

Floquet prethermal order by disorder

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Frustrated magnets can have accidental ground state degeneracies which may be lifted by various forms of disorder, for example in the form of thermal or quantum fluctuations. This order by disorder (ObD) paradigm is well established in equilibrium and here is generalized to Floquet many-body systems. Investigating a periodically driven XXZ-compass model on the square lattice, we show that in a prethermal regime, dynamical fluctuations induced by high-frequency drives select a discrete set of states out of a degenerate ground state manifold of the lowest order Floquet Hamiltonian chosen as initial states. Remarkably, prior to the ObD selection, an unusual fluctuating regime emerges leading to a prethermalization timescale scaling linearly with the drive frequency. We argue that prethermal ObD with its unusual approach to the selected states is a generic phenomenon of driven frustrated systems and confirm it in the paradigmatic J_1 - J_2 XX model.

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Introduction. Frustration in many-body systems originates from the presence of competing interactions [1–4] and can lead to exotic phenomena such as classical spin liquids, e.g., spin ices [5,6], or topologically ordered ground states in quantum spin liquids [7–12]. Typically, a system subject to frustration can exhibit accidental degeneracies of the classical ground-state (g.s.) manifold, unprotected by symmetry. A central concept in the field is the order by disorder (ObD) mechanism [13–16], which entails that the classical degeneracy can be lifted by introducing fluctuations (or entropy). Although the latter normally favor featureless disordered states, via ObD they turn out to induce long-range ordering.

In equilibrium there are typically three distinct forms of ObD: (i) Originally, Villain and collaborators showed that disorder in the form of site dilution can induce long-range order in a fully frustrated Ising model [13]. Generically, quenched disorders can induce a discrete set of minima in an otherwise degenerate manifold of the free energy [17]. (ii) Disorder as thermal fluctuations can lead to an entropic selection of specific states from the degenerate manifold, eventually restoring long-range order at finite temperatures [18–20]. (iii) Zero point quantum fluctuations can similarly promote ordered states even at zero temperature [18,21]. Up to now, most studies of ObD have been confined to the realm of equilibrium physics. Notable exceptions are works on tuning the free energy landscape via time-dependent modulations [22,23] and, very recently, the generalization of ObD to dynamical systems with nonreciprocal interactions [24], both

of which incorporate stochastic noise for stabilizing ObD-selected steady states.

In this paper, we address the question of whether ObD can be generalized to driven systems without noise and whether qualitatively new features can arise with time dependence. We provide affirmative answers by focusing on Floquet frustrated magnets. Floquet engineering under periodic driving has recently become a versatile tool for realizing novel phases of matter beyond thermal equilibrium [25–28]. Even in the absence of external noise, driven systems can settle—after a short typically frequency independent time scale τ_{pth} —into quasisteady, so-called prethermal, states. These appear for large drive frequency ω and can stabilize quasiequilibrium and nontrivial dynamical phases up to exponentially long (in ω) thermalization timescales, after which the system heats up to a featureless infinite temperature state [29–38]. While this has been first worked out for quantum many-body systems [31,33], similar effects appear in classical systems where they can be efficiently simulated [39–42]. It allows us to show that the fluctuations induced by a periodic drive can select a long-range ordered prethermal phase, a phenomenon we dub Floquet prethermal ObD. Remarkably, the transition in time to the prethermal plateau is itself highly unusual. Namely, starting from states close to the degenerate g.s. manifold (of the Floquet Hamiltonian in high-frequency limit), a long fluctuating regime appears scaling linear in frequency, $\tau_{\text{pth}} \propto \omega$. The characteristic scaling appears because of a hierarchical symmetry reduction of the effective Floquet Hamiltonian as recently introduced in Ref. [43]. Intuitively, the systems need sufficient time for the drive-induced fluctuations to explore the whole phase space to generate the entropy for prethermal ObD selection.

To illustrate this phenomenon, we consider a classical frustrated compass spin model on the square lattice which displays an accidentally $O(2)$ degenerate g.s. manifold. It

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was recently shown that in equilibrium, four discrete states are selected via thermal ObD [44]. We here generalize the model to a periodically driven version. We show that a characteristic dependence of the prethermal mean energy and thermalization time appears for initial states chosen from the original $O(2)$ manifold. Independent of the initial state, each trajectory settles into four discrete states and the selection mechanism is always accompanied by a long fluctuating regime. To explain the latter, we develop an analytical theory of harmonic fluctuations around the mean-field transient dynamics. Frequency-dependent corrections to the effective Floquet Hamiltonian lead to an increase of fluctuations in time, thereby accounting for the unusual prethermalization timescale $\tau_{\text{pth}} \propto \omega$.

