integral will relax a general nonequilibrium distribution to the local equilibrium distribution $f_0(\mathbf{r}, \mathbf{p}) = (\exp\{[\varepsilon(\mathbf{p}) - \mu]/T(\mathbf{r})\} + 1)^{-1}$, which depends on a chemical potential μ and a local electronic temperature $T(\mathbf{r})$ that varies in space. Using the chain rule, $\partial_{\mathbf{r}} f_0 = -(\varepsilon - \mu)(-\partial_{\varepsilon} f_0)\mathcal{E}^T$, we see that the temperature gradient $\mathcal{E}^T$ induces a deviation from local equilibrium $\delta f(t, \mathbf{r}, \mathbf{p}) = f(t, \mathbf{r}, \mathbf{p}) - f_0(\mathbf{r}, \mathbf{p})$, which in turn generates a thermoelectric current.

We analyze the kinetic Eq. (7) using a relaxation-time approximation. The leading deformation δf solves [56,63,64]

$$\partial_t \delta f + \hat{\mathcal{J}}\{\delta f\} = e\mathcal{E} \cdot (\partial_{\mathbf{p}} f_0) - \left[\mathbf{v} - \frac{e}{\hbar} \mathcal{E} \times \boldsymbol{\Omega} \right] \cdot (\partial_{\mathbf{r}} f_0), \quad (8)$$

which is sourced by the three terms on the right-hand side. Here, the first term describes the external Coulomb force, the second term is induced by the thermal gradient that couples to the band velocity, and the third term, which couples the temperature gradient to the anomalous velocity, is of topological origin. Note that, in 2D, the Berry curvature is normal to the 2D metal surface, $\boldsymbol{\Omega}(\mathbf{p}) = \Omega(\mathbf{p})\hat{\mathbf{z}}$, and the motion is always in plane. The relaxation-time approximation assumes that each source term ψ is an eigenmode of the collision integral with a characteristic decay rate $\hat{\mathcal{J}}\{\psi\} = -\tau^{-1}\psi$. For an interaction-dominated FL, these rates will in general be different: The Coulomb force induces a shift in the center-of-mass momentum (as is seen by using the chain rule), which as discussed will not relax by binary collisions since they conserve the canonical momentum, and has thus infinite lifetime in a clean system. The second term in Eq. (8) induces a thermal energy current, which is a higher-order current mode with standard FL scaling $\tau_{\text{FL}} \sim 1/T^2$ (where we assume the interaction-dominated limit $\omega\tau_{\text{FL}} \simeq 1$). The last term, however, describes the topological deformation. In anisotropic FLs, this mode does not induce a thermal current and will thus decay anomalously slowly with general $\tau_{\text{AN}} \sim 1/T^4$ (for which $\omega\tau_{\text{AN}} \gg 1$ or, equivalently, $\tau_{\text{AN}}/\tau_{\text{FL}} \gg 1$).

Assuming a time-dependent electric field $\mathcal{E}_a(t) = \text{Re}\{\mathcal{E}_a(\omega)e^{i\omega t}\}$, a solution of the kinetic Eq. (8) gives a deviation from equilibrium $\delta f = \text{Re}\{\delta f^0 + \delta f^\omega e^{i\omega t}\}$ with [56]

$$\begin{aligned} \delta f^0 &= +\tau_{\text{FL}}(\varepsilon - \mu)(\mathbf{v} \cdot \mathcal{E}^T)(-\partial_{\varepsilon} f_0), \\ \delta f^\omega &= -\frac{e}{i\omega}[\mathbf{v} \cdot \mathcal{E}(\omega)](-\partial_{\varepsilon} f_0) \\ &\quad - \frac{e}{\hbar}\tau_{\text{AN}}(\omega)(\varepsilon - \mu)[\mathcal{E}(\omega) \times \boldsymbol{\Omega}] \cdot \mathcal{E}^T(-\partial_{\varepsilon} f_0). \end{aligned} \quad (9)$$

All of these contributions have odd parity. They induce a nonlinear current in Eq. (6) when combined with the anomalous velocity, and for the topological deformation (the second term in δf^ω) also in combination with the band velocity. In principle, the leading deformation δf , which is linear in the position and momentum derivatives, acts as a source for a second-order correction δf_2 that is obtained from an iterative solution of the kinetic equation. We do not state this term since it has even parity, and accordingly, its contribution to the nonlinear current vanishes, as discussed after Eq. (6). Note that the currents considered are nonlinear in the electric field $\mathcal{E}$ but linear in $\mathcal{E}^T$. Nonlinear temperature effects, such as nonlinear Nernst currents [65], are thus subleading in this counting. Note also that the third-order correction δf_3 has again odd parity and will give rise to a finite current. Since this contribution is of nontopological origin [56], it will not involve an anomalously enhanced lifetime τ_{AN} . Moreover, this nontopological current is subleading in the hydrodynamic regime, where τ_{FL} is the shortest time scale, because it is proportional to τ_{FL}^3 and thus can be neglected in comparison with currents linearly proportional to the electron lifetime.