Finally, we argue that Floquet prethermal ObD is a generic phenomenon originating from the interplay of accidental degeneracies and periodic driving. We confirm that it also appears in the paradigmatic J_1 - J_2 model on the square lattice and discuss its experimental relevance.

Model. We study a trinary Floquet Hamiltonian $H(t)$ on the square lattice, subjected to a periodic modulation at frequency $\omega = 2\pi/T$ (with period T),

$$H(t) = \begin{cases} H_x \equiv -\sum_{r,\delta} J_\delta^x S_r^x S_{r+\delta}^x & \text{for } t \in [0, T/3) \\ H_y \equiv -\sum_{r,\delta} J_\delta^y S_r^y S_{r+\delta}^y & \text{for } t \in [T/3, 2T/3) \\ H_z \equiv -\sum_{r,\delta} J_\delta^z S_r^z S_{r+\delta}^z & \text{for } t \in [2T/3, T), \end{cases} \quad (1)$$

where r is the lattice site and $\delta = \hat{x}, \hat{y}$ denote the lattice vector along the x and y directions, respectively. Specifically, the bond-oriented XXZ-compass interaction J_δ^a is graphically illustrated as [45]

$$J_\delta^x = \begin{array}{|c|c|} \hline \text{---} & \text{---} \\ \hline \end{array}, \quad J_\delta^y = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}, \quad \text{for } \cdots \equiv J + K, \quad \text{---} \equiv J.$$

Here, we use $J = 1$ as the energy unit and restrict to $K \geq 0$ and $0 \leq J^z < 1$. The spin dynamics is described by the standard Hamilton equations of motion $\partial \tilde{S}_r / \partial t = \{\tilde{S}_r, H(t)\}$. It leads to a stroboscopic evolution function which can be analytically integrated [45,46]. Know that the stroboscopic time dependence is chosen for numerical convenience but a continuous drive would lead to similar results.

Given the stroboscopic evolution function, the effective prethermal Hamiltonian H_{eff} is defined as $e^{-T(H_{\text{eff}}, \cdot)} \equiv e^{-\frac{T}{3}(H_z, \cdot)} e^{-\frac{T}{3}(H_y, \cdot)} e^{-\frac{T}{3}(H_x, \cdot)}$. In the high-frequency limit, we can expand the above equation and approximate the effective Hamiltonian up to the first order as [36] $H_{\text{eff}} = H_{\text{eff}}^{(0)} + H_{\text{eff}}^{(1)} + O(\omega^{-2})$. The zeroth order, $H_{\text{eff}}^{(0)}$, is just the XXZ-compass model studied in Ref. [44]

$$H_{\text{eff}}^{(0)} = (H_x + H_y + H_z)/3. \quad (2)$$

The subleading order, which comes from the Poisson brackets $H_{ab} \equiv \{H_a, H_b\}$, can be expressed as

$$H_{\text{eff}}^{(1)} = -\frac{\pi}{9\omega} (H_{zx} + H_{zy} + H_{yx}), \quad (3)$$

where $H_{ab} = \sum_{r,\delta=\pm\hat{x},\pm\hat{y}} \sum_{\delta'\neq\delta} J_\delta^a J_{\delta'}^b \epsilon^{abc} S_{r+\delta}^a S_{r+\delta'}^b S_r^c + \sum_{r,\delta=\pm\hat{x},\pm\hat{y}} J_\delta^a J_\delta^b \epsilon^{abc} (S_r^a S_{r+\delta}^b S_{r+\delta}^c + S_{r+\delta}^a S_{r+\delta}^b S_r^c)$. In the absence of the compass anisotropy, $K = 0$, $H_{\text{eff}}^{(0)}$ in Eq. (2) exhibits in-plane $O(2)$ symmetry about the z axis. The g.s. is given

by ferromagnetic (FM) aligned spins within the XY plane, characterized by an azimuthal angle $\phi \in [0, 2\pi)$. For $K > 0$, the $O(2)$ symmetry of $H_{\text{eff}}^{(0)}$ is reduced into a discrete Z_4 one. However, the g.s. energy remains invariant under arbitrary global in-plane spin rotations, manifesting an accidentally $O(2)$ -degenerate g.s. manifold, see Fig. 1(a). In equilibrium, fluctuations of thermal ObD lift the accidental degeneracy and the free energy has four minima at half-integer π [44,45]. In the driven case, we observe that $H_{\text{eff}}^{(1)}$ reduces the Z_4 symmetry to Z_2 , which is an example of the hierarchical symmetry reduction of Floquet systems [43].

We first explore the dynamical model in Eq. (1) via numerical simulations on a finite system with $N = L^2$ spins. Since $H_{\text{eff}}^{(0)}$ exhibits thermal ObD, it is natural to explore whether the corresponding nonequilibrium dynamics can establish a similar state selection mechanism via prethermalization. Therefore, we initialize the simulations with the FM ordered g.s. of $H_{\text{eff}}^{(0)}$, characterized by a spin orientation $(\cos \phi_{\text{in}}, \sin \phi_{\text{in}}, 0)$, where ϕ_{in} denotes the initial azimuthal angle, see Fig. 1(a). To bring the many-body character of the system into play, we add perturbations on top of this FM order such that $\phi_r = \phi_{\text{in}} + \delta\phi_r$, with $\delta\phi_r$'s a random Gaussian noise with a standard deviation of $2\pi G$. Nonzero G increases the energy density of initial states and tunes the effective temperature of the expected prethermal plateau.

Prethermalization timescale. Generic periodically driven many-body systems with large drive frequencies are governed by two timescales. The first one, τ_{pth} , captures the initial equilibration dynamics after which the system enters the prethermal plateau. Normally, as the short-time dynamics is primarily governed by $H_{\text{eff}}^{(0)}$, the prethermalization time is independent of frequency and governed by the local energy scale, $\tau_{\text{pth}} \propto 1/J$. However, as a key finding, we report that in our setup, τ_{pth} has an unusual dependence on ω as $\tau_{\text{pth}} \propto \omega$, see inset of Fig. 1(b). To probe τ_{pth} , we introduce the decorrelator $d(t)$ quantifying the distance between two initially close replicas of the system [39,41,45,47,48], $\tilde{S}_r(r)$ and $\tilde{S}'_r(t)$, as $d(t) = \sqrt{\frac{1}{N} \sum_r [\tilde{S}_r(t) - \tilde{S}'_r(t)]^2}$. During the initial fluctuating dynamics the spins move together and the value of the decorrelator remains small. Eventually, it grows exponentially from its tiny value around $d(t=0)$, and this onset of many-body chaos means that the system explores the many-body phase space (the distribution of initially almost aligned spins spreads). Note, at infinite temperature characterized by only trivial correlations, the typical value of decorrelator is $d_\infty = \sqrt{2}$ [39]. The prethermalization timescale τ_{pth} is quantitatively determined by the time at which the decorrelator crosses 10% of its infinite temperature from a tiny initial value.

Intuitively, τ_{pth} quantifies the time when the dynamics of systems deviates from its motion as a single large spin, that is, from the mean-field solution $\tilde{S}^a(t)$ given by the generalized Landau-Lifshitz-Gilbert (LLG) equation

$$\partial_t \tilde{S}^a(t) = \{\tilde{S}^a(t), H_{\text{eff}}^{(0)}\} + \{\tilde{S}^a(t), H_{\text{eff}}^{(1)}\} + O(\omega^{-2}). \quad (4)$$

We allow fluctuations as perturbations to the mean-field solution such as $\tilde{S}^a \rightarrow M^{-1}(\tilde{S}^a + \delta\tilde{S}^a)$ with M the normalization factor. We would like to examine the dynamics of fluctuations to gain insights into τ_{pth} since $d \sim \delta\tilde{S}$ in the simplest approximation. We introduce $\tilde{\phi}(t)$ and $\tilde{\theta}(t)$ such