Anomalous field-induced thermoelectricity. From Eq. (6) and using the solution in Eq. (9), the nonlinear charge current is expressed as

$$j_a = \text{Re}\{j_a^0 + j_a^\omega e^{i\omega t} + j_a^{2\omega} e^{2i\omega t}\}, \quad (10)$$

where j_a^0 is the (rectified) direct current, j_a^ω corresponds to the first harmonic, and $j_a^{2\omega}$ denotes the second-harmonic generation. Neglecting the thermal gradient in Eq. (9), $\mathcal{E}^T = 0$, we recover the Drude linear response and the known nonlinear Hall effect characterized by the Berry curvature dipole D_a [40]. However, since this contribution is not sensitive to electron interactions, here, we focus on the interplay of the thermal gradient and the external electric field.

We first discuss the rectified field-induced nonlinear thermoelectric current, which arises from the interplay of anomalous velocity and the second component of δf^ω , which gives

$$j_a^0 = \chi_{abcd}^{\mathcal{E}\mathcal{E}^T} \mathcal{E}_b(\omega) \mathcal{E}_c^*(\omega) \mathcal{E}_d^T, \quad (11)$$

with a response function proportional to τ_{AN} and C_Ω :

$$\chi_{abcd}^{\mathcal{E}\mathcal{E}^T} = T \frac{e^3}{2\hbar^2} \epsilon_{acz} \epsilon_{bdz} \tau_{\text{AN}}(\omega) C_\Omega, \quad (12)$$

where the Berry curvature capacity C_Ω is defined in Eq. (2), ϵ_{abc} is the antisymmetric tensor, and we sum over repeated indices. The Berry curvature capacity also sets the magnitude of the second-harmonic field-induced thermoelectric current $j_a^{2\omega} = \chi_{abcd}^{\mathcal{E}\mathcal{E}^T} \mathcal{E}_b(\omega) \mathcal{E}_c(\omega) \mathcal{E}_d^T$. Note that there is a field-induced current that arises from the combination of a third-order deformation δf_3 (which again has odd parity) with the band velocity term. This correction, however, is not of topological origin and distinct from the response in Eq. (12) in that it is of third order in a relaxation time. To establish the experimental signatures of the field-induced thermoelectric current discussed in the introduction, we decompose the electric fields in Eq. (11) for circularly and linearly polarized light using $\mathcal{E}_b(\omega) \mathcal{E}_c^*(\omega) = \text{Re}[\mathcal{E}_b(\omega) \mathcal{E}_c^*(\omega)] + \epsilon_{bcl} \mathcal{F}_l(\omega)/2i$, which in coordinate-free form gives Eq. (4).

In addition to the rectified field-induced thermoelectric current, there is also a dynamical current:

$$j_a^\omega = \chi_{abc}^{\mathcal{E}\nabla T} \mathcal{E}_b(\omega) \mathcal{E}_c^T, \quad (13)$$

which scales linearly with both electric field and thermal gradient. The corresponding response function is

$$\chi_{abc}^{\mathcal{E}\nabla T} = -T \frac{e^2}{\hbar} \{ \epsilon_{abz} \tau_{\text{FL}} C_{v_c} - \epsilon_{cbz} \tau_{\text{AN}}(\omega) C_{v_a} \}, \quad (14)$$

where the velocity-curvature capacity C_{v_a} is defined in Eq. (3). The first contribution stems from δf^0 in combination with the anomalous velocity, and the second contribution arises from δf^ω combined with the band velocity. In coordinate-free form, we obtain Eq. (5).

We proceed to establish the anomalous temperature scaling of the nonlinear response functions: Both the Berry curvature capacity and the velocity-curvature capacity are linear functions of temperature. Formally, this is seen using a Sommerfeld expansion [66], which yields an angular integral over the Berry curvature:

$$\begin{bmatrix} C_\Omega \\ C_{v_a} \end{bmatrix} \approx -\frac{\pi^2 T}{3} \frac{\partial}{\partial \mu} \int_0^{2\pi} \frac{d\theta}{2\pi} \mathcal{D}(\mu, \theta) \begin{bmatrix} \Omega^2(\mu, \theta) \\ v_a(\mu, \theta) \Omega(\mu, \theta) \end{bmatrix}, \quad (15)$$

where $\mathbf{v}(\mu, \theta)$ is the Fermi velocity at a given angle θ , and $\mathcal{D}(\mu, \theta)$ is the angle-resolved density of states. For FL lifetimes $\tau_{\text{FL}} \sim 1/T^2$, this temperature dependence cancels to yield $\chi^{\mathcal{E}\nabla T}, \chi^{\mathcal{E}\mathcal{E}\nabla T} \sim T^0$. By contrast, an anomalous odd-parity lifetime $\tau_{\text{AN}} \sim 1/T^4$ gives a much larger magnitude that scales instead as $\chi^{\mathcal{E}\nabla T}, \chi^{\mathcal{E}\mathcal{E}\nabla T} \sim 1/T^2$. This enhancement serves as a distinct and explicit signature of odd-parity dynamics.