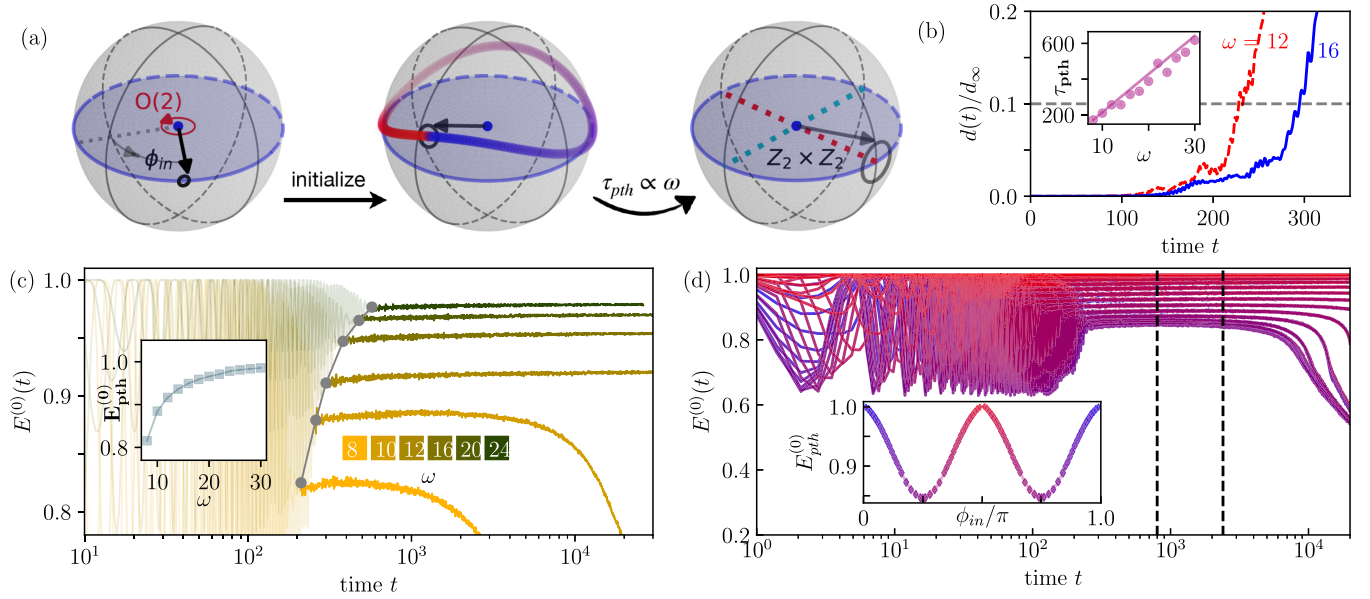


FIG. 1. Prethermal order by disorder. (a) The dynamics of an individual O(3) spin (arrow), and the circles positioned atop the arrows strand for the averaged fluctuations. Left: The system has an accidentally O(2)-degenerate g.s. manifold about the z axis, w.r.t. the zeroth-order effective Hamiltonian H_{eff}^0 in Eq. (2). The initial state manifests as an in-plane FM-ordered state with an azimuthal angle ϕ_{in} . Middle: Before the prethermal phase, all spins evolve as a single spin, tracing a trajectory represented by colorful dots (blue to red over time). The fluctuations (circle) increase as $e^{t/\tau_{\text{pth}}}$. Right: In the prethermal phase, fluctuations are relatively strong and almost stop increasing, leading to a FM-ordered prethermal state selected by fluctuations. (b) The decorrelation initially grows exponentially. The cross of the dashed line at 0.1 and $d(t)$ determines the prethermalization timescale τ_{pth} , which is proportional to drive frequency ω , see inset. (c) Dynamics of the zeroth order energy $E^{(0)}(t)$. The transparent (solid) regime represents the timescale before (after) τ_{pth} . Inset: The zeroth order prethermal energy $E_{\text{pth}}^{(0)}$ as a function of drive frequency ω . (d) $E^{(0)}(t)$ for various initial azimuthal angles ϕ_{in} , varying from $\phi_{\text{in}} = 0$ to $\phi_{\text{in}} = \pi/2$ (blue to red). Note that the blue-side line ($\phi_{\text{in}}^b \in [0, \pi/4]$) and corresponding red-side line $\phi_{\text{in}}^r = \pi/2 - \phi_{\text{in}}^b$ converge into a single line in the prethermal regime. Inset: The zeroth-order prethermal energy $E_{\text{pth}}^{(0)}$ as a function of ϕ_{in} . $E_{\text{pth}}^{(0)}$ is calculated as an average over the time interval of $[10^3, 3 \times 10^3]$, corresponding to the time window delimited by two vertical dashed lines. The size of the marker is magnified by a factor of 10^3 times the standard deviation. Here we used $L = 72$, $K = 1$, $J^z = 0$, and $G = 10^{-3}$ [$\phi_{\text{in}} = 0.2\pi$ for (b) and (c), and $\omega = 9$ for (d)].

that $(\bar{S}^x, \bar{S}^y, \bar{S}^z) = (\cos \bar{\phi} \sin \bar{\theta}, \sin \bar{\phi} \sin \bar{\theta}, \cos \bar{\theta})$. The fluctuations are now denoted as $\bar{\phi}(t) \rightarrow \bar{\phi}(t) + \delta\bar{\phi}(t)$ and $\bar{\theta}(t) \rightarrow \bar{\theta}(t) + \delta\bar{\theta}(t)$. Expanding Eq. (4) to leading order of $\delta\bar{\phi}$ and $\delta\bar{\theta}$, we obtain [45]

$$\begin{pmatrix} \partial_t & \\ & \partial_t \end{pmatrix} \begin{pmatrix} \delta\bar{\phi} \\ \delta\bar{\theta} \end{pmatrix} = \frac{\bar{\mathcal{E}}(t)}{\omega} \begin{pmatrix} \delta\bar{\phi} \\ \delta\bar{\theta} \end{pmatrix}, \quad (5)$$

where $\bar{\mathcal{E}}$ is a ω -independent 2×2 matrix given by the mean-field solution of $\bar{S}^a(t)$. Therefore, the fluctuations grow exponentially as $\sim e^{t\varepsilon/\omega}$ with ε the eigenvalues of $\bar{\mathcal{E}}$. Since the increase of fluctuations leads to the onset of the prethermal phase, it follows straightforwardly that $\tau_{\text{pth}} \propto \omega$. The argument should hold for generic frustrated Floquet systems initialized within the g.s. manifold of its zeroth-order Floquet Hamiltonian, and does not depend on the form of the introduced noise. It is attributed to the fact that, for these initial states, the zeroth order term in the equation of motion is negligible. Instead, the LLG equation is governed by the first-order effective Hamiltonian which is linear in ω^{-1} . Consequently, the significance of $H_{\text{eff}}^{(1)}$ is to induce the dynamical fluctuations, resulting in a prethermalization timescale proportional to the drive frequency.

Prethermal ObD. Subsequent to τ_{pth} , the system enters a prethermal regime over a thermalization timescale τ_{th} before eventually heating up to infinite temperature. In accordance

with the prethermalization paradigm, the nonequilibrium system can be described by a thermal Gibbs ensemble at a certain temperature β_{pth} [32,34,49]. As our second key result, we find that the prethermal ObD selected phase exhibits a strong dependence on the initial azimuthal angle ϕ_{in} .

To characterize the thermal phase and to track the energy absorption, we investigate the evolution of normalized energy $E^{(0)}(t) = H_{\text{eff}}^{(0)}(t)/E_0$, where $E_0 = -(2J + K)N/3$ is the g.s. energy of H_{eff}^0 . As demonstrated in Fig. 1(c), in the prethermal phase, fluctuations are small, and $E^{(0)}(t)$ exhibits a plateau value, persisting for an exponentially long time as $\tau_{\text{th}} \sim e^{c\omega}$ [35,50]. Fixing ω , the manifestation of prethermal ObD can be directly elucidated by the significant dependence of thermalization timescales τ_{th} on initial azimuthal angles ϕ_{in} . As ϕ_{in} approaches values around $m\pi/2$ ($m = 0, 1, \dots$, corresponding to $\hat{x}$ and $\hat{y}$ directions), the associated timescales τ_{th} are much longer compared to those with ϕ_{in} 's being close to $\pm\pi/4$, see Fig. 1(d). For instance, for $\phi_{\text{in}} = \pi/10$, $E^{(0)}(t)$ remains around its plateau value up to $\tau_{\text{th}} \sim 10^6$. In contrast, for $\phi_{\text{in}} = \pi/4$, it already deviates from the plateau toward its infinite-temperature value around $\tau_{\text{th}} \sim 10^4$.