Material realization. In the remainder of this letter, we quantify the strength of field-induced thermoelectric effects by evaluating the parameters C_Ω and C_{v_a} in a model system for topological insulator surface states such as Bi_2Te_3 , which supports a protected Dirac fermion quasiparticle at the boundary [67]. Our choice of material is motivated by the exceptional thermoelectric properties of bismuth chalcogenides, which exhibit the highest thermoelectric figure [68–74], and the recent observation of the nonlinear Hall effect in this bismuth-based compound [75,76]. The thermoelectric properties and their interplay with electronic topology have attracted much theoretical interest [77–80]. Indeed, this system has a hexagonal Fermi surface and a Berry curvature with threefold symmetry, which does not couple to the center-of-mass current and thus has anomalously large lifetime τ_{AN} .

The surface states are described by a two-band effective Hamiltonian [81,82]:

$$\hat{\mathcal{H}} = \gamma_x p_x \hat{\sigma}_y - \gamma_y p_y \hat{\sigma}_x + \lambda (p_x^3 - 3p_x p_y^2) \hat{\sigma}_z, \quad (16)$$

where $\gamma_x = (1 - \alpha)v_F$ and $\gamma_y = (1 + \alpha)v_F$, with v a Fermi velocity parameter, $\hat{\sigma}_{x,y,z}$ are spin Pauli matrices, and we include a phenomenological parameter α to include uniaxial strain that breaks the full threefold rotational symmetry. Note that we have neglected a particle-hole asymmetry term which does not contribute to the Berry curvature. For numerical calculations, we use the Bi_2Te_3 parameters $v = 2.55 \text{ eV \AA}$ and $\lambda = 250 \text{ eV \AA}^3$ [81]. The dispersion

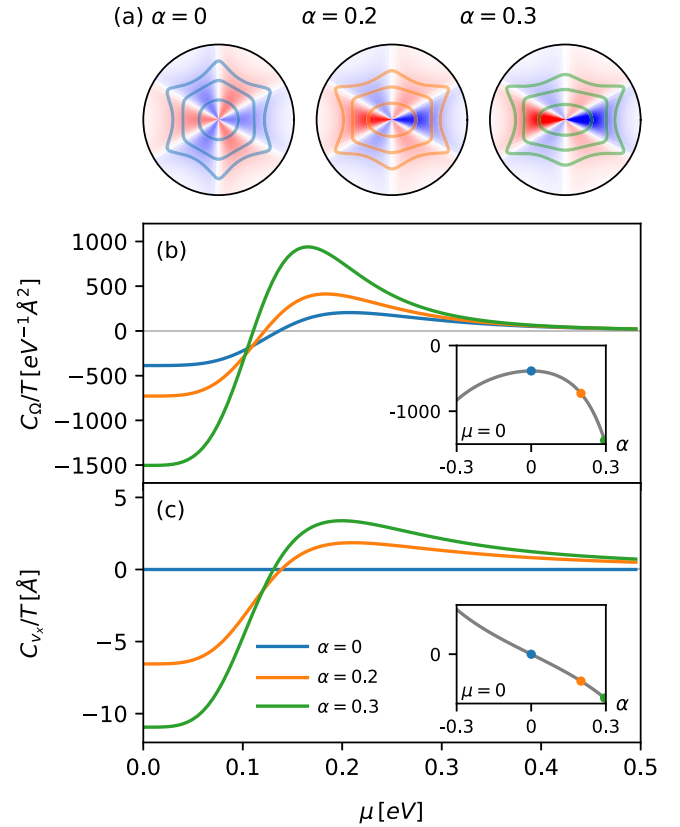


FIG. 2. (a) Density plot of the Berry curvature $\Omega(\mathbf{p})$ for three strain parameters $\alpha = 0, 0.2$, and 0.3 . Continuous lines show the Fermi surface for three chemical potentials $\mu = 0.1, 0.2$, and 0.3 eV. The Berry curvature has odd parity and increases with increasing strain. (b) Berry curvature capacity C_Ω and (c) velocity-curvature capacity component C_{v_x} as a function of chemical potential. Inset: Intrinsic values at $\mu = 0$ as a function of the strain parameters α , where points mark values in the main plot.