The prethermal ObD also appears in the relationship between prethermal energy $E_{\text{pth}}^{(0)}$ and initial azimuthal angles ϕ_{in} , where $E_{\text{pth}}^{(0)}$ is the average of $E^{(0)}(t)$ over a time window in the prethermal regime; see inset of Fig. 1(d). We find that

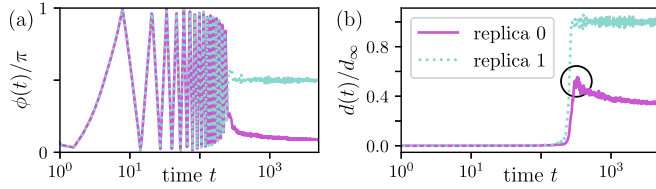


FIG. 2. State selection. The dynamics of (a) averaged azimuthal angles $\phi(t)$ and (b) decorrelators $d(t)$ for two different replicas. Here we used $L = 72, K = 1, J^z = 0, G = 10^{-3}, \phi_{\text{in}} = 0.2\pi$, and $\omega = 12$.

$E_{\text{pth}}^{(0)}$ exhibits a clear $\pi/2$ periodicity on ϕ_{in} . Note that $E_{\text{pth}}^{(0)}$ is normalized by a negative constant E_0 . Therefore, its maxima (minima) at $\phi_{\text{in}} = 0, \pi$ ($\phi_{\text{in}} = \pi/4$ modulo $\pi/2$) signify the lowest (highest) points in the landscape of the real energy. Indeed, the behavior of $E_{\text{pth}}^{(0)}$ clearly resembles the free energy of $H_{\text{eff}}^{(0)}$ versus ϕ_{in} under the thermal ObD mechanism [18,45]. We emphasize that this prethermal ObD effect vanishes for $K = 0$.

It is well known that the thermal ObD appears only for finite but small temperatures. A similar effect can also be observed in prethermal ObD, where the inverse drive frequency $1/\omega$ can be thought of as tuning the effective temperature. When $1/\omega$ is zero (zero temperature), the system stops absorbing energy and cannot prethermalize. When $1/\omega$ is sufficiently large (high temperature), the system absorbs energy too quickly to sustain a prethermal regime. Moreover, the oscillation amplitude of $E_{\text{pth}}^{(0)}$ w.r.t. ϕ_{in} decreases with increasing ω , by noting that $E_{\text{pth}}^{(0)}$ at $\phi_{\text{in}} = 0.2\pi$ increases as ω increases and that at $\phi_{\text{in}} = 0$ slightly changes with ω (not shown); see the inset of Fig. 1(c).

One of the crucial aspects of thermal ObD is that it is the entropy that selects the ordered states. Here, in the absence of entropy for individual trajectories, we examine the averaged azimuthal angle, $\phi(t) = N^{-1} \sum_r \cos^{-1}(S_r^x(t)/\sqrt{1 - [S_r^z(t)]^2})$. For different initial angles ϕ_{in} , we find that $\phi(t)$, after τ_{pth} , reaches several prethermal azimuthal angles such as $\phi(t) \approx 0$ (modulo $\pi/2$); see Figs. 2(a) and 3(a). This is consistent with the idea of prethermal ObD, e.g., the prethermal angle is selected by the fluctuations generated by the periodic drive. This is corroborated by the dynamics of the decorrelator $d(t)$ in Fig. 2(b). After its exponential increase to a relatively large value at τ_{pth} , $d(t)$ either (i) drops to a relatively smaller plateau value, or (ii) further grows up and stays at a larger plateau value of $d \approx 1$. Note that there are four degenerate states selected by the ObD mechanism. Then, two initial states can end up in the same prethermal state (for instance, both are along the $\hat{x}$ direction) leading to a reduction in the value of $d(t)$. On the other hand, they can occupy two different states (for instance, one along the $\hat{x}$ direction and the other along the $\hat{y}$ direction). Then it can cause a larger plateau value of $d \approx \sqrt{1^2 + 1^2}/d_\infty \approx 1$.

According to the Z_4 symmetry of $H_{\text{eff}}^{(0)}$, one would have expected that the prethermal state with $\phi(t) \approx 0$ (along the $\hat{x}$ direction) and that with $\phi(t) \approx \pi/2$ (along the $\hat{y}$ direction) are exactly degenerate. However, one can find that the prethermal angles $\phi(t)$ deviate slightly from these values; see Fig. 1(a).

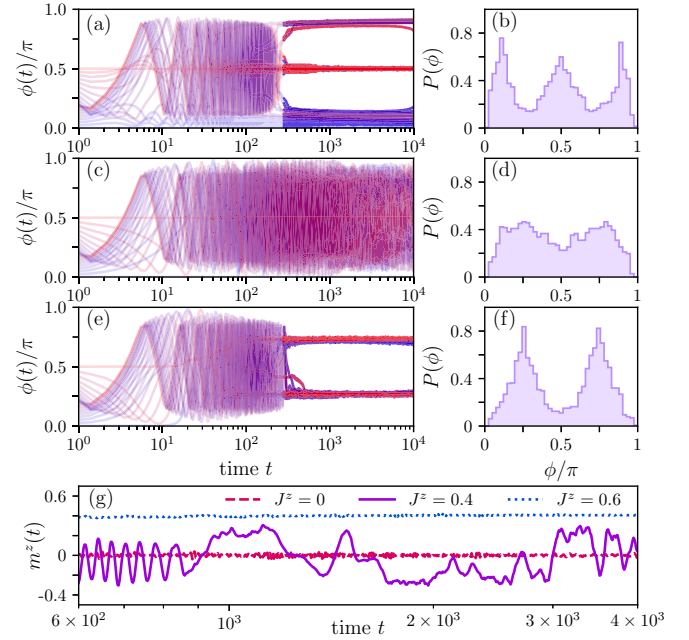


FIG. 3. Thermal and prethermal phase transition by J^z in $H_{\text{eff}}^{(0)} + H_{\text{eff}}^{(1)}$. (a), (c), (e) Stroboscopic dynamics of averaged azimuthal angles $\phi(t)$ versus initial azimuthal angles ϕ_{in} varying from $\phi_{\text{in}} = 0$ to $\phi_{\text{in}} = \pi/4$ (blue to red) for (a) $J^z = 0$, (c) $J^z = 0.4$, and (e) $J^z = 0.8$. The solid regimes denote the prethermal phases. Here we have used $L = 72, K = 1, G = 10^{-3}$, and $\omega = 9$. (b), (d), (f) Probability distribution $P(\phi)$ for the azimuthal angle obtained by using Monte Carlo simulations with $L = 32, K = 1, T = 0.08, \beta = 5.0$, and (b) $J^z = 0$, (d) $J^z = 0.4$, as well as (f) $J^z = 0.8$. (g) The dynamics of out-of-plane magnetization $m^z(t)$ for various J^z . The rest model parameters are the same as those in (a), (c), and (e).

This can be resolved by taking into account the subleading effective Hamiltonian $H_{\text{eff}}^{(1)}$ (3) breaking Z_4 into Z_2 . We carry out Monte Carlo (MC) simulations for the static Hamiltonian $H_{\text{eff}}^{(0)} + H_{\text{eff}}^{(1)}$ to sample the thermal ensemble, by which we construct a probability distribution for the azimuthal angle $P(\phi)$. As shown in Fig. 3(b), the maxima in $P(\phi)$ are consistent with the plateau values in the prethermal regime shown in Fig. 3(a). Note that the peak at $\phi = \pi/2$ is broader than the other two peaks, explaining the fact that the fluctuations of $\phi(t)$ around the plateau value of $\pi/2$ are a little bit stronger than those around the other two plateau values. Finally, for comparison, we also perform MC simulations for $H_{\text{eff}}^{(0)}$ only, leading to a probability distribution $P(\phi)$ preserving the Z_4 symmetry.

Dynamical phase diagram. The first-order effective Hamiltonian $H_{\text{eff}}^{(1)}$ favors noncoplanar order but its contribution to the prethermal energy is small when $J^z = 0$ and K is moderately large. As observed above, the dynamics is still governed by prethermal ObD, in which an in-plane FM order is selected. However, increasing J^z leads to a nonvanishing H_{zx} and H_{zy} in Eq. (3), competing with the lowest order $H_{\text{eff}}^{(0)}$.