is $\varepsilon(\mathbf{p}) = \pm[v^2(\theta)p^2 + \lambda^2 p^6 \cos^2(3\theta)]^{1/2}$, with $v(\theta) = v[1 + \alpha^2 - 2\alpha \cos(2\theta)]^{1/2}$, which has a hexagonal Fermi surface that is stretched in the x direction for a positive strain parameter,² with a corresponding angle-resolved density of states $\mathcal{D}(\mu, \theta) = \mu \{4\pi[v^2(\theta) + 3\lambda^2 p_F^4(\mu, \theta) \cos^2(3\theta)]\}^{-1}$, with $p_F(\mu, \theta)$ being the Fermi wave vector. The Berry curvature reads $\Omega(\mathbf{p}) = \Omega(\varepsilon, \theta) = \frac{\lambda(1-\alpha^2)}{v} (\frac{v_F}{\varepsilon})^3 \cos(3\theta)$, which is shown in Fig. 2(a) as a density plot in momentum space for three values of the strain parameter $\alpha = 0, 0.2$, and 0.3 . The Berry curvature inherits the hexagonal symmetry but has odd parity, and it is enhanced by adding strain. We also include in each figure the Fermi surface contour for $\mu = 0.1, 0.2$, and 0.3 eV, with apparent hexagonal symmetry.

Figure 2(b) shows the leading temperature coefficient of the Berry curvature capacity C_Ω , Eq. (15), as a function of the chemical potential μ for three values of the strain parameter $\alpha = 0, 0.2$, and 0.3 (blue, orange, and green curves,

²Note that we consider the effect of strain on the Fermi velocity and do not include a possible shift of Dirac cone in the surface Brillouin zone.

respectively). Notably, at the Dirac point at small μ , the Berry curvature is constant and the density of states vanishes as a linear function of μ , such that C_Ω is finite with an intrinsic value $C_\Omega = -\pi\lambda^2 T/12v^4$ without strain. This is distinct from other topological transport coefficients like the Berry curvature dipole, which vanishes at the Dirac point [40]. The inset shows the intrinsic value as a function of the strain parameter, which is widely tunable. As the chemical potential increases, the Berry curvature capacity increases as well and changes sign, which is determined by an inflection point in the density of states along C_{3v} symmetry axes.

Figure 2(c) shows the x component C_{v_x} of the velocity-curvature capacity for the same parameters as in Fig. 2(b). Quite generally, the mirror symmetry of the system forces C_v to vanish for any value λ (blue curve), but the C_{v_x} component is nonzero if the threefold rotational symmetry is broken by strain, while C_{v_y} remains zero. In this regard, the symmetry requirement of the velocity-curvature capacity C_v is like that of the Berry curvature dipole $\mathbf{D}$ [83]. The intrinsic plateau of C_{v_x} shows an almost linear dependence on the strain parameter α as depicted in the inset of Fig. 2(c). At finite chemical potential, the functional dependence of C_{v_x} is again linked to the density of states and like the Berry curvature capacity. These unique characteristics make these two quantities C_Ω and C_v promising for experimental investigations of the

odd-parity relaxation dynamics in optically assisted thermoelectric spectroscopy.

To estimate the strength of the nonlinear light-induced electrothermal currents, we obtain the ratio:

$$\frac{|j^{\mathcal{E}\nabla T}|}{|j^{\nabla T}|} = \frac{\alpha}{2\pi n} \frac{\tau_{\text{AN}}}{\tau_{\text{FL}}} \frac{I}{I_0}, \quad (17)$$

which is valid near the intrinsic limit. Here, α is the fine-structure constant, n is the refraction index, and I is the intensity of incident light relative to the characteristic intensity $I_0 = v^2/\hbar\lambda^2 = 1.65 \times 10^{13} \text{ W m}^{-2}$. For typical intensities $I = 10^{11}\text{--}10^{13} \text{ W m}^{-2}$ and an enhancement $\tau_{\text{AN}}/\tau_{\text{FL}} \approx 10^0\text{--}10^3$, a thermoelectric current $j^{\nabla T} = 1 \text{ mA}$ gives a nonlinear current $j^{\mathcal{E}\nabla T} = 70 \text{ nA--}0.7 \text{ mA}$, well within the range of current experiments. To investigate these currents, the driving laser frequency should be smaller than the optical gap $\hbar\omega < 2|\mu|$ to avoid photoelectron generation and minimize light-induced heating [53–55,75,76]. In addition, an external magnetic field breaks time-reversal invariance and decreases the nonlinear field-induced current, providing a further diagnostic tool of the anomalous dynamics.

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