As a function of increasing J_z , we can identify two phases, with a phase transition point at $J^{z,*} \approx 0.4$, by monitoring the dynamics of the out-of-plane magnetizations, $m^z(t) = \frac{1}{N} \sum_r S_r^z(t)$. (i) When $J^z < J^{z,*}$ the system clearly manifests prethermal ObD phenomenon, e.g., the prethermal

energetics are governed by $H_{\text{eff}}^{(0)}$ and the main effect of $H_{\text{eff}}^{(1)}$ is to introduce fluctuations as discussed above. In this prethermal ObD phase, $m^z(t)$ vanishes, see Fig. 3(g). (ii) Around the transition point $J^z \approx J^{z,*}$, the system is unable to prethermalize, e.g., the timescale of $\tau_{\text{pth}} > 10^4$ is extremely long; see Figs. 3(c) and 3(g). Neither the azimuthal angles $\phi(t)$ nor the out-of-plane $m^z(t)$ can manifest a well-defined plateau value; instead, they keep displaying strong fluctuations. (iii) For larger $J^z > J^{z,*}$, the [111] phase is stabilized, in which a finite out-of-plane magnetization emerges with a plateau value of $m^z(t) \approx \pm 0.5$. Meanwhile, the azimuthal angle $\phi(t)$ settles to a plateau value shifted toward $\approx \pi/4$ (modulo $\pi/2$); see Fig. 3(e). Remarkably, this phase transition indicates that the prethermal ObD does not necessarily follow from the thermal ObD, as the latter remains valid for $J^{z,*} < J^z < 1$.

These nonequilibrium dynamics can be understood better with the help of MC simulations for $H_{\text{eff}}^{(0)} + H_{\text{eff}}^{(1)}$. For the transition point, the probability distribution for the azimuthal angle ϕ (and also the out-of-plane magnetization m^z , not shown) has no obvious peaks and valleys, as shown in Fig. 3(d). Therefore, the transition point is akin to a critical point with huge thermal entropy, which does not support any orders. This entropy, manifesting in the nonequilibrium dynamics, prevents the system from rapid prethermalization. On the other hand, in the [111] phase, we find that the probability distribution $P(\phi)$ exhibits two peaks at $\phi = \pi/4$ and $\phi = 3\pi/4$, respectively; see Fig. 3(f). It is consistent with the prethermal FM order observed in Fig. 3(e), along the [111] direction or its Z_2 -symmetric counterparts. We note that the maxima of the prethermal energy $E_{\text{pth}}^{(0)}$ are still around $\phi = 0$ and $\phi = \pi/2$ [45], which are not the typical prethermal angle of the [111] order. It implies that the prethermal [111] order is not selected by the ObD mechanism of $H_{\text{eff}}^{(0)}$, but selected by $H_{\text{eff}}^{(0)} + H_{\text{eff}}^{(1)}$.

Discussion and conclusion. In addition to known types of “disorder” in equilibrium ObD—e.g., quenched lattice disorder, thermal, and quantum fluctuations—we showed that fluctuations induced by a periodic drive can select a discrete set of states starting with initial conditions within the accidental degenerate g.s. manifold of the lowest order Floquet Hamiltonian. Focusing on the Floquet XXZ-compass model, we reveal the prethermal ObD through the analysis of the prethermal energy and thermalization time dependence on the initial state direction. We argue that the concept of prethermal ObD is generic and not sensitive to specific driving protocols. Indeed, we have confirmed similar results for a driven J_1 - J_2 antiferromagnetic XX model [45].

An unexpected finding is that the prethermal ObD can only be observed after an unusual prethermalization timescale linearly in the drive frequency ω . The reason is that the family of initial states lie within the accidental degenerate g.s. manifold of $H_{\text{eff}}^{(0)}$. It is then the subleading ω -dependent correction that introduces the fluctuations necessary for the entropic ObD selection in the time window that would have been governed by $H_{\text{eff}}^{(0)}$ in standard prethermalization. As a side effect, we also observe that for large J^z these corrections can trigger a prethermal transition to a different state with out-of-plane magnetization.

For future work, it will be very worthwhile to study other examples of prethermal ObD, e.g., those with an extensive number of accidental symmetries of the g.s. manifold like the antiferromagnetic Heisenberg model on the kagome lattice. In addition, our set-up is a natural platform for investigating the (accidental) symmetry-breaking hierarchy which can be engineered in Floquet systems [43]. Another intriguing possibility is that different choices of the periodic drive can activate different types of fluctuations which could lead to distinct orders in the prethermal plateau. It will also be interesting to establish the general relation between thermal, quantum, and prethermal ObDs. Since entropy plays a pivotal role in achieving prethermal ObD, it will be helpful to develop numerical tools and concepts to quantify entropy production in driven systems. Similarly, it would be worthwhile to think of different experimental platforms for its realization.

In conclusion, we expect that prethermal ObD will provide a versatile playground for intriguing nonequilibrium physics and adds another aspect to one of the paradigms of frustrated systems.

